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The Erdős-Hajnal Conjecture for Paths and Antipaths

N. Bousquet\textsuperscript{a}, A. Lagoutte\textsuperscript{b,1,*}, S. Thomassé\textsuperscript{b,1}

\textsuperscript{a}AlGoCo project-team, CNRS, LIRMM, 161 rue Ada, 34392 Montpellier France.
\textsuperscript{b}LIP, UMR 5668 ENS Lyon - CNRS - UCBL - INRIA, Université de Lyon, 46, allée d’Italie, 69364 Lyon France.

Abstract

We prove that for every $k$, there exists $c_k > 0$ such that every graph $G$ on $n$ vertices with no induced path $P_k$ or its complement $\overline{P_k}$ contains a clique or a stable set of size $n^{c_k}$.

Keywords: Erdős-Hajnal, path, antipath, Ramsey

An $n$-graph is a graph on $n$ vertices. For every vertex $x$, $N(x)$ denotes the neighborhood of $x$, that is the set of vertices $y$ such that $xy$ is an edge. The degree $\deg(x)$ is the size of $N(x)$. In this note, we only consider classes of graphs that are closed under induced subgraphs. Moreover a class $C$ is strict if it does not contain all graphs. It is said to have the (weak) Erdős-Hajnal property if there exists some $c > 0$ such that every graph of $C$ contains a clique or a stable set of size $n^c$ where $n$ is the size of $G$. The Erdős-Hajnal conjecture [8] asserts that every strict class of graphs has the Erdős-Hajnal property; see [3] for a survey.

This fascinating question is open even for graphs not inducing a cycle of length five. When excluding a single graph $H$, Alon, Pach and Solymosi showed in [2] that it suffices to consider prime $H$, namely graphs without nontrivial modules (a module is a subset $V'$ of vertices such that for every $x, y \in V'$, $N(x) \setminus V' = N(y) \setminus V'$).

A natural approach is then to study classes of graphs with intermediate difficulty, hoping to get a proof scheme which could be extended. A natural prime candidate to forbid is certainly the path. Unfortunately, even excluding the path on five vertices seems already hard. Chudnovsky and Zwols studied the class $C_k$ of graphs not inducing the path $P_k$ on $k$ vertices or its complement $\overline{P_k}$. They proved the Erdős-Hajnal property for $P_5$ and $\overline{P_5}$-free graphs [7]. This was extended for $P_5$ and $\overline{P_5}$-free graphs by Chudnovsky and Seymour [6]. Moreover structural results have been provided for $C_5$ [4, 5]. We show in this note that for every fixed $k$, the class $C_k$ has the Erdős-Hajnal property. An $n$-graph is an $\varepsilon$-stable set if it has at most $\varepsilon(n^2)$ edges. The complement of an $\varepsilon$-stable set is an $\varepsilon$-clique. Fox and Sudakov [11] proved the following:

**Theorem 1** ([11]). For every positive integer $k$ and every $\varepsilon \in (0, 1/2)$, there exists $\delta > 0$ such that every $n$-graph $G$ satisfies one of the following:

- $G$ induces all graphs on $k$ vertices.
- $G$ contains an $\varepsilon$-stable set of size at least $\delta n$.
- $G$ contains an $\varepsilon$-clique of size at least $\delta n$.

Note that a stronger result was previously showed by Rödl [14] using Szemerédi’s regularity lemma, but Fox and Sudakov’s proof provides a much better quantitative estimate ($\delta = 2^{-ck(\log 1/\varepsilon)^2}$ for some constant $c$). They further conjecture that a polynomial estimate should hold, which would imply the Erdős-Hajnal conjecture.

*Corresponding author

Email addresses: nicolas.bousquet@lirmm.fr (N. Bousquet), aurelie.lagoutte@ens-lyon.fr (A. Lagoutte\textsuperscript{1}), stephan.thomasse@ens-lyon.fr (S. Thomassé\textsuperscript{1})

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In a graph $G$, a biclique of size $t$ is a (not necessarily induced) complete bipartite subgraph $(X, Y)$ such that both $|X|, |Y| \geq t$. Observe that it does not require any condition inside $X$ or inside $Y$. Erdős, Hajnal and Pach proved in [9] that for every strict class $\mathcal{C}$, there exists some $c > 0$ such that for every $n$-graph $G$ in $\mathcal{C}$, $G$ or its complement $\overline{G}$ contains a biclique of size $cn$. This "half" version of the conjecture was improved to a "three quarter" version by Fox and Sudakov [10], where they show the existence of a polynomial size stable set or biclique. Following the notations of [12], a class $C$ of graphs has the strong Erdős-Hajnal property if there exists a constant $c$ such that for every $n$-graph $G$ in $C$, $G$ or $\overline{G}$ contains a biclique of size $cn$. It was proved that having the strong Erdős-Hajnal property implies having the (weak) Erdős-Hajnal property:

**Theorem 2 ([1, 12]).** If $C$ is a class of graphs having the strong Erdős-Hajnal property, then $C$ has the weak Erdős-Hajnal property.

Proof. (sketch) Let $c$ be the constant of the strong Erdős-Hajnal property, meaning that for every $n$-graph $G$ in $C$, $G$ or $\overline{G}$ contains a biclique of size $cn$. Let $c' > 0$ be such that $c'^2 \geq 1/2$. We prove by induction that every $n$-graph $G$ in $C$ induces a $P_4$-free graph of size $n^{c'}$. By our hypothesis on $C$, there exists, say, a biclique $(X, Y)$ of size $cn$ in $G$. Applying the induction hypothesis inside both $X$ and $Y$, we form a $P_4$-free graph on $2(cn)^{c'} \geq n^{c'}$ vertices. The Erdős-Hajnal property of $C$ follows from the fact that every $P_4$-free $n^{c'}$-graph has a clique or a stable set of size at least $n^{c'/2}$.

We now prove our main result. The key lemma is an adaptation of Gyárfás’ proof of the $\chi$-boundedness of $P_k$-free graphs, see [13].

**Lemma 3.** For every $k \geq 2$, there exists $\varepsilon_k > 0$ and $c_k$ (with $0 < c_k \leq 1/2$) such that every connected $n$-graph $G$ with $n \geq 2$ satisfies one of the following:

- There exists a vertex of degree more than $\varepsilon_k n$.
- For every vertex $v$, $G$ contains an induced $P_k$ starting at $v$.
- The complement $\overline{G}$ of $G$ contains a biclique of size $c_k n$.

Proof. We proceed by induction on $k$. For $k = 2$, since $G$ is connected, every vertex is the endpoint of an edge (that is, a $P_2$). Thus we can arbitrarily define $\varepsilon_2 = c_2 = 1/2$.

If $k > 2$, let $\varepsilon_k = \frac{\varepsilon_{k-1} - 1}{2 + \varepsilon_{k-1}}$ and $c_k = \frac{\varepsilon_{k-1}(1-\varepsilon_{k-1})}{2}$. Let us assume that the first item is false. We will show that the second or the third item is true. Let $v_1$ be any vertex and $S = V(G) \setminus (N(v_1) \cup \{v_1\})$. The size $s$ of $S$ is at least $(1-\varepsilon_k)n - 1$. If $S$ have only small connected components, meaning of size at most $s/2$, then one can divide the connected components into two parts with at least $(s+1)/4$ vertices each, and no edges between both parts. This gives in $\overline{G}$ a biclique of size $(s+1)/4 \geq \frac{1-\varepsilon_k}{4}n$, thus of size at least $c_k n$ since $c_k \leq \frac{1-\varepsilon_k}{4}$. Otherwise, $S$ has a giant connected component $S'$, meaning of size $s'$ more than $s/2$. Let $v_2$ be a vertex adjacent both to $v_1$ and to some vertex in $S'$. Observe that $v_2$ exists since $G$ is connected. Consider now the graph $G_2$ induced by $S' \cup \{v_2\}$. The maximum degree in $G_2$ is still at most $\varepsilon_k n = \varepsilon_{k-1}(1-\varepsilon_{k}) n/2 \leq \varepsilon_{k-1}(s'+1)$. By the induction hypothesis, either the second or the third item is true for $G_2$ with parameter $k-1$. The second item gives an induced $P_{k-1}$ in $G_2$ starting at $v_2$, thus an induced $P_k$ in $G$ starting at $v_1$. The third item gives a biclique of size $c_{k-1} |G_2|$ in $\overline{G_2}$. Since $|G_2| = s'+1 \geq \frac{1-\varepsilon_k}{2}n$, this gives a biclique of size at least $c_{k-1}(1-\varepsilon_{k}) n = c_k n$ and concludes the proof.

**Theorem 4.** For every $k \geq 2$, $C_k$ has the strong Erdős-Hajnal property. Thus, by Theorem 2, the class $C_k$ has the (weak) Erdős-Hajnal property.

Proof. Let $\varepsilon_k$ be as defined in Lemma 3 and $\varepsilon = \varepsilon_k/8 > 0$. By Theorem 1, there exists $\delta > 0$ such that every graph $G$ not inducing $P_k$ or $P_k$ does contain an $\varepsilon$-stable set or an $\varepsilon$-clique of size at least $\delta n$. Free to consider the complement of $G$, we can assume that $G$ contains an $\varepsilon$-stable set $S_0$ of size $\delta n$. We start by deleting in $S_0$ all the vertices with degree in $S_0$ at least $2\varepsilon s_0$ where $s_0$ is the size of $S_0$. Since the average degree in $S_0$
is at most $\varepsilon s_0$, we do not delete more than half of the vertices. We call $S$ the remaining subgraph which is a $4\varepsilon$-stable set of size $s \geq \delta n/2$ with maximum degree less than $4\varepsilon s$.

Let $G_S$ be the graph induced by $S$. Our goal is to find a constant $c$ such that $G_S$ have a biclique of size $cs$, which gives a biclique in $G$ of size at least $c\delta n/2$ and concludes the proof. Assume first that $G_S$ only has small connected components, meaning of size less than $s/2$. Then one can partition the connected components of $G_S$ in order to get a biclique in $G_S$ of size $s/4$. Otherwise, $G_S$ has a connected component $S'$ of size $s' \geq s/2$. The degree of every vertex in $S'$ is at most $8\varepsilon s' = \varepsilon k s'$, and $S'$ does not contain any induced $P_k$ since $G$ does not. By Lemma 3, there exists a biclique of size $c_k s' \geq c_k s/2$ in the complement of the graph induced by $S'$, thus in $G_S$. 


