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## Occupation times for single-file diffusion

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# Occupation times for single-file diffusion

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**Abstract.** We consider a file of identical Brownian particles moving on the same axis  $x'Ox$  without crossing each other. They all start from the origin  $O$  at time  $t = 0$  and are stopped at some time  $t$ . Denoting by  $T$  the time spent on the half-line  $[Ox)$  by a given particle of the line, we establish analytical formulae for the first two moments  $\langle T \rangle$  and  $\langle T^2 \rangle$ . In particular, considering the limit of an infinite number of particles, we get, for the ‘middle’ particle

$$\left\langle \left( \frac{T}{t} \right)^2 \right\rangle = \frac{3}{8} + \frac{1}{8} J_0(\pi) = 0.336\,96\dots$$

( $J_0$  is a Bessel function). This result (and also numerical simulations) shows that the distribution of  $T$ , though being close to it, is not fully a constant one.

**Keywords:** stochastic processes (theory), Brownian motion, diffusion

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**1. Introduction**

Single file diffusion is defined as the diffusion of a tracer particle in a one dimensional system of hard-core Brownian particles. This type of model is used in many physical, chemical, and biological processes to analyze the dynamics of particles in crowded quasi-dimensional systems. Experimentally, it has been evidenced by passive microrheology in zeolites, transport of confined colloidal particles, or charged spheres in circular channels [1–4]. Due to the hard-core interactions, the order of particles is unchanged with time, which results when infinitely many hard-core particles are present in a subdiffusive behavior of the mean-square displacement of the tracer particle  $\langle X_t^2 \rangle \propto \sqrt{t}$ . This result, first discovered analytically by Harris [5], has been extended in several directions, both for discrete and continuous variants of the model. Examples include the determination of the full propagator for the tracer particle [6], the treatment of the case where the motion of the bath particles is subdiffusive [7], the case where the tracer particle performs a biased diffusion [8] or the determination of first-passage properties [9]. In contrast, the analysis of occupation times for single file diffusion—the cumulative time spent by the tracer particle on the positive half-line up to a given observation time—does not seem to have been considered so far.

Let us first recall that the study of occupation times for an *isolated* Brownian motion (BM) traces back to the famous Levy’s arc-sine law [10, 11]. It states that, for a

one-dimensional Brownian motion ( $\text{BM}(\mathbb{R})$ ) starting at the origin at  $t = 0$  and stopped at  $t$ , the time  $T$  spent when  $x > 0$  satisfies the probability law ( $0 < t_1 < t$ ):

$$P(T < t_1) = \frac{2}{\pi} \arcsin \sqrt{\frac{t_1}{t}} \quad (1)$$

with the symmetric U-shape density:

$$\mathcal{P}(T) = \frac{1}{\pi} \frac{1}{\sqrt{T(t-T)}} \quad (2)$$

Barlow, Pitman and Yor [12] have extended Levy's result to a BM on a set of  $n$  semi-infinite lines originating from  $O$  (a kind of special graph).  $\{T_i\}$  being the set of occupation times on all the lines, the authors computed explicitly the Laplace transform of the  $\{T_i\}$  distribution. This study has itself been extended more recently [13] to a BM on a quite general graph. A step further was done in [14] where the exit and occupation times were studied, still on graphs, for a BM with general drift and boundary conditions.

On another hand, theoretical physicists [15] and also mathematicians [16] were interested in the quadrant occupation time  $T$  for the planar BM. In particular, the first four moments of this occupation time are computed in [15], with the result<sup>3</sup>:

$$\left\langle \left( \frac{T}{t} \right) \right\rangle = \frac{1}{4} \quad (3)$$

$$\left\langle \left( \frac{T}{t} \right)^2 \right\rangle = \frac{5}{32} - \frac{1}{8\pi^2} = 0.14358\dots \approx \frac{1}{7} \quad (4)$$

$$\left\langle \left( \frac{T}{t} \right)^3 \right\rangle = \frac{63}{512} - \frac{7}{32\pi^2} = 0.10088\dots \approx \frac{1}{10} \quad (5)$$

$$\left\langle \left( \frac{T}{t} \right)^4 \right\rangle = \frac{907}{8192} - \frac{1}{64\pi} - \frac{631}{2304\pi^2} - \frac{7}{384\pi^4} = 0.0778\dots \approx \frac{1}{13} \quad (6)$$

Despite the importance of the problem, the determination of occupation times for *single file diffusion* does not seem to have been investigated so far. Here we consider hard-core one-dimensional Brownian particles that all start from the origin  $O$  at time  $t = 0$  and are stopped at some time  $t$ . We will be interested in the time  $T$  spent on the half-line  $[Ox)$  by a given particle of the line, both for a finite and infinite number of Brownian particles. Note that, while the evolution of the full system is Markovian, the dynamics of the only tracer is not. These memory effects make the determination of the occupation time  $T$  a non-trivial problem. Here, we focus on the two first moments  $\langle T \rangle$  and  $\langle T^2 \rangle$ .

The outline of the paper is more precisely as follows. We first define the dynamics of the system and show that the determination of the occupation time of a given particle of the line can be mapped onto a single particle problem. We next consider the particular case when the tracer particle is the 'middle' particle of the line. Analytical formulae for the two first moments of the occupation time are derived for an arbitrary (odd) number of particles on the line. The limit of an infinite number of particles is then analyzed in detail. Finally, these results are extended to the case of a tracer of arbitrary position in the line.

<sup>3</sup> Equation (2) is the beta law  $B_{1/2,1/2}$ . It is worth noticing that, despite the complication of the results equations (3)–(6), those moments are very close to the ones of the beta law  $B_{1,1/3}$  that are, respectively,  $\frac{1}{4}$ ,  $\frac{1}{7}$ ,  $\frac{1}{10}$ ,  $\frac{1}{13}$ . We will come back to those results in section 4.

## 2. Basic setup

We adopt here a continuous space description of the system and consider a file of  $n = 2q + 1$  identical Brownian particles  $p_l$  moving on a line  $x'Ox$ . Those particles are labelled by their rank  $l$  ( $= 1, 2, \dots, 2q + 1$ ) and cannot cross each other.  $l = 1$  labels the rightmost particle,  $l = 2q + 1$  the leftmost particle and  $l = q + 1$  the ‘middle’ particle.

Two adjacent particles can experience elastic collisions. During such collisions, they simply exchange their velocities because they are identical. Alternatively, we can consider that the particles are free to cross each other, only exchanging their ranks at each collision. Pushing this picture further, we are left with a unique Brownian particle  $P$ , freely moving in  $\mathbb{R}^n$ , whose coordinates  $x_i$  represent the positions of the previous particles.

For the sake of simplicity, we will consider that  $P$  starts its motion at the origin at  $t = 0$  and stops somewhere at time  $t$ . This implies, in particular, that, at  $t = 0$ , all the particles  $p_l$  are at the same point  $O$ . This could seem unrealistic because the particles are mutually impenetrable. Nevertheless we think this is not the case. Indeed, this is equivalent to supposing that at  $t = 0$  the particles are well separated (but chosen arbitrarily close to their neighbours because they are all point-like) and that we stop the motion at sufficiently large  $t$ .

We are interested in the time  $T$  spent by the particle of rank  $l$  on the half line  $x > 0$ .  $T$  is equal to the time spent by  $P$  in all the orthants of  $\mathbb{R}^n$  with, **at least**,  $l$  positive coordinates. From general scaling properties of the Brownian motion starting from  $O$ , we deduce that  $T$  scales like  $t$  and define<sup>4</sup> the random variable  $u = \frac{T}{t}$ . In the following, we will be especially interested in the computation of the first two moments of  $u$ .

## 3. Occupation time of the middle particle (rank $l = q + 1$ ) of a file of $n = 2q + 1$ particles

### 3.1. The three particle case ( $q = 1$ )

Let us first consider the particular case of three particles and show how  $\langle T^2 \rangle$  can be computed for the rank 2 particle.

Following the previous section, we consider a particle  $P$  freely wandering in  $\mathbb{R}^3$ . Among the  $2^3$  orthants in  $\mathbb{R}^3$ , only four of them will be relevant for the computation of the moments<sup>5</sup>  $\langle T^k \rangle$ :  $(x_1, x_2, x_3 > 0) \equiv \mathcal{U}_1$ ,  $(x_1, x_2 > 0, x_3 < 0) \equiv \mathcal{U}_2$ ,  $(x_1 > 0, x_2 < 0, x_3 > 0) \equiv \mathcal{U}_3$ ,  $(x_1 < 0, x_2, x_3 > 0) \equiv \mathcal{U}_4$ . Let us call  $\mathcal{U}$  the sum of these four orthants.

The moments  $\langle T^k \rangle$  are given by an expansion of the following identity in powers of  $\lambda$ :

$$\langle e^{-\lambda T} \rangle = \int_{\mathbb{R}^3} d^3 \vec{r} \langle \vec{r} | e^{-H_\lambda t} | \vec{0} \rangle \quad (7)$$

$$\text{where} \quad H_\lambda(\vec{r}) = -\frac{1}{2} \Delta + \lambda 1_{\mathcal{U}}(\vec{r}) \quad (8)$$

( $1_{\mathcal{U}}(\vec{r}) = 1$  if  $\vec{r} \in \mathcal{U}$ ,  $= 0$  otherwise).

<sup>4</sup> We will see that the random variable  $u$  appears naturally in the following computations.

<sup>5</sup> We immediately deduce the obvious result  $\langle u \rangle = 1/2$  for the middle particle.

The second order in  $\lambda$  leads to:

$$\left\langle \left( \frac{T^2}{2} \right) \right\rangle = \int_0^t dt_2 \int_0^{t_2} dt_1 \int_{\mathcal{U}} dx_1 dx_2 dx_3 \int_{\mathcal{U}} dx'_1 dx'_2 dx'_3 \times \mathcal{G}(x_1, x_2, x_3; t_1) \mathcal{G}(x'_1 - x_1, x'_2 - x_2, x'_3 - x_3; t_2 - t_1) \quad (9)$$

where  $\mathcal{G}$  is the 3D-propagator:

$$\mathcal{G}(x_1, x_2, x_3; t) = G(x_1; t)G(x_2; t)G(x_3; t) \quad (10)$$

$$\text{with } G(x; t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \quad (11)$$

Writing the spatial integral

$$\int_{\mathcal{U}} dx_1 dx_2 dx_3 \int_{\mathcal{U}} dx'_1 dx'_2 dx'_3 \mathcal{G}(x_1, x_2, x_3; t_1) \mathcal{G}(x'_1 - x_1, x'_2 - x_2, x'_3 - x_3; t_2 - t_1)$$

as  $\int_{\mathcal{U}} \int_{\mathcal{U}} \mathcal{G} \mathcal{G}$ , we get, with standard manipulations:

$$\int_{\mathcal{U}} \int_{\mathcal{U}} \mathcal{G} \mathcal{G} = \left( 4 \int_{\mathcal{U}_1} \int_{\mathcal{U}_1} + 6 \int_{\mathcal{U}_1} \int_{\mathcal{U}_2} + 6 \int_{\mathcal{U}_2} \int_{\mathcal{U}_3} \right) \mathcal{G} \mathcal{G} \quad (12)$$

Now, let us introduce the integrals

$$J = \int_0^\infty dx \int_0^\infty dx' G(x; t_1) G(x' - x; t_2 - t_1) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \sqrt{\frac{t_1}{t_2}} \equiv \frac{1}{4\pi} (\pi + \phi) \quad (13)$$

$$J' = \int_0^\infty dx \int_{-\infty}^0 dx' G(x; t_1) G(x' - x; t_2 - t_1) = \frac{1}{4} - \frac{1}{2\pi} \arcsin \sqrt{\frac{t_1}{t_2}} \equiv \frac{1}{4\pi} (\pi - \phi) \quad (14)$$

where we have set  $\frac{t_1}{t_2} = \sin^2(\phi/2)$ . It is worth noticing that a factor  $(\pi + \phi)$  is generated when  $xx' > 0$  and  $(\pi - \phi)$  when  $xx' < 0$ .

So, we can immediately write:

$$\int_{\mathcal{U}_1} \int_{\mathcal{U}_1} \mathcal{G} \mathcal{G} = J^3 \quad (15)$$

$$\int_{\mathcal{U}_1} \int_{\mathcal{U}_2} \mathcal{G} \mathcal{G} = J^2 J' \quad (16)$$

$$\int_{\mathcal{U}_2} \int_{\mathcal{U}_3} \mathcal{G} \mathcal{G} = J J'^2 \quad (17)$$

Finally, we get:

$$\left\langle \left( \frac{T^2}{2} \right) \right\rangle = \frac{t^2}{2} \int_0^\pi d\phi \frac{\sin \phi}{2(4\pi)^3} (4(\pi + \phi)^3 + 6(\pi + \phi)^2(\pi - \phi) + 6(\pi + \phi)(\pi - \phi)^2) \quad (18)$$

$$\langle u^2 \rangle = \frac{3}{8} - \frac{3}{16\pi^2} = 0.35600\dots \quad (19)$$

Recall that for a single particle ( $q = 0$ ), we should have  $\langle u^2 \rangle = \frac{3}{8} = 0.375$ . We observe that, for the middle particle,  $\langle u^2 \rangle$  decreases when  $n$  increases from 1 to 3. We will come back to this point later in section 4.2.1.

**Table 1.** The second moment,  $\langle u^2 \rangle$ , for the middle particle of a file of  $n$  particles.

$q$	$n$	$l$	$\langle u^2 \rangle$
0	1	1	$\frac{3}{8} = 0.375$
1	3	2	$\frac{3}{8} - \frac{3}{16\pi^2} = 0.356\ 00\dots$
2	5	3	$\frac{3}{8} - \frac{15}{32\pi^2} + \frac{135}{64\pi^4} = 0.349\ 16\dots$
3	7	4	$\frac{3}{8} - \frac{105}{128\pi^2} + \frac{735}{64\pi^4} - \frac{7875}{128\pi^6} = 0.345\ 78\dots$
7	15	8	0.341 01...
50	101	51	0.337 53...

### 3.2. General $q$ values

We now extend the previous results to the case of an arbitrary (odd) number of particles in the file. Following the previous section 3.1, we have to consider two positions of the particle  $P$  with, at least,  $(q+1)$  positive coordinates and we must distinguish the coordinates that will keep their sign from the other ones. More precisely, we consider the configuration built with:

- (a) a first position with  $(m+k)$  negative coordinates
- (b) a second position with  $(m+r)$  negative coordinates

such that  $m$  coordinates will stay negative and  $(k+r)$  will change of sign. So,  $2q+1-(k+r)$  coordinates will keep their sign. The contribution of such a configuration to  $\langle u^2 \rangle$  will be equal to:

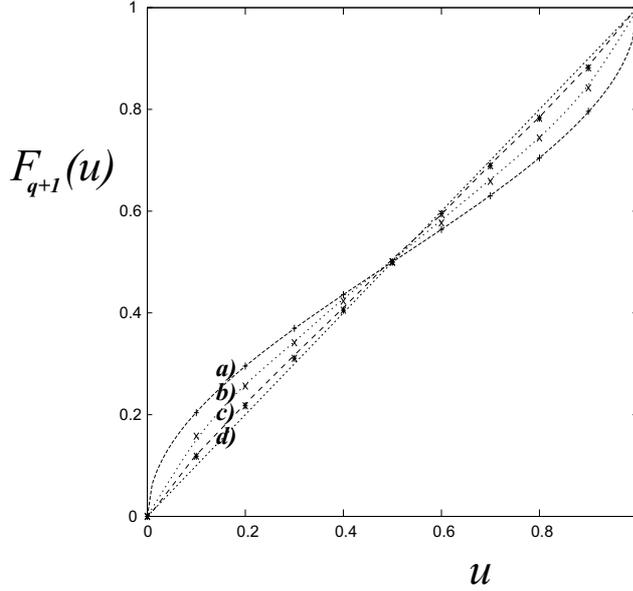
$$\frac{1}{2(4\pi)^{2q+1}} \int_0^\pi d\phi \sin \phi (\pi + \phi)^{2q+1-k-r} (\pi - \phi)^{k+r}$$

Now, the number of configurations with given  $m, k, r$  is simply  $\frac{(2q+1)!}{m!k!r!(2q+1-m-k-r)!}$ . Observing that  $m+k \leq q$  and  $m+r \leq q$ , we get the general expression for  $\langle u^2 \rangle$ :

$$\langle u^2 \rangle = \frac{1}{2(4\pi)^{2q+1}} \int_0^\pi d\phi \sin \phi \sum_{m=0}^q \sum_{k=0}^{q-m} \sum_{r=0}^{q-m} \frac{(2q+1)! (\pi + \phi)^{2q+1-k-r} (\pi - \phi)^{k+r}}{m!k!r!(2q+1-m-k-r)!} \quad (20)$$

The results for different  $q$  values are quoted in table 1.

We observe that  $\langle u^2 \rangle$  is written as a series in  $\frac{1}{\pi^2}$  starting with the term  $\frac{3}{8}$ . Moreover,  $\langle u^2 \rangle$  decreases when  $q$  increases and seems to have a limit close to  $\frac{1}{3}$  that could be consistent with a constant distribution  $P_{q+1}(u)$ . The same indication comes out from numerical simulations that show that this distribution becomes flatter and flatter as long as  $q$  increases (see figure 1 where the integrated probability density  $F_{q+1}(u) = \int_0^u P_{q+1}(v) dv$  is plotted). All those considerations motivate us to study the limit of  $\langle u^2 \rangle$  when  $q \rightarrow \infty$ .



**Figure 1.** The evolution of the integrated distribution  $F_{q+1}(u)$  for the middle particle when the number of particles in the file increases. Numerical simulations ( $5 \times 10^6$  events,  $5 \times 10^5$  gaussian steps) for (a) a single particle ( $q = 0$ ): + (the curve is the arc-sine law); (b) a file of three particles ( $q = 1$ ):  $\times$ ; (c) a file of 11 particles ( $q = 5$ ): \*. (In (b) and (c), the lines are drawn to guide the eye.) In (d), we display (dotted line) the diagonal  $F_{q+1}(u) = u$  that should appear for a flat distribution  $P_{q+1}(u) = 1$ .

### 3.3. The limit $q \rightarrow \infty$

To study  $\langle u^2 \rangle$  in that limit, we first set:  $m \equiv qX$ ,  $k \equiv qY$ ,  $r \equiv qZ$ , and write the summations in equation (20) as integrals:

$$\sum_{m=0}^q \sum_{k=0}^{q-m} \sum_{r=0}^{q-m-k} \frac{(2q+1)! (\pi + \phi)^{2q+1-k-r} (\pi - \phi)^{k+r}}{m! k! r! (2q+1-m-k-r)!} \approx q^3 \int_0^1 dX \int_0^{1-X} dY \int_0^{1-X-Y} dZ \exp(I(X, Y, Z)) \quad (21)$$

The integration domain  $\mathcal{D}$  is displayed in figure 2(a): we integrate inside the pyramid AOCBD.

Now, using the Stirling formula, we apply the saddle-point method ( $\phi$  is fixed). With the conditions

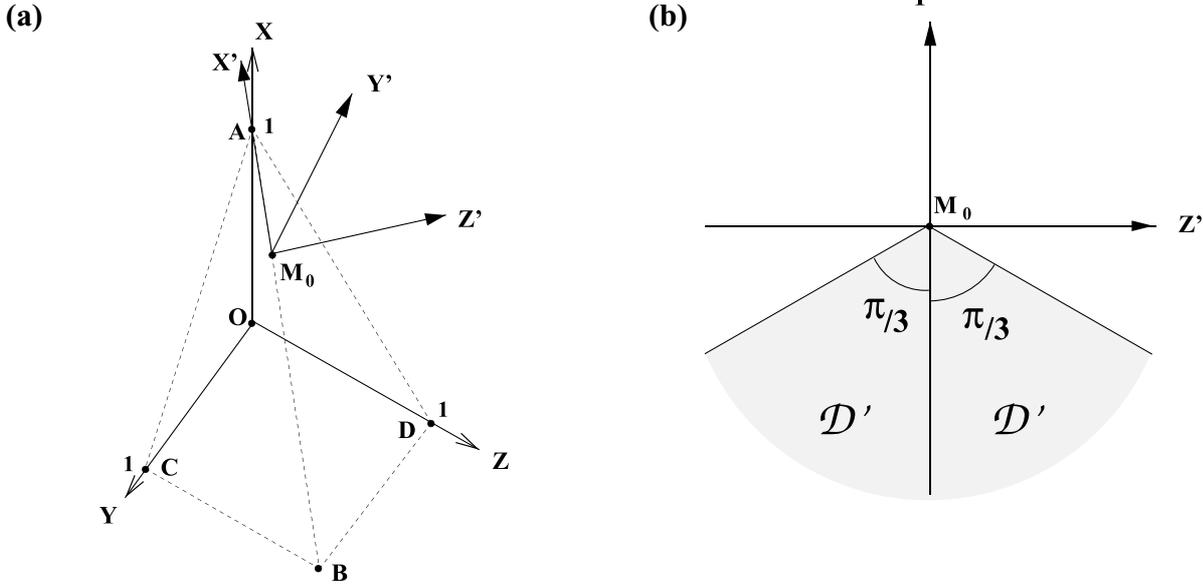
$$\left. \frac{\partial I}{\partial X} \right|_{X_0, Y_0, Z_0} = \left. \frac{\partial I}{\partial Y} \right|_{X_0, Y_0, Z_0} = \left. \frac{\partial I}{\partial Z} \right|_{X_0, Y_0, Z_0} = 0$$

we get the saddle-point  $M_0(X_0, Y_0, Z_0)$  that lies between A and B (see figure 2(a)):

$$X_0 = \frac{1}{2} + \frac{\phi}{2\pi} \quad (22)$$

$$Y_0 = Z_0 = \frac{1}{2} - \frac{\phi}{2\pi} = 1 - X_0 \quad (23)$$

$$\exp(I(X_0, Y_0, Z_0)) = \frac{(4\pi)^{2q+1} 2\sqrt{\pi}}{q^{3/2} (\pi - \phi)(\pi + \phi)} \quad (24)$$



**Figure 2.** (a) The integration domain  $\mathcal{D}$ —the interior of the pyramid AOCBD, the saddle-point  $M_0$  and the three perpendicular axes  $\overrightarrow{M_0X'}$ ,  $\overrightarrow{M_0Y'}$ ,  $\overrightarrow{M_0Z'}$ ; (b) the integration domain  $\mathcal{D}'$ .

Then,  $\langle u^2 \rangle$  is written as:

$$\langle u^2 \rangle = \sqrt{\pi} q^{3/2} \int_0^\pi \frac{d\phi \sin \phi}{(\pi - \phi)(\pi + \phi)} \int_{\mathcal{D}} dX dY dZ \exp(V) \quad (25)$$

$$V = -\frac{q}{X_0} \left( (X - X_0)^2 + \frac{(Y - Y_0)^2 + (Z - Z_0)^2}{2(1 - X_0)} \right) - \frac{q}{X_0} ((X - X_0)(Y - Y_0) + (X - X_0)(Z - Z_0) + (Z - Z_0)(Y - Y_0)) \quad (26)$$

Let us focus, for the moment, on the spatial (domain  $\mathcal{D}$ ) integration.

Introducing (see figure 2(a)) the three perpendicular axes  $(\overrightarrow{M_0X'}, \overrightarrow{M_0Y'}, \overrightarrow{M_0Z'})$  such that  $\overrightarrow{M_0X'} \parallel \overrightarrow{BA}$  and such that  $\overrightarrow{M_0Y'}$  lies in the plane  $Y = Z$ , we get:

$$V = -q \left( \frac{X'^2}{3X_0(1 - X_0)} + \frac{(10 - 9X_0)Y'^2}{6X_0} + \frac{Z'^2}{2(1 - X_0)} - \frac{2X'Y'}{3\sqrt{2}X_0(1 - X_0)} \right)$$

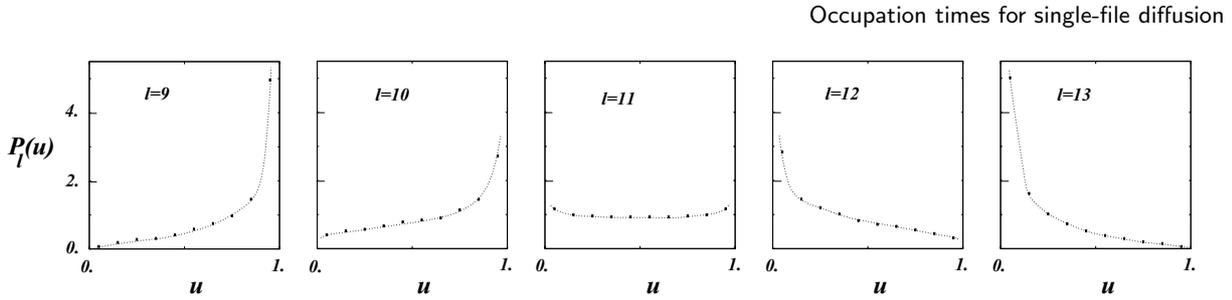
$q$  being large, we can perform the  $X'$  integration and get

$$\sqrt{\frac{3\pi X_0(1 - X_0)}{q}} \times I'$$

where

$$I' \equiv \int_{\mathcal{D}'} dY' dZ' \exp \left( -q \left( \frac{3Y'^2}{2X_0} + \frac{Z'^2}{2(1 - X_0)} \right) \right) \quad (27)$$

The integration domain  $\mathcal{D}'$ , for  $Y'$  and  $Z'$  is displayed in figure 2(b) (the angle between the two planes  $X + Y = 1$  and  $X + Z = 1$  is equal to  $2\pi/3$ ).



**Figure 3.** Numerical simulations for a file of 21 ( $= 2q + 1$ ) particles. The distribution  $P_l(u)$  is displayed for the particles of ranks  $9 \leq l \leq 13$ . For  $l = 11$  (middle particle),  $P_l(u)$  still has a (flattened) symmetric U-shape. But  $P_l(u)$  has a very different shape for the particles surrounding the middle one.

In the large  $q$  limit and using polar coordinates, we can compute

$$I' = \frac{2\sqrt{X_0(1-X_0)}}{q\sqrt{3}} \arcsin \sqrt{X_0}$$

Now, with the equations (22), (23) and (25), we get:

$$\langle u^2 \rangle = \frac{1}{2\pi} \int_0^\pi d\phi \sin \phi \arcsin \sqrt{\frac{1}{2} + \frac{\phi}{2\pi}} \tag{28}$$

$$= \frac{1}{2\pi} \int_0^\pi d\phi \sin \phi \left( \frac{\pi}{4} + \frac{1}{2} \arcsin \frac{\phi}{\pi} \right) \tag{29}$$

Finally, we obtain the analytical result for  $q \rightarrow \infty$ :

$$\langle u^2 \rangle = \frac{3}{8} + \frac{1}{8} J_0(\pi) = 0.33696... \tag{30}$$

where  $J_0$  is a standard Bessel function.

Interestingly, it is found that  $\langle u^2 \rangle \neq \frac{1}{3}$ . We thus conclude that the distribution of  $u$  is not constant.

#### 4. Occupation time of a particle of general rank $l$

We now show that the previous results can be extended to determine the first two moments of the occupation time of a particle of arbitrary rank  $l$ .

Figure 3 displays the distribution  $P_l(u)$  for different particles of rank  $l$  belonging to a file of  $n = 2q + 1 = 21$  particles. We note that the shape of  $P_l(u)$  is very sensitive to the value of  $l$ , especially for  $l$  close to 11 ( $= q + 1$ ). We also remark that, obviously,  $P_{10}(u) = P_{12}(1 - u)$ ,  $P_9(u) = P_{13}(1 - u)$ , ... and, more generally,

$$P_l(u) = P_{2q+2-l}(1 - u) = P_{n+1-l}(1 - u) \tag{31}$$

This reflects the symmetry with respect to the middle particle of the file. We will come back to this point later.

In the following, we will consider a particle of general rank  $l$  in a file of  $n = 2q + 1$  particles<sup>6</sup>. In particular, we will focus our attention on the limit  $q \rightarrow \infty$ .

#### 4.1. The first moment $\langle u \rangle$

To get  $\langle u \rangle$ , it is enough to count the number of orthants of  $\mathbb{R}^{2q+1}$  with, at least,  $l$  positive coordinates:

$$\langle u \rangle = \frac{\sum_{r=l}^{2q+1} \binom{2q+1}{r}}{2^{2q+1}} \quad (32)$$

With the Stirling formula and the replacement of the sum by an integral, we get, for large  $q$  values:

$$\langle u \rangle = \frac{1}{2} \operatorname{erfc}(a) \quad (33)$$

where  $\operatorname{erfc}$  is the complementary error function:

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt = 1 - \operatorname{erf}(z) \quad (34)$$

and

$$a = \sqrt{\frac{1}{q}} (l - (q + 1)), \quad 1 \leq l \leq 2q + 1 \quad (35)$$

( $a = 0$  corresponds to the middle particle.)

As shown in figure 4(a), the agreement between equations (32) (for a file of  $n = 21$  particles) and (33) is quite good.

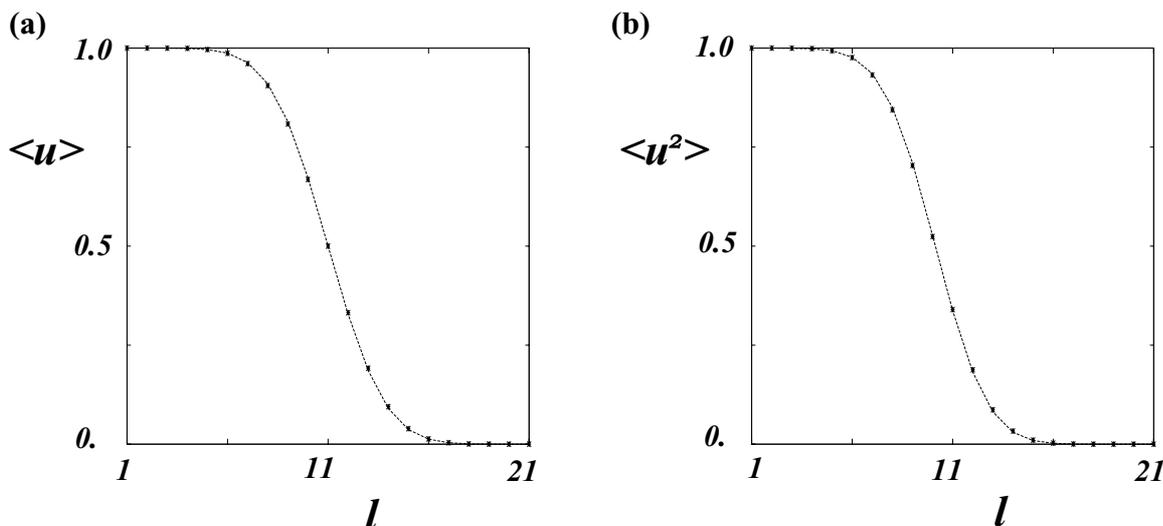
#### 4.2. The second moment $\langle u^2 \rangle$

Following the same lines as in section 3.2, we get the general result

$$\langle u^2 \rangle = \frac{1}{2(4\pi)^{2q+1}} \int_0^\pi d\phi \sin \phi \sum_{m=0}^{2q+1-l} \sum_{k=0}^{2q+1-l-m} \sum_{r=0}^{2q+1-l-m} \frac{(2q+1)! (\pi + \phi)^{2q+1-k-r} (\pi - \phi)^{k+r}}{m! k! r! (2q+1-m-k-r)!} \quad (36)$$

We remark that this expression is the same as equation (20) except for a change in the upper bounds of the summations; (20) is recovered when we set  $l = q + 1$ .

<sup>6</sup> For  $n$  even, we will simply replace  $q$  by  $(n - 1)/2$  in our results. For a file of  $n = 2$  particles, we can take advantage of the results obtained for the quadrant occupation time of the planar BM. Those results are recalled in section 1. Considering the particle of rank  $l = 2$ , i.e. the particle at the left end of the file, we immediately see that the four first moments  $\langle u^k \rangle$  are known and given by equations (3)–(6). We deduce the corresponding moments for the rank  $l = 1$  particle by using the equation (31).



**Figure 4.** For a file of  $n = 21$  particles, computation of the moments  $\langle u \rangle$  and  $\langle u^2 \rangle$  as a function of the particle rank  $l$  ( $1 \leq l \leq 21$ ); (a) dots: equation (32), dashed line: the erfc function, equation (33); (b) dots: equation (36), dashed line: equation (40). We observe the agreement between the exact computation (dots) and the computation done in the limit of an infinite file of particles (dashed lines). The behaviours of  $\langle u \rangle$  and  $\langle u^2 \rangle$  look very similar. However, for  $l = 11$ , we have  $\langle u \rangle = 1/2$  and  $\langle u^2 \rangle = 0.339\,81\dots \approx 3/8 + (1/8)J_0(\pi) \approx 0.337$ .

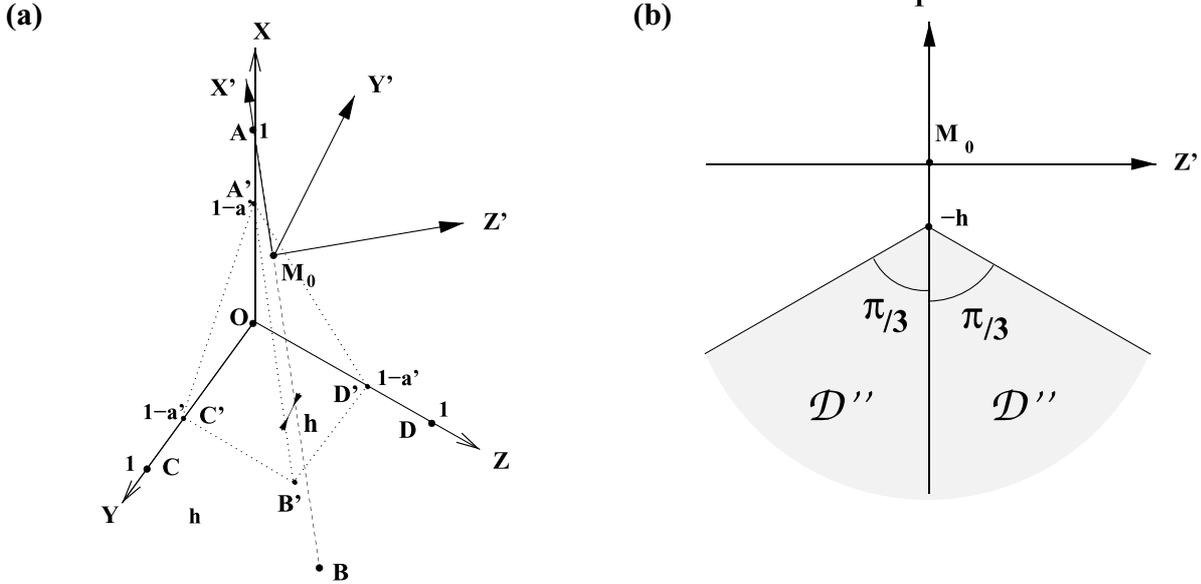
**Table 2.** The two first moments,  $\langle u \rangle$  and  $\langle u^2 \rangle$ , for a particle of rank  $l$  belonging to a file of three particles.

$l$	$\langle u \rangle$	$\langle u^2 \rangle$
1	$\frac{7}{8}$	$\frac{1}{128} \left( 105 - \frac{18}{\pi^2} \right) = 0.806\,06\dots$
2	$\frac{1}{2}$	$\frac{3}{8} - \frac{3}{16\pi^2} = 0.356\,00\dots$
3	$\frac{1}{8}$	$\frac{1}{128} \left( 9 - \frac{18}{\pi^2} \right) = 0.056\,06\dots$

*4.2.1. The three particle case.* For  $q = 1$ , equations (32) and (36) lead to the results quoted in table 2.

Looking at the values of  $\langle u \rangle$  and  $\langle u^2 \rangle$ , we observe that the right-end ( $l = 1$ ) particle spends most of its time on the  $x > 0$  half-line. This is due to the middle particle ( $l = 2$ ) that acts on it as a fluctuating and purely reflecting barrier [17]. The same argument holds for the left-end ( $l = 3$ ) particle that spends most of its time on the  $x < 0$  half-line. Let us now consider the middle ( $l = 2$ ) particle. This one is subjected to the  $l = 1$  and  $l = 3$  particles that act as two fluctuating barriers that are typically located, respectively, in the  $x > 0$  and  $x < 0$  half spaces. As a consequence of the reflections, the middle particle is repelled towards the origin  $O$ , the U-shape distribution  $P_2(u)$  is flattened and  $\langle u^2 \rangle$  is decreased when compared to a single Brownian particle<sup>7</sup>.

<sup>7</sup> In [17], the author shows that, for the middle particle, the diffusion constant  $D$  decreases when the number of particles in the file increases. However, we think that this decrease of  $D$  cannot be *directly* invoked to explain the decrease of  $\langle u^2 \rangle$ . Indeed, if we consider a single Brownian particle, its diffusion constant plays no role in the arc-sine law.



**Figure 5.** (a) The new integration domain—the interior of the pyramid  $A'OC'B'D'$ . The saddle-point  $M_0$  and the three perpendicular axes  $\overrightarrow{M_0X'}$ ,  $\overrightarrow{M_0Y'}$ ,  $\overrightarrow{M_0Z'}$  are unchanged;  $h$  is the distance between the parallel lines  $(AB)$  and  $(A'B')$ ; (b) the integration domain  $\mathcal{D}''$  obtained by a translation of the domain  $\mathcal{D}'$  (displayed in figure 2(b)) along  $\overrightarrow{M_0Y'}$ .

4.2.2. *The limit  $q \rightarrow \infty$ .* To study this limit, we proceed as in section 3.3, equation (21):

$$\sum_{m=0}^{2q+1-l} \sum_{k=0}^{2q+1-l-m} \sum_{r=0}^{2q+1-l-m} \frac{(2q+1)!(\pi+\phi)^{2q+1-k-r}(\pi-\phi)^{k+r}}{m!k!r!(2q+1-m-k-r)!} \approx q^3 \int_0^{1-a'} dX \int_0^{1-a'-X} dY \int_0^{1-a'-X} dZ \exp(I(X, Y, Z)) \quad (37)$$

where

$$a' = \frac{a}{\sqrt{q}} = \frac{l-(q+1)}{q}; \quad -1 \leq a' \leq 1$$

The new integration domain for  $X$ ,  $Y$ ,  $Z$  is shown in figure 5(a) and the saddle-point is unchanged.

Let us define  $h$ , the distance between the two parallel lines  $(AB)$  and  $(A'B')$ :

$$h = a' \sqrt{\frac{2}{3}} = \frac{a\sqrt{2}}{\sqrt{3q}}.$$

Still following the same lines as in section 3.3, we replace the integral  $I'$ , equation (27), by:

$$I'' = \int_{\mathcal{D}''} dY' dZ' \exp\left(-q \left(\frac{3Y'^2}{2X_0} + \frac{Z'^2}{2(1-X_0)}\right)\right) \quad (38)$$

$$= 2 \int_{-\infty}^{-h} dY' \int_0^{(-h-Y')\sqrt{3}} dZ' \exp\left(-q \left(\frac{3Y'^2}{2X_0} + \frac{Z'^2}{2(1-X_0)}\right)\right) \quad (39)$$

The new integration domain  $\mathcal{D}''$ , for  $Y'$  and  $Z'$ , is displayed in figure 5(b): this is simply the domain  $\mathcal{D}'$  (see figure 2(b)) shifted by  $-h$  along the axis  $\overrightarrow{M_0 Y'}$ .

Finally, after some algebra, we get:

$$\langle u^2 \rangle \equiv f(a) = \sqrt{\pi} \int_0^{1/2} dx \frac{\sin(2\pi x)}{\sqrt{1-x}} \int_0^\infty dy \operatorname{erf}\left(\frac{y}{\sqrt{x}}\right) e^{-\frac{(y+a)^2}{(1-x)}} \quad (40)$$

where  $a$  has been defined in equation (35) and  $\operatorname{erf}$  is the error function (see equation (34)).

Using the property equation (31), the result (33) and, also, observing that the change  $l \rightarrow 2q+2-l$  only amounts to the change  $a \rightarrow -a$ , we get the relationship:  $f(a) - f(-a) = -\operatorname{erf}(a)$ . So, the odd part of  $f(a)$  is simply  $-\frac{1}{2}\operatorname{erf}(a)$ . Writing  $f(a) = \sum_0^\infty c_n a^n$ , we immediately deduce the odd coefficients

$$c_{2n+1} = \frac{(-1)^{n+1}}{\sqrt{\pi} n! (2n+1)} \quad (41)$$

Unfortunately, the situation is more tricky for the even coefficients. We only got the analytical results:

$$c_0 = \frac{3}{8} + \frac{1}{8} J_0(\pi) = 0.336\,96\dots \quad (42)$$

$$c_2 = \frac{\pi}{4} (H_0(\pi) - J_1(\pi)) = 0.183\,16\dots \quad (43)$$

$J_0$  and  $J_1$  are Bessel functions and  $H_0$  is a Struve function [18] ( $H_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \sin(x \cos u) du$ ).

$c_0$  ( $= f(0)$ ), equation (42), is simply the result (30) obtained for the middle particle in section 3.3.

In figure 4(b), we show the good agreement between the results (36) (for a file of  $n = 21$  particles) and (40). We also observe that the distribution  $P_l(u)$  is peaked at 0 (1) as soon as  $(l - (q + 1))$  is of the order of  $\sqrt{q}$  ( $-\sqrt{q}$ ).

## 5. Conclusion

In conclusion, we have studied analytically the occupation time  $T$  of the half-line  $[Ox)$  of a tracer particle among  $n$  hard-core Brownian particles starting all initially from  $O$ . These particles are labelled by their rank  $l$  and cannot cross each other. While the dynamics of the entire system is Markovian, the evolution of the only tracer is not, which makes this question a non-trivial problem. However, this problem can be mapped onto a single particle problem in  $\mathbb{R}^n$ , which allowed us to obtain explicit results for the two first moments of the occupation time  $T$ . Analytical results have been obtained for an arbitrary number  $n$  of Brownian particles and an arbitrary choice of the rank  $l$  of the tracer particle. The limit of an infinite number of particles  $n \rightarrow \infty$  has been analyzed in detail.

Natural extensions of this work concern the case of arbitrary initial conditions and the determination of higher order moments. Another important generalization would be to reconsider the problem when the tracer particle is different from the other bath particles, and for example is the only particle to experience the effect of a bias.

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