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# Random walks on quasi-periodic tilings

*B. de Loynes*

*Institut de Recherches Mathématiques Avancées  
Université de Strasbourg  
deloynes@unistra.fr*

## Abstract

In this paper, we study random walks on quasi-periodic graphs induced by tiling the standard real vector space  $\mathbb{R}^d$  using the cut-and-project method. We first show a dichotomy of Pólya type, namely we prove that the simple random walk on a cut-and-project graph is recurrent if  $d \leq 2$  and transient otherwise. Nonetheless the aperiodic graphs we consider here are no longer the Cayley graph of a group but of a groupoid. Secondly, we prove the asymptotic entropy of such random walks is zero.

## 1 Introduction and motivations

Starting with the theorem of Pólya stating that the simple random walk on the integer lattice  $\mathbb{Z}^d$  is recurrent if and only if  $d \leq 2$  and transient, otherwise, random walks on finitely generated groups has been intensively studied in the literature. As a matter of fact, a random walk on a group belongs to the class of random processes with independent and stationary increments. These two properties make the study of random walks more tractable. However, the notion of random walk need not to be reduced to the group structure. It appears that many questions related to random walks on weaker algebraic structures like groupoids or semi-groupoids are still open. Loosing the independence and stationarity properties, there is no reason *a priori* that those structures carry the same theory of random walks. For instance, in [CP03], the simple random walk on an example of sub-semigroupoid on vertices of  $\mathbb{Z}^2$  is shown to be transient.

The class of graphs (or groupoids) considered in this paper are obtained by tiling the standard real vector space  $\mathbb{R}^d$  with the help of the cut-and-project scheme. More precisely, let  $E$  be a  $d$ -dimensional vector subspace of  $\mathbb{R}^N$ , named the *real space*, and set  $E_{\text{int}} = E^\perp$  the orthogonal complement of  $E$ , called the *internal space*. Let  $K$  be the unit cube in  $\mathbb{Z}^N$ . An edge in the Cayley graph of  $\mathbb{Z}^N$  is accepted and projected on  $E$  (orthogonally) if it can be translated by a vector of  $E$  in the unit cube  $K + t$ ,  $t \in E_{\text{int}}$ . Under suitable assumptions this method gives rise to a family of tilings  $\mathcal{T}_t$  of the space  $E$  whose prototiles are the projections of the  $d$ -dimensional facets of the  $N$ -dimensional unit cube  $K$ . Moreover, depending on the orientations of the space  $E$  and  $E_{\text{int}}$  those tilings will be periodic or aperiodic — the group of translations is given by  $E \cap \mathbb{Z}^N$ . Such a tiling naturally defines a connected graph embedded in the space  $\mathbb{R}^d$ , called *the cut-and-project graph* — the vertex and edge sets are respectively the sets of vertices and sides defining the tiles. An important example which can be constructed using the cut-and-project scheme (see section 2.1 for the details or [ODK88] for the original statement) is the Penrose's third tiling of  $\mathbb{R}^2$  with two types of rhombs — thin and thick — which has been initially defined by Penrose using matching rules forcing the tiling to be aperiodic. Another interesting example is the icosahedral tiling of  $\mathbb{R}^3$  (see section 2.1 for its definition) because of its connection with quasi-crystals. The quasi-crystals have been discovered by Shechtman in 1982 observing that

the diffraction patterns of an alloy of Al-Mn has a 10-fold symmetry which contradicts the classical theory of crystallography. The theoretical description of this discovery can be found in the seminal paper [SBGC84] of D. Shechtman, I. Blech and J. W. Cahn. The icosahedral tiling appears to be the mathematical description of this alloy (see also [KD86]).

Avoiding technical assumptions, which are generic, a first theorem related to the type problem in the context of random walks on cut-and-project graphs can be written as follows (see theorem 2.3 in section 2.3 for further details).

**Theorem 1.1.** *Denoting by  $P^n(x, y)$  the  $n$ -step transition probability between  $x$  and  $y \in \mathbb{G}^0$ , and setting  $d = \dim E$ , generically, the following asymptotic estimates hold :*

- $P^{2n}(x, x) \geq C_0(n \log n)^{-d/2}$ , and
- $P^n(x, y) \leq C_1 n^{-d/2}$ ,

*for some constants  $C_0, C_1 > 0$ . Consequently, the simple random walk on the cut-and-project graph is recurrent if  $d \leq 2$  and transient otherwise.*

The proof of this theorem involves an estimate of the growth rate of balls for the statement related to the lower bound of the return probability, whereas we establish an  $d$ -dimensional isoperimetric inequality to obtain the upper bound of the  $n$ -step transition probabilities. As far as we are considering the type problem, many technics such as “civilized” embedding — see [DS84] — or Nash-Williams inequalities — see [Woe00] for a modern approach — are available. However, in the sequel we obtain finer informations on the decreasing of the heat kernel. In particular, the establishment of a Nash-Williams inequality is often at least as difficult as finding isoperimetric inequalities while the former provides with a weaker result than the latter. Since, these techniques are part of the folklore of the theory of reversible Markov chains, the proof of those heat kernel estimates are given at the end of this paper in section 4.

Among properties about random walks we shall be interested in, one of them is related to the asymptotic entropy. For a general Markov chain, the asymptotic entropy is intimately connected with the tail — or asymptotic —  $\sigma$ -algebra. Namely, the tail  $\sigma$ -algebra is trivial, *i.e.* it only contains measurable sets of probability 0 or 1, if and only if the asymptotic entropy is zero. This kind of result can be seen as an analogue of the Kolmogorov 0-1 law for independent and identically distributed random variables. In fact, the triviality of the tail  $\sigma$ -algebra can be interpreted as a certain asymptotic independence. Notice that the invariant  $\sigma$ -algebra is always contained in the tail  $\sigma$ -algebra. Consequently, the triviality of the latter implies the triviality of the former.

For a general Markov chain, the tail  $\sigma$ -algebra is connected to the so-called tail boundary which is itself isometrically isomorphic to the space of sequences of bounded harmonic functions on the state space — for this general context, see [Kai92]. Also, the invariant  $\sigma$ -algebra is intimately related to the Poisson boundary which is isometrically isomorphic to the space of bounded harmonic functions. Thus, the tail (*resp.* the invariant)  $\sigma$ -algebra is trivial if and only if there is no non constant sequences of bounded harmonic functions (*resp.* bounded harmonic functions).

It turns out that the two  $\sigma$ -algebras always coincide for a random walk on a locally compact topological group provided the starting distribution is a Dirac mass. As a consequence, the Poisson boundary is trivial — or equivalently the random walk is Liouville — if and only if the asymptotic entropy is zero. But in the general case of Markov chains, it can happen that the Poisson is trivial but the asymptotic entropy is positive (see [Ben13] for an example of this fact). Consequently, the strategy consisting in proving the triviality of the Poisson boundary, or the Liouville property, fails to prove the entropy is zero in full generality, and fails to give the

triviality of the tail  $\sigma$ -algebra. Therefore, we can expect an estimate of the asymptotic entropy to give valuable informations about the asymptotic behavior of the random walk.

In the context of random walks on groups — see [KV83, Der86] and references therein — or random walks on homogeneous spaces — see [KW02] — the asymptotic entropy satisfies the so-called fundamental inequality  $h \leq \ell \cdot v$  where  $h$  is the asymptotic entropy of the random walk,  $\ell$  the linear rate of escape and  $v$  the exponential growth rate of the group considered. This inequality no longer holds in the general case of Markov chains for mainly two reasons. First, the asymptotic entropy of a Markov chain heavily depends on the initial distribution considered which is not the case in the context of random walks on groups — the Markov transition kernel related to such a random walk is actually invariant under the group action. In particular, this prevents from identifying the asymptotic entropy to the more tractable Avez entropy (see [Ave72]). Secondly, the space of sequences of increments of random walks is structurally an ergodic Bernoulli shift, and from this observation a Shannon-McMillan-Breiman type theorem can be stated. In [KKR04], this inequality is extended to the context of *Random Walk with Random Transition Probabilities*<sup>1</sup>, that is a random process whose increments are still independent but no longer identically distributed. And the distribution of an increment is chosen accordingly to the configuration of an ergodic dynamical system which permits to recover, somehow, the stationarity of increments as in the context of random walk on groups.

It turns out that our random walks on cut-and-project graphs can be seen as a RW RTP, more specifically it can be seen as a *Random Walk with Internal Degree of Freedom*<sup>2</sup> — see for instance [KS83]. Consequently, we may estimate the asymptotic entropy of this RW RTP by taking advantage of the estimate of [KKR04]. However, note that the following theorem is not a direct corollary, since, as it will be clear in the sequel, the underlying probability preserving dynamical system actually depends on the random walk itself. Once again, for the sake of simplicity, we do not enter in the details of the assumptions which are also generic in a sense which will be made precise in section 3.2.

**Theorem 1.2.** *Generically, the asymptotic entropy of the simple random walk on a cut-and-project is zero. Consequently, the tail and invariant  $\sigma$ -algebras are trivial.*

The section 2.1 is devoted to the basic notation and definitions related to tilings and the description of the cut-and-project method. We also construct the induced cut-and-project graph on which we study the simple random walk. At the end of this section, we give the statement of the theorem related to the type problem precisizing the assumptions hidden behind the term “generically”. The proof of this theorem is given in the last section. In the section 3, we present the main object of this paper, that is RW RTP and RWIDF. More precisely, we show how the simple random walk on a cut-and-project graph can be seen as a RWIDF. We end this section by proving the asymptotic entropy is zero.

## 2 Cut-and-project graph and the type problem

### 2.1 Tiling the standard $d$ -dimensional real vector space

We start with the description of the cut-and-project scheme to tile the real line. We consider the standard integer lattice  $\mathbb{Z}^2$  of  $\mathbb{R}^2$ . Let  $E$  be an irrational line in  $\mathbb{R}^2$ , *i.e.* satisfying  $E \cap \mathbb{Z}^N = \{0\}$  and  $E_{\text{int}}$  be the line orthogonal to  $E$ . We denote by  $K$  the unit square in  $\mathbb{R}^2$ . Thus, the translation of  $K$  along  $E$  defines a strip (see figure 1).

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<sup>1</sup> RW RTP

<sup>2</sup> RWIDF

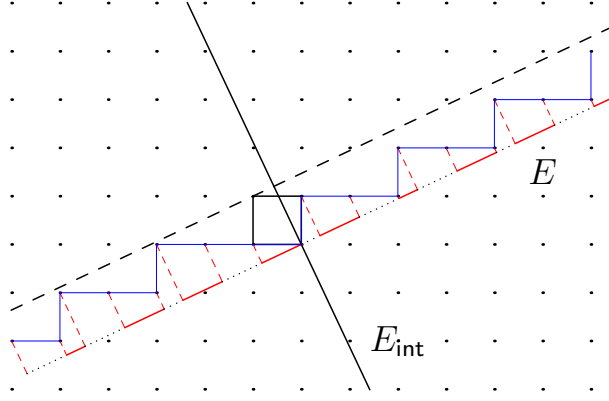


Fig. 1: Quasi-periodic tiling of the real line within the cut-and-project scheme.

Consequently, we obtain a tiling of the space  $E$  with two types of segments (short and long). The short and long segments correspond to the projections of vertical and horizontal sides of the unit square which are entirely contained in the strip. Actually, it can be noticed that there is an ambiguity in the example of the figure 1 since two opposite sides of the same unit square are completely contained in the strip so that we have to choose which one we project. However, the strip can be translated by a vector  $t \in E_{\text{int}}$  in such a way that there is no ambiguity. And since the projection of the lattice  $\mathbb{Z}^2$  on the internal space is countable, there is no ambiguity for all but countably many  $t \in E_{\text{int}}$ .

Finally, in a non-ambiguous case, we observe that there is a unique broken line which is completely contained in the strip. This is the theorem of [ODK88], recalled at the end of this section, for the case of dimension 1.

**Definition 2.1.** A subset  $F$  of  $\mathbb{R}^d$  is termed *regular* — for the usual topology of  $\mathbb{R}^d$  — if it is bounded, has a non-empty interior  $\text{Int}(F)$ , and is such that its closure  $\text{Clos}(F)$  coincides with  $\text{Clos}(\text{Int}(F))$  and its interior  $\text{Int}(F)$  equals  $\text{Int}(\text{Clos}(F))$ .

Two regular subsets  $F_1$  and  $F_2$  are termed *congruent* if  $F_1 = F_2$  up to a translation. The property of congruence induces an equivalence relation on the set of tiles; an equivalence class is termed a *prototile*.

Let  $\mathcal{T}$  be a set of regular subsets of  $\mathbb{R}^d$ , we denote by  $\mathcal{P}$  the corresponding set of prototiles, *i.e.* the factor set of  $\mathcal{T}$  with respect the equivalence relation of congruence.

**Definition 2.2.** A denumerable set  $\mathcal{T} = \{F_i\}_{i \in I}$  of regular subsets is a *tiling* of  $\mathbb{R}^d$  if

- the corresponding set  $\mathcal{P}$  of prototiles is finite,
- $\mathbb{R}^d = \bigcup_{i \in I} F_i$ , and
- $\text{interior}(F_i) \cap \text{interior}(F_j) = \emptyset$  if  $i \neq j \in I$ .

If  $\mathcal{T}$  is a tiling, then a regular set  $F \in \mathcal{T}$  is called a *tile*.

Let  $E$  be a  $d$ -dimensional subspace of  $\mathbb{R}^N$ , and  $E_{\text{int}}$  be its orthogonal supplement in  $\mathbb{R}^N$ . The spaces  $E$  and  $E_{\text{int}}$  are termed respectively *real space* and *internal space*. We will denote by  $\mathfrak{p}$  and  $\mathfrak{p}_{\text{int}}$  the canonical projections from  $\mathbb{R}^N = E \oplus E_{\text{int}}$  to  $E$  and from  $E \oplus E_{\text{int}}$  to  $E_{\text{int}}$ . Thus we have the following

$$E \xleftarrow{\mathfrak{p}} E \oplus E_{\text{int}} \xrightarrow{\mathfrak{p}_{\text{int}}} E_{\text{int}} .$$

We denote by  $K$  the unit cube in  $\mathbb{Z}^N$ , namely

$$K = \left\{ \sum_{i=1}^N \alpha_i \varepsilon_i : 0 \leq \alpha_i \leq 1 \right\},$$

where  $(\varepsilon_1, \dots, \varepsilon_N)$  is the canonical basis of  $\mathbb{R}^N$ .

Let  $p$  be an integer  $0 \leq p \leq N$  and  $M_p = \{I = (i_1, \dots, i_p) \subset \{1, \dots, N\}\}$  be the set of index sets with  $p$  elements. The  $p$ -facets of the unit cube are indexed by  $M_p$  as follows :

$$K_I = \left\{ \sum_{i \in I} \alpha_i \varepsilon_i : \alpha_i \in [0, 1] \right\} \text{ for all } I \in M_p \text{ and } p > 0,$$

and  $K_\emptyset = \{0\}$ . Obviously, the unit cube  $K$  admits the decomposition  $K = K_I + K_{I^c}$ .

**Assumption 1.** We assume that the configuration  $\mathbb{R}^N = E \oplus E^\perp$  is *non degenerated*, i.e. for any  $I = \{i_1, \dots, i_d\} \in M_d$ , the system  $\{\mathbf{p}(\varepsilon_i), i \in I\}$  is of rank  $d$  and the system  $\{\mathbf{p}_{\text{int}}(\varepsilon_i), i \in I^c\}$  is of rank  $N - d$ .

Actually, the two systems of vectors are simultaneously maximal or not maximal since  $E_{\text{int}} = E^\perp$ . Moreover, maximality is a generic property and under this condition the  $d$ -facets of the unit cube are isomorphic to their projections on  $E$  by  $\mathbf{p}$ , and also, the  $(N - d)$ -facets are isomorphic to their projections on  $\mathbf{p}_{\text{int}}$ . We will denote by  $D_I$  the projection  $\mathbf{p}(K_I)$  of the  $d$ -facet related to  $I \in M_d$ . According to [ODK88] the set

$$\mathcal{T}_t = \left\{ x + D_I : x = \mathbf{p}(\xi), \mathbf{p}_{\text{int}}(\xi) \in \mathbf{p}_{\text{int}}(K_{I^c} + t), I \in M_d \right\}$$

is a tiling for every non ambiguous  $t \in E_{\text{int}}$  whose group of translations is given by  $E \cap \mathbb{Z}^N$ . Moreover,  $\mathcal{T}_t$  is the projection of the unique  $d$ -dimensional faceted manifold entirely contained in the strip  $\mathcal{K}_t = K + E + t$  — for almost every  $t \in E_{\text{int}}$ .

The Penrose's third tiling is obtained by the cut-and-project method if we consider the real space  $E$  in  $\mathbb{R}^5$  spanned by the two following vectors (see [ODK88])

$$v_1 = (1, \cos(2\pi/5), -\cos(\pi/5), -\cos(\pi/5), \cos(2\pi/5)),$$

and,

$$v_2 = (0, \sin(2\pi/5), \sin(\pi/5), -\sin(\pi/5), -\sin(2\pi/5)).$$

For the icosahedral tiling of  $\mathbb{R}^3$  — see [KD86] —, the vector subspaces  $E$  and  $E_{\text{int}}$  of  $\mathbb{R}^6$  are defined with the help of projectors (which are both given here for the sake of simplicity)

$$\mathbf{p} = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & 1 & -1 & -1 & 1 & 1 \\ 1 & \sqrt{5} & 1 & -1 & -1 & 1 \\ -1 & 1 & \sqrt{5} & 1 & -1 & 1 \\ -1 & -1 & 1 & \sqrt{5} & 1 & 1 \\ -1 & -1 & -1 & 1 & \sqrt{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{5} \end{pmatrix} \text{ and } \mathbf{p}_{\text{int}} = \frac{1}{2\sqrt{5}} \begin{pmatrix} \sqrt{5} & -1 & 1 & 1 & -1 & -1 \\ -1 & \sqrt{5} & -1 & 1 & 1 & -1 \\ 1 & -1 & \sqrt{5} & -1 & 1 & -1 \\ 1 & 1 & -1 & \sqrt{5} & -1 & -1 \\ -1 & 1 & 1 & -1 & \sqrt{5} & -1 \\ -1 & -1 & -1 & -1 & -1 & \sqrt{5} \end{pmatrix}.$$

## 2.2 Constructive definition of a cut-and-project tiling

We shall generalise the definition of  $M_d$  by introducing the set  $M_d^\pm$  of every subset with  $d$  indices in  $\{\pm 1, \dots, \pm N\}$ , namely

$$M_d^\pm = \left\{ \{i_1, \dots, i_d\}, i_l \in \{\pm 1, \dots, \pm N\}, l = 1, \dots, d \right\}.$$

With this notation, we will denote the  $d$ -dimensional facet of index  $I$  by

$$K_I = \left\{ \sum_{l=1}^d \alpha_l \varepsilon_l : \alpha_l \in [0, 1], l = 1, \dots, d \right\}$$

where  $\varepsilon_{-|i|}$  has to be understood as  $-\varepsilon_{|i|}$ .

Let  $t \in E_{\text{int}}$  be non ambiguous and let  $\xi \in \mathbb{Z}^N \cap \mathcal{K}_t$ . We will say that a  $d$ -dimensional facet  $K_I$  is admissible with respect to the point  $\xi$  (or equivalently that  $I$  is admissible) if and only if

$$\mathfrak{p}_{\text{int}}\left(\xi - \sum_{i \in I: i_l < 0} \varepsilon_{|i|}\right) \in \mathfrak{p}_{\text{int}}(K_{|I|} + t),$$

where  $|I| = \{|i|, i \in I, l = 1, \dots, d\}$  so that  $|I| \in M_d$ , the complement being defined as in the previous section. The figure 2 clarifies this notion of admissibility and more precisely the condition above.

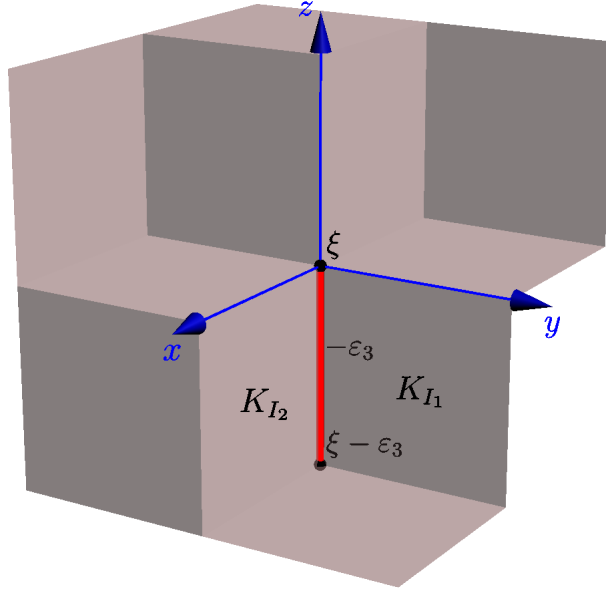


Fig. 2: Local patch of the cut manifold.

Denoting by  $\mathcal{S}_{\pm} = \{\pm \varepsilon_1, \dots, \pm \varepsilon_d\}$ , we will say that  $\xi$  is connected to  $\xi + \varepsilon_{i_0}$  if and only if there exist  $I_1, \dots, I_d \in M_d^{\pm}$  admissible with respect to  $\xi$  such that

$$I_1 \cap \dots \cap I_d = \{i_0\}.$$

As a matter of fact that the connection of  $\xi$  to  $\xi + \varepsilon_{i_0}$  is equivalent to the connection of  $\xi + \varepsilon_{i_0}$  to  $\xi$ . Such a connection will be denoted by  $\xi \leftrightarrow \xi + \varepsilon_{i_0}$ .

We set  $\mathbb{X}_t = \mathfrak{p}_{\text{int}}(K + t)$  and denote by  $\bar{\mathcal{S}}_{\pm}$  the projection of the set  $\mathcal{S}_{\pm}$  on the space  $E_{\text{int}}$ . For facets  $I_1, \dots, I_d \in M_d^{\pm}$ , and a point  $x = \mathfrak{p}_{\text{int}}(\xi)$  for some  $\xi \in \mathbb{Z}^N \cap \mathcal{K}_t$ , we define

$$\chi_t(I_1, \dots, I_d, x) = \prod_{l=1}^d \mathbf{1}_{\mathfrak{p}_{\text{int}}(K_{|I_l|} + t)}\left(x - \sum_{i \in I_l: i_l < 0} \bar{s}_{|i|}\right).$$

This map is  $\{0, 1\}$ -valued and is equal to 1 if and only if every set  $I_1, \dots, I_d$  are admissible with respect to  $\xi$  which is projected on  $x \in E_{\text{int}}$ <sup>3</sup>.

Now, we can give a constructive definition of a tiling by setting

$$\mathbf{T}_0^t = \left\{ \xi + K_I : I \in M_d^\pm, \mathbf{p}_{\text{int}} \left( \xi - \sum_{i \in I: i < 0} \varepsilon_{|i|} \right) \in \mathbf{p}_{\text{int}}(K_{|I|^c} + t) \right\},$$

$\mathbf{V}_0^t = \{\xi\}$ ,  $\mathbf{E}_0^t = \emptyset$ . For  $n \geq 0$ , we define

$$\mathbf{V}_{n+1}^t = \bigcup_{\xi \in \mathbf{V}_n} \left\{ \xi + \varepsilon_i : \{\varepsilon_i\} = \bigcap_{l=1}^d K_{I_l} : \chi_t(I_1, \dots, I_d, \mathbf{p}_{\text{int}}(\xi)) = 1 \right\},$$

$$\mathbf{E}_{n+1}^t = \bigcup_{\xi \in \mathbf{V}_n} \left\{ (\xi, \xi + \varepsilon_i) : \{\varepsilon_i\} = \bigcap_{l=1}^d K_{I_l} : \chi_t(I_1, \dots, I_d, \mathbf{p}_{\text{int}}(\xi)) = 1 \right\}$$

and

$$\mathbf{T}_{n+1}^t = \bigcup_{\xi \in \mathbf{V}_{n+1}} \left\{ \xi + K_I : I \in M_d^\pm, \mathbf{p}_{\text{int}} \left( \xi - \sum_{i \in I: i < 0} \varepsilon_{|i|} \right) \in \mathbf{p}_{\text{int}}(K_{|I|^c} + t) \right\}.$$

Then by construction,  $\mathbf{p} \left( \bigcup_{n \geq 0} \mathbf{T}_n^t \right)$  is the cut-and-project tiling indexed by  $t \in E_{\text{int}}$ . In addition, the cut-and-project, denoted by  $\mathbb{G}_t$ , is the graph whose vertex and edge sets are given respectively by

$$\mathbb{G}_t^0 = \mathbf{p} \left( \bigcup_{n \geq 0} \mathbf{V}_n^t \right) \text{ and } \mathbb{G}_t^1 = \mathbf{p} \left( \bigcup_{n \geq 0} \mathbf{E}_n^t \right).$$

Note that there is a natural action of  $\mathbb{Z}^N \times \mathbb{Z} \cap E$  on the tilings  $\mathcal{T}_t$  and the cut-and-project graphs  $\mathbb{G}_t$ :

$$(u, v) \cdot \mathcal{T}_t = \mathcal{T}_{t + \mathbf{p}_{\text{int}}(u)} + v + \mathbf{p}(u) \text{ and } (u, v) \cdot \mathbb{G}_t = \mathbb{G}_{t + \mathbf{p}_{\text{int}}(u)} + v + \mathbf{p}(u),$$

for  $(u, v) \in \mathbb{Z}^N \times \mathbb{Z} \cap E$ . We say that two graphs or two tilings are congruent if they belong to the same orbit with respect to this action.

The simple random walk on the graph  $\mathbb{G}$  is the Markov chain  $(M_n)_{n \geq 0}$  whose state space is the set  $\mathbb{G}^0$  and the transition probabilities are defined for all  $x, y \in \mathbb{G}^0$  by

$$P(x, y) = \begin{cases} P(x, y) = \frac{1}{\deg(x)} & \text{if } (x, y) \in \mathbb{G}^1, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\deg(x)$  denotes the degree of point  $x \in \mathbb{G}^0$ , i.e.  $\deg(x) = \text{card}\{y \in \mathbb{G}^0 : (x, y) \in \mathbb{G}^1\}$ .

We can also define the graph embedded in  $\mathbb{Z}^N$ , denoted by  $\tilde{\mathbb{G}}_t$ , whose vertex and edge sets are respectively given by

$$\tilde{\mathbb{G}}_t^0 = \bigcup_{n \geq 0} \mathbf{V}_n^t \text{ and } \tilde{\mathbb{G}}_t^1 = \bigcup_{n \geq 0} \mathbf{E}_n^t.$$

In the sequel, we will denote by  $(\tilde{M}_n)_{n \geq 0}$  the simple random walk on the graph  $\tilde{\mathbb{G}}_t$ .

**Assumption 2.** The restriction of  $\mathbf{p} : E \oplus E_{\text{int}} \rightarrow E$  to the subset  $\mathbb{Z}^N \cap \mathcal{K}_t$  is bijective.

The assumption 2 implies that  $\tilde{M}_n = \mathbf{p}^{-1} M_n$  for all  $n \geq 0$ . This assumption is equivalent, in our model, to the condition  $E_{\text{int}} \cap \mathbb{Z}^N = \{0\}$ . Furthermore, it turns out that this assumption is generic in the sense it is satisfied for all but countably many configurations  $E \oplus E_{\text{int}}$ .

<sup>3</sup> Note that this maps is well defined in the sense that the admissibility condition depends only on the relative position of  $x \in \mathbb{X}_t$ , i.e. on its convex coordinates. Thus, if  $x \in \mathbb{X}_t$  has no preimage in  $\mathbb{Z}^N \cap \mathcal{K}_t$ , this notion of admissibility still makes sense.

### 2.3 The type problem for the simple random walk on a cut-and-project graph.

A first theorem dealing with heat kernel estimates allows to deduce a Pólya dichotomy type results.

**Theorem 2.3.** *Set  $d = \dim E \leq N$  and assume that  $\mathbb{R}^N = E \oplus E_{\text{int}} = E \oplus E^\perp$ , and the non degeneracy assumption 1. Consider the simple random walk  $(M_n)_{n \geq 0}$  on the cut-and-project graph  $\mathbb{G}$  induced by the tiling  $\mathcal{T}_t$  for a generic  $t \in E_{\text{int}}$ . If  $E_{\text{int}} \cap \mathbb{Z}^N = \{0\}$ , then the following estimates hold for  $x, y \in \mathbb{G}^0$ ,*

- $P^{2n}(x, x) \geq C_0(n \log n)^{-d/2}$ , and
- $P^n(x, y) \leq C_1 n^{-d/2}$ ,

for some constants  $C_0, C_1 > 0$ . Obviously, we obtain the following dichotomy:

- if  $\dim E \leq 2$  then  $(M_n)_{n \geq 0}$  is recurrent,
- if  $\dim E \geq 3$  then  $(M_n)_{n \geq 0}$  is transient.

Even though the theorem is stated for simple random walk, it can be trivially extended to strongly reversible random walks, uniformly irreducible, and with bounded range (see [Anc90] for instance).

In the example of the icosahedral tiling, it is easy to check that  $E_{\text{int}} \cap \mathbb{Z}^6 = \{0\}$  and that the non-degeneracy hypothesis is fulfilled. Thus, the theorem applies and for all generic  $t \in E_{\text{int}}$  the simple random walk is transient. Unfortunately, it is obvious that the vector  $(1, 1, 1, 1, 1)$  belongs to  $E_{\text{int}}$  in the configuration considered for the Penrose's third tiling so that we might ask for weaker hypothesis.

The proof of theorem 2.3 is postponed to section 4.

## 3 The asymptotic entropy estimate

### 3.1 Random walk with internal degree of freedom

**Definition 3.1.** A point  $x \in \mathbb{X}_0$  is ambiguous if

$$\mathfrak{p}_{\text{int}}(\mathbb{Z}^N) \cap \{x + u\} \neq \emptyset$$

implies that  $u$  is ambiguous in  $E_{\text{int}}$  for the cut-and-project scheme. We denote by  $\mathcal{N}$  the set of ambiguous points.

Let  $x \in \mathbb{X}_0 \setminus \mathcal{N}$ , and define the conductance kernel for  $t \in E_{\text{int}}$  non ambiguous

$$C_t(x, dy) = \sum_{\substack{I_1, \dots, I_d \in M_d^\pm: \\ \text{card} \bigcap_{l=1}^d I_l = 1}} \chi_t(I_1, \dots, I_d, x) \sum_{s \in \mathcal{S}_\pm} \delta_{x+\bar{s}}(dy) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s).$$

Setting  $m_t(x) = \int_{\mathbb{X}_t} C_t(x, dy)$ , we define the Markov transition kernel by

$$Q_t(x, dy) = \frac{C_t(x, dy)}{m_t(x)}.$$

**Proposition 3.2.** *Let  $t \in E_{\text{int}}$  non ambiguous and  $x \in \mathbb{X}_0 \setminus \mathcal{N}$ , then*

$$Q_t(x, dy) = Q_0(x - t, dy - t).$$

*Proof.* This is a consequence of the relation

$$\chi_t(I_1, \dots, I_d, x) = \chi_0(I_1, \dots, I_d, x - t)$$

for  $t \in E_{\text{int}}$  non ambiguous,  $x \in \mathbb{X}_0$  and  $I_1, \dots, I_d \in M_d^\pm$ .  $\square$

Because of this proposition, all the useful informations are actually contained in  $C_0$  and  $Q_0$  <sup>4</sup>.

On the state space  $\mathbb{X}_0 \times \mathbb{Z}^N$ , we define the Markov chain  $(X_n, \tilde{M}_n)_{n \geq 0}$  whose initial distribution is  $\delta_x \otimes \delta_0$  for some  $x \in \mathbb{X}_0 \setminus \mathcal{N}$  and transition kernel  $R$  is given by

$$R((x, z), (y, z')) = Q_0(x, y) \mu^{x, y}(z' - z),$$

where  $(\mu^{x, y})_{x, y \in \mathbb{X}_0}$  is the family of probability measures defined by

$$\mu^{x, y} = \frac{\sum_{s \in \mathcal{S}^\pm : \mathfrak{p}_{\text{int}}(s) = y - x} \delta_s}{\text{card}\{s \in \mathcal{S}^\pm : \mathfrak{p}_{\text{int}}(s) = y - x\}}.$$

The second coordinate  $(\tilde{M}_n)_{n \geq 0}$  is named a *random walk with internal degree of freedom* <sup>5</sup>, whereas the first coordinate is the Markov chain governing the internal degrees of freedom.

Let us define the following finite measure on  $\mathbb{X}_0 \setminus \mathcal{N}$

$$\tilde{\pi}(dx) = \mathbf{1}_{\mathbb{X}_0}(x) m_0(x) \text{Leb}(dx)$$

where  $\text{Leb}$  is the Lebesgue measure on  $\mathbb{X}_0$ , and denote by  $\pi$  the normalised probability measure.

**Proposition 3.3.** *The probability measure  $\pi$  is stationary with respect to  $Q$ .*

*Proof.* Let  $f : \mathbb{X}_0 \rightarrow \mathbb{R}$  be a non negative real measurable function and compute

$$\begin{aligned} \tilde{\pi} Q f &= \int \tilde{\pi}(dx) \int Q(x, dy) f(y) = \int \mathbf{1}_{\mathbb{X}_0}(x) \text{Leb}(dx) \int C(x, dy) f(y) \\ &= \int \sum_{\substack{I_1, \dots, I_d : \\ \text{card} \bigcap_{l=1}^d I_l = 1}} \sum_{s \in \mathcal{S}_\pm} \chi(I_1, \dots, I_d, x) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s) f(x + \bar{s}) \mathbf{1}_{\mathbb{X}_0}(x) \text{Leb}(dx). \end{aligned}$$

Exchange the sums and the integral,

$$\begin{aligned} &\int \sum_{\substack{I_1, \dots, I_d : \\ \text{card} \bigcap_{l=1}^d I_l = 1}} \sum_{s \in \mathcal{S}_\pm} \chi(I_1, \dots, I_d, x) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s) f(x + \bar{s}) \mathbf{1}_{\mathbb{X}_0}(x) \text{Leb}(dx) \\ &= \sum_{\substack{I_1, \dots, I_d : \\ \text{card} \bigcap_{l=1}^d I_l = 1}} \sum_{s \in \mathcal{S}_\pm} \int \chi(I_1, \dots, I_d, x) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s) f(x + \bar{s}) \mathbf{1}_{\mathbb{X}_0}(x) \text{Leb}(dx), \end{aligned}$$

set in each integral terms  $z = x + \bar{s}$ ,

$$\begin{aligned} &= \sum_{\substack{I_1, \dots, I_d : \\ \text{card} \bigcap_{l=1}^d I_l = 1}} \sum_{s \in \mathcal{S}_\pm} \int \text{Leb}(dz) f(z) \mathbf{1}_{\mathbb{X}_0}(z - \bar{s}) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s) \chi(I_1, \dots, I_d, z - \bar{s}) \\ &= \int \text{Leb}(dz) f(z) \sum_{\substack{I_1, \dots, I_d : \\ \text{card} \bigcap_{l=1}^d I_l = 1}} \sum_{s \in \mathcal{S}_\pm} \mathbf{1}_{\mathbb{X}_0}(z - \bar{s}) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s) \chi(I_1, \dots, I_d, z - \bar{s}) \\ &= \int \text{Leb}(dz) f(z) m(z) \mathbf{1}_{\mathbb{X}_0}(z) = \tilde{\pi} f, \end{aligned}$$

<sup>4</sup> In the sequel, we shall forget the zero indices if there is no ambiguity.

<sup>5</sup> RWIDF

since

$$\mathbf{1}_{\mathbb{X}_0}(z - \bar{s}) \mathbf{1}_{\bigcap_{l=1}^d K_{I_l}}(s) \chi(I_1, \dots, I_d, z - \bar{s})$$

is non zero if and only if  $z - \bar{s}$  and  $z$  are connected *via*  $\bar{s}$  and in this case  $\mathbf{1}_{\mathbb{X}_0}(z - \bar{s}) = \mathbf{1}_{\mathbb{X}_0}(z)$  by construction.  $\square$

As a consequence of this proposition (see [KKR04]), the associated *random walk with random transition probabilities*, RW RTP for short, is determined by the path space  $\Omega = \mathbb{X}_0^{\mathbb{Z}}$  with the shift-invariant Markov measure  $\mathbf{Q}^\pi$  and the map

$$(\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \mapsto \mu^{\omega_0, \omega_1}.$$

Introduce the following notation for the convolution of  $\mu^\omega, \dots, \mu^{T^n \omega}$ :

$$\mu^\omega * \dots * \mu^{T^n \omega} = \mu_{0,n}^\omega.$$

Under the assumption 2, the map  $\mathbf{p}_{\text{int}} \circ \mathbf{p}^{-1}$  is well defined for  $x \in \mathbb{G}_t^0$  and  $t \in E_{\text{int}}$  non ambiguous. Therefore we shall introduce the notation  $\tilde{x} = \mathbf{p}^{-1}(x)$  and  $\bar{x} = \mathbf{p}_{\text{int}}(\tilde{x}) - t$ . In addition, if  $f : \mathbb{G}_t^0 \rightarrow \mathbb{R}$  is a function satisfying  $f(x) = f(y)$  if  $\bar{x} = \bar{y}$ , we will denote by  $\bar{f} : \mathbb{X}_0 \rightarrow \mathbb{R}$  the function such that  $f(x) = \bar{f}(\bar{x})$  for all  $x \in \mathbb{G}_t^0$ . With this notation we can now state the following proposition which makes precise the connection between the RWIDF and the original simple random walk on a cut-and-project graph.

**Proposition 3.4.** *Let  $t \in E_{\text{int}}$  be non ambiguous,  $\mathbb{G}_t$  the corresponding cut-and-project graph and  $x \in \mathbb{G}_t^0$ . Then, for any  $y \in \mathbb{G}_t^0$ , the following holds:*

- (i)  $P(x, y) = \mathbf{E}^{\bar{x}}(\mu^\omega(\tilde{y} - \tilde{x}))$ ,
- (ii) for  $n \geq 1$ ,  $\delta_x P^n(y) = \mathbf{E}^{\bar{x}}(\mu_{0,n-1}^\omega(\tilde{y} - \tilde{x}))$ ,
- (iii)  $Pf(x) = Q\bar{f}(\bar{x})$ .

where  $\mathbf{E}^{\bar{x}}$  is the expectation with respect to  $\mathbf{Q}^{\bar{x}}$ .

*Proof.* By construction,  $\mathbf{E}^{\bar{x}}(\mu^\omega)$  is the uniform probability on admissible neighbours  $\tilde{x} + \varepsilon_i$  of  $\tilde{x}$ . Then, the equality (i) comes from the assumption 2.

By equality (i), the equality (ii) holds for  $n = 1$ . We start with

$$P^{n+1}(x, y) = \sum_{\xi \in \mathbb{Z}^N} P(x, \mathbf{p}(\xi)) P^n(\mathbf{p}(\xi), y).$$

By the point (i) and the induction assumption, it follows

$$P^{n+1}(x, y) = \sum_{\xi \in \mathbb{Z}^N} Q(\bar{x}, \bar{\xi}) \mu^{\bar{x}, \bar{\xi}}(\xi - \tilde{x}) \mathbf{E}^{\bar{\xi}}(\mu_{0,n-1}^\omega(\tilde{y} - \xi)).$$

Remark that for  $z \in \mathbb{Z}^N$ , we have the following equivalence

$$\mu^{\bar{x}, \bar{z}}(\xi - \tilde{x}) \neq 0 \iff \bar{\xi} = \bar{z}. \quad (1)$$

Consequently, the simple sum above can be rewritten as the following double sum

$$\sum_{z \in \mathbb{Z}^N} \sum_{\xi \in \mathbb{Z}^N} Q(\bar{x}, \bar{z}) \mu^{\bar{x}, \bar{z}}(\xi - \tilde{x}) \mathbf{E}^{\bar{z}}(\mu_{0,n-1}^\omega(\tilde{y} - \xi)),$$

which, by Markov property, is nothing but

$$\sum_{z, \xi \in \mathbb{Z}^N} Q(\bar{x}, \bar{z}) \mathbf{E}^{\bar{x}}(\mu^\omega(\xi - \tilde{x}) | X_1 = \bar{z}) \mathbf{E}^{\bar{x}}(\mu_{0,n-1}^{T\omega}(\tilde{y} - \xi) | X_1 = \bar{z}).$$

Since past and future of a Markov chain, conditionally to the present, are independent, the sum becomes

$$\sum_{z \in \mathbb{Z}^N} Q(\bar{x}, \bar{z}) \mathbf{E}^{\bar{x}}(\mu_{0,n}^\omega(\tilde{y} - \tilde{x}) | X_1 = \bar{z}),$$

that is

$$\mathbf{E}^{\bar{x}}(\mu_{0,n}^\omega(\tilde{y} - \tilde{x})),$$

which ends the proof of (ii).

The last point can be proved almost similarly, in fact we write

$$\begin{aligned} Pf(x) &= \sum_{\xi \in \mathbb{Z}^N} P(x, \mathfrak{p}(\xi)) f(\mathfrak{p}(\xi)) \\ &= \sum_{\xi \in \mathbb{Z}^N} Q(\bar{x}, \bar{\xi}) \mu^{\bar{x}, \bar{\xi}}(\xi - \tilde{x}) \bar{f}(\bar{\xi}). \end{aligned}$$

But with the equivalence 1, this sum is equal to the double sum

$$\sum_{z, \xi \in \mathbb{Z}^N} Q(\bar{x}, \bar{z}) \mu^{\bar{x}, \bar{z}}(\xi - \tilde{x}) \bar{f}(\bar{z}).$$

Since  $\mu^{\bar{x}, \bar{z}}$  is a probability, the result follows. □

### 3.2 Entropy estimate of RWIDF

We now have the ingredients to state a generic estimate of the asymptotic entropy of the simple random walk on a cut-and-project graph. The notion of genericity will be made precise at the end of this section.

For a discrete measure  $\mu$ , we define its *entropy*  $H(\mu)$  by

$$H(\mu) = - \sum_{z \in \mathbb{Z}^N} \mu(z) \log \mu(z).$$

The asymptotic entropy of the simple random walk on a cut-and-project is given (see [Kai92] for instance for a definition in the context of Markov chains) for any  $x \in \mathbb{G}_0$  by

$$h(x) = \lim_{n \rightarrow \infty} H(\delta_x P^n) - \sum_{y \in \mathbb{G}_0} P(x, y) H(\delta_y P^{n-1}).$$

**Proposition 3.5.** *There exists a function  $\bar{h} : \mathbb{X}_0 \rightarrow \mathbb{R}_+$  such that for all  $t \in E_{\text{int}}$  non ambiguous and  $x \in \mathbb{G}_t^0$  a vertex of the corresponding cut-and-project graph, the following equality holds*

$$h(x) = \bar{h}(\bar{x})$$

where  $\bar{x} = \mathfrak{p}_{\text{int}}(\mathfrak{p}^{-1}x) - t$ .

*Proof.* By proposition 3.4 we get that

$$H(\delta_y P^{n-1}) = H\left(\int_{\Omega} \mathbf{Q}^{\bar{y}}(d\omega) \mu_{0,n-2}^{\omega}\right),$$

and also that

$$h(x) = \lim_{n \rightarrow \infty} H\left(\int_{\Omega} \mathbf{Q}^{\bar{x}}(d\omega) \mu_{0,n-1}^{\omega}\right) - \sum_{\bar{y} \in \mathbb{Z}^N} Q(\bar{x}, \bar{y}) H\left(\int_{\Omega} \mathbf{Q}^{\bar{y}}(d\omega) \mu_{0,n-2}^{\omega}\right).$$

This form only depends on  $\bar{x} \in \mathbb{X}_0$ . □

**Theorem 3.6.** *The function  $\bar{h}$  is zero  $\pi$ -a.s. .*

*Proof.* Integrating with respect to the stationary measure  $\pi$ , and exchanging the limit and the integral (the quantity inside the limit is non negative), we get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{X}_0} \pi(d\bar{x}) H\left(\int_{\Omega} \mathbf{Q}^{\bar{x}}(d\omega) \mu_{0,n-1}^{\omega}\right) - \int_{\mathbb{X}_0} Q(\bar{x}, d\bar{y}) H\left(\int_{\Omega} \mathbf{Q}^{\bar{y}}(d\omega) \mu_{0,n-2}^{\omega}\right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{X}_0} \pi(d\bar{x}) H\left(\int_{\Omega} \mathbf{Q}^{\bar{x}}(d\omega) \mu_{0,n-1}^{\omega}\right) - H\left(\int_{\Omega} \mathbf{Q}^{\bar{x}}(d\omega) \mu_{0,n-2}^{\omega}\right) \\ &= \lim_{n \rightarrow \infty} \frac{\int_{\mathbb{X}_0} d\pi(d\bar{x}) H\left(\int_{\Omega} \mathbf{Q}^{\bar{x}}(d\omega) \mu_{0,n-1}^{\omega}\right)}{n}, \end{aligned}$$

by stationarity of  $\pi$ . The following majoration is immediate

$$H\left(\int_{\Omega} \mathbf{Q}^{\bar{x}}(d\omega) \mu_{0,n-1}^{\omega}\right) \leq N \log(n).$$

Consequently,  $\bar{h} = 0$ ,  $\pi$ -a.s. □

This theorem gives the existence of a subset  $\mathcal{N}_{\pi}$  of  $\mathbb{X}_0$ , such that  $\pi(\mathcal{N}_{\pi}) = 1$  and  $\bar{h} = 0$  on  $\mathcal{N}_{\pi}$ . Let  $t \in E_{\text{int}}$  non ambiguous, and  $x \in \mathbb{G}_t^0$ , then proposition 3.5 implies  $h(x) = 0$  if and only if  $\bar{x} \in \mathcal{N}_{\pi}$ . We summarize the aforementioned results into the following.

**Theorem 3.7.** *The asymptotic entropy of the simple random walk on a cut-and-project graph is generically zero, that is  $h(x) = 0$  for all  $x \in \mathbb{G}_t^0$  with  $t \in E_{\text{int}}$  non ambiguous such that  $\bar{x} \in \mathcal{N}_{\pi}$ .*

In the context of Markov chains, the asymptotic entropy is zero if and only its tail  $\sigma$ -algebra is trivial, *i.e.* its measurable sets have measure 0 or 1. The invariant  $\sigma$ -algebra is also trivial since the latter is contained in the former. The converse is not true in general; there exists Markov chains for which the invariant  $\sigma$ -algebra is trivial and its asymptotic entropy strictly positive.

## 4 A proof of the recurrence and transience

The proof of theorem 2.3 involves an equirepartition theorem of points in the cut-and-project graph. This can be deduced from a theorem of Schlottmann on model sets. The notion of model sets involved in the theorem of Schlottmann appears in a slightly different context which is recalled below.

## 4.1 Model sets and uniform distribution

We still denote by  $E$  a  $d$ -dimensional vector subspace of the real standard vector space  $\mathbb{R}^N$ , and by  $E_{\text{int}}$  the orthogonal complement of  $E$ . We also denote by  $\mathbf{p}$  and  $\mathbf{p}_{\text{int}}$  the canonical projections on  $E$  and  $E_{\text{int}}$  respectively. However, we need not restrict ourselves to the integer lattice  $\mathbb{Z}^N$ .

**Definition 4.1.** A subset  $\Lambda \subset E \oplus E_{\text{int}}$  is called a *lattice* if  $\Lambda$  is a discrete Abelian subgroup of  $E \oplus E_{\text{int}}$  such that there exists a compact  $K \subset E \oplus E_{\text{int}}$  satisfying  $\Lambda + K = E \oplus E_{\text{int}}$  ( $\Lambda$  is said relatively dense).

In the context of model sets, it is usually assumed that  $\mathbf{p}_{\text{int}}(\Lambda)$  is dense in  $E_{\text{int}}$ , and that  $\mathbf{p}$  restricted to  $\Lambda$  is bijective on its image  $\mathbf{p}(\Lambda)$ . Under such assumptions, we will say that the spaces  $E$  and  $E_{\text{int}}$  are in a *standard configuration*.

We denote by  $\mu$  and  $\mu_{\text{int}}$  the Lebesgue measure on  $E$  and  $E_{\text{int}}$  respectively. A *window* is a bounded subset  $W$  of  $E_{\text{int}}$  which is the closure of its interior with zero measure boundary (with respect to the Lebesgue measure  $\mu_{\text{int}}$ ). The set  $L = \{\mathbf{p}(x) : x \in \Lambda, \mathbf{p}_{\text{int}}(x) \in W\}$  is termed a *regular cut-and-project set* (or a regular *model set*). If the real space  $E$  and the internal space  $E_{\text{int}}$  are in a standard configuration, it can be shown that a regular cut-and-project set is a Delone set (see [Moo00] for instance).

**Definition 4.2.** A subset  $S$  of  $\mathbb{R}^N$  is a *Delone* (Delaunay) set if the following holds

1.  $\Lambda$  is *relatively dense*: there exists a non empty open set  $\mathcal{O}$  such that, for any  $v \in E$ ,  $v + \mathcal{O}$  contains a point of  $\Lambda$  ;
2.  $\Lambda$  is *uniformly discrete*: there exists a non empty open set  $\mathcal{O}'$  such that, for any  $v \in E$ ,  $v + \mathcal{O}'$  contains at most one point of  $\Lambda$ .

Fixing a basis of  $E$ , we denote by  $\|\cdot\|_p$  the standard  $p$ -norm on  $E$ , namely for  $x \in E$ ,

$$\|x\|_p = \left( \sum_{i=1}^d |x_i|^p \right)^{1/p}.$$

In addition, the  $p$ -metric induced by the  $p$ -norm is denoted by  $d_p$ . We denote by  $B_p(x, r)$  the ball of radius  $r > 0$  centered at  $x \in \mathbb{R}^d$ , and by  $\partial B_p(x, r)$  the corresponding sphere. We simply write  $B$  and  $\partial B$ , without subscripts, if the choice of a specific metric is irrelevant to the statement of the result.

The version given here can be found in [Sch98], but a similar statement is shown in [Hof98].

**Theorem 4.3.** Let  $E \cong \mathbb{R}^d$ ,  $E_{\text{int}} \cong \mathbb{R}^{(N-d)}$ . Let  $\Lambda$  be a lattice in  $E \oplus E_{\text{int}}$ . Assume that the space  $E$  and  $E_{\text{int}}$  are in a standard configuration. Then, uniformly in  $t \in E \oplus E_{\text{int}}$ ,

$$\lim_{r \rightarrow \infty} \frac{\text{card}(\Lambda \cap (t + B(0, r) + W))}{\mu(B(0, r))} = \frac{\mu_{\text{int}}(W)}{\mu \otimes \mu_{\text{int}}(\tilde{\Lambda})}$$

where  $\tilde{\Lambda}$  is a fundamental domain of  $\Lambda$ .

Combining arguments of [Hof98], we can deduce that when  $\Lambda$  is supposed to be the standard integer lattice  $\mathbb{Z}^N$ , the density of  $\mathbf{p}_{\text{int}}(\mathbb{Z}^N)$  in  $E_{\text{int}}$  can be substituted with the simpler condition  $\mathbb{Z}^N \cap E_{\text{int}} = \{0\}$ , which is itself equivalent to the injectivity of  $\mathbf{p}$  restricted to the integer lattice  $\mathbb{Z}^N$ . Moreover, the window we consider in the sequel is given by  $W = W_t = \mathbf{p}_{\text{int}}(K + t)$ , for  $t \in E_{\text{int}}$ .

## 4.2 Isoperimetric inequalities, reversible random walks

A Markov chain  $(M_n)_{n \geq 0}$  on an undirected graph  $\mathbb{G} = (\mathbb{G}^0, \mathbb{G}^1)$  is reversible if there exists a measure — the total conductance —  $m : \mathbb{G}^0 \mapsto (0, \infty)$  such that

$$m(x)p(x, y) = m(y)p(y, x) \quad (2)$$

for all  $x, y \in \mathbb{G}^0$ . Since the graph is undirected, the degree  $(\deg(x))_{x \in \mathbb{G}^0}$  is a measure satisfying (2) for the simple random walk. We view  $m$  indiscernably as a measure or as a combinatorial object.

Let us denote by  $d_{\mathbb{G}}$  the usual *graph metric* on the cut-and-project graph  $\mathbb{G}$  and by  $B_{\mathbb{G}}(x, n)$  the ball of radius  $n$  centered at  $x$ , *i.e.*

$$B_{\mathbb{G}}(x, n) = \{y \in \mathbb{G}^0 : d_{\mathbb{G}}(x, y) \leq n\}.$$

For a finite subset  $A \subset \mathbb{G}^0$  we will denote by  $\partial A$  the boundary of  $A$  defined as

$$\partial A = \{x \in A : \exists y \notin A \text{ with } d_{\mathbb{G}}(x, y) = 1\}.$$

The boundary of a ball  $B_{\mathbb{G}}(x, n)$  will be denoted by  $\partial B_{\mathbb{G}}(x, n)$ . The *growth function* of  $(\mathbb{G}, P)$  at the point  $x$  is given by  $V_P(x, n) = m(B_{\mathbb{G}}(x, n))$ . We set

$$V_P(n) = \inf_{x \in \mathbb{G}^0} V_P(x, n).$$

We omit the index  $P$  when the operator  $P$  defines the simple random walk.

To prove the statement on recurrence of theorem 2.3 we will use the following which can be found in [LP95], but see [Woe00] for an equivalent statement.

**Theorem 4.4.** *Suppose that  $V_P(x, n) \leq Cn^d$  and that the invariant measure  $m$  satisfies  $\inf_{x \in \mathbb{G}^0} m(x) > 0$ . Then*

$$p^{(2n)}(x, x) \geq C(n \log n)^{-d/2},$$

for some  $C > 0$ .

Note that the graph metric of the Cayley graph of  $\Lambda = \mathbb{Z}^n$  is nothing but the metric  $d_{\Lambda}$  induced by the 1-norm on  $\mathbb{R}^N$  in the canonical basis which is denoted in the sequel  $\|\cdot\|_{\Lambda}$ . This allows us to compare the graph metrics  $d_{\mathbb{G}}$  and  $d_{\Lambda}$ .

**Lemma 4.5.** *For all  $x, y \in \mathbb{G}^0$ ,*

$$d_{\Lambda}(\xi, \eta) = d_{\mathbb{G}}(x, y),$$

where  $(\xi, \eta) \in \Lambda^2$  is the unique pair in the strip  $\mathcal{K}_t = K + E + t$ , for  $t \in E_{\text{int}}$ , such that  $\mathbf{p}(\xi) = x$  and  $\mathbf{p}(\eta) = y$ .

This lemma states that a geodesic path in the graph can not be the projection of a non geodesic path of the lattice  $\Lambda$ .

*Proof.* This lemma is a direct consequence of the fact, due to [ODK88], that the tiling  $\mathcal{T}_t$  is the projection of a unique  $d$ -dimensional faceted manifold entirely contained in the strip  $K + E + t$ .  $\square$

Consequently, the theorem of Schlottmann, [Sch98], with lemma 4.5 yields the following ball growth estimates.

**Proposition 4.6.** *Under the assumptions of theorem 2.3, the following estimate is satisfied for all  $x \in \mathbb{G}^0$*

$$\mathbf{k}^{-1}l^d \leq \text{card}B_{\mathbb{G}}(x, l) \leq \mathbf{k}l^d$$

for a constant  $\mathbf{k} > 1$  independent of  $x \in \mathbb{G}^0$ .

*Proof.* Let  $x, y \in \mathbb{G}^0$  and let  $(\xi, \eta)^2 \in (\Lambda \cap \mathcal{K}_t)^2$  be the unique pair of points such that  $\mathbf{p}(\xi) = x$  and  $\mathbf{p}(\eta) = y$ . On one hand, we obtain

$$d_2(x, y) \leq \|\mathbf{p}\|d_{\Lambda}(\xi, \eta)$$

where  $\|\mathbf{p}\|$  is the matrix norm defined by

$$\|\mathbf{p}\| = \sup_{y \in E \oplus E_{\text{int}}: \|y\|_{\Lambda} \leq 1} \frac{\|\mathbf{p}(y)\|_2}{\|y\|_{\Lambda}}.$$

On the other hand, there exist  $u, v \in W \subset E_{\text{int}}$  such that  $\xi = x + u$  and  $\eta = y + u$  (and obviously these  $u, v$  are uniquely determined). Thus, we get the following obvious inequality :

$$d_{\Lambda}(\xi, \eta) = \|\xi - \eta\|_{\Lambda} \leq \|x - y\|_{\Lambda} + \|u - v\|_{\Lambda} \leq c_0\|x - y\|_2 + \text{diam}(W).$$

Consequently, by lemma 4.5, we get

$$\|\mathbf{p}\|^{-1}d_2(x, y) \leq d_{\mathbb{G}}(x, y) \leq c_0d_2(x, y) + \text{diam}(W).$$

Applying theorem 4.3 and remarking that

$$B_2(x, c_0^{-1}(n - \text{diam}(W))) \subset B_{\mathbb{G}}(x, n) \subset B_2(x, n\|\mathbf{p}\|),$$

we get the inequality of the proposition.  $\square$

Denote by  $a$  the conductance defined by  $a(x, y) = m(x)p(x, y)$ . For every  $f \in c_0(\mathbb{G}^0)$  — the space of functions with compact support on  $\mathbb{G}^0$  — we define

$$\|f\|_D^2 = \frac{1}{2} \sum_{x, y \in \mathbb{G}^0} a(x, y) |f(x) - f(y)|^2$$

the Dirichlet norm of  $f$ . Recall the following.

**Theorem 4.7** (Theorem 1 of [Var85]). *Let  $\alpha \geq 2$ , if for every  $f \in c_0(\mathbb{G}^0)$*

$$\|f\|_r \leq C\|f\|_D \text{ where } r = \frac{2\alpha}{\alpha - 2}, \quad (3)$$

$C > 0$  is independent of  $f$ , and  $\|\cdot\|_r$  is the standard norm in  $l^r(\mathbb{G}^0, m)$ . Then we have,

$$\sup_{x, y \in \mathbb{G}^0} \frac{p^n(x, y)}{m(y)} = \mathcal{O}(n^{-\alpha/2}).$$

The isoperimetric inequality does not imply, in full generality, the inequality of theorem 4.7. However, for a function  $f \in c_0(\mathbb{G}^0)$ , the Sobolev norm is defined by

$$\|f\|_S = \sum_{x, y \in \mathbb{G}^0} a(x, y) |f(x) - f(y)|,$$

and according to proposition of section 3 of [Var85], we have the following.

**Proposition 4.8.** *Let  $\alpha > 2$  and assume that there exists  $C > 0$  such that for all  $f \in c_0(\mathbb{G}^0)$  the Sobolev inequality holds*

$$\|f\|_{\alpha/(\alpha-1)} \leq C\|f\|_S.$$

*Then, there exists  $C' > 0$  such that for all  $f \in c_0(\mathbb{G}^0)$  the following holds*

$$\|f\|_{2\alpha/(\alpha-2)} \leq C'\|f\|_D.$$

As a matter of fact, a  $d$ -dimensional isoperimetric inequality is equivalent to a Sobolev inequality with  $\alpha = d$  (see [Woe00], proposition (4.3), p. 40). Because of technical difficulties, we will not be able to prove a  $d$ -dimensional isoperimetric inequality for the initial graph but only for its  $k$ -fuzz. Nonetheless, the  $k$ -fuzz construction leaving the type of the simple random walk invariant, the conclusion will be immediate.

If  $\mathbb{G}$  is a graph, the  $k$ -fuzz of  $\mathbb{G}$ , denoted by  $\text{Fuzz}_k(\mathbb{G})$ , has the same set of vertices as  $\mathbb{G}$  and  $(x, y)$  is an edge in  $\text{Fuzz}_k(\mathbb{G})$  if and only if  $1 \leq d_{\mathbb{G}}(x, y) \leq k$ . We note  $\rho$  the graph metric on  $\text{Fuzz}_k(\mathbb{G})$ . It is well known that the balls in the two graphs can be compared as well as the spheres, namely

$$B_\rho(x, n) = B_{\mathbb{G}}(x, kn) \text{ and } \partial B_\rho(x, n) = \bigcup_{l=kn-k+1}^{kn} \partial B_{\mathbb{G}}(x, l).$$

**Proposition 4.9.** *Let  $d = \dim E$ . The  $k$ -fuzz  $\text{Fuzz}_k(\mathbb{G})$  satisfies a  $d$ -dimensional isoperimetric inequality for  $k$  large enough, i.e.*

$$\text{card} B_\rho(x, n) \leq k \text{card} \partial B_\rho(x, n)^{d/(d-1)}$$

for some  $k > 0$ .

*Proof.* According to proposition 4.6 there exist  $C_-, C_+ > 0$  such that for all  $n \geq 1$

$$C_-(kn)^d \leq \text{card} B_\rho(x, n) \leq C_+(kn)^d.$$

Hence, we need a lower bound of  $\text{card} \partial B_\rho(x, n)$ , namely, we have to show that

$$\text{card} \partial B_\rho(x, n) \geq \kappa(kn)^{d-1}.$$

By lemma 4.5, and from the proof of 4.6, we get

$$\|\mathbf{p}\|^{-1} d_2(x, y) \leq d_{\mathbb{G}}(x, y) \leq c_0 d_2(x, y) + \text{diam}(W).$$

Consequently, a point  $y \in \partial B_\rho(x, n)$  satisfies

$$c_0^{-1}(kn - k + 1 - \text{diam}(W)) \leq d_2(x, y) \leq \|\mathbf{p}\|kn,$$

and in terms of balls we get

$$B_2(x, \|\mathbf{p}\|kn) \setminus B_2(x, c_0^{-1}(kn - k + 1 - \text{diam}(W))) \subset \partial B_\rho(x, n).$$

Since  $c_0\|\mathbf{p}\| \geq 1$ , it is obvious that for any  $k \geq 1$

$$B_2(x, c_0^{-1}kn) \setminus B_2(x, c_0^{-1}kn - c_0^{-1}(k - 1 + \text{diam}(W))) \subset \partial B_\rho(x, n).$$

In the sequel, we need to adapt the proof of proposition 2.1 in [Sch98]. Setting  $r = c_0^{-1}kn$  and  $w = w(k) = c_0^{-1}(k - 1 + \text{diam}(W))$ , and defining

$$N(r, w, x, W) = \frac{\text{card}(\Lambda \cap (B_2(x, r) \setminus B_2(x, r - w)) + W)}{\mu(B_2(x, r) \setminus B_2(x, r - w))},$$

we want to show that the inequality

$$\alpha\mu_{\text{int}}(W) \leq N(r, w, x, W) \leq (1 - \alpha)\mu_{\text{int}}(W) \quad (4)$$

holds for some  $\alpha > 0$ . Obviously, we have,

$$\begin{aligned} N(r, w, x, W) &= \beta_r \frac{\text{card}(\Lambda \cap (B_2(x, r) + W))}{\mu(B_2(x, r))} \\ &\quad + (1 - \beta_r) \frac{\text{card}(\Lambda \cap (B_2(x, r - w) + W))}{\mu(B_2(x, r - w))}, \end{aligned}$$

where  $\beta_r = \frac{\mu(B_2(x, r))}{\mu(B_2(x, r) \setminus B_2(x, r - w))}$ . Consequently, we can majorize

$$\begin{aligned} |N(r, w, x, W) - \mu_{\text{int}}(W)| &\leq \beta_r \left| \frac{\text{card}(\Lambda \cap (B_2(x, r) + W))}{\mu(B_2(x, r))} - \mu_{\text{int}}(W) \right| \\ &\quad + (\beta_r - 1) \left| \frac{\text{card}(\Lambda \cap (B_2(x, r - w) + W))}{\mu(B_2(x, r - w))} - \mu_{\text{int}}(W) \right|. \end{aligned}$$

It follows from relations (3.20) and (3.23) in [Sch98] that

$$\begin{aligned} |N(r, w, x, W) - \mu_{\text{int}}(W)| &\leq \mu_{\text{int}}(W) \left[ \beta_r \frac{\mu\{B_2(x, r + \delta + \epsilon) \setminus B_2(x, r - \delta - \epsilon)\}}{\mu(B_2(x, r))} \right. \\ &\quad \left. + (\beta_r - 1) \frac{\mu\{B_2(x, r - w + \delta + \epsilon) \setminus B_2(x, r - w - \delta - \epsilon)\}}{\mu(B_2(x, r - w))} \right], \end{aligned}$$

where  $\delta > 0$  depends on the window  $W$  and  $\epsilon > 0$  only depends on the lattice  $\Lambda$ . Obviously, for  $r$  large enough, there exists  $\kappa_0 > 0$  such that  $\beta_r \leq \kappa_0 \frac{r}{dw}$  and  $\kappa_1 > 0$  such that

$$\beta_r \frac{\mu\{B_2(x, r + \delta + \epsilon) \setminus B_2(x, r - \delta - \epsilon)\}}{\mu(B_2(x, r))} \leq \kappa_1 \frac{\delta + \epsilon}{w(k)}.$$

Since  $w$  can be made arbitrarily large with  $k \geq 1$ , the quantity  $\kappa_1 \frac{\delta + \epsilon}{w}$  can be made strictly smaller than 1, and we conclude that

$$|N(r, w, x, W) - \mu_{\text{int}}(W)| \leq (1 - \alpha)\mu_{\text{int}}(W),$$

for some  $\alpha = \alpha(w) = \alpha(k) > 0$ . Consequently, the following holds for large enough  $k$

$$\text{card} \partial B_\rho(x, n) \geq \text{card}(\Lambda \cap (B_2(x, r) \setminus B_2(x, r - w)) + W) \geq \kappa(kn)^{d-1},$$

and the  $k$ -fuzz  $\text{Fuzz}_k(\mathbb{G})$  satisfies a  $d$ -dimensional isoperimetric inequality.  $\square$

*Proof of theorem 2.3.* Assume that  $\dim E = d$  then the proposition 4.6 with the theorem 4.4, remarking that  $1 \leq \deg(x) \leq 2N$ ,  $x \in \mathbb{G}^0$ , imply that

$$p^{(2n)}(x, x) \geq C_0(n \log n)^{-d/2}.$$

The  $k$ -fuzz graph  $\mathbb{G}^k$  satisfies a  $d$ -dimensional isoperimetric inequality by proposition 4.9. Obviously, the original graph satisfies, also, a  $d$ -dimensional isoperimetric inequality (theorem 4.7 in [Woe00]). Then, according to theorem of [Var85], it satisfies a Dirichlet inequality with parameter  $\alpha = d$ , thus theorem 4.7 implies the estimate on the  $n$ -step transition probabilities :

$$p^{(n)}(x, y) \leq C_1 n^{-d/2},$$

for some  $C_1 > 0$ .  $\square$

## 5 Conclusions

As examples in [CP03] suggest, when the graph considered is no longer the Cayley graph of group, we can observe new behaviors of simple random walks. More precisely, a 2-dimensional random walk appears to be transient. It turns out that the random walk considered by Campanino and Petritis can, as in our context, be seen as RWIDF. On this family of examples, we can remark that when the internal Markov chain driving the RWIDF admits a stationary probability measure, the resulting process is recurrent, otherwise it is transient. Somehow, the internal Markov chain introduces a notion of memory. The existence of an invariant probability implies that this memory is bounded in a certain sense so that the resulting process behaves as in the memoryless case (*i.e.* as a random walk with i.i.d. increments).

For a transient Markov chain, we can ask for the determination of the Martin boundary. Generally speaking, the study of Martin boundary involves sharp estimates on the asymptotic of the Green function. In [Uch07, KU08], such estimates are given in the context of Markov additive processes which is just a RWIDF on the lattice  $\mathbb{Z}^N$ . These results involve the Doeblin condition for the internal Markov chain. Such condition seems to be quite difficult to prove in our context so that the determination of the Martin boundary is still an open problem even though there is no reason to expect a different result compared to the standard case of random walk on the lattice.

Since the asymptotic entropy is shown to be zero, we shall expect to state a Strong Law of Large Number and a Central Limit Theorem. However, it would require a precise description of ergodic properties of the invariant probability measure given in this paper for identifying the limit in the law of large number and the covariance matrix in the central limit theorem.

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