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# The Erdős-Hajnal Conjecture for Paths and Antipaths

N. Bousquet<sup>a</sup>, A. Lagoutte<sup>b,1,\*</sup>, S. Thomassé<sup>b,1</sup>

<sup>a</sup>*AIGCo project-team, CNRS, LIRMM, 161 rue Ada, 34392 Montpellier France.*

<sup>b</sup>*LIP, UMR 5668 ENS Lyon - CNRS - UCBL - INRIA, Université de Lyon, 46, allée d'Italie, 69364 Lyon France.*

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## Abstract

We prove that for every  $k$ , there exists  $c_k > 0$  such that every graph  $G$  on  $n$  vertices with no induced path  $P_k$  or its complement  $\overline{P}_k$  contains a clique or a stable set of size  $n^{c_k}$ .

*Keywords:* Erdős-Hajnal, path, antipath, Ramsey

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An  $n$ -graph is a graph on  $n$  vertices. For every vertex  $x$ ,  $N(x)$  denotes the neighborhood of  $x$ , that is the set of vertices  $y$  such that  $xy$  is an edge. The degree  $\deg(x)$  is the size of  $N(x)$ . In this note, we only consider classes of graphs that are closed under induced subgraphs. Moreover a class  $\mathcal{C}$  is *strict* if it does not contain all graphs. It is said to have the (*weak*) *Erdős-Hajnal property* if there exists some  $c > 0$  such that every graph of  $\mathcal{C}$  contains a clique or a stable set of size  $n^c$  where  $n$  is the size of  $G$ . The Erdős-Hajnal conjecture [8] asserts that every strict class of graphs has the Erdős-Hajnal property; see [3] for a survey. This fascinating question is open even for graphs not inducing a cycle of length five. When excluding a single graph  $H$ , Alon, Pach and Solymosi showed in [2] that it suffices to consider *prime*  $H$ , namely graphs without nontrivial modules (a *module* is a subset  $V'$  of vertices such that for every  $x, y \in V'$ ,  $N(x) \setminus V' = N(y) \setminus V'$ ). A natural approach is then to study classes of graphs with intermediate difficulty, hoping to get a proof scheme which could be extended. A natural prime candidate to forbid is certainly the path. Unfortunately, even excluding the path on five vertices seems already hard. Chudnovsky and Zwols studied the class  $\mathcal{C}_k$  of graphs not inducing the path  $P_k$  on  $k$  vertices or its complement  $\overline{P}_k$ . They proved the Erdős-Hajnal property for  $P_5$  and  $\overline{P}_6$ -free graphs [7]. This was extended for  $P_5$  and  $\overline{P}_7$ -free graphs by Chudnovsky and Seymour [6]. Moreover structural results have been provided for  $\mathcal{C}_5$  [4, 5]. We show in this note that for every fixed  $k$ , the class  $\mathcal{C}_k$  has the Erdős-Hajnal property. An  $n$ -graph is an  $\varepsilon$ -stable set if it has at most  $\varepsilon \binom{n}{2}$  edges. The complement of an  $\varepsilon$ -stable set is an  $\varepsilon$ -clique. Fox and Sudakov [11] proved the following:

**Theorem 1** ([11]). *For every positive integer  $k$  and every  $\varepsilon \in (0, 1/2)$ , there exists  $\delta > 0$  such that every  $n$ -graph  $G$  satisfies one of the following:*

- $G$  induces all graphs on  $k$  vertices.
- $G$  contains an  $\varepsilon$ -stable set of size at least  $\delta n$ .
- $G$  contains an  $\varepsilon$ -clique of size at least  $\delta n$ .

Note that a stronger result was previously showed by Rödl [14] using Szemerédi's regularity lemma, but Fox and Sudakov's proof provides a much better quantitative estimate ( $\delta = 2^{-ck(\log 1/\varepsilon)^2}$  for some constant  $c$ ). They further conjecture that a polynomial estimate should hold, which would imply the Erdős-Hajnal conjecture.

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\*Corresponding author

*Email addresses:* nicolas.bousquet@lirmm.fr (N. Bousquet), aurelie.lagoutte@ens-lyon.fr (A. Lagoutte), stephan.thomasse@ens-lyon.fr (S. Thomassé)

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In a graph  $G$ , a *biclique of size  $t$*  is a (not necessarily induced) complete bipartite subgraph  $(X, Y)$  such that both  $|X|, |Y| \geq t$ . Observe that it does not require any condition inside  $X$  or inside  $Y$ . Erdős, Hajnal and Pach proved in [9] that for every strict class  $\mathcal{C}$ , there exists some  $c > 0$  such that for every  $n$ -graph  $G$  in  $\mathcal{C}$ ,  $G$  or its complement  $\overline{G}$  contains a biclique of size  $n^c$ . This "half" version of the conjecture was improved to a "three quarter" version by Fox and Sudakov [10], where they show the existence of a polynomial size stable set or biclique. Following the notations of [12], a class  $\mathcal{C}$  of graphs has the *strong Erdős-Hajnal property* if there exists a constant  $c$  such that for every  $n$ -graph  $G$  in  $\mathcal{C}$ ,  $G$  or  $\overline{G}$  contains a biclique of size  $cn$ . It was proved that having the strong Erdős-Hajnal property implies having the (weak) Erdős-Hajnal property:

**Theorem 2** ([1, 12]). *If  $\mathcal{C}$  is a class of graphs having the strong Erdős-Hajnal property, then  $\mathcal{C}$  has the weak Erdős-Hajnal property.*

*Proof.* (sketch) Let  $c$  be the constant of the strong Erdős-Hajnal property, meaning that for every  $n$ -graph  $G$  in  $\mathcal{C}$ ,  $G$  or  $\overline{G}$  contains a biclique of size  $cn$ . Let  $c' > 0$  be such that  $c^{c'} \geq 1/2$ . We prove by induction that every  $n$ -graph  $G$  in  $\mathcal{C}$  induces a  $P_4$ -free graph of size  $n^{c'}$ . By our hypothesis on  $\mathcal{C}$ , there exists, say, a biclique  $(X, Y)$  of size  $cn$  in  $G$ . Applying the induction hypothesis inside both  $X$  and  $Y$ , we form a  $P_4$ -free graph on  $2(cn)^{c'} \geq n^{c'}$  vertices. The Erdős-Hajnal property of  $\mathcal{C}$  follows from the fact that every  $P_4$ -free  $n^{c'}$ -graph has a clique or a stable set of size at least  $n^{c'/2}$ .  $\square$

We now prove our main result. The key lemma is an adaptation of Gyárfás' proof of the  $\chi$ -boundedness of  $P_k$ -free graphs, see [13].

**Lemma 3.** *For every  $k \geq 2$ , there exists  $\varepsilon_k > 0$  and  $c_k$  (with  $0 < c_k \leq 1/2$ ) such that every connected  $n$ -graph  $G$  with  $n \geq 2$  satisfies one of the following:*

- *There exists a vertex of degree more than  $\varepsilon_k n$ .*
- *For every vertex  $v$ ,  $G$  contains an induced  $P_k$  starting at  $v$ .*
- *The complement  $\overline{G}$  of  $G$  contains a biclique of size  $c_k n$ .*

*Proof.* We proceed by induction on  $k$ . For  $k = 2$ , since  $G$  is connected, every vertex is the endpoint of an edge (that is, a  $P_2$ ). Thus we can arbitrarily define  $\varepsilon_2 = c_2 = 1/2$ .

If  $k > 2$ , let  $\varepsilon_k = \frac{\varepsilon_{k-1}}{(2+\varepsilon_{k-1})}$  and  $c_k = \frac{c_{k-1}(1-\varepsilon_k)}{2}$ . Let us assume that the first item is false. We will show that the second or the third item is true. Let  $v_1$  be any vertex and  $S = V(G) \setminus (N(v_1) \cup \{v_1\})$ . The size  $s$  of  $S$  is at least  $(1 - \varepsilon_k)n - 1$ . If  $S$  have only *small* connected components, meaning of size at most  $s/2$ , then one can divide the connected components into two parts with at least  $(s + 1)/4$  vertices each, and no edges between both parts. This gives in  $\overline{G}$  a biclique of size  $(s + 1)/4 \geq \frac{(1-\varepsilon_k)n}{4}$ , thus of size at least  $c_k n$  since  $c_k \leq \frac{1-\varepsilon_k}{4}$ . Otherwise,  $S$  has a *giant* connected component  $S'$ , meaning of size  $s'$  more than  $s/2$ . Let  $v_2$  be a vertex adjacent both to  $v_1$  and to some vertex in  $S'$ . Observe that  $v_2$  exists since  $G$  is connected. Consider now the graph  $G_2$  induced by  $S' \cup \{v_2\}$ . The maximum degree in  $G_2$  is still at most  $\varepsilon_k n = \varepsilon_{k-1}(1-\varepsilon_k)n/2 \leq \varepsilon_{k-1}(s'+1)$ . By the induction hypothesis, either the second or the third item is true for  $G_2$  with parameter  $k - 1$ . The second item gives an induced  $P_{k-1}$  in  $G_2$  starting at  $v_2$ , thus an induced  $P_k$  in  $G$  starting at  $v_1$ . The third item gives a biclique of size  $c_{k-1}|G_2|$  in  $\overline{G_2}$ . Since  $|G_2| = s' + 1 \geq \frac{1-\varepsilon_k}{2}n$ , this gives a biclique of size at least  $\frac{c_{k-1}(1-\varepsilon_k)}{2}n = c_k n$  and concludes the proof.  $\square$

**Theorem 4.** *For every  $k \geq 2$ ,  $\mathcal{C}_k$  has the strong Erdős-Hajnal property. Thus, by Theorem 2, the class  $\mathcal{C}_k$  has the (weak) Erdős-Hajnal property.*

*Proof.* Let  $\varepsilon_k$  be as defined in Lemma 3 and  $\varepsilon = \varepsilon_k/8 > 0$ . By Theorem 1, there exists  $\delta > 0$  such that every graph  $G$  not inducing  $P_k$  or  $\overline{P}_k$  does contain an  $\varepsilon$ -stable set or an  $\varepsilon$ -clique of size at least  $\delta n$ . Free to consider the complement of  $G$ , we can assume that  $G$  contains an  $\varepsilon$ -stable set  $S_0$  of size  $\delta n$ . We start by deleting in  $S_0$  all the vertices with degree in  $S_0$  at least  $2\varepsilon s_0$  where  $s_0$  is the size of  $S_0$ . Since the average degree in  $S_0$

is at most  $\varepsilon s_0$ , we do not delete more than half of the vertices. We call  $S$  the remaining subgraph which is a  $4\varepsilon$ -stable set of size  $s \geq \delta n/2$  with maximum degree less than  $4\varepsilon s$ .

Let  $G_S$  be the graph induced by  $S$ . Our goal is to find a constant  $c$  such that  $\overline{G_S}$  have a biclique of size  $cs$ , which gives a biclique in  $\overline{G}$  of size at least  $c\delta n/2$  and concludes the proof. Assume first that  $G_S$  only has *small* connected components, meaning of size less than  $s/2$ . Then one can partition the connected components of  $G_S$  in order to get a biclique in  $\overline{G_S}$  of size  $s/4$ . Otherwise,  $G_S$  has a connected component  $S'$  of size  $s' \geq s/2$ . The degree of every vertex in  $S'$  is at most  $8\varepsilon s' = \varepsilon_k s'$ , and  $S'$  does not contain any induced  $P_k$  since  $G$  does not. By Lemma 3, there exists a biclique of size  $c_k s' \geq c_k s/2$  in the complement of the graph induced by  $S'$ , thus in  $\overline{G_S}$ .  $\square$

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