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# ON A FUNCTIONAL INEQUALITY ARISING IN THE ANALYSIS OF FINITE-VOLUME METHODS

P BOUSQUET <sup>\*</sup>, F. BOYER <sup>†</sup>, AND F. NABET<sup>†</sup>

**Abstract.** We establish a Poincaré–Wirtinger type inequality on some particular domains with a precise estimate of the constant depending only on the geometry of the domain. This type of inequality arises, for instance, in the analysis of finite volume (FV) numerical methods.

As an application of our result, we prove uniform a priori bounds for the FV approximate solutions of the heat equation with Ventcell boundary conditions in the natural energy space defined as the set of those functions in  $H^1(\Omega)$  whose traces belong to  $H^1(\partial\Omega)$ . The main difficulty here comes from the fact that the approximation is performed on non-polygonal control volumes since the domain itself is non-polygonal.

**Key words.** Poincaré–Wirtinger inequality - Finite Volume methods - Ventcell boundary condition

**AMS subject classifications.** 35A23, 35K05, 65M08

**1. Introduction.** The main goal of this paper is to study functional inequalities of the following form

$$(1.1) \quad \left| \frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_\mathcal{K}} \int_\mathcal{K} u \right| \leq C \text{diam}(\mathcal{K}) \frac{1}{m_\mathcal{K}} \int_\mathcal{K} |Du|, \quad \forall u \in W^{1,1}(\mathbb{R}^n),$$

where  $\mathcal{K}$  is a connected Lipschitz domain in  $\mathbb{R}^n$  and  $\sigma$  a non empty open subset of  $\partial\mathcal{K}$ . We assume that  $\mathcal{K}$  is bounded and connected. We have denoted by  $m_\mathcal{K}$  the volume of  $\mathcal{K}$  and  $m_\sigma$  its surface measure, namely its  $(n-1)$ -dimensional Hausdorff measure. Those notations will be used all along this paper.

The fact that such an inequality holds is straightforward, for instance by applying the Bramble–Hilbert lemma. Our main purpose is to estimate the dependence of the constant  $C$  with respect to the geometry of the domain  $\mathcal{K}$ . In the particular case of a convex domain  $\mathcal{K}$  the mean-value inequality immediately implies that

$$(1.2) \quad \left| \frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_\mathcal{K}} \int_\mathcal{K} u \right| \leq \text{diam}(\mathcal{K}) \left( \sup_\mathcal{K} |Du| \right), \quad \forall u \in W^{1,\infty}(\mathbb{R}^n),$$

with no geometric constant in the right-hand side. The inequality (1.1) has to be seen as a generalisation of (1.2) to less regular functions  $u$ . This loss of regularity induces that the constant in the inequality may depend on the shape of  $\mathcal{K}$ . Observe that, if  $\mathcal{K}$  is not convex, one has to replace  $\sup_\mathcal{K} |Du|$  by  $\sup_{\text{Conv}(\mathcal{K})} |Du|$  in (1.2), where  $\text{Conv}(\mathcal{K})$  is the convex hull of  $\mathcal{K}$ .

Inequalities of the form (1.1) play an important role in the analysis of finite volume numerical methods for elliptic or parabolic equations on general meshes, which is our main motivation. They are meant to be applied to each cell (control volume)  $\mathcal{K}$  in a mesh of a given computational domain. They allow to prove stability estimates in (discrete) Sobolev spaces for the natural  $L^2$  projections of the functions defined on  $\Omega$  and the projections of their traces. To our knowledge, such inequalities have only been established up to now in the framework of polygonal sets  $\mathcal{K}$ . However, for more complex situations, like for the discretisation of the heat equation with dynamic Ventcell boundary conditions, we are interested in proving such inequalities for non polygonal domains. We detail such an application in Section 6.

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Let us mention some references where such inequalities are proved and/or used in the finite volume framework.

- In [6, Lemma 3.4] (see also [3, Lemma 7.2] and [4, Lemmas 6.2 and 6.3]), (1.1) is proved (in 2D for simplicity) when  $\mathcal{K}$  is polygonal and convex, with a constant  $C$  depending only on the number of edges/faces of  $\mathcal{K}$ , and on the shape-regularity ratio  $(\text{diam}(\mathcal{K}))^2/m_{\mathcal{K}}$ .
- In [5, Lemma 6.6], the inequality is slightly generalized to a polygonal  $\mathcal{K}$  which is simply supposed to be star-shaped with respect to a suitable ball.
- In [1], such inequalities are used for the convergence and error analysis of some approximation of non-linear Leray-Lions type operators (a model of which is the  $p$ -Laplace problem).

Finally, we refer to [8] for an example of analysis of a more complex model of a non-linear evolution equation associated with a non-linear dynamical boundary condition. This reference was in fact our main motivation for the present work. Indeed, compared to the other references above, the numerical method in [8] is derived on non-polygonal control volumes, so that an inequality like (1.1) is needed on non-polygonal open sets  $\mathcal{K}$ , see also Section 6.

The outline of the paper is the following. In Section 2, we state our main result (Theorem 2.1 in 2D) and the geometric assumptions that we shall work with in the sequel. Section 3 is devoted to the proof of the main result whereas in Section 4, we state and prove a sort of Poincaré-Wirtinger inequality related to the functional inequality proved in Theorem 2.1. Section 5 is dedicated to the extension of our main inequality in the higher dimensional case (i.e. in  $\mathbb{R}^n$ , for  $n \geq 3$ ). Finally, in order to illustrate this work, we provide an application, as simple as possible, of Theorem 2.1 to the proof of uniform discrete energy estimates for a finite volume approximation of a toy system on a non-polygonal domain  $\Omega$ .

**2. Main result.** Given a  $C^1$  curve  $\sigma \subset \mathbb{R}^2$  and a point  $z_* \in \mathbb{R}^2 \setminus \sigma$ , we consider the following domain  $T$ :

$$T = \{z_* + t(\gamma(\theta) - z_*) : t \in ]0, 1[, \theta \in ]0, 1[ \},$$

where  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$  is a  $C^1$  parametrization of  $\sigma$ :  $\gamma([0, 1]) = \sigma$ ,  $\gamma$  is one-to-one and  $|\gamma'|$  does not vanish. Without loss of generality, we choose the parametrization  $\gamma$  in such a way that  $|\gamma'(\theta)| = m_\sigma$  for every  $\theta \in [0, 1]$ .

We say that  $T$  is a pseudo-triangle if for every  $x \in \sigma$ ,

$$(2.1) \quad \{z_* + t(x - z_*) : t \geq 0\} \cap \sigma = \{x\}.$$



FIG. 2.1. The pseudo-triangle  $T$  with its curved edge  $\sigma$  and one of its sub-triangle

Without loss of generality, we shall assume that the vertex  $z_*$  of  $T$  opposite to  $\sigma$  is the origin  $(0, 0)$  of  $\mathbb{R}^2$ .

**THEOREM 2.1.** *Let  $T$  be a pseudo-triangle as above. We assume that there exist  $\mu, \nu > 0$  such that for any sub-arc  $\tilde{\sigma} \subset \sigma$ , the corresponding sub-triangle  $T_{\tilde{\sigma}}$  (see Figure 2.1) satisfies*

$$(2.2) \quad \mu \leq \frac{m_{T_{\tilde{\sigma}}}}{m_{\tilde{\sigma}}} \leq \nu.$$

Then for every  $p \in [1, +\infty[$  and every  $u \in W^{1,p}(T)$ ,

$$(2.3) \quad \left| \frac{1}{m_{\sigma}} \int_{\sigma} u - \frac{1}{m_T} \int_T u \right|^p \leq C(m_{\sigma} + \text{diam}(T))^p \frac{1}{m_T} \int_T |Du|^p,$$

where  $C$  only depends on  $p$  and on the ratio  $\frac{\nu}{\mu}$ .

As already noticed in the introduction, the main issue is to understand how the constant in the inequality depends on the geometry of this pseudo-triangle  $T$ .

REMARK 2.1. For a real flat triangle  $T$ , the quantity  $\frac{m_{T_{\tilde{\sigma}}}}{m_{\tilde{\sigma}}}$  does not depend on  $\tilde{\sigma}$  and is equal to  $\frac{m_T}{m_{\sigma}}$ . In this particular case,  $\mu = \nu$  and  $m_{\sigma} \leq \text{diam}(T)$ . We recover exactly in this particular case the inequality proved in [6].

PROPOSITION 2.2. Under assumption (2.1), the map  $\theta \mapsto \det(\gamma(\theta), \gamma'(\theta))$  is either nonnegative everywhere or nonpositive everywhere.

In the sequel, we assume that the orientation is chosen such that  $\det(\gamma, \gamma') \geq 0$ . Then, assumption (2.2) is equivalent to the following inequality

$$(2.4) \quad 2\mu m_{\sigma} \leq \det(\gamma(\theta), \gamma'(\theta)) \leq 2\nu m_{\sigma}, \quad \forall \theta \in [0, 1].$$

*Proof.* One can assume without loss of generality that  $\gamma([0, 1])$  is contained in the half plane  $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ . There exists a  $C^1$  map  $\varphi : [0, 1] \rightarrow ]-\pi/2, \pi/2[$  such that for every  $\theta \in [0, 1]$ ,

$$\frac{\gamma(\theta)}{|\gamma(\theta)|} = (\cos \varphi(\theta), \sin \varphi(\theta)).$$

By (2.1), the map  $\varphi$  is one-to-one. Hence, it is either strictly increasing or strictly decreasing. Assume for instance that  $\varphi$  is strictly increasing.

This implies that for every  $\theta \in [0, 1]$ , for every  $h > 0$  such that  $\theta + h \in [0, 1]$ ,

$$\begin{aligned} \det\left(\gamma(\theta), \frac{\gamma(\theta + h) - \gamma(\theta)}{h}\right) &= \frac{1}{h} \det(\gamma(\theta), \gamma(\theta + h)) \\ &= \frac{1}{h} |\gamma(\theta)| |\gamma(\theta + h)| \sin(\varphi(\theta + h) - \varphi(\theta)) \geq 0. \end{aligned}$$

Passing to the limit  $h \rightarrow 0$  yields the desired result.

Assume now that (2.2) holds true, then let  $0 \leq a < b \leq 1$ , and consider  $\tilde{\sigma} = \gamma([a, b])$ . Then

$$m_{\tilde{\sigma}} = (b - a)m_{\sigma} \text{ and } m_{T_{\tilde{\sigma}}} = \frac{1}{2} \int_a^b \det(\gamma(\theta), \gamma'(\theta)) d\theta,$$

thus thanks to (2.2),

$$\mu \leq \frac{\int_a^b \det(\gamma(\theta), \gamma'(\theta)) d\theta}{2(b - a)m_{\sigma}} \leq \nu,$$

and we obtain (2.4) when  $b$  tends to  $a$ . Conversely, assume that (2.4) holds. Then (2.2) follows by integration of (2.4) on the segment  $[a, b]$ .  $\square$

REMARK 2.2. *For some particular cases, we can estimate the constant in the inequality (2.3) even if the pseudo-triangle  $T$  does not satisfy assumption (2.2). In order to illustrate such a situation, we consider the pseudo-triangle  $T$  defined as follows:*

$$T = \{t\gamma(\theta) : 0 < t < 1, -\theta_0 < \theta < \pi + \theta_0\}, \text{ with } \gamma(\theta) = (R \cos \theta, R \sin \theta + R/\sin(\theta_0)),$$

where  $0 < \theta_0 < \frac{\pi}{2}$ ,  $R > 0$ . Observe that  $\gamma$  and  $\gamma'$  are colinear for  $\theta = -\theta_0$  or  $\theta = \pi + \theta_0$  so that assumption (2.2) is not satisfied here. We decompose the pseudo-triangle  $T$  into the piece of disk  $P$  of radius  $R$  and center  $A = (0, R/\sin(\theta_0))$  and the quadrilateral  $Q$  defined by:  $Q = T \setminus P$ .

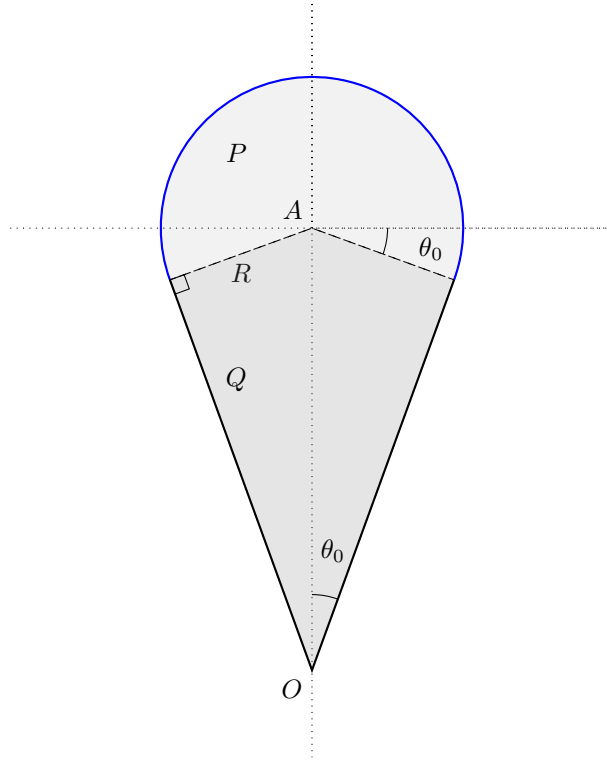


FIG. 2.2. A particular case which does not satisfy assumption (2.2)

First, we remark that assumption (2.2) is satisfied for the pseudo-triangle  $P$  with the ratio  $\frac{\nu}{\mu}$  equal to 1 (see Remark 2.1). Then we can apply Theorem 2.1 to the pseudo-triangle  $P$ , so that there exists a constant  $C_0 > 0$  which does not depend on  $R$  and  $\theta_0$  such that

$$\left| \frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_P} \int_P u \right| \leq C_0 R \frac{1}{m_P} \int_P |Du|.$$

Moreover, since  $P \subset T$  and  $T$  is convex, we can apply [3, Lemma 7.1],

$$\left| \frac{1}{m_T} \int_T u - \frac{1}{m_P} \int_P u \right| \leq \widetilde{C}_0 \frac{(\text{diam}(T))^3}{m_P m_T} \int_T |Du|.$$

Now, we want to control the volume of the pseudo-triangle  $T$  by the volume of  $P$ . We note that

$$m_Q = \frac{R^2}{\tan \theta_0} \quad , \quad m_P = \frac{R^2}{2}(\pi + 2\theta_0) \quad , \quad \text{diam}(T) = R \left(1 + \frac{1}{\sin \theta_0}\right).$$

Hence

$$m_Q \leq \frac{\text{diam}(T)}{R} \frac{2}{\pi} m_P$$

and then using that  $R \leq \text{diam}(T)$ ,

$$m_P \geq \frac{\pi R}{\pi R + 2\text{diam}(T)} m_T \geq C_1 \frac{R}{\text{diam}(T)} m_T.$$

This implies

$$\left| \frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_P} \int_P u \right| \leq \frac{C_0}{C_1} \text{diam}(T) \frac{1}{m_T} \int_T |Du|,$$

and

$$\left| \frac{1}{m_T} \int_T u - \frac{1}{m_P} \int_P u \right| \leq \frac{\widetilde{C}_0}{C_1} \frac{(\text{diam}(T))^4}{R m_T^2} \int_T |Du|.$$

Since  $R \leq \text{diam}(T)$ ,  $m_T \leq (\text{diam}(T))^2$ , the above two inequalities yield

$$\begin{aligned} \left| \frac{1}{m_T} \int_T u - \frac{1}{m_\sigma} \int_\sigma u \right| &\leq C \frac{(\text{diam}(T))^4}{R m_T^2} \int_T |Du| \\ &\leq \frac{C'}{\sin(\theta_0)^2} \text{diam}(T) \frac{1}{m_T} \int_T |Du|, \end{aligned}$$

where  $C'$  is a universal constant. As expected we observe that the above inequality does not depend on  $R$  and blows up when  $\theta_0$  goes to 0.

**3. Proof of Theorem 2.1.** By Jensen's inequality, we only need to establish the case  $p = 1$ . We begin by proving that the result follows from the existence of a suitable diffeomorphism between  $T$  and the unit cube. We notice that the existence of such a diffeomorphism is ensured by a general result of [2] but we have to be able to estimate the derivatives of this diffeomorphism in function of the geometry of  $T$ . That is why we resume explicitly the steps of [2] that allow us to control all the constants involved in the estimates.

In the sequel, we denote by  $Q^2 = ]0, 1[^2$  the unit cube. By a standard approximation argument, one can assume that  $\gamma \in C^2(0, 1)$ .

**LEMMA 3.1.** *Assume that there exists a Lipschitz continuous map  $\Phi : \overline{Q^2} \rightarrow \overline{T}$  such that*

1.  $\Phi$  is a  $C^1$  diffeomorphism from  $Q^2$  onto  $T$ ,
2.  $\Phi(0, \theta) = (0, 0)$ ,
3.  $\Phi(1, \theta) = \gamma(\theta)$ ,
4.  $\text{Jac } \Phi(s, \theta) = 2m_T s$ .

Then for every  $u \in W^{1,1}(T)$ , we have

$$\frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_T} \int_T u = \frac{1}{2m_T} \int_T Du(x, y) [s \partial_s \Phi(s, \theta)]_{(s, \theta) = \Phi^{-1}(x, y)} dx dy.$$

*Proof.* It follows from (2.4) that the pseudo-triangle  $T$  is biLipschitz homeomorphic to a (true) triangle, see the proof of Lemma 4.2 in section 4. In particular,  $T$  is a Lipschitz domain. By a standard density argument, we can thus assume that  $u \in C^1(\overline{T})$ . Let

$$v(s, \theta) = u \circ \Phi(s, \theta) \text{Jac } \Phi(s, \theta).$$

Then for every  $(t, \theta) \in Q^2$ ,

$$v(1, \theta) - v(t, \theta) = \int_t^1 \partial_s v(s, \theta) ds.$$

Hence, thanks to the assumptions on  $\Phi$ , we obtain

$$2m_T u(\gamma(\theta)) - u \circ \Phi(t, \theta) \text{Jac } \Phi(t, \theta) = \int_t^1 (\partial_s(u \circ \Phi) \text{Jac } \Phi + 2m_T(u \circ \Phi)) ds.$$

By integrating this equality on  $Q^2$  and using an obvious change of variables, we get

$$(3.1) \quad 2 \frac{m_T}{m_\sigma} \int_\sigma u - \int_T u = \int_0^1 d\theta \int_0^1 dt \int_t^1 (\partial_s(u \circ \Phi) \text{Jac } \Phi + 2m_T(u \circ \Phi)) ds.$$

By Fubini theorem,

$$\begin{aligned} \int_0^1 d\theta \int_0^1 dt \int_t^1 (\partial_s(u \circ \Phi) \text{Jac } \Phi + 2m_T(u \circ \Phi)) ds \\ = \int_{Q^2} s (\partial_s(u \circ \Phi) \text{Jac } \Phi + 2m_T(u \circ \Phi)) ds d\theta. \end{aligned}$$

Since  $\text{Jac } \Phi(s, \theta) = 2m_T s$ , the same change of variables gives

$$\begin{aligned} \int_{Q^2} s (\partial_s(u \circ \Phi) \text{Jac } \Phi + 2m_T(u \circ \Phi)) ds d\theta \\ = \int_T Du(x, y) [s \partial_s \Phi(s, \theta)]_{(s, \theta) = \Phi^{-1}(x, y)} dx dy + \int_T u. \end{aligned}$$

The lemma now follows from this identity together with (3.1).  $\square$

In the case when  $T$  is a real triangle, namely when  $\sigma$  is a segment, we can observe that the map

$$\phi_2 : (s, \theta) \in \overline{Q^2} \mapsto s\gamma(\theta) \in \overline{T},$$

satisfies the assumptions of the previous lemma.

In the general case where  $\sigma$  is curved, we have

$$\text{Jac } \phi_2(s, \theta) = s \det(\gamma(\theta), \gamma'(\theta)),$$

and this quantity depends on  $\theta$ : we cannot choose  $\Phi = \phi_2$ . Thus, we are going to compose  $\phi_2$  with a diffeomorphism of the unit cube to construct a homeomorphism  $\Phi : \overline{Q^2} \rightarrow \overline{T}$  satisfying the assumptions of the above lemma, see Figure 3.1.

To simplify the notation, we define  $g$  by

$$g : (s, \theta) \in \overline{Q^2} \mapsto \text{Jac } \phi_2(s, \theta) = s \det(\gamma, \gamma')(\theta).$$

Then, we construct a first diffeomorphism  $\phi_1$  of  $Q^2$  such that  $\phi_2 \circ \phi_1$  satisfies the first two assumptions of Lemma 3.1 and a weaker version (integrated with respect to  $s$ ) of the third assumption.

LEMMA 3.2. *There exists a  $C^1$  diffeomorphism  $\phi_1 : \overline{Q^2} \rightarrow \overline{Q^2}$  such that*

1. *for every  $x \in \partial Q^2$ ,  $\phi_1(x) = x$ ,*
2. *for every  $\theta \in [0, 1]$ ,*

$$(3.2) \quad \int_0^1 \text{Jac}(\phi_2 \circ \phi_1)(s, \theta) ds = \int_0^1 g \circ \phi_1(s, \theta) \text{Jac } \phi_1(s, \theta) ds = m_T.$$

3. *for every  $(s, \theta) \in \overline{Q^2}$ ,*

$$(3.3) \quad \frac{\mu}{4\nu} \leq \text{Jac } \phi_1(s, \theta) \leq \frac{4\nu}{\mu}.$$

*Proof.* Let

$$(3.4) \quad \varepsilon = \frac{\mu}{10\nu}$$

and  $\zeta \in C_c^\infty(0, 1)$  be a cut-off function such that  $0 \leq \zeta \leq 1 + \varepsilon$ ,  $|\zeta'|_{L^\infty} \leq \frac{10}{\varepsilon}$  and

$$(3.5) \quad \int_0^1 \zeta(s) ds = 1 \quad \text{and} \quad \int_0^1 |\zeta(s) - 1| ds < \varepsilon.$$

We introduce the map

$$G : (a, b) \mapsto \int_0^1 s ds \int_0^{a+\zeta(s)b} \det(\gamma, \gamma')(\theta) d\theta.$$

Then  $G$  is well-defined and  $C^2$  on the set  $\{(a, b) : 0 \leq a \leq 1, \frac{-a}{1+\varepsilon} \leq b \leq \frac{1-a}{1+\varepsilon}\}$  (here, we use the fact that  $\gamma \in C^2([0, 1])$  so that  $\det(\gamma, \gamma')$  is  $C^1([0, 1])$ ). Moreover, we have

$$\partial_a G(a, b) = \int_0^1 \det(\gamma, \gamma')(a + \zeta(s)b) s ds$$

$$\partial_b G(a, b) = \int_0^1 \det(\gamma, \gamma')(a + \zeta(s)b) \zeta(s) s ds.$$

By (2.4),

$$(3.6) \quad \partial_b G(a, b) \geq 2m_\sigma \mu \int_0^1 \zeta(s) s ds > 0.$$

We claim that

$$(3.7) \quad G\left(a, \frac{-a}{1+\varepsilon}\right) \leq m_T a.$$



Indeed, by (2.4),

$$G\left(a, \frac{-a}{1+\varepsilon}\right) \leq 2m_\sigma \nu \int_0^1 s\left(a - \frac{a}{1+\varepsilon} \zeta(s)\right) ds \leq \frac{2m_\sigma \nu a}{1+\varepsilon} \int_0^1 (1+\varepsilon - \zeta(s)) ds.$$

By (3.5), this implies

$$G\left(a, \frac{-a}{1+\varepsilon}\right) \leq \frac{2m_\sigma \nu a \varepsilon}{1+\varepsilon} \leq 2m_\sigma \nu a \varepsilon.$$

Since

$$m_T = \frac{1}{2} \int_0^1 \det(\gamma, \gamma')(\theta) d\theta \geq m_\sigma \mu,$$

it follows from (3.4) that

$$G\left(a, \frac{-a}{1+\varepsilon}\right) \leq m_T a,$$

which proves our claim.

Similarly, one can prove that

$$\int_0^1 s ds \int_{a+\zeta(s)\frac{1-a}{1+\varepsilon}}^1 \det(\gamma, \gamma')(\theta) d\theta \leq m_T(1-a).$$

This can be written as

$$G(1, 0) - G\left(a, \frac{1-a}{1+\varepsilon}\right) \leq m_T(1-a).$$

Since

$$G(1, 0) = \int_0^1 s ds \int_0^1 \det(\gamma, \gamma')(\theta) d\theta = m_T,$$

this implies

$$(3.8) \quad G\left(a, \frac{1-a}{1+\varepsilon}\right) \geq m_T a.$$

We deduce from (3.6), (3.7) and (3.8) that for every  $a \in [0, 1]$ , there exists a unique  $w(a) \in [-a/(1+\varepsilon), (1-a)/(1+\varepsilon)]$  such that

$$G(a, w(a)) = m_T a.$$

By the implicit function theorem, the function  $w$  is  $C^2$  on  $[0, 1]$  and satisfies

$$(3.9) \quad \partial_a G(a, w(a)) + \partial_b G(a, w(a)) w'(a) = m_T.$$

Since  $G(0, 0) = 0$  and  $G(1, 0) = m_T$ , we have  $w(0) = 0 = w(1)$ .

We claim that for every  $s \in [0, 1]$ , for every  $a \in [0, 1]$ ,

$$1 + \zeta(s) w'(a) > 0.$$

We only need to prove that  $\partial_b G(a, w(a))(1 + \zeta(s)w'(a)) > 0$ . By (3.9),

$$\begin{aligned}\partial_b G(a, w(a))(1 + \zeta(s)w'(a)) &= \partial_b G(a, w(a)) + \zeta(s)(-\partial_a G(a, w(a)) + m_T) \\ &= \partial_b G(a, w(a)) - \partial_a G(a, w(a)) + \zeta(s)m_T \\ &\quad + (1 - \zeta(s))\partial_a G(a, w(a)).\end{aligned}$$

But

$$\begin{aligned}\partial_b G(a, w(a)) - \partial_a G(a, w(a)) &= \int_0^1 \det(\gamma, \gamma')(a + \zeta(s)w(a))(\zeta(s) - 1)s \, ds \\ &\geq -2\nu m_\sigma \int_0^1 |\zeta - 1| \geq -2\nu m_\sigma \varepsilon = -\frac{\mu m_\sigma}{5}.\end{aligned}$$

In the last inequality, we have used (3.5). Moreover, since  $0 \leq \zeta \leq 1 + \varepsilon$ ,

$$\begin{aligned}\zeta(s)m_T + (1 - \zeta(s))\partial_a G(a, w(a)) \\ \geq \min(\partial_a G(a, w(a)), (1 + \varepsilon)m_T - \varepsilon\partial_a G(a, w(a))).\end{aligned}$$

Since  $m_\sigma \mu \leq \partial_a G(a, w(a)) \leq m_\sigma \nu$  and  $m_T \geq m_\sigma \mu$ , we get by (3.4)

$$\zeta(s)m_T + (1 - \zeta(s))\partial_a G(a, w(a)) \geq \frac{9m_\sigma \mu}{10}.$$

This implies

$$(3.10) \quad \partial_b G(a, w(a))(1 + \zeta(s)w'(a)) \geq \frac{m_\sigma \mu}{2}.$$

We now define

$$\phi_1(s, \theta) = (s, \theta + \zeta(s)w(\theta)) \quad , \quad (s, \theta) \in \overline{Q^2}.$$

Then  $\phi_1$  is  $C^1$  on  $Q^2$  and satisfies  $\text{Jac } \phi_1(s, \theta) = 1 + \zeta(s)w'(\theta) > 0$ . Since for every  $s \in [0, 1]$ , the function  $\theta \mapsto \theta + \zeta(s)w(\theta)$  is continuous and increasing, it maps  $[0, 1]$  onto  $[0, 1]$ . Hence  $\phi_1$  is a  $C^1$  diffeomorphism from  $\overline{Q^2}$  onto  $\overline{Q^2}$ . Moreover,  $\phi_1$  agrees with the identity map on  $\partial Q^2$ . We now turn to the proof of (3.2).

Let  $a \in [0, 1]$ . By definition of  $w$  and  $g$ ,

$$\int_0^1 ds \int_0^{a+\zeta(s)w(a)} g(s, \theta) \, d\theta = m_T a.$$

By definition of  $\phi_1$ , this can be written

$$\int_{\phi_1((0,1) \times (0,a))} g = m_T a.$$

By the change of variables formula, this gives

$$\int_0^a d\theta \int_0^1 g \circ \phi_1(s, \theta) \text{Jac } \phi_1(s, \theta) \, ds = m_T a.$$

This implies that for every  $\theta \in (0, 1)$ ,

$$\int_0^1 g \circ \phi_1(s, \theta) \text{Jac } \phi_1(s, \theta) \, ds = m_T,$$

which completes the proof of (3.2). It remains to prove (3.3). Similarly to (3.10),

$$\partial_b G(a, w(a))(1 + \zeta(s)w'(a)) \leq 2m_\sigma \nu.$$

Now,

$$2m_\sigma \mu \int_0^1 \zeta(s)s \, ds \leq \partial_b G(a, b) \leq 2m_\sigma \nu \int_0^1 \zeta(s)s \, ds.$$

We can estimate the right hand side by  $2m_\sigma \nu$  while

$$\begin{aligned} 2m_\sigma \mu \int_0^1 \zeta(s)s \, ds &\geq 2m_\sigma \mu \left( \int_0^1 s \, ds - \int_0^1 |\zeta(s) - 1| \, ds \right) \\ &\geq 2m_\sigma \mu \left( \int_0^1 s \, ds - \varepsilon \right) \geq \frac{4m_\sigma \mu}{5}. \end{aligned}$$

Hence

$$\frac{4m_\sigma \mu}{5} \leq \partial_b G(a, b) \leq 2m_\sigma \nu$$

and thus

$$(3.11) \quad \frac{\mu}{4\nu} \leq 1 + \zeta(s)w'(a) \leq 4\frac{\nu}{\mu}.$$

Since  $\text{Jac } \phi_1(s, \theta) = 1 + \zeta(s)w'(\theta)$ , this completes the proof of (3.3). The lemma is proven.

□

We proceed with the construction of the diffeomorphism  $\Phi$  that we search in the form  $\phi_2 \circ \phi_1 \circ \phi_0$ . To simplify the notation, we consider the map  $g_1 : Q^2 \rightarrow \mathbb{R}$  defined as follows:

$$\begin{aligned} g_1(s, \theta) &= \text{Jac } (\phi_2 \circ \phi_1)(s, \theta) \\ &= g \circ \phi_1(s, \theta) \text{Jac } \phi_1(s, \theta) \\ &= s \det(\gamma, \gamma')(\theta + \zeta(s)w(\theta))(1 + \zeta(s)w'(\theta)). \end{aligned}$$

Observe that for every  $s \in (0, 1]$ ,  $\theta \in [0, 1]$ ,  $g_1(s, \theta) > 0$ .

**LEMMA 3.3.** *There exists a Lipschitz homeomorphism  $\phi_0 : \overline{Q^2} \rightarrow \overline{Q^2}$  which is  $C^1$  on  $Q^2$  and such that*

1. *for every  $\theta \in [0, 1]$ ,  $\phi_0(0, \theta) = (0, \theta)$  and  $\phi_0(1, \theta) = (1, \theta)$ ,*
2. *for every  $(s, \theta) \in Q^2$ ,*

$$\text{Jac } (\phi_2 \circ \phi_1 \circ \phi_0)(s, \theta) = g_1 \circ \phi_0(s, \theta) \text{Jac } \phi_0(s, \theta) = 2m_T s.$$

3. *for every  $(s, \theta) \in Q^2$ ,*

$$|\partial_s \phi_0(s, \theta)| \leq C,$$

where  $C$  only depends on  $\nu/\mu$ .

*Proof.* For every  $(s, \theta) \in \overline{Q^2}$ , we denote by  $v(s, \theta)$  the unique element of  $[0, 1]$  such that

$$(3.12) \quad \int_0^{v(s, \theta)} g_1(s', \theta) \, ds' = m_T s^2.$$

The map  $v$  is well-defined since  $g_1(s, \theta) > 0$  for every  $s \in (0, 1]$ ,  $\theta \in [0, 1]$  and also because

$$\int_0^1 g_1(s', \theta) ds' = m_T.$$

This is exactly the reason why we constructed  $g_1$  in Lemma 3.2. Moreover,  $v(0, \theta) = 0$  and  $v(1, \theta) = 1$ . By the implicit function theorem,  $v$  is  $C^1$  on  $(0, 1] \times [0, 1]$  and satisfies  $g_1(v(s, \theta), \theta) \partial_s v(s, \theta) = 2m_T s$ ; that is,

$$(3.13) \quad v(s, \theta) \det(\gamma, \gamma')(\theta + \zeta \circ v(s, \theta) w(\theta))(1 + \zeta \circ v(s, \theta) w'(\theta)) \partial_s v(s, \theta) = 2m_T s.$$

In particular, for every  $(s, \theta) \in Q^2$ ,  $\partial_s v(s, \theta) > 0$ .

We deduce from Lemma 3.2 that for every  $(s, \theta) \in Q^2$ ,

$$m_\sigma \frac{\mu^2}{2\nu} s \leq g_1(s, \theta) \leq \frac{8m_\sigma \nu^2}{\mu} s.$$

Since  $m_\sigma \mu \leq m_T \leq m_\sigma \nu$ , it follows from (3.12) that

$$(3.14) \quad \frac{\mu}{2\nu} s \leq v(s, \theta) \leq 2 \frac{\nu}{\mu} s.$$

From (3.13) and (3.11),

$$(3.15) \quad 0 < v(s, \theta) \partial_s v(s, \theta) \leq 4 \frac{\nu^2}{\mu^2} s.$$

In view of (3.14), this also implies that

$$(3.16) \quad 0 < \partial_s v(s, \theta) \leq 8 \frac{\nu^3}{\mu^3}.$$

A similar argument proves that  $\partial_\theta v \in L^\infty(Q^2)$ . We now define

$$\phi_0(s, \theta) = (v(s, \theta), \theta).$$

Then  $\phi_0$  is a homeomorphism from  $\overline{Q^2}$  onto  $\overline{Q^2}$  which is  $C^1$  on  $Q^2$ . Moreover,  $\phi_0(0, \theta) = (0, \theta)$ ,  $\phi_0(1, \theta) = (1, \theta)$  and  $\text{Jac } \phi_0(s, \theta) = \partial_s v(s, \theta)$ . By differentiation of (3.12), we get

$$2m_T s = g_1(v(s, \theta), \theta) \partial_s v(s, \theta) = g_1 \circ \phi_0(s, \theta) \text{Jac } \phi_0(s, \theta).$$

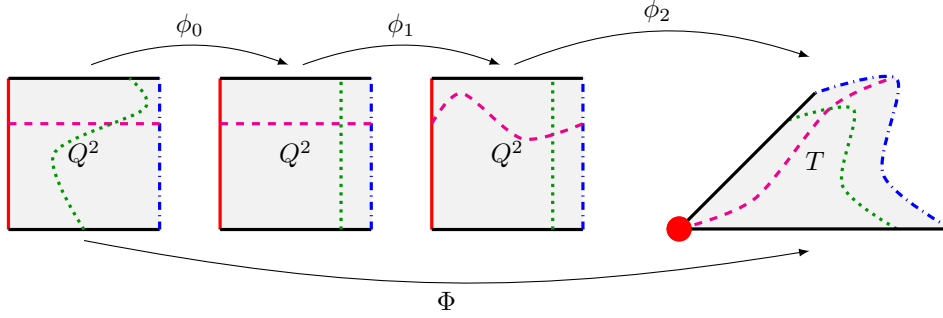
This completes the proof of the lemma.  $\square$

Lemmas 3.2, 3.3 clearly show that, as announced, the map

$$\Phi = \phi_2 \circ \phi_1 \circ \phi_0,$$

satisfies all the assumptions of Lemma 3.1. The structure is summarized in Figure 3.1:

- The side  $\{s = 0\}$  of  $Q^2$  (in red solid on the figure) is pointwise preserved by  $\phi_0$  and  $\phi_1$  and mapped to the vertex of  $T$  (also in red) by  $\phi_2$ .
- The side  $\{s = 1\}$  of  $Q^2$  (in blue dashdotted on the figure) is pointwise preserved by  $\phi_0$  and  $\phi_1$  and mapped to  $\sigma$  by  $\phi_2$ .
- The horizontal segments  $\{\theta = \text{cte}\}$  of  $Q^2$  (in magenta dashed) are preserved as a whole by  $\phi_0$  then deformed by  $\phi_1$  and  $\phi_2$ .

FIG. 3.1. Construction of the diffeomorphism  $\Phi$ 

- The vertical segments  $\{s = \text{cte}\}$  (in green dotted) are preserved as a whole by  $\phi_1$  and deformed by  $\phi_0^{-1}$  and  $\phi_2$ .

In order to complete the proof of Theorem 2.1, it remains to prove the following estimate.

LEMMA 3.4. *There exists a constant  $C > 0$  such that*

$$|\partial_s \Phi(s, \theta)| \leq C \frac{\nu^3}{\mu^3} (\text{diam}(T) + m_\sigma).$$

*Proof.* By definition, we have

$$\Phi(s, \theta) = v(s, \theta) \gamma(\theta + \zeta \circ v(s, \theta) w(\theta)).$$

Hence,

$$\begin{aligned} \partial_s \Phi(s, \theta) &= (\partial_s v(s, \theta)) \gamma(\theta + \zeta \circ v(s, \theta) w(\theta)) \\ &\quad + v(s, \theta) \partial_s v(s, \theta) \zeta'(v(s, \theta)) w(\theta) \gamma'(\theta + \zeta \circ v(s, \theta) w(\theta)). \end{aligned}$$

By construction,  $|\zeta'|_{L^\infty} \leq 10/\varepsilon = 100\nu/\mu$  and  $|w|_{L^\infty} \leq 1$ . It then follows from Lemma 3.3, see (3.15) and (3.16), that

$$|\partial_s \Phi(s, \theta)| \leq C \frac{\nu^3}{\mu^3} (\text{diam}(T) + m_\sigma).$$

□

Lemma 3.4 together with Lemma 3.1 yield the desired inequality (2.3).

**4. A Poincaré inequality.** In this section, we derive a Poincaré inequality related to Theorem 2.1.

THEOREM 4.1. *We consider the same assumption (2.2) as in Theorem 2.1. Then for every  $p \in [1, +\infty[$  and every  $u \in W^{1,p}(T)$ ,*

$$\frac{1}{m_T} \int_T \left| u - \frac{1}{m_\sigma} \int_\sigma u \right|^p \leq C (\text{diam}(T) + m_\sigma)^p \frac{1}{m_T} \int_T |Du|^p,$$

where  $C$  only depends on  $p$  and on the ratio  $\frac{\nu}{\mu}$ .

This result is a consequence of the inequality proved in the previous section and of the following lemma.

LEMMA 4.2. *Under the assumption (2.2), for every  $u \in W^{1,p}(T)$ , we have*

$$\frac{1}{m_T^2} \int_T \int_T |u(x) - u(x')|^p dx dx' \leq C(\text{diam}(T) + m_\sigma)^p \frac{1}{m_T} \int_T |Du|^p,$$

where  $C$  only depends on  $p$  and on the ratio  $\frac{\nu}{\mu}$ .

*Proof.* Let us introduce the reference unit triangle  $T_0$  defined by

$$T_0 = \{(a, b) \in ]0, 1[^2, \ b < a\}.$$

On this domain, the following inequality classically holds

$$(4.1) \quad \int_{T_0} \int_{T_0} |v(y) - v(y')|^p dy dy' \leq C_p \int_{T_0} |Dv(y)|^p dy, \quad \forall v \in W^{1,p}(T_0),$$

with a value of  $C_p > 0$  depending only on  $p$ .

We introduce (see Figure 4.1) the following diffeomorphism from  $T_0$  onto  $T$

$$\Psi : (a, b) \in T_0 \mapsto a\gamma(b/a) \in T.$$

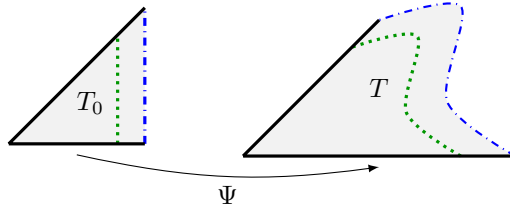


FIG. 4.1. The diffeomorphism  $\Psi$

We proceed now with the estimate of the derivatives of  $\Psi$ . An immediate computation shows that

$$D\Psi(a, b) = (\gamma(b/a) - (b/a)\gamma'(b/a), \gamma'(b/a)),$$

so that we get

$$\text{Jac } \Psi(a, b) = \det(\gamma(b/a), \gamma'(b/a)),$$

and thus, by (2.4) (which is a consequence of (2.2)), we deduce that

$$(4.2) \quad 0 < 2\mu m_\sigma \leq \text{Jac } \Psi(a, b) \leq 2\nu m_\sigma, \quad \forall (a, b) \in T_0,$$

and

$$(4.3) \quad \|D\Psi\|_\infty \leq (\|\gamma\|_\infty + \|\gamma'\|_\infty) \leq (\text{diam}(T) + m_\sigma).$$

For any  $u \in C^1(\bar{T})$  we set  $v = u \circ \Psi \in W^{1,p}(T_0)$  and we use  $\Psi$  as a change of variables

$$\begin{aligned} \int_T \int_T |u(x) - u(x')|^p dx dx' &= \int_{T_0} \int_{T_0} |v(y) - v(y')|^p \text{Jac } \Psi(y) \text{Jac } \Psi(y') dy dy' \\ &\leq \|\text{Jac } \Psi\|_\infty^2 \int_{T_0} \int_{T_0} |v(y) - v(y')|^p dy dy'. \end{aligned}$$

Then by (4.1) and the change of variables  $x = \Psi(y)$  again, we get

$$\begin{aligned}
\int_T \int_T |u(x) - u(x')|^p dx dx' &\leq C_p \|\text{Jac } \Psi\|_\infty^2 \int_{T_0} |Dv(y)|^p dy \\
&\leq C_p \frac{\|\text{Jac } \Psi\|_\infty^2}{\inf_{T_0} \text{Jac } \Psi} \int_{T_0} |Dv(y)|^p \text{Jac } \Psi(y) dy \\
&= C_p \frac{\|\text{Jac } \Psi\|_\infty^2}{\inf_{T_0} \text{Jac } \Psi} \int_T |(Dv)(\Psi^{-1}(x))|^p dx \\
&= C_p \frac{\|\text{Jac } \Psi\|_\infty^2}{\inf_{T_0} \text{Jac } \Psi} \int_T |Du(x)|^p \|D\Psi(\Psi^{-1}(x))\|^p dx \\
&\leq C_p \frac{\|\text{Jac } \Psi\|_\infty^2 \|D\Psi\|_\infty^p}{\inf_{T_0} \text{Jac } \Psi} \int_T |Du(x)|^p dx.
\end{aligned}$$

By using the previous estimates (4.2) and (4.3) we conclude that

$$\begin{aligned}
\int_T \int_T |u(x) - u(x')|^p dx dx' &\leq C_p \frac{2m_\sigma \nu^2}{\mu} (\text{diam}(T) + m_\sigma)^p \int_T |Du(x)|^p dx \\
&\leq C_p \frac{2m_\sigma \nu^2}{\mu^2} \mu (\text{diam}(T) + m_\sigma)^p \int_T |Du(x)|^p dx.
\end{aligned}$$

Since  $\mu m_\sigma \leq m_T$ , we finally obtain

$$\int_T \int_T |u(x) - u(x')|^p dx dx' \leq C m_T (\text{diam}(T) + m_\sigma)^p \int_T |Du(x)|^p dx$$

for a  $C$  depending only on  $p$  and  $\nu/\mu$ . Dividing this inequality by  $m_T^2$  gives the claim.  $\square$

We can now prove the claimed Poincaré inequality.

*Proof of Theorem 4.1.* By the triangle inequality,

$$\left| \frac{1}{m_T} \int_T u - \frac{1}{m_\sigma} \int_\sigma u \right|^p \leq C_p \frac{1}{m_T} \int_T \left| u - \frac{1}{m_T} \int_T u \right|^p + C_p \left| \frac{1}{m_T} \int_T u - \frac{1}{m_\sigma} \int_\sigma u \right|^p.$$

One can estimate the second term with Theorem 2.1:

$$\left| \frac{1}{m_T} \int_T u - \frac{1}{m_\sigma} \int_\sigma u \right|^p \leq C (m_\sigma + \text{diam}(T))^p \frac{1}{m_T} \int_T |Du|^p,$$

where  $C$  only depends on the ratio  $\frac{\nu}{\mu}$ . By Jensen's inequality, the first term is not larger than the quantity

$$\frac{1}{m_T^2} \int_T \int_T |u(x) - u(x')|^p dx dx',$$

which is, in turn, estimated by using Lemma 4.2.  $\square$

**5. The higher dimensional case.** Let  $n \geq 2$ . Let  $\gamma : \overline{Q^{n-1}} \rightarrow \mathbb{R}^n$  be a  $C^1$  map on the closure of the unit cube  $Q^{n-1} = (0, 1)^{n-1}$  such that  $\gamma$  is one-to-one and  $|\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma| > 0$  on  $\overline{Q^{n-1}}$ . We denote by  $\sigma = \gamma(Q^{n-1})$  the corresponding hypersurface and by  $T$  the set:

$$T = \{s\gamma(\theta) : s \in (0, 1), \theta \in Q^{n-1}\}.$$

We assume that for every  $\theta \in \overline{Q^{n-1}}$ ,

$$\{s\gamma(\theta), s \geq 0\} \cap \sigma = \gamma(\theta).$$

THEOREM 5.1. *We assume that there exist  $\mu, \nu > 0$  such that for every  $\theta \in Q^{n-1}$ ,*

$$(5.1) \quad \mu \leq \frac{\det(\gamma, \partial_1 \gamma, \dots, \partial_{n-1} \gamma)(\theta)}{n|\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma|(\theta)} \leq \nu.$$

*Then for every  $p \in [1, +\infty[$  and every  $u \in W^{1,p}(T)$ ,*

$$\left| \frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_T} \int_T u \right|^p \leq C(\|D\gamma\|_{L^\infty} + \text{diam}(T))^p \frac{1}{m_T} \int_T |Du|^p,$$

*where  $C$  only depends on the ratio  $\frac{\nu}{\mu}$ .*

REMARK 5.1. *Observe that the quantity in (5.1) is invariant with respect to the parametrization of  $\sigma$ . More precisely, let  $\psi : Q^{n-1} \rightarrow Q^{n-1}$  be a  $C^1$  map such that  $|\text{Jac } \psi| > 0$  everywhere and  $\tilde{\gamma} = \gamma \circ \psi$ . Then*

$$\partial_1 \tilde{\gamma} \wedge \dots \wedge \partial_{n-1} \tilde{\gamma} = ((\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma) \circ \psi) \text{Jac } \psi.$$

*This implies*

$$\begin{aligned} \det(\tilde{\gamma}, \partial_1 \tilde{\gamma}, \dots, \partial_{n-1} \tilde{\gamma}) &= \langle \tilde{\gamma}, \partial_1 \tilde{\gamma} \wedge \dots \wedge \partial_{n-1} \tilde{\gamma} \rangle \\ &= \langle \tilde{\gamma}, ((\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma) \circ \psi) \text{Jac } \psi \rangle \\ &= (\det(\gamma, \partial_1 \gamma, \dots, \partial_{n-1} \gamma) \circ \psi) \text{Jac } \psi. \end{aligned}$$

*Hence,*

$$\frac{\det(\tilde{\gamma}, \partial_1 \tilde{\gamma}, \dots, \partial_{n-1} \tilde{\gamma})}{|\partial_1 \tilde{\gamma} \wedge \dots \wedge \partial_{n-1} \tilde{\gamma}|} = \frac{\det(\gamma, \partial_1 \gamma, \dots, \partial_{n-1} \gamma) \circ \psi}{|\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma| \circ \psi}.$$

*In particular, when  $\psi$  is a  $C^1$  diffeomorphism,  $\gamma$  satisfies (5.1) if and only if  $\tilde{\gamma}$  satisfies (5.1).*

In view of Lemma A.1 given in Appendix, one can assume without loss of generality that

$$(5.2) \quad |\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma|(\theta) = m_\sigma, \quad \forall \theta \in Q^{n-1}.$$

Exactly as in the 2 dimensional case, the proof of Theorem 5.1 is a consequence of the following lemmas.

LEMMA 5.2. *Assume that there exists a Lipschitz continuous map  $\Phi : [0, 1] \times \overline{Q^{n-1}} \rightarrow \overline{T}$  such that*

1.  $\Phi$  is a  $C^1$  diffeomorphism from  $(0, 1) \times Q^{n-1}$  onto  $T$ ,
2.  $\Phi(0, \theta) = (0, 0)$ ,
3.  $\Phi(1, \theta) = \gamma(\theta)$ ,
4.  $\text{Jac } \Phi(s, \theta) = nm_T s^{n-1}$ .

*Then for every  $u \in W^{1,1}(T)$ , we have*

$$\frac{1}{m_\sigma} \int_\sigma u - \frac{1}{m_T} \int_T u = \frac{1}{nm_T} \int_T Du(x, y) [s \partial_s \Phi(s, \theta)]_{(s, \theta) = \Phi^{-1}(x, y)} dx dy.$$



The proof is very similar to the proof of Lemma 3.1 and we omit it. In particular, we observe that by (5.2), we have

$$\int_{\sigma} u = \int_{Q^{n-1}} u(\gamma(\theta)) |\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma|(\theta) d\theta = m_{\sigma} \int_{Q^{n-1}} u(\gamma(\theta)) d\theta = m_{\sigma} \int_{Q^{n-1}} u(\Phi(1, \theta)) d\theta.$$

The construction of a suitable  $\Phi$  then follows the lines of the two dimensional case. We proceed to indicate the major changes. We still denote by  $\phi_2$  the map

$$\phi_2 : (s, \theta) \in [0, 1] \times \overline{Q^{n-1}} \rightarrow s\gamma(\theta) \in \bar{T}.$$

Then  $\text{Jac } \phi_2(s, \theta) = s^{n-1} \det(\gamma, \partial_1 \gamma, \dots, \partial_{n-1} \gamma)(\theta)$ , and we shall look for  $\Phi$  under the form  $\Phi = \phi_2 \circ \phi_2 \circ \phi_0$ .

LEMMA 5.3. *There exists a  $C^1$  diffeomorphism  $\phi_1 : [0, 1] \times \overline{Q^{n-1}} \rightarrow [0, 1] \times \overline{Q^{n-1}}$  such that*

1. *for every  $\theta \in Q^{n-1}$ ,  $\phi_1(0, \theta) = (0, \theta)$  and  $\phi_1(1, \theta) = (1, \theta)$ ,*
2. *for every  $\theta \in Q^{n-1}$ ,*

$$\int_0^1 \text{Jac}(\phi_2 \circ \phi_1)(s, \theta) ds = m_T.$$

3. *for every  $s \in [0, 1]$  and every  $\theta \in \overline{Q^{n-1}}$ ,*

$$(5.3) \quad |\partial_s \phi_1(s, \theta)| \leq C \quad , \quad C' \leq \text{Jac } \phi_1(s, \theta) \leq C''$$

where  $C, C'$  and  $C'' > 0$  only depend on the ratio  $\nu/\mu$ .

*Proof.*

#### Step 1: Construction of an auxiliary function

Fix  $1 \leq k \leq n-1$ . Given  $\theta \in \overline{Q^{n-1}}$ , we use the notation  $\theta^{k-1} = (\theta_1, \dots, \theta_{k-1})$  and  $\theta' = (\theta_{k+1}, \dots, \theta_{n-1})$ . Assume that there exists a  $C^1$  function  $h : [0, 1] \times \overline{Q^{n-1}} \rightarrow \mathbb{R}$  such that

1. *for every  $\theta' = (\theta_{k+1}, \dots, \theta_{n-1}) \in \overline{Q^{n-k-1}}$ ,*

$$\int_{(0,1) \times Q^k} h(s, \theta^k, \theta') ds d\theta^k = m_T,$$

2. *there exist  $C < 1 < C'$  only depending on  $\nu/\mu$  such that for every  $\theta' \in \overline{Q^{n-k-1}}$ ,*

$$C n m_{\sigma} \mu s^{n-1} \leq h \leq C' n m_{\sigma} \nu s^{n-1}.$$

We introduce

$$\varepsilon = \frac{C\mu}{2nC'\nu}$$

and a cut-off function:  $\zeta \in C_c^\infty((0, 1) \times Q^{k-1})$  such that  $0 \leq \zeta \leq 1 + \varepsilon$  and

$$\int_{(0,1) \times Q^{k-1}} \zeta(s, \theta^{k-1}) ds d\theta^{k-1} = 1 \quad , \quad \int_{(0,1) \times Q^{k-1}} |\zeta(s, \theta^{k-1}) - 1| ds d\theta^{k-1} < \varepsilon.$$

We can further require that

$$\|D\zeta\|_{L^\infty} \leq \frac{C_n}{\varepsilon}$$

for some constant  $C_n$  which only depends on  $n$ . Consider the map

$$G_{\theta'} : (a, b) \mapsto \int_{(0,1) \times Q^{k-1}} ds d\theta^{k-1} \int_0^{a+\zeta(s, \theta^{k-1})b} h(s, \theta^k, \theta') d\theta_k.$$

Then  $G_{\theta'}$  is well-defined and  $C^2$  on the set  $\{(a, b) : 0 \leq a \leq 1, \frac{-a}{1+\varepsilon} \leq b \leq \frac{1-a}{1+\varepsilon}\}$ . As in the proof of Lemma 3.2, there exists a  $C^2$  map  $w : [0, 1] \times \overline{Q^{n-k-1}} \mapsto [-1, 1]$  such that for every  $a \in [0, 1]$  and every  $\theta' \in \overline{Q^{n-k-1}}$ ,  $G_{\theta'}(a, w(a, \theta')) = m_T a$ ; that is,

$$\int_{(0,1) \times Q^{k-1}} ds d\theta^{k-1} \int_0^{a+\zeta(s, \theta^{k-1})w(a, \theta')} h(s, \theta^k, \theta') d\theta_k = m_T a.$$

Moreover,  $w(0, \theta') = 0 = w(1, \theta')$  and by differentiation of the above identity with respect to  $a$ , one gets

$$(5.4) \quad \int_{(0,1) \times Q^{k-1}} h(s, \theta^{k-1}, a + \zeta(s, \theta^{k-1})w(a, \theta'), \theta') \\ (1 + \zeta(s, \theta^{k-1})\partial_a w(a, \theta')) ds d\theta^{k-1} = m_T.$$

As in the proof of Lemma 3.2, this leads to the following estimate:

$$(5.5) \quad D \leq 1 + \zeta(s, \theta^{k-1})\partial_a w(a, \theta') \leq D'$$

for some constants  $D, D' > 0$  which only depend on  $\nu/\mu$ . We omit the details.

**Step 2 : construction of  $\phi_1$**

Let  $\psi_{n+1} = id$  and  $h_{n+1} = h_n = \text{Jac } \phi_2$ . We construct by induction on  $k = n, \dots, 0$  two sequences of maps

$$\psi_{k+1} : [0, 1] \times \overline{Q^{n-1}} \rightarrow [0, 1] \times \overline{Q^{n-1}} \quad , \quad h_k : [0, 1] \times \overline{Q^{n-1}} \rightarrow \mathbb{R}$$

such that for  $k = 0, \dots, n$ ,

1. The map  $\psi_{k+1}$  is a  $C^2$  diffeomorphism from  $[0, 1] \times \overline{Q^{n-1}}$  onto  $[0, 1] \times \overline{Q^{n-1}}$ ,
2. For every  $\theta \in Q^{n-1}$ ,  $\psi_{k+1}(0, \theta) = (0, \theta)$  and  $\psi_{k+1}(1, \theta) = (1, \theta)$ ,
3. We have

$$h_k = (h_{k+1} \circ \psi_{k+1}) \text{Jac } \psi_{k+1},$$

4. There exist two constants  $0 < C_k < 1 < C'_k$  depending only on  $\nu/\mu$  such that for every  $s \in [0, 1]$  and every  $\theta \in \overline{Q^{n-1}}$ ,

$$(5.6) \quad C_k \leq \text{Jac } \psi_{k+1}(s, \theta) \leq C'_k, \\ C_k n m_\sigma \mu s^{n-1} \leq h_k(s, \theta) \leq C'_k n m_\sigma \nu s^{n-1}.$$

5. We have

$$\int_{(0,1) \times Q^k} h_k(s, \theta^k, \theta') ds d\theta^k = m_T,$$

where  $\theta^k = (\theta_1, \dots, \theta_k)$  and  $\theta' = (\theta_{k+1}, \dots, \theta_{n-1})$ .

Observe that these conditions are satisfied for  $k = n$ . We assume that for some  $k \geq 1$ ,  $\psi_{n+1}, \psi_n, \dots, \psi_{k+1}$ , and thus  $h_n, \dots, h_k$  are already constructed and satisfy the above properties. Let  $\varepsilon_k = \frac{C_k \mu}{2n C'_k \nu}$  and  $\zeta_k$  a function which satisfies the properties of the function  $\zeta$  introduced in Step 1, with  $\varepsilon = \varepsilon_k$ . We apply Step 1 to the function  $h_k$  with  $\zeta_k$  and  $C' = C'_k$ ,  $C' = C'_k$ . This gives a function  $w_k : [0, 1] \times \overline{Q^{n-k-1}} \rightarrow [-1, 1]$  satisfying the properties enumerated in Step 1.

We then construct  $\psi_k$  as follows

$$\psi_k(s, \theta_1, \dots, \theta_{n-1}) = (s, \theta^{k-1}, v_k(s, \theta), \theta') = (s, \theta_1, \dots, \theta_{k-1}, v_k(s, \theta), \theta_{k+1}, \dots, \theta_{n-1})$$

with

$$v_k(s, \theta) = \theta_k + \zeta_k(s, \theta^{k-1}) w_k(\theta_k, \theta').$$

Since  $\text{Jac } \psi_k = 1 + \zeta_k \partial_{\theta_k} w_k$ , (5.5) implies

$$C_{k-1} \leq \text{Jac } \psi_k \leq C'_{k-1},$$

where  $0 < C_{k-1} < 1 < C'_{k-1}$  only depend on  $\nu/\mu$ .

Let  $h_{k-1} = (h_k \circ \psi_k) \text{Jac } \psi_k$ . By (5.4),

$$\begin{aligned} \int_{(0,1) \times Q^{k-1}} h_{k-1} \\ &= \int_{(0,1) \times Q^{k-1}} h_k(s, \theta^{k-1}, \theta_k + \zeta_k(s, \theta^{k-1}) w_k(\theta_k, \theta'), \theta') \text{Jac } \psi_k(s, \theta) ds d\theta^{k-1} \\ &= m_T. \end{aligned}$$

As in the proof of Lemma 3.2, one can check that  $h_{k-1}$  and  $\psi_k$  satisfy all the remaining properties, even if it means changing the actual value of the constants  $C_{k-1}$  and  $C'_{k-1}$ . This completes the construction by induction of  $h_0, \dots, h_n$  and  $\psi_1, \dots, \psi_{n+1}$ . We now define

$$\phi_1 = \psi_n \circ \psi_{n-1} \circ \dots \circ \psi_1.$$

Then  $\phi_1$  is a  $C^1$  diffeomorphism from  $[0, 1] \times \overline{Q^{n-1}}$  onto itself and  $\phi_1$  coincides with the identity on  $\{0\} \times \overline{Q^{n-1}}$  and  $\{1\} \times \overline{Q^{n-1}}$ . By construction,

$$\begin{aligned} h_0 &= (h_1 \circ \psi_1) \text{Jac } \psi_1 = (((h_2 \circ \psi_2) \text{Jac } \psi_2) \circ \psi_1) \text{Jac } \psi_1 \\ &= (h_2 \circ \psi_2 \circ \psi_1) \text{Jac } (\psi_2 \circ \psi_1) = \dots \\ (5.7) \quad &= (h_n \circ \psi_n \circ \dots \circ \psi_1) \text{Jac } (\psi_n \circ \dots \circ \psi_1) \\ &= (h_n \circ \phi_1) \text{Jac } \phi_1 \\ &= (\text{Jac } \phi_2 \circ \phi_1) \text{Jac } \phi_1 = \text{Jac } (\phi_2 \circ \phi_1). \end{aligned}$$

In particular, we deduce that

$$m_T = \int_{(0,1)} h_0(s, \theta') ds = \int_{(0,1)} \text{Jac } (\phi_2 \circ \phi_1)(s, \theta') ds.$$

### Step 3: Proof of (5.3)

For every  $k = 1, \dots, n-1$ , and every  $(s, \theta) \in [0, 1] \times \overline{Q^{n-1}}$ , let us introduce the notation:

$$[(s, \theta)]^k = \theta^k = (\theta_1, \dots, \theta_k).$$

Then

$$\psi_1(s, \theta) = (s, [\psi_1(s, \theta)]^1, \theta') = (s, \theta_1 + \zeta_1(s)w_1(\theta_1, \theta'), \theta')$$

with  $\theta' = (\theta_2, \dots, \theta_{n-1})$ , and for every  $k = 1, \dots, n$ ,

$$\begin{aligned} & \psi_k \circ \psi_{k-1} \circ \dots \circ \psi_1(s, \theta) \\ &= (s, [\psi_{k-1} \circ \dots \circ \psi_1(s, \theta)]^{k-1}, \theta_k + \zeta_k(s, [\psi_{k-1} \circ \dots \circ \psi_1(s, \theta)]^{k-1})w_k(\theta_k, \theta'), \theta') \end{aligned}$$

where  $\theta' = (\theta_{k+1}, \dots, \theta_n)$ .

Since  $\|w_k\|_{L^\infty} \leq 1$ , it follows by induction on  $k = 1, \dots, n$  that there exists  $A_k > 0$  which depends only on  $\nu/\mu$  such that

$$\|\partial_s(\psi_k \circ \dots \circ \psi_1)\|_{L^\infty} \leq A_k.$$

In particular,  $\|\partial_s \phi_1\|_{L^\infty} \leq A_n$ . The fact that  $C \leq \text{Jac } \phi_1 \leq C'$ , for some  $C, C' > 0$  depending only on  $\nu/\mu$ , follows from (5.6) and the identity

$$\text{Jac } \phi_1 = \prod_{k=1}^n \text{Jac } \psi_{k+1}.$$

The lemma is proved.  $\square$

LEMMA 5.4. *There exists an homeomorphism  $\phi_0 : [0, 1] \times \overline{Q^{n-1}} \rightarrow [0, 1] \times \overline{Q^{n-1}}$  which is  $C^1$  on  $(0, 1) \times Q^{n-1}$  and such that*

1. *for every  $\theta \in (0, 1)$ ,  $\phi_0(0, \theta) = (0, \theta)$  and  $\phi_0(1, \theta) = (1, \theta)$ ,*
2. *for every  $(s, \theta) \in (0, 1) \times Q^{n-1}$ ,*

$$\text{Jac } (\phi_2 \circ \phi_1 \circ \phi_0)(s, \theta) = nm_T s^{n-1},$$

3. *there exists a  $C^1$  map  $v : (0, 1) \times Q^{n-1} \rightarrow [0, 1]$  such that for every  $(s, \theta) \in (0, 1) \times Q^{n-1}$ ,  $\phi_0(s, \theta) = (v(s, \theta), \theta)$  and*

$$|v(s, \theta)| \leq Cs \quad , \quad |\partial_s v(s, \theta)| \leq C$$

where  $C$  only depends on  $\nu/\mu$ .

The proof is essentially the same as the proof of Lemma 3.3. The only difference is that now by (5.6) for  $k = 0$  and (5.7)

$$Cnm_\sigma \mu s^{n-1} \leq \text{Jac } (\phi_2 \circ \phi_1) \leq C'nm_\sigma \nu s^{n-1}.$$

The rest of the proof is the same and we omit it.

Lemma 5.3 and Lemma 5.4 show that the map

$$\Phi = \phi_2 \circ \phi_1 \circ \phi_0,$$

satisfies all the assumptions of Lemma 5.2. The analogue of Lemma 3.4 is given by the following:

LEMMA 5.5. *There exists a constant  $C > 0$  such that*

$$|\partial_s \Phi(s, \theta)| \leq C(\text{diam}(T) + \|D\gamma\|_{L^\infty})$$

where  $C$  only depends on  $\nu/\mu$ .

*Proof.* By construction,

$$\Phi(s, \theta) = \phi_2 \circ \phi_1 \circ \phi_0(s, \theta) = \phi_2(\phi_1(v(s, \theta), \theta)).$$

Hence,

$$\partial_s \Phi(s, \theta) = D\phi_2(\phi_1(v(s, \theta), \theta)) \cdot \partial_s \phi_1(v(s, \theta), \theta) \cdot \partial_s v(s, \theta).$$

By Lemma 5.4,  $|\partial_s v|, |v| \leq C$  and by Lemma 5.3, we have  $|\partial_s \phi_1| \leq C$ . Hence

$$|\partial_s \Phi(s, \theta)| \leq C(\text{diam}(T) + \|D\gamma\|_{L^\infty})$$

where  $C$  only depends on  $\nu/\mu$ .  $\square$

We can now proceed to the proof of the main theorem.

*Proof of Theorem 5.1.* By Jensen's inequality, we only need to prove the case  $p = 1$ . By a standard approximation argument, one can further assume that  $\gamma \in C^2(\overline{Q^{n-1}})$ .

The required inequality is then a consequence of the equality given by Lemma 5.2, and by the construction and estimate of a suitable  $\Phi$  given by Lemmas 5.3, 5.4 and 5.5.  $\square$

## 6. Applications to the analysis of some finite volume methods.

**6.1. Regular families of meshes of a smooth domain.** Let  $\Omega$  be a bounded domain of  $\mathbb{R}^2$  with a  $C^2$  boundary. A finite volume mesh  $\mathfrak{M}$  of  $\Omega$  is a finite family of compact subsets of  $\overline{\Omega}$  with non-empty interiors usually referred to as control volumes and denoted by the letter  $\mathcal{K}$ . This family is supposed to satisfy

$$\mathring{\mathcal{K}} \cap \mathring{\mathcal{L}} = \emptyset, \quad \forall \mathcal{K}, \mathcal{L} \in \mathfrak{M}, \mathcal{K} \neq \mathcal{L},$$

$$\overline{\Omega} = \bigcup_{\mathcal{K} \in \mathfrak{M}} \mathcal{K}.$$

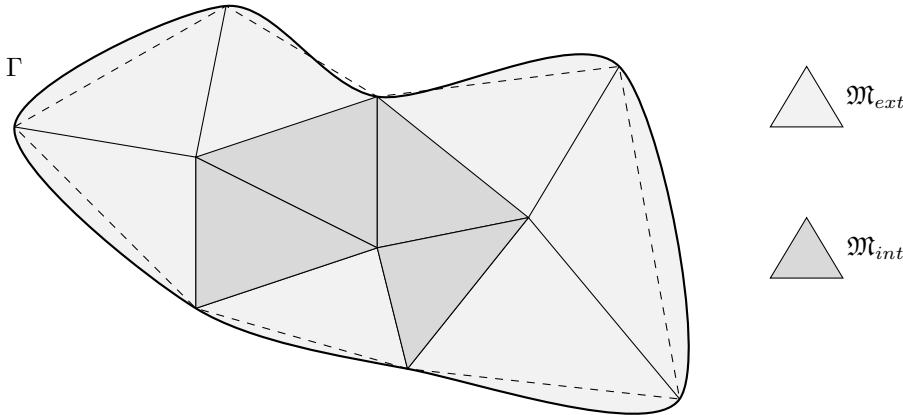


FIG. 6.1. The non-polygonal mesh  $\mathfrak{M}$  of  $\Omega$  and the two submeshes  $\mathfrak{M}_{int}$  and  $\mathfrak{M}_{ext}$

We assume that  $\mathfrak{M}$  can be split into two disjoint subsets (see Figure 6.1) as follows:

- The set of polygonal control volumes  $\mathfrak{M}_{int}$  that satisfy: for any  $\mathcal{K} \in \mathfrak{M}_{int}$ ,  $\mathcal{K}$  is polygonal and  $\mathcal{K} \cap \partial\Omega$  contains at most a finite number of points.

- The set of curved control volumes  $\mathfrak{M}_{ext}$  that satisfy : for any  $\mathcal{K} \in \mathfrak{M}_{ext}$ ,  $\mathcal{K}$  is a pseudo-triangle whose curved edge is contained in the boundary of the domain  $\Omega$ . With any such curved control volume  $\mathcal{K}$ , we associate the (real) triangle  $\tilde{\mathcal{K}}$  which possesses the same vertices as  $\mathcal{K}$  (see the dashed lines in Figure 6.1). Observe that  $\mathcal{K}$  may not be included in  $\bar{\Omega}$ .

We may now define the approximate mesh to be the following set of control volumes

$$\widetilde{\mathfrak{M}} = \bigcup_{\mathcal{K} \in \mathfrak{M}_{int}} \{\mathcal{K}\} \cup \bigcup_{\mathcal{K} \in \mathfrak{M}_{ext}} \{\tilde{\mathcal{K}}\}.$$

This is a finite volume mesh made of polygonal control volumes.

The size and the regularity of such a mesh are measured by the quantities

$$\text{size}(\mathfrak{M}) = \max_{\sigma \in \mathcal{E}} m_{\sigma}, \text{ and } \text{reg}_1(\mathfrak{M}) = \max_{\mathcal{L} \in \mathfrak{M}} \frac{\text{diam}(\mathcal{L})^2}{m_{\mathcal{L}}},$$

where  $\mathcal{E}$  is the set of the edges  $\sigma$  of all the control volumes in the mesh  $\mathfrak{M}$ . Usual convergence results in the finite volume framework assume that  $\text{size}(\mathfrak{M})$  goes to 0 and that  $\text{reg}_1(\mathfrak{M})$  remains bounded. This means that control volumes are not allowed to become flat while the mesh is refined.

The main objective of this section is to prove that, if one builds a mesh  $\mathfrak{M}$  of  $\Omega$  as described previously such that  $\text{size}(\mathfrak{M})$  is small enough, then each boundary curved control volumes  $\mathcal{K} \in \mathfrak{M}_{ext}$  satisfies the assumptions of Theorem 2.1 with a ratio  $\nu_{\mathcal{K}}/\mu_{\mathcal{K}}$  which is independent of  $\mathcal{K}$ . In other words, on such curved elements, the inequality (2.3) holds with a constant  $C$  uniformly bounded as the mesh is refined.

**PROPOSITION 6.1.** *Let  $\Omega$  be a bounded domain of class  $C^2$  in  $\mathbb{R}^2$  and  $\xi_0 > 0$ . There exists  $h_0 > 0$  depending only on  $\Omega$  and  $\xi_0$ , such that for any finite volume mesh  $\mathfrak{M}$  as described above, if*

$$(6.1) \quad \text{reg}_1(\mathfrak{M}) \leq \xi_0 \text{ and } \text{size}(\mathfrak{M}) \leq h_0,$$

*then any exterior control volume  $\mathcal{K} \in \mathfrak{M}_{ext}$  (which is a pseudo-triangle) satisfies the assumption (2.4) with two values of  $\mu$  and  $\nu$  that satisfy  $\nu/\mu = 3$ .*

*Proof.* The exterior control volume  $\mathcal{K}$  can be written in the following form (see Figure 6.2)

$$\mathcal{K} = \{(1-s)\gamma(t) : s \in [0, 1], t \in [0, h]\},$$

where the opposite vertex which is supposed to be the origin  $(0, 0)$  lies inside  $\Omega$ , and  $\gamma : [0, h] \rightarrow \mathbb{R}^2$  is a normal parametrization of the curved edge  $\sigma \subset \Gamma$ :  $\|\gamma'(t)\| = 1$  for every  $t \in [0, h]$ .

We also introduce the associated real triangle  $\tilde{\mathcal{K}}$  with vertices  $(0, 0)$ ,  $\gamma(0)$  and  $\gamma(h)$ .

First, we claim that if we assume that

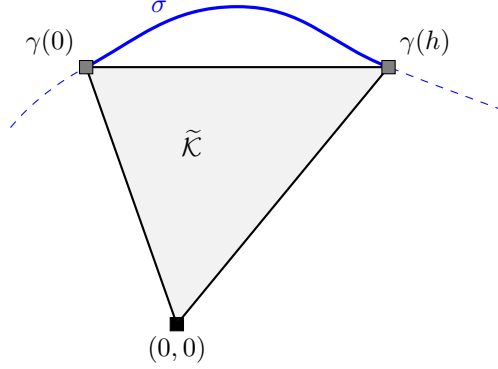
$$(6.2) \quad h_0 < \frac{1}{2\|\gamma''\|_{\infty}},$$

then we have

$$(6.3) \quad m_{\sigma} = h \leq 2\text{diam}(\tilde{\mathcal{K}}).$$

Indeed, let  $t \in [0, h]$ . By the mean value inequality, there exists  $\xi_t \in [0, h]$  such that

$$\begin{aligned} \langle \gamma'(t), \gamma(h) - \gamma(0) \rangle &= h \langle \gamma'(t), \gamma'(\xi_t) \rangle \\ &\geq h - h^2 \|\gamma'\|_{\infty} \|\gamma''\|_{\infty} = h - h^2 \|\gamma''\|_{\infty} \\ &\geq \frac{h}{2}. \end{aligned}$$

FIG. 6.2. A control volume  $\mathcal{K} \in \mathfrak{M}_{ext}$  with a curved edge  $\sigma \subset \Gamma$ 

The last inequality follows from (6.1) and (6.2). The conclusion follows from the Cauchy-Schwarz inequality and the fact that the parametrization  $\gamma$  is normal and satisfies  $\|\gamma(h) - \gamma(0)\| \leq \text{diam}(\tilde{\mathcal{K}})$ .

Then, we are going to prove relation (2.4). For any  $t \in [0, h]$ , we write the term  $\det(\gamma(t), \gamma'(t))$  as follows:

$$\begin{aligned}
 \det(\gamma(t), \gamma'(t)) &= \det\left(\gamma(0), \frac{\gamma(h) - \gamma(0)}{h}\right) \\
 &+ \det\left(\gamma(0), \gamma'(t) - \frac{\gamma(h) - \gamma(0)}{h}\right) \\
 &+ \det(\gamma(t) - \gamma(0), \gamma'(t)) \\
 &:= I_1 + I_2 + I_3.
 \end{aligned}
 \tag{6.4}$$

Now, we have to control the terms  $I_j$ ,  $j = 1, 2, 3$ .

We begin with the term  $I_1$ . Clearly, we have

$$I_1 = \frac{2}{h} m_{\tilde{\mathcal{K}}}.$$

As regards the second term in (6.4), there exists  $\zeta_t \in [0, h]$  such that,

$$|I_2| = |\det(\gamma(0), \gamma'(t) - \gamma'(\zeta_t))| \leq \|\gamma(0)\| \|\gamma''\|_\infty h \leq 2 \|\gamma''\|_\infty \text{diam}(\tilde{\mathcal{K}})^2.$$

Now, we are concerned by the last term in (6.4). There exists  $\tilde{\zeta}_t \in [0, h]$  such that,

$$|I_3| = \left| \det\left(-t\gamma'(t) + \frac{t^2}{2}\gamma''(\tilde{\zeta}_t), \gamma'(t)\right) \right| \leq \frac{h^2}{2} \|\gamma'\|_\infty \|\gamma''\|_\infty,$$

and since the parametrization is normal, we deduce by (6.3) that

$$|I_3| \leq \frac{h^2}{2} \|\gamma'\|_\infty \|\gamma''\|_\infty \leq 2 \|\gamma''\|_\infty (\text{diam}(\tilde{\mathcal{K}}))^2.$$

Gathering relations (6.4)-(6.7) we get,

$$\left| \frac{2}{h} m_{\tilde{\mathcal{K}}} - \det(\gamma(t), \gamma'(t)) \right| \leq 4 \|\gamma''\|_\infty (\text{diam}(\tilde{\mathcal{K}}))^2.$$

Thanks to the definition of  $\text{reg}_1(\mathfrak{M})$  and assumption (6.1) on  $\text{reg}_1(\mathfrak{M})$  we have,

$$\left| \det(\gamma(t), \gamma'(t)) - \frac{2}{h} m_{\tilde{\kappa}} \right| \leq 4 \|\gamma''\|_{\infty} \xi_0 m_{\tilde{\kappa}}.$$

Then, we obtain

$$m_{\tilde{\kappa}} \left( \frac{2}{h} - 4 \|\gamma''\|_{\infty} \xi_0 \right) \leq \det(\gamma(t), \gamma'(t)) \leq m_{\tilde{\kappa}} \left( \frac{2}{h} + 4 \|\gamma''\|_{\infty} \xi_0 \right).$$

Assuming that  $h_0$  satisfies, additionally to (6.2), the condition

$$(6.8) \quad h_0 < \frac{1}{4 \|\gamma''\|_{\infty} \xi_0}$$

we finally proved

$$\frac{1}{h} m_{\tilde{\kappa}} \leq \det(\gamma(t), \gamma'(t)) \leq \frac{3}{h} m_{\tilde{\kappa}},$$

which exactly gives (2.4) with a ratio  $\nu/\mu$  equal to 3 (observe that in (2.4), the parametrization is not normal but satisfies  $|\gamma'| = m_{\sigma}$  which does not change anything to the ratio  $\nu/\mu$ ). The claim is proved provided one chooses a  $h_0$  that satisfies (6.2) and (6.8).  $\square$

**6.2. Example of application : the heat equation with dynamic Ventcell boundary conditions.** As an illustration of the previous discussion we shall briefly describe a finite volume approximation of the following model problem

$$\begin{aligned} (6.9a) \quad & \partial_t u - \Delta u = 0, \quad \text{in } ]0, T[ \times \Omega, \\ (6.9b) \quad & \alpha \partial_t u|_{\Gamma} - \Delta_{\Gamma} u|_{\Gamma} + u|_{\Gamma} + \partial_n u = 0, \quad \text{in } ]0, T[ \times \Gamma, \\ (6.9c) \quad & u(0, \cdot) = u^0, \quad \text{in } \Omega. \end{aligned}$$

Here,  $\alpha \geq 0$  is a parameter,  $u|_{\Gamma}$  denotes the trace of  $u$  on the boundary  $\Gamma = \partial\Omega$  and  $\Delta_{\Gamma}$  denotes the Laplace-Beltrami operator on  $\Gamma$ .

REMARK 6.1. *The second equation of this system has to be understood as a boundary condition associated with the heat equation. It is usually referred to as a (dynamic, if  $\alpha > 0$ ) Ventcell boundary condition, see for instance [9] for a recent work on this kind of problem.*

*We also refer to [8] where the result of the present paper was used as an important tool to give a complete convergence result (and as by-product a well-posedness result) for a much more complex model. This model is known as the Cahn-Hilliard equation with dynamic boundary condition. It is a fourth-order non-linear parabolic equation assorted with a non-linear dynamic boundary condition.*

The natural energy space for the problem (6.9) is the space

$$H_{\Gamma}^1 = \{u \in H^1(\Omega), \quad u|_{\Gamma} \in H^1(\Gamma)\},$$

endowed with the norm

$$\|u\|_{H_{\Gamma}^1} = (\|\nabla u\|_{L^2(\Omega)}^2 + \|u|_{\Gamma}\|_{L^2(\Gamma)}^2 + \|\nabla_{\Gamma} u|_{\Gamma}\|_{L^2}^2)^{\frac{1}{2}},$$

where  $\nabla_{\Gamma}$  denotes the tangential gradient on  $\Gamma$ .



A well-posedness result can be proved in this space, the main ingredient being the following formal energy estimate, obtained by multiplying the first equation by  $\partial_t u$  and the boundary condition by  $\partial_t u|_\Gamma$

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{H_\Gamma^1}^2 &= \frac{d}{dt} \left( \int_\Omega |\nabla u(t, \cdot)|^2 + \int_\Gamma |u|_\Gamma(t, \cdot)|^2 + \int_\Gamma |\nabla_\Gamma u|_\Gamma(t, \cdot)|^2 \right) \\ &= -2 \int_\Omega |\partial_t u|^2 - 2\alpha \int_\Gamma |\partial_t u|_\Gamma|^2. \end{aligned}$$

Let  $\mathfrak{M}$  be a finite volume mesh of  $\Omega$  as defined in Section 6.1. We recall here the main notations of the mesh  $\mathfrak{M}$  (see Figure 6.3) used to obtain the finite volume scheme and we refer the reader to [6], for example, for more details.

We decompose  $\mathcal{E}$  (the set of all the edges in the mesh) into the subset of exterior edges  $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E} : \sigma \subset \Gamma\}$  and the subset of interior edges  $\mathcal{E}_{int} = \{\sigma \in \mathcal{E} : \sigma \not\subset \Gamma\}$ . Similarly we use the notations  $\mathcal{E}_\kappa^{int}$  and  $\mathcal{E}_\kappa^{ext}$  for the edges of a given control volume  $\mathcal{K} \in \mathfrak{M}$ . If  $\sigma$  is an interior edge which separates the control volumes  $\mathcal{K}$  and  $\mathcal{L}$ , we note  $\sigma = \kappa|\mathcal{L}$ . For any neighboring exterior edges  $\sigma, \tilde{\sigma} \in \mathcal{E}_{ext}$ , we note  $\mathbf{v} = \sigma|\tilde{\sigma}$  their common vertex (that belongs to  $\Gamma$ ).

Let us remark that we have to solve an equation on the boundary  $\Gamma$ , thus we have to define boundary unknowns. In this context, we define a boundary mesh  $\partial\mathfrak{M}$  which is in fact equal to the set of exterior edges of the initial mesh  $\mathfrak{M}$ . Thus, when we want to refer to the set of exterior edges we will note  $\mathcal{E}_{ext}$  and we want to refer to the set of boundary control volumes we will note  $\partial\mathfrak{M}$ . At each control volume  $\mathcal{K} \in \mathfrak{M}$  we associate a point  $x_\mathcal{K} \in \mathcal{K}$  called the center of the control volume  $\mathcal{K}$  and at each edge  $\sigma \in \mathcal{E}$  we associate a center  $x_\sigma \in \sigma$ . We assume that they satisfy the following orthogonality condition:

$$\begin{aligned} [x_\mathcal{K}, x_\mathcal{L}] \perp \sigma, \quad \text{and } x_\sigma &= [x_\mathcal{K}, x_\mathcal{L}] \cap \sigma, \quad \forall \sigma = \kappa|\mathcal{L} \in \mathcal{E}_{int}, \\ [x_\mathcal{K}, x_\sigma] &\perp e_\sigma, \quad \forall \sigma \in \mathcal{E}_\kappa^{ext}, \mathcal{K} \in \mathfrak{M}_{ext}, \end{aligned}$$

where  $e_\sigma$  is the chord associated with  $\sigma$  in the second case.

For  $\mathcal{K} \in \mathfrak{M}$  and any edge  $\sigma \in \mathcal{E}_\mathcal{K}$ , we note  $d_{\mathcal{K},\sigma}$  the distance between the center  $x_\mathcal{K}$  and the center  $x_\sigma$ , and for interior edges  $\sigma = \kappa|\mathcal{L} \in \mathcal{E}_{int}$ , we set  $d_{\mathcal{K},\mathcal{L}} = d_{\mathcal{K},\sigma} + d_{\mathcal{L},\sigma}$ .

For any vertex  $\mathbf{v} = \sigma|\tilde{\sigma}$ , we define  $d_{\sigma,\tilde{\sigma}}$  as the length of the arc included in  $\Gamma$  whose ends are  $x_\sigma, x_{\tilde{\sigma}}$  and passing through  $\mathbf{v} = \sigma|\tilde{\sigma}$  (drawn with larger dashes on figure 6.3).

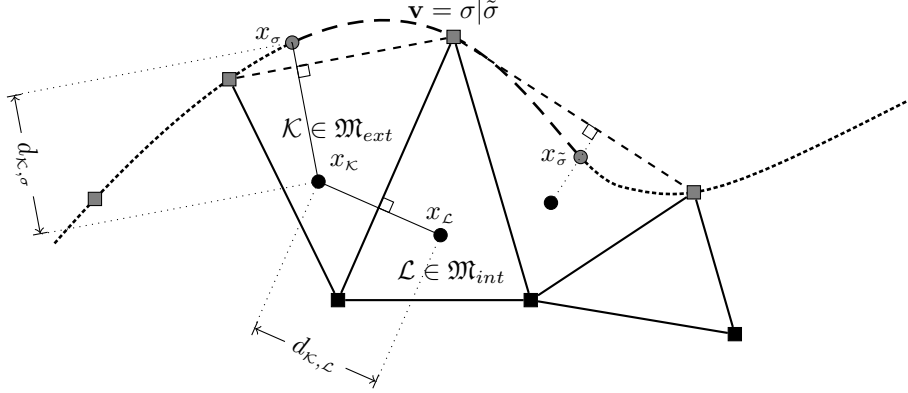
With these new notation, we can now measure the regularity of the mesh with respect to the position of the centers in each control volume and each edge by the following quantity

$$\text{reg}_2(\mathfrak{M}) = \max \left( \max_{\substack{\mathcal{K} \in \mathfrak{M} \\ \sigma \in \mathcal{E}_\mathcal{K}}} \frac{m_\sigma}{d_{\mathcal{K},\sigma}}, \max_{\mathbf{v} = \sigma|\tilde{\sigma}} \frac{m_\sigma + m_{\tilde{\sigma}}}{d_{\sigma,\tilde{\sigma}}} \right).$$

Finally, for simplicity, we shall assume that the interior control volumes are triangles but the approach can easily be generalized to more general convex polygonal interior control volumes.

In order to obtain the semi-discrete finite volume scheme associated with problem (6.9) we integrate equation (6.9a) on all control volumes  $\mathcal{K} \in \mathfrak{M}$  and we integrate equation (6.9b) on all boundary control volumes  $\sigma \in \partial\mathfrak{M}$ . Then we use a consistent two-point flux approximation for the Laplace operator in  $\Omega$  and for the Laplace-Beltrami operator on  $\Gamma$ . A solution of this scheme is thus a set of time-dependent unknowns

$$\mathbf{u}(t) = \left( (u_\mathcal{K}(t))_{\mathcal{K} \in \mathfrak{M}}, (u_\sigma(t))_{\sigma \in \partial\mathfrak{M}} \right) \in \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\partial\mathfrak{M}}.$$

FIG. 6.3. Finite volume mesh  $\mathfrak{M}$  of  $\Omega$ 

The scheme reads as follows: Find  $t \mapsto \mathbf{u}(t) \in \mathbb{R}^{\mathfrak{M}} \times \mathbb{R}^{\partial\mathfrak{M}}$  such that,

$$(6.10) \quad \begin{cases} m_K \partial_t u_K + \sum_{\sigma=K|\mathcal{L} \in \mathcal{E}_K^{int}} m_\sigma \frac{u_K - u_{\mathcal{L}}}{d_{K,\mathcal{L}}} + \sum_{\sigma \in \mathcal{E}_K^{ext}} m_\sigma \frac{u_K - u_\sigma}{d_{K,\sigma}} = 0, \quad \forall K \in \mathfrak{M}, \\ \alpha m_\sigma \partial_t u_\sigma + \sum_{\mathbf{v}=\sigma|\tilde{\sigma}} \frac{u_\sigma - u_{\tilde{\sigma}}}{d_{\sigma,\tilde{\sigma}}} + m_\sigma u_\sigma - m_\sigma \frac{u_K - u_\sigma}{d_{K,\sigma}} = 0, \quad \forall \sigma \in \partial\mathfrak{M}. \end{cases}$$

We postpone the important discussion on the choice of the discrete initial condition  $\mathbf{u}(0) = \mathbf{u}^0$  to Theorem 6.2.

The discrete version of the  $H_\Gamma^1$  norm is defined as follows

$$\|\mathbf{u}\|_{1,\mathfrak{M},\partial\mathfrak{M}} = \left( \|\mathbf{u}\|_{1,\mathfrak{M}}^2 + \|\mathbf{u}\|_{0,\partial\mathfrak{M}}^2 + \|\mathbf{u}\|_{1,\partial\mathfrak{M}}^2 \right)^{\frac{1}{2}},$$

where each term is given by

$$\|\mathbf{u}\|_{1,\mathfrak{M}}^2 = \sum_{\sigma=K|\mathcal{L} \in \mathcal{E}_K^{int}} m_\sigma d_{K,\mathcal{L}} \left( \frac{u_K - u_{\mathcal{L}}}{d_{K,\mathcal{L}}} \right)^2 + \sum_{K|\sigma \in \mathcal{E}_K^{ext}} m_\sigma d_{K,\sigma} \left( \frac{u_K - u_\sigma}{d_{K,\sigma}} \right)^2,$$

$$\|\mathbf{u}\|_{0,\partial\mathfrak{M}}^2 = \sum_{\sigma \in \partial\mathfrak{M}} m_\sigma (u_\sigma)^2, \quad \text{and} \quad \|\mathbf{u}\|_{1,\partial\mathfrak{M}}^2 = \sum_{\mathbf{v}=\sigma|\tilde{\sigma}} d_{\sigma,\tilde{\sigma}} \left( \frac{u_\sigma - u_{\tilde{\sigma}}}{d_{\sigma,\tilde{\sigma}}} \right)^2.$$

A discrete energy estimate is obtained by multiplying the first equation in (6.10) by  $\partial_t u_K$ , the second equation by  $\partial_t u_\sigma$  and by summing the resulting equalities on  $\mathfrak{M}$  and  $\partial\mathfrak{M}$ . We obtain

$$\frac{d}{dt} \|\mathbf{u}(t)\|_{1,\mathfrak{M},\partial\mathfrak{M}}^2 \leq 0,$$

where we did not specify the form of the dissipation terms, since it is not important for our purpose.

This estimate shows that the discrete  $H_\Gamma^1$  norm of the approximate solution decreases along the time and thus satisfies

$$\sup_{t \in [0,T]} \|\mathbf{u}(t)\|_{1,\mathfrak{M},\partial\mathfrak{M}} \leq \|\mathbf{u}^0\|_{1,\mathfrak{M},\partial\mathfrak{M}}.$$

This *a priori* estimate is the main tool to prove the convergence of the numerical method. However, in order to be useful, we see that the discrete initial data  $\mathbf{u}^0$  needs to be a stable approximation of  $u^0$  in the sense that  $\|\mathbf{u}^0\|_{1,\mathfrak{M},\partial\mathfrak{M}}$  has to be bounded uniformly with respect to the mesh size, for any  $u^0 \in H_\Gamma^1$ .

In this framework, the inequality we proved in this paper leads to the following stability result, which was our main motivation.

**THEOREM 6.2.** *Let  $\xi_0 > 0$  and  $h_0 > 0$  given by Proposition 6.1. There exists a  $C > 0$  such that for any finite volume mesh  $\mathfrak{M}$  of  $\Omega$  satisfying*

$$\text{reg}_1(\mathfrak{M}) \leq \xi_0, \quad \text{reg}_2(\mathfrak{M}) \leq \xi_0, \quad \text{and} \quad \text{size}(\mathfrak{M}) \leq h_0,$$

*and for any  $u^0 \in H_\Gamma^1$ , we have*

$$\|\mathbf{u}^0\|_{1,\mathfrak{M},\partial\mathfrak{M}} \leq C \|u^0\|_{H_\Gamma^1},$$

where  $\mathbf{u}^0 = \left( (u_\kappa^0)_{\kappa \in \mathfrak{M}}, (u_\sigma^0)_{\sigma \in \partial\mathfrak{M}} \right)$  is defined by

$$(6.11) \quad \begin{cases} u_\kappa^0 = \frac{1}{m_\kappa} \int_\kappa u^0(x) dx, \quad \forall \kappa \in \mathfrak{M}, \\ u_\sigma^0 = \frac{1}{m_\sigma} \int_\sigma u^0|_\Gamma dx, \quad \forall \sigma \in \partial\mathfrak{M}. \end{cases}$$

Notice first that, in order to take advantage of the assumed regularity of the trace of  $u^0$  on  $\Gamma$  in the estimate of the tangential gradient term  $\|\mathbf{u}^0\|_{1,\partial\mathfrak{M}}$ , we absolutely need to define the boundary terms  $\mathbf{u}_\sigma^0$  by using only the values of the trace of  $u^0$  on  $\Gamma$  and not, for instance, the values of  $u^0$  on the chords associated with each boundary control volume  $\sigma$ .

*Proof.*

- The estimate of the  $L^2$  term  $\|\mathbf{u}^0\|_{0,\partial\mathfrak{M}}$  is a straightforward consequence of Jensen's inequality.
- For any two neighboring boundary control volumes  $\sigma$  and  $\tilde{\sigma}$ , one can easily prove by using a Taylor formula on the manifold  $\Gamma$ , that

$$\left| \frac{1}{m_\sigma} \int_\sigma u^0 - \frac{1}{m_{\tilde{\sigma}}} \int_{\tilde{\sigma}} u^0 \right|^2 \leq (m_\sigma + m_{\tilde{\sigma}}) \int_{\sigma \cup \tilde{\sigma}} |\nabla_\Gamma u^0|^2.$$

It follows that

$$\begin{aligned} \|\mathbf{u}^0\|_{1,\partial\mathfrak{M}}^2 &= \sum_{\mathbf{v}=\sigma|\tilde{\sigma}} \frac{1}{d_{\sigma,\tilde{\sigma}}} \left( \frac{1}{m_\sigma} \int_\sigma u^0 - \frac{1}{m_{\tilde{\sigma}}} \int_{\tilde{\sigma}} u^0 \right)^2 \\ &\leq \sum_{\mathbf{v}=\sigma|\tilde{\sigma}} \frac{m_\sigma + m_{\tilde{\sigma}}}{d_{\sigma,\tilde{\sigma}}} \int_{\sigma \cup \tilde{\sigma}} |\nabla_\Gamma u^0|^2, \\ &\leq 2\text{reg}_2(\mathfrak{M}) \int_\Gamma |\nabla_\Gamma u^0|^2. \end{aligned}$$

- It remains to estimate the term  $\|\mathbf{u}^0\|_{1,\mathfrak{M}}^2$ . To this end, we first estimate the term corresponding to the interior edges as follows

$$\begin{aligned} \sum_{\sigma=\mathcal{K}|\mathcal{L} \in \mathcal{E}_{int}} m_\sigma d_{\mathcal{K},\mathcal{L}} \left( \frac{u_\kappa^0 - u_\mathcal{L}^0}{d_{\mathcal{K},\mathcal{L}}} \right)^2 &= \sum_{\sigma=\mathcal{K}|\mathcal{L} \in \mathcal{E}_{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} (u_\kappa^0 - u_\mathcal{L}^0)^2 \\ &\leq 2 \sum_{\sigma=\mathcal{K}|\mathcal{L} \in \mathcal{E}_{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}} [(u_\kappa^0 - u_\sigma^0)^2 + (u_\sigma^0 - u_\mathcal{L}^0)^2], \end{aligned}$$

where we have introduced the mean-values on the edges  $u_\sigma^0$  as in (6.11) but for interior edges now.

Gathering this computation with the other term in  $\|\mathbf{u}^0\|_{1,\mathfrak{M}}^2$ , we obtain

$$\|\mathbf{u}^0\|_{1,\mathfrak{M}}^2 \leq 2 \sum_{\kappa \in \mathfrak{M}} \sum_{\sigma \in \mathcal{E}_\kappa} \frac{m_\sigma}{d_{\kappa,\sigma}} (u_\kappa^0 - u_\sigma^0)^2.$$

We can now use Theorem 2.1 and Proposition 6.1, to obtain

$$\begin{aligned} \|\mathbf{u}^0\|_{1,\mathfrak{M}}^2 &\leq C_{\xi_0} \sum_{\kappa \in \mathfrak{M}} \sum_{\sigma \in \mathcal{E}_\kappa} \frac{m_\sigma}{d_{\kappa,\sigma}} (m_\sigma + \text{diam}(\kappa))^2 \frac{1}{m_\kappa} \int_\kappa |\nabla u^0|^2 \\ &\leq C_{\xi_0} \text{reg}_2(\mathfrak{M}) (1 + \text{reg}_2(\mathfrak{M}))^2 \sum_{\kappa \in \mathfrak{M}} \sum_{\sigma \in \mathcal{E}_\kappa} \text{diam}(\kappa)^2 \frac{1}{m_\kappa} \int_\kappa |\nabla u^0|^2 \\ &\leq 3C_{\xi_0} \text{reg}_2(\mathfrak{M}) (1 + \text{reg}_2(\mathfrak{M}))^2 \text{reg}_1(\mathfrak{M}) \int_\Omega |\nabla u^0|^2, \end{aligned}$$

and the claim is proved. Notice that the assumptions of Theorem 2.1 are satisfied with a uniform ratio  $\nu/\mu$  thanks to Proposition 6.1, and to the fact that for interior control volumes  $\kappa$ , which are real triangles, the ratio  $\nu/\mu$  is equal to 1 (see Remark 2.1).  $\square$

#### Appendix A. An intermediate result.

LEMMA A.1. *There exists a  $C^1$  diffeomorphism  $\psi : \overline{Q^{n-1}} \rightarrow \overline{Q^{n-1}}$  such that, setting  $\tilde{\gamma} = \gamma \circ \psi$ , for every  $\theta \in \overline{Q^{n-1}}$ ,*

$$|\partial_1 \tilde{\gamma} \wedge \cdots \wedge \partial_{n-1} \tilde{\gamma}|(\theta) = m_\sigma.$$

*Proof.* This follows from the proof of [7, Lemma 2], see also [2, Theorem 7, Proposition A.2]. However, in the former reference, all the data are assumed to be smooth. In the latter (which gives a result on more general domains than a cube), the map  $\psi$  is merely  $C^1$  on  $Q^{n-1}$  instead of  $\overline{Q^{n-1}}$ . For the convenience of the reader, we detail the proof.

Let  $f = \frac{1}{m_\sigma} |\partial_1 \gamma \wedge \cdots \wedge \partial_{n-1} \gamma|$ . Then  $\int_{Q^{n-1}} f = 1$ .

For every  $k = 1, \dots, n-1$ , there exists a  $C^1$  map  $f_k : \overline{Q^k} \rightarrow \mathbb{R}$  such that  $f = f_1 \dots f_{n-1}$  and

$$(A.1) \quad \int_0^1 f_k(x_1, \dots, x_{k-1}, t) dt = 1 \quad , \quad (x_1, \dots, x_{k-1}) \in \overline{Q^{k-1}}.$$

Indeed, let

$$f_1(x_1) = \int_{Q^{n-2}} f(x_1, t_2, \dots, t_{n-1}) dt_2 \dots dt_{n-1} \quad , \quad x_1 \in [0, 1]$$

and then define by induction on  $1 \leq k \leq n-1$ , the map  $f_k$  by

$$f_1(x_1) \dots f_k(x_1, \dots, x_k) = \int_{Q^{n-1-k}} f(x_1, \dots, x_k, t_{k+1}, \dots, t_{n-1}) dt_{k+1} \dots dt_{n-1}.$$

(When  $k = n-1$ , the right-hand side is simply  $f(x_1, \dots, x_{n-1})$ .) One easily checks that (A.1) is satisfied. We now define the map  $\rho = (\rho_1, \dots, \rho_{n-1})$  by

$$\rho_i(x_1, \dots, x_i) = \int_0^{x_i} f_i(x_1, \dots, x_{i-1}, t) dt \quad , \quad 1 \leq i \leq n-1.$$

Then  $x_i \mapsto \rho_i(x_1, \dots, x_i)$  maps diffeomorphically  $[0, 1]$  onto  $[0, 1]$ . Moreover,

$$\text{Jac } \rho = \prod_{i=1}^{n-1} \frac{\partial \rho_i}{\partial x_i} = \prod_{i=1}^{n-1} f_i = f.$$

We then define  $\psi = \rho^{-1}$ . Then

$$1 = \text{Jac } (\rho \circ \psi) = ((\text{Jac } \rho) \circ \psi) \text{Jac } \psi = f \circ \psi \text{Jac } \psi.$$

Since

$$\partial_1 \tilde{\gamma} \wedge \dots \wedge \partial_{n-1} \tilde{\gamma} = ((\partial_1 \gamma \wedge \dots \wedge \partial_{n-1} \gamma) \circ \psi) \text{Jac } \psi,$$

this completes the proof of the lemma. □

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