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Transition waves for Fisher-KPP equations with general time-heterogeneous and space-periodic coefficients

Grégoire Nadin *† Luca Rossi ‡

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Abstract

This paper is devoted to existence and non-existence results for generalized transition waves solutions of space-time heterogeneous Fisher-KPP equations. When the coefficients of the equation are periodic in space but otherwise depend in a fairly general fashion on time, we prove that such waves exist as soon as their speed is sufficiently large in a sense. When this speed is too small, transition waves do not exist anymore, this result holds without assuming periodicity in space. These necessary and sufficient conditions are proved to be optimal when the coefficients are periodic both in space and time. Our method is quite robust and extends to general non-periodic space-time heterogeneous coefficients, showing that transition waves solutions of the nonlinear equation exist as soon as one can construct appropriate solutions of a given linearized equation.

Key words: Fisher-KPP equation, reaction-diffusion, travelling waves, generalized transition waves, generalized principal eigenvalues.
2010 Mathematical Subject Classification: 35B40, 35K57, 35B51, 35K10, 35P05.

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1 Introduction

We are concerned with transition waves solutions of the space-time heterogeneous reaction-diffusion equation

$$\partial_t u - \mathrm{Tr}(A(x,t)D^2u) + q(x,t)\cdot Du = f(x,t,u), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{1}$$

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Here D and D^2 denote respectively the gradient and the Hessian with respect to the space variables. We assume that the terms in the equation are periodic in x, with the same period. The matrix field A is uniformly elliptic and the nonlinearity $f(x,t,\cdot)$ vanishes at 0 and 1. The steady states 0 and 1 are respectively unstable and stable.

When the coefficients do not depend on (x,t), equation (1) becomes a classical homogeneous monostable reaction-diffusion equation. The pioneering works on such equations are due to Kolmogorov, Petrovski and Piskunov [12] and Fisher [11] in the 30's, when $f(u) = u(1-u)$. They investigated the existence of travelling wave solutions, that is, solutions of the form $u(x,t) = \phi(x \cdot e - ct)$, with $\phi(-\infty) = 1$, $\phi(+\infty) = 0$, $\phi > 0$. The quantity $c \in \mathbb{R}$ is the speed of the wave and $e \in \mathbb{S}^{N-1}$ is its direction. Kolmogorov, Petrovski and Piskunov [12] proved that when $A = I_N$, $q \equiv 0$ and $f = u(1-u)$, there exists $c^* > 0$ such that (1) admits travelling waves of speed c if and only if $c \geq c^*$. This property was extended to more general monostable nonlinearities by Aronson and Weinberger [2]. The properties (uniqueness, stability, attractivity, decay at infinity) of these waves have been extensively studied since then.

An increasing attention has been paid to heterogeneous reaction-diffusion since the 2000's. In particular, the existence of appropriate generalizations of travelling waves solutions has been proved for various classes of heterogeneities such as shear [9], time periodic [1], space periodic [3, 7, 29], space-time periodic [20, 16], time almost periodic [24] and time uniquely ergodic [26] ones, under several types of hypotheses on the nonlinearity. Now, the topical question is to understand whether reaction-diffusion equations with general heterogeneous coefficients admit wave-like solutions or not. A generalization of the notion of travelling waves has been given by Berestycki and Hamel [4, 5].

Definition 1.1. [4, 5] A *generalized transition wave* (in the direction $e \in S^{N-1}$) is a positive time-global solution u of (1) such that there exists a function $c \in L^\infty(\mathbb{R})$ satisfying

$$\lim_{x\cdot e\to-\infty} u\Big(x + e\int_0^t c(s)ds, t\Big) = 1, \qquad \lim_{x\cdot e\to+\infty} u\Big(x + e\int_0^t c(s)ds, t\Big) = 0, \tag{2}$$

uniformly with respect to $t \in \mathbb{R}$ [1]. The function c is called the *speed* of the generalized transition wave u, and $\phi(x,t) := u(x + e\int_0^t c(s)ds, t)$ is the associated *profile*.

The profile of a generalized transition wave satisfies

$$\lim_{x\cdot e\to-\infty} \phi(x,t) = 1, \qquad \lim_{x\cdot e\to+\infty} \phi(x,t) = 0, \quad \text{uniformly with respect to } t \in \mathbb{R}.$$

It is clear that any perturbation of c obtained by adding a function with bounded integral is still a speed of u, with a different profile. Reciprocally, if $\tilde{c}$ is another speed associated with

[1] For a given function $g = g(x,t)$, the condition $\lim_{x\cdot e\to\pm\infty} g(x,t) = l$ uniformly with respect to $t \in \mathbb{R}$ means that

$$\lim_{r\to+\infty} \sup_{\pm x\cdot e>r,\ t\in\mathbb{R}} |g(x,t) - l| = 0.$$

u, then it is easy to check that $t \mapsto \int_0^t (c - \tilde{c})$ is bounded. Obviously, all the notions of waves used previously when the coefficients belong to particular classes of heterogeneities can be viewed as transition waves.

The existence of such waves have been proved for one-dimensional space heterogeneous reaction-diffusion with ignition-type nonlinearities (that is, $f(x,u) = 0$ if $u \in [0,\theta) \cup \{1\}$ and $f(x,u) > 0$ if $u \in (\theta, 1)$) in parallel ways by Nolen and Ryzhik [21] and Mellet, Roquejoffre and Sire [15], and their stability was proved in [14]. For space heterogeneous monostable nonlinearities, when $f(x,u) > 0$ when $u \in (0,1)$ and $f(x,0) = f(x,1) = 0$, transition waves might not exist [19] in general. Let mention that this justified the introduction of the alternative notion of *critical travelling wave* by the first author in [17] for one-dimensional equations. Some existence results have also been obtained by Zlatos for partially periodic multi-dimensional equations of ignition-type [30].

When the coefficients only depend on t in a general way, the existence of transition waves was first proved by Shen for bistable nonlinearities [25] (that is, nonlinearities vanishing at $u = 0$ and $u = 1$ but negative near these two equilibria) and for monostable equations with time uniquely ergodic coefficients [26]. The case of general time heterogeneous monostable equations was investigated by the two authors in [18]. As observed in [18], the notions of least and upper mean play a crucial role in such frameworks.

Definition 1.2. The *least mean* (resp. the *upper mean*) *over* $\mathbb{R}$ of a function $g \in L^\infty(\mathbb{R})$ is given by

$$\lfloor g \rfloor := \lim_{T \to +\infty} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} g(s)ds, \qquad \Big(\text{resp. } \lceil g \rceil := \lim_{T \to +\infty} \sup_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} g(s)ds \Big).$$

As shown in Proposition 3.1 of [18], the definitions of $\lfloor g \rfloor$, $\lceil g \rceil$ do not change if one replaces $\lim_{T \to +\infty}$ with $\sup_{T>0}$ and $\inf_{T>0}$ respectively in the above expressions; this shows that $\lfloor g \rfloor$, $\lceil g \rceil$ are well defined for any $g \in L^\infty(\mathbb{R})$. Notice that g admits a uniform mean $\langle g \rangle$, that is $\langle g \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} g(s)ds$ exists uniformly with respect to $t \in \mathbb{R}$, if and only if $\lfloor g \rfloor = \lceil g \rceil = \langle g \rangle$. This is the case in particular when the coefficients are uniquely ergodic.

Note that if c and $\tilde{c}$ are two speeds associated with the same wave u, then $c - \tilde{c}$ has a bounded integral and thus $\lfloor c \rfloor = \lfloor \tilde{c} \rfloor$.

The two authors proved in [18] that when $A \equiv I_N$, $q \equiv 0$ and f only depends on (t,u) and is concave and positive with respect to $u \in (0,1)$, there exists a speed $c_* > 0$ such that for all $\gamma > c_*$ and $|e| = 1$, equation (1) admits a generalized transition wave with speed $c = c(t)$ in the direction e such that $\lfloor c \rfloor = \gamma$, while no such waves exist when $\gamma < c_*$.

When the coefficients not only depend on t in a general way but also on x periodically, some of the above results have been extended. Assuming in addition that the coefficients are uniquely ergodic and recurrent with respect to t and that $A \equiv I_N$, Shen [28] proved the existence of a speed c_* such that for all $\gamma > c_*$, there exists a generalized transition waves for monostable equations with speed c

The case of space periodic and time general monostable equations was first studied by the second author and Ryzhik [23], under the additional assumption that the dependences in t and x are separated, in the sense that A and q only depend on x, periodically, while f

only depends on (t,u). They proved both the existence of generalized transition waves of speed c such that $\lfloor c \rfloor > c_*$ and the non-existence of such waves with $\lfloor c \rfloor < c_*$. Moreover, they provided a more general non-existence result, without assuming that the dependence on x of A and q is periodic.

The aim of the present paper is to consider the general case of coefficients depending on both x and t. As in [23], we assume the periodicity in x only for the existence result.

2 Hypotheses and results

2.1 Statement of the main results

Throughout the paper, the terms in (1) will always be assumed to satisfy the following (classical) regularity hypotheses:

$$\begin{cases} A \text{ is symmetric, uniformly continuous,} \\ \exists\, 0 < \underline{\alpha} \leq \overline{\alpha}, \quad \forall (x,t) \in \mathbb{R}^{N+1}, \quad \underline{\alpha} I \leq A(x,t) \leq \overline{\alpha} I, \end{cases} \tag{3}$$

$$q \text{ is bounded and uniformly continuous on } \mathbb{R}^{N+1}, \tag{4}$$

$$\begin{cases} f \text{ is a Caratheodory function on } \mathbb{R}^{N+1} \times [0,1], \\ \exists \delta > 0,\ f(x,t,\cdot) \in W^{1,\infty}([0,1]) \cap C^1([0,\delta)), \text{ uniformly in } (x,t) \in \mathbb{R}^{N+1}. \end{cases} \tag{5}$$

The assumption that q is uniformly continuous is a technical hypothesis used in the proofs to pass to the limit in sequences of translations of the equation. It could be replaced by $\operatorname{div} q = 0$. We further assume that f is of monostable type, 0 being the unstable equilibrium and 1 being the stable one. Namely,

$$\forall (x,t) \in \mathbb{R}^{N+1}, \quad f(x,t,0) = 0, \tag{6}$$

$$\forall (x,t) \in \mathbb{R}^{N+1}, \quad f(x,t,1) = 0, \tag{7}$$

$$\forall u \in (0,1), \qquad \inf_{(x,t) \in \mathbb{R}^{N+1}} f(x,t,u) > 0. \tag{8}$$

In order to derive the existence result, we need some additional hypotheses. The first one is the standard KPP condition:

$$\forall (x,t) \in \mathbb{R}^{N+1},\ u \in [0,1], \quad f(x,t,u) \leq \mu(x,t) u, \tag{9}$$

where, here and in the sequel, μ denotes the function defined as follows:

$$\mu(x,t) := \partial_u f(x,t,0).$$

Conditions (8), (9) imply that $\inf \mu > 0$. The second condition is

$$\exists C > 0,\ \delta, \nu \in (0,1], \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R},\ u \in (0,\delta), \quad f(x,t,u) \geq \mu(x,t) u - C u^{1+\nu}. \tag{10}$$

Note that a sufficient condition for (10) to hold is $f(x,t,\cdot) \in C^{1+\nu}([0,\delta])$, uniformly with respect to x, t. The last condition for the existence result is

$$\exists l = (l_1, \dots, l_N) \in \mathbb{R}^N_+, \quad \forall t \in \mathbb{R},\ u \in (0,1), \quad A,\ q,\ f \ \text{ are } l\text{-periodic in } x, \tag{11}$$

where a function g is said to be l-periodic in x if it satisfies

$$\forall j \in \{1, \dots, N\},\ \forall x \in \mathbb{R}^N, \quad g(x + l_j e_j) = g(x),$$

$(e_1, \dots, e_N)$ being the canonical basis of $\mathbb{R}^N$.

When we say that a function is a (sub, super) solution of (1) we always mean that it is between 0 and 1. We deal with strong solutions whose derivatives ∂_t, D, D^2 belong to some $L^p(\mathbb{R}^{N+1})$, $p \in (1, \infty)$. Many of our statements and equations, such as (1), are understood to hold a.e., even if we omit to specify it, and inf, sup are used in place of ess inf, ess sup.

The main results of this paper consist in a sufficient and a necessary conditions for the existence of generalized transition waves, expressed in terms of their speeds.

Theorem 2.1. *Under the assumptions* (3)-(11)*, for all $e \in S^{N-1}$, there exists $c_* \in \mathbb{R}$ such that for every $\gamma > c_*$, there is a generalized transition wave in the direction e with a speed c such that $\lfloor c \rfloor = \gamma$.*

The minimal speed c_* we construct is explicitly given by (29), (34) and (37). A natural question is to determine whether our construction gives an optimal speed or not, that is, do generalized transition waves with speed c such that $\lfloor c \rfloor < c_*$ exist? One naturally starts with checking if our c_* coincides with the optimal speed known to exist in some particular cases, such as space-time periodic or space independent. In Section 2.3 we show that this is the case. The answer in the general, non space-periodic, case is only partial. It is contained in the next theorem, where, however, we can relax the monostability hypotheses (8)-(9) by

$$\left\lfloor \inf_{x \in \mathbb{R}^N} \mu(x, \cdot) \right\rfloor > 0, \tag{12}$$

and we can drop (7), (10) as well as (11). We actually need an extra regularity assumption on A:

$$A \text{ is uniformly Hölder-continuous in } x, \text{ uniformly with respect to } t. \tag{13}$$

This ensures the validity of some a priori Lipschitz estimates quoted from [22], that will be needed in the sequel. It is not clear to us if such estimates hold without (13).

Theorem 2.2. *Under the assumptions* (3)-(6), (12)-(13)*, for all $e \in S^{N-1}$, there exists $c^* \in \mathbb{R}$ such that if c is the speed of a generalized transition wave in the direction e then $\lfloor c \rfloor \geq c^*$.*

We point out that no spatial-periodicity condition is assumed in the previous statement. In order to prove Theorem 2.2 we derive a characterization of the least mean - Proposition 4.1 below - that we believe being of independent interest. The definition of c^* is given in Section 4. Of course, $c^* \leq c_*$ if the hypotheses of both Theorems 2.1 and 2.2 are fulfilled. We do not

know if, in general, $c_* = c^*$, that is, if the speed c_* is minimal, in the sense that there does not exist any wave with a speed having a smaller least mean. When the coefficients are periodic in space and time or only depend on time, we could identify the speed c_* more explicitly (see Section 2.3 below). Indeed, we recover in these frameworks some characterizations of the speeds identified in earlier papers [16, 18, 23], which were proved to be minimal. In the general framework, we leave this question open.

Finally, we leave as an open problem the case $\lfloor c \rfloor = c_*$, for which we believe that generalized transition waves still exist.

2.2 Optimality of the monostability assumption

The assumption (8) implies that 0 and 1 are respectively unstable and stable. Let us discuss the meaning and the optimality of this hypothesis, which might seem strong. Actually, as we do not make any additional assumption on the coefficients, we can consider much more general asymptotic states $p_- = p_-(x,t) < p_+ = p_+(x,t)$ in place of 0 and 1, and try to construct generalized transition waves v connecting p_- to p_+. Indeed, if $p_\pm$ are solutions to (1), with $p_+ - p_-$ bounded and having positive infimum, then the change of variables

$$u(x,t) := \frac{v(x,t) - p_-(x,t)}{p_+(x,t) - p_-(x,t)}$$

leads to an equation of the same form, with reaction term

$$\tilde{f}(x,t,u) := \frac{f(x,t,up_+ + (1-u)p_-) - uf(x,t,p_+) - (1-u)f(x,t,p_-)}{p_+ - p_-}.$$

The new equation admits the steady states 0 and 1. Moreover, assuming that $u \mapsto f(x,t,u)$ is strictly concave, then $\tilde{f}$ satisfies conditions (8), (9), the latter following from the inequality

$$\forall u \in (0,1), \quad u(p_+ - p_-)\partial_u f(x,t,p_-) \geq f(x,t,up_+ + (1-u)p_-) - f(x,t,p_-).$$

This shows that, somehow, the concavity hypothesis of the nonlinearity with respect to u is stronger, up to some change of variables, than the positivity hypothesis of the nonlinearity.

Let us illustrate the above procedure with an explicit example where $p_- \equiv 0$. Consider the equation

$$\partial_t v = \Delta v + \mu(x,t)v - v^2, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \tag{14}$$

with μ periodic in x, bounded and such that $\inf \mu > 0$. The later condition implies that the solution 0 is linearly unstable (actually, it can be relaxed by (12), see the discussion below). Then one can check that there is a time-global solution $p = p(x,t)$ which is bounded, has a positive infimum and is periodic in x. Let $u := v/p$. This function satisfies

$$\partial_t u = \Delta u + 2\frac{\nabla p}{p} \cdot \nabla u + p(x,t)u(1-u),$$

which is an equation of the form (1) for which (9)-(11) hold, at least if, for instance, μ is uniformly Hölder-continuous, since then ∇p is bounded by Schauder's parabolic estimates, and $\inf p > 0$.

Following this example, one can wonder whether (8) is an optimal condition (up to some change of variables) for the existence of transition waves. It is well-known that other classes of nonlinearities, such as bistable or ignition ones, could still give rise to transition waves (see for instance [3]). Thus, this question only makes sense if one reduces to the class of nonlinearities which are monostable, in a sense. Let us assume that f satisfies (6), (7) and that 0 is linearly unstable, in the weak sense that (12) holds. Then, using the properties of the least mean derived in [18], one can construct arbitrarily small subsolutions $\underline{u} = \underline{u}(t)$ and thus, as 1 is a positive solution, there exists a minimal solution p of (1) in the class of bounded solutions with positive infimum. One could then check that our proof still works and gives rise to generalized transition waves connecting 0 to p. Indeed, condition (8) only ensures that $p \equiv 1$. As a conclusion, the positivity hypothesis (8) is not optimal: one could replace it by (12) but then the generalized transition waves we construct connect 0 to the minimal time-global solution, which might not be 1.

Since for the existence of positive solutions it is sufficient to require (12) rather than $\inf \mu > 0$, one may argue that, in order to guarantee that 1 is the minimal time-global solution with positive infimum, hypothesis (8) could be relaxed by

$$\forall u \in (0,1), \quad \left\lfloor \min_{x \in \mathbb{R}^N} f(x, \cdot, u) \right\rfloor > 0. \tag{15}$$

This is not true, as shown by the following example. Let $p \in C^1(\mathbb{R})$ be a strictly decreasing function such that $p(\pm\infty) \in (0,1)$. Let f satisfy $f(t, p(t)) = p'(t)$. It is clear that f can be extended in such a way that (15) holds; however p is a time-global solution of $\partial_t u = f(t,u)$ with positive infimum which is smaller than 1.

Finally, if 0 is linearly stable, in the sense that

$$\left\lceil \sup_{x \in \mathbb{R}^N} \mu(x, \cdot) \right\rceil < 0 \tag{16}$$

holds, and (9) is satisfied, then there do not exist generalized transition waves at all, and, more generally, solutions to the Cauchy problem with bounded initial data converge uniformly to 0 as $t \to \infty$. Indeed, as an easy application of the property of the least (and upper) mean (39), one can construct a supersolution $\overline{u} = e^{\sigma(t) - \varepsilon t}$, for some $\sigma \in W^{1,\infty}(\mathbb{R})$ and $\varepsilon > 0$. The convergence to 0 of bounded solutions then follows from the comparison principle.

2.3 Description of the method and application to particular cases

The starting point of the construction of generalized transition waves consists in finding an explicit expression for the speed. This is not a trivial task in the case of mixed space-time dependence considered in this paper. We achieve it by an heuristic argument, that we now illustrate.

Suppose that u is a generalized transition wave in a direction $e \in S^{N-1}$. Its tail at large $x \cdot e$ is close from being a solution of the linearized equation around 0:

$$\partial_t u - \mathrm{Tr}(A(x,t) D^2 u) + q(x,t) \cdot Du = \mu(x,t) u. \tag{17}$$

It is natural to expect the tail of u to decays exponentially. Thus, since the equation is spatially periodic, we look for (the tail of) u under the form

$$u(x,t) = e^{-\lambda x\cdot e}\eta_\lambda(x,t), \qquad \text{with } \eta_\lambda \text{ positive and } l\text{-periodic in } x. \tag{18}$$

Rewriting this expression as

$$u(x,t) = \exp\left(-\lambda\left(x\cdot e - \frac{1}{\lambda}\ln\eta_\lambda(x,t)\right)\right),$$

shows that the speed of u, namely, a function c for which (2) holds, should satisfy

$$\left|\int_0^t c(s)ds - \frac{1}{\lambda}\ln\eta_\lambda(x,t)\right| \le C,$$

for some C independent of $(x,t)\in\mathbb{R}^N\times\mathbb{R}$. Clearly, this can hold true only if the ratio between maximum and minimum of $\eta_\lambda(\cdot,t)$ is bounded uniformly on t. This property follows from a Harnack-type inequality, Lemma 3.1 below, which is the keystone of our proof and actually the only step where the periodicity in x really plays a role. It would be then natural to define $c(t) := \frac{1}{\lambda}\frac{d}{dt}\ln\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R})}$. The problem is that we do not know if this function is bounded, since it is not clear whether $\partial_t\eta_\lambda \in L^\infty(\mathbb{R}^{N+1})$ or not. We overcome this difficulty by showing that there exists a Lipschitz continuous function S_λ such that

$$\exists\beta>0, \quad \forall t\in\mathbb{R}, \quad \left|S_\lambda(t) - \frac{1}{\lambda}\ln\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}\right| \le \beta. \tag{19}$$

We deduce that the function c defined (almost everywhere) by $c := S'_\lambda$ is bounded and it is an admissible speed for the wave u. The method described above provides, for any given $\lambda>0$, a wave with speed $c=c_\lambda$ for the *linearized* equation which decays with exponential rate λ. It is known -for instance in the case of constant coefficients- that only decaying rates which are "not too fast" are admissible for waves of the nonlinear reaction-diffusion equation. In Section 3.3, we identify a threshold rate λ_*. In the following section we construct generalized transition waves for any $\lambda<\lambda_*$, recovering with the least mean of their speeds the whole interval $(\lfloor c_{\lambda_*}\rfloor, +\infty)$. We do not know if the critical speed $c_* := \lfloor c_{\lambda_*}\rfloor$ is optimal, nor if an optimal speed does exist. However, we show below that this is the case if one applies the above procedure to some particular classes of heterogeneities already investigated in the literature.

In the case where the coefficients are periodic in time too, the class of admissible speeds has been characterized by the first author in [16] (see also [6]). Following the method described above, we see that an entire solution of (17) in the form (18) is given by $\eta_\lambda(x,t) = e^{k(\lambda)t}\varphi_\lambda(x,t)$, where $\big(k(\lambda),\varphi_\lambda\big)$ are the principal eigenelements of the problem[2]

$$\begin{cases} \partial_t\varphi_\lambda - \mathrm{Tr}(AD^2\varphi_\lambda) + (q+2\lambda Ae)D\varphi_\lambda - (\mu+\lambda^2 eAe + \lambda q\cdot e)\varphi_\lambda + k(\lambda)\varphi_\lambda = 0 \text{ in } \mathbb{R}^N\times\mathbb{R}, \\ \varphi_\lambda > 0, \\ \varphi_\lambda \text{ is periodic in } t \text{ and } x. \end{cases} \tag{20}$$

[2] The properties of these eigenelements, which are unique (up to a multiplicative constant in the case of φ_λ) are described in [16] for instance.

Actually, the uniqueness up to a multiplicative constant of solutions of (17) in the form (18), provided by Lemma 3.1 (proved without assuming the time-periodicity), implies that η_λ has necessarily this form. Thus, $S_\lambda(t) := \frac{k(\lambda)}{\lambda} t$ satisfies (19), whence the speed of the wave for the linearized equation with decaying rate λ is $c_\lambda \equiv k(\lambda)/\lambda$. Since the c_λ are constant (and then they have uniform mean), it turns out that the threshold λ_* we obtain for the decaying rates coincides with the minimum point of $\lambda \mapsto c_\lambda$ (see Remark 1 below). We eventually derive the existence of a generalized transition wave for any speed larger than $c_* := \min_{\lambda>0} k(\lambda)/\lambda$, which is exactly the sharp critical speed for pulsating travelling fronts obtained in [16]. To sum up, our construction of the minimal speed c_* is optimal in the space-time periodic framework. On the other hand, the definition of c^* in Section 4 below could also easily be identified to this speed and thus Theorem 2.2 implies that there do not exist generalized transition waves with a speed c such that $\lfloor c \rfloor < \min_{\lambda>0} k(\lambda)/\lambda$. We therefore recover also the non-existence result for pulsating travelling fronts. Only the existence of fronts with critical speed is not recovered.

In the case investigated by the authors in [18], namely, when $A \equiv I_N$, $q \equiv 0$ and f does not depend on x, one can easily check that $\eta_\lambda(t) = e^{\int_0^t \mu(s)ds + \lambda^2 t}$. As a function S_λ we can simply take $\frac{1}{\lambda} \ln \|\eta_\lambda(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = \frac{1}{\lambda} \int_0^t \mu(s)ds + \lambda t$, which is Lipschitz continuous. Whence $c_\lambda(t) := \lambda + \frac{\mu(t)}{\lambda}$ is a speed of a wave with decaying rate λ. In this case the critical decaying rate λ_* is equal to $\sqrt{\lfloor \mu \rfloor}$ (see again Remark 1) and thus we have $c_* = 2\sqrt{\lfloor \mu \rfloor}$. This is the same speed c_* as in [18], which was proved to be minimal.

Under the assumptions made by Ryzhik and the second author in [23], that is, A and q only depend on x (periodically) and f only depend on (t, u), the speeds c_* derived in the present paper and in [23] coincide, and thus it is minimal, in the sense that there do not exist any generalized transition wave with a lower speed.

When $A \equiv I_N$ and q, f are periodic in x and uniquely ergodic in t, then one can prove that the same holds true for the function $\partial_t \eta_\lambda / \eta_\lambda$ by uniqueness, and thus $\alpha \lfloor c_\alpha \rfloor$ could be identified with the Lyapounov exponent $\lambda(\alpha, \xi)$ used by Shen in [28], where ξ is the direction of propagation. We thus recover the same speed c_* as in [28] in this framework, which was not proved to be minimal since Shen did not investigate the nonexistence of transition waves with lower speed in [28]. Note that this identification is not completely obvious. However, as the formalism of the present paper and [28] are very different, we leave these computations to the reader.

Lastly, let consider the following example, where one could indeed construct directly the generalized transition waves:

$$\partial_t u - \partial_{xx} u - q(t) \partial_x u = \mu_0 u(1-u), \tag{21}$$

with q bounded and uniformly continuous and $\mu_0 > 0$. This equation satisfies assumptions (3)-(12). The change of variables $v(x,t) := u(x - \int_0^t q, t)$ leads to the classical homogeneous Fisher-KPP equation $\partial_t v - \partial_{xx} v = \mu_0 v(1-v)$. This equation admits travelling wave solutions of the form $v(x,t) = \phi_c(x - ct)$, with $\phi_c(-\infty) = 1$ and $\phi_c(+\infty) = 0$, for all $c \geq 2\sqrt{\mu_0}$. Hence, equation (21) admits generalized transition waves $u(x,t) = \phi_c(x - ct + \int_0^t q, t)$ of speed $c - q(t)$ if and only if $c \geq 2\sqrt{\mu_0}$. That is, the set of least mean of admissible speeds is

$[2\sqrt{\mu_0} - \lceil q \rceil, +\infty)$. Computing c_* in this case, one easily gets

$$\eta_\lambda = \eta_\lambda(t) = e^{\lambda^2 t - \lambda \int_0^t q(s)ds + \mu_0 t}, \quad c_\lambda(t) = \lambda - q(t) + \mu_0/\lambda \quad \text{and} \quad c_* = 2\sqrt{\mu_0} - \lceil q \rceil .$$

One could check that c^* coincides with this value too, meaning that Theorems 2.1 and 2.2 fully characterize the possible least means for admissible speeds, except for the critical one.

3 Existence result

Throughout this section, we fix $e \in S^{N-1}$ and we assume that conditions (3)-(11) hold. Actually, condition (8) could be weakened by (12), except for the arguments in the very last part of the proof in Section 3.4. As already mentioned in Section 2.2, these arguments could be easily adapted to the case where (8) is replaced by (12), leading to transition waves connecting 0 to the minimal solution with positive infimum.

3.1 Solving the linearized equation

We focus on solutions with prescribed spatial exponential decay.

Lemma 3.1. *For all $\lambda > 0$, the equation* (17) *admits a time-global solution of the form* (18)*. Moreover, η_λ is unique up to a multiplicative constant and satisfies, for all $t \in \mathbb{R}$, $T \geq 0$,*

$$\max_{x\in\mathbb{R}^N} \eta_\lambda(x, t+T) \leq \max_{x\in\mathbb{R}^N} \eta_\lambda(x,t) \exp\left((\overline{\alpha}\lambda + \sup_{\mathbb{R}^{N+1}} |q|)\lambda T + \int_t^{t+T} \max_{x\in\mathbb{R}^N} \mu(x,s)ds \right), \tag{22}$$

$$\min_{x\in\mathbb{R}^N} \eta_\lambda(x, t+T) \geq C \max_{x\in\mathbb{R}^N} \eta_\lambda(x,t) \exp\left((\underline{\alpha}\lambda - \sup_{\mathbb{R}^{N+1}} |q|)\lambda T + \int_t^{t+T} \min_{x\in\mathbb{R}^N} \mu(x,s)ds \right), \tag{23}$$

with $C > 0$ only depending on a constant bounding $|\lambda|$, $|l|$, $\underline{\alpha}^{-1}$, $\overline{\alpha}$, N and the L^∞ norms of μ and q.

The function $(x,t) \mapsto e^{-\lambda x\cdot e}\eta_\lambda(x,t)$ is a solution of the linearization of (1) near the unstable equilibrium. We will show in the next section that it is somehow a transition wave solution of the linearized equation, in the sense that it moves in the direction e with a certain speed. Due to hypothesis (9), we could use it as a supersolution of the nonlinear equation. Then, in Section 3.3, in order to construct an appropriate subsolution, we will need to restrict to exponents λ less than some threshold λ_*. We will eventually derive the existence of transition waves in Section 3.4.

As mentioned in Section 2.3, Lemma 3.1 is the only point where the spatial periodicity hypothesis (11) is used. If the coefficients were depending in a general way on both x and t and if one was able to construct a solution η_λ of equation (25) for which there exists $C > 0$ such that for all $T > 0$, $(x,t) \in \mathbb{R}^{N+1}$, one has:

$$\frac{1}{C}\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} e^{-CT} \leq \eta_\lambda(x,t+T) \leq C\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} e^{CT}, \tag{24}$$

then the forthcoming other steps of the proof still apply and it is possible to construct a generalized transition wave solution of the nonlinear equation (1). We describe this extension in Section 3.5 below. It would be very useful to determine optimal conditions on the coefficients enabling the derivation of a global Harnack-type inequality (24) for the linearized equation. We leave this question as an open problem.

Proof of Lemma 3.1. The problem for η_λ is

$$\partial_t \eta_\lambda = \mathrm{Tr}(AD^2\eta_\lambda) - (q + 2\lambda Ae)\cdot D\eta_\lambda + (\mu + \lambda^2 eAe + \lambda q\cdot e)\eta_\lambda, \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}. \tag{25}$$

We find a positive, l-periodic solution to (25) as the locally uniform limit of (a subsequence of) solutions η^n of the problem in $\mathbb{R}^N\times(-n,+\infty)$, with initial datum $\eta^n(-n,\cdot)\equiv m_n$, where m_n is a positive constant chosen in such a way that, say, $\sup_{x\in\mathbb{R}^N}\eta^n(0,x)=1$.

Let us show that any l-periodic solution η_λ to (25) satisfies (22) and (23). For given $t_0\in\mathbb{R}$, the function

$$\max_{x\in\mathbb{R}^N}\eta_\lambda(x,t_0)\,\exp\left((\overline{\alpha}\lambda^2+\sup_{\mathbb{R}^{N+1}}|q|\lambda)(t-t_0)+\int_{t_0}^{t}\max_{x\in\mathbb{R}^N}\mu(x,s)ds\right)$$

is a supersolution of (25) larger than η_λ at time t_0. Since η_λ is bounded, we can apply the parabolic comparison principle and derive (22). Let $\mathcal{C}$ denote the periodicity cell $\prod_{j=1}^N[0,l_j]$. By parabolic Harnack's inequality (see, e.g., Corollary 7.42 in [13]), we have that

$$\forall t\in\mathbb{R},\quad \max_{x\in\mathcal{C}}\eta_\lambda(x,t-1)\le\tilde{C}\min_{x\in\mathcal{C}}\eta_\lambda(x,t), \tag{26}$$

for some $\tilde{C}>0$ depending on a constant bounding $|\lambda|$, $|l|$, $\underline{\alpha}^{-1}$, $\overline{\alpha}$, N and the L^∞ norms of μ and q, and not on t. On the other hand, the comparison principle yields, for $T\ge 0$,

$$\min_{x\in\mathbb{R}^N}\eta_\lambda(x,t+T)\ge\min_{x\in\mathbb{R}^N}\eta_\lambda(x,t)\,\exp\left((\underline{\alpha}\lambda^2-\sup_{\mathbb{R}^{N+1}}|q|\lambda)T+\int_t^{t+T}\min_{x\in\mathbb{R}^N}\mu(x,s)ds\right).$$

Combining this inequality with (26) we eventually derive

$$\min_{x\in\mathbb{R}^N}\eta_\lambda(x,t+T)\ge\tilde{C}^{-1}\max_{x\in\mathbb{R}^N}\eta_\lambda(x,t-1)\,\exp\left((\underline{\alpha}\lambda^2-\sup_{\mathbb{R}^{N+1}}|q|\lambda)T+\int_t^{t+T}\min_{x\in\mathbb{R}^N}\mu(x,s)ds\right),$$

from which (23) follows by (22).

It remains to prove the uniqueness result. Assume that (17) admits two positive, l-periodic in x solutions η^1, η^2. As shown before, we know that they both satisfy (22) and (23). We first claim that there exists $K>1$ such that

$$\forall t\in\mathbb{R},\ x\in\mathbb{R}^N,\quad K^{-1}\eta^2(x,t)\le\eta^1(x,t)\le K\eta^2(x,t). \tag{27}$$

Let $h>0$ be such that $\eta^1\le h\eta^2$ at $t=0$. It follows that, for $t\le 0$, $\min_{x\in\mathbb{R}^N}\eta^1(x,t)\le h\max_{x\in\mathbb{R}^N}\eta^2(x,t)$, because otherwise the parabolic strong maximum

principle would imply $\eta^1 > h\eta^2$ at $t = 0$. Whence, applying (23) with $T = 0$ to both η^1 and η^2, we find a positive constant K such that

$$\forall t < 0, \quad \max_{x\in\mathbb{R}^N} \eta^1(x,t) \le K \min_{x\in\mathbb{R}^N} \eta^2(x,t).$$

This proves the second inequality in (27), for $t < 0$, whence for all $t \in \mathbb{R}$ by the maximum principle. The first inequality, with a possibly larger K, is obtained by exchanging the roles of η^1 and η^2. Now, call

$$k := \limsup_{t\to-\infty} \max_{x\in\mathbb{R}^N} \frac{\eta^1(x,t)}{\eta^2(x,t)}.$$

We know from (27) that $k \in [K^{-1}, K]$. Consider a sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} t_n = -\infty, \qquad \lim_{n\to\infty} \max_{x\in\mathbb{R}^N} \frac{\eta^1(x,t_n)}{\eta^2(x,t_n)} = k.$$

Define the sequences of functions $(\eta^1_n)_{n\in\mathbb{N}}$, $(\eta^2_n)_{n\in\mathbb{N}}$ as follows:

$$\forall i \in \{1,2\},\ n \in \mathbb{N}, \quad \eta^i_n(x,t) := \frac{\eta^i(x,t+t_n)}{\max_{y\in\mathbb{R}^N} \eta^1(y,t_n)}.$$

We deduce from (22) and (23) that the $(\eta^1_n)_{n\in\mathbb{N}}$ are uniformly bounded from above and uniformly bounded from below away from 0 in, say, $\mathbb{R}^N \times [-2,2]$. The same is true for $(\eta^2_n)_{n\in\mathbb{N}}$ by (27). Thus, by parabolic estimates and periodicity in x, the sequences $(\eta^i_n)_n$, $(\partial_t\eta^i_n)_n$, $(D\eta^i_n)_n$ and $(D^2\eta^i_n)_n$ converge, up to subsequences, in $L^p_{loc}(\mathbb{R}^{N+1})$. Morrey's inequality yields that the sequences $(\eta^1_n)_n$ and $(\eta^2_n)_n$ converge locally uniformly to some functions $\tilde\eta^1$ and $\tilde\eta^2$ respectively.

Call $A_n := A(\cdot, \cdot + t_n)$, $q_n := q(\cdot, \cdot + t_n)$, $\mu_n := \mu(\cdot, \cdot + t_n)$. As A and q are uniformly continuous, $(A_n)_n$ and $(q_n)_n$ converge (up to subsequences) to some functions $\tilde A$ and $\tilde q$ in $L^\infty_{loc}(\mathbb{R}^{N+1})$, whereas $(\mu_n)_n$ converges to some $\tilde\mu$ in the $L^\infty(\mathbb{R}^{N+1})$ weak-$\star$ topology. Hence, passing to the weak $L^p_{loc}(\mathbb{R}^{N+1})$ limit in the equations satisfied by the $(\eta^i_n)_{n\in\mathbb{N}}$, we get

$$\partial_t\tilde\eta^i = \mathrm{Tr}(\tilde A D^2\tilde\eta^i) - (\tilde q + 2\lambda\tilde A e)D\tilde\eta^i + (\tilde\mu + \lambda^2 e\tilde A e + \lambda\tilde q\cdot e)\tilde\eta^i, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

Clearly, these equations hold almost everywhere because all the terms are measurable functions. That is, the $\tilde\eta^i$ are strong solutions. Moreover,

$$\tilde\eta^1 \le k\tilde\eta^2, \qquad \max_{x\in\mathbb{R}^N} \frac{\tilde\eta^1(x,0)}{\tilde\eta^2(x,0)} = k.$$

The strong maximum principle then yields $\tilde\eta^1 \equiv k\tilde\eta^2$. As a consequence, for any $\varepsilon > 0$, we can find $n_\varepsilon \in \mathbb{N}$ such that, for $n \ge n_\varepsilon$, $(k-\varepsilon)\eta^2_n < \eta^1_n < (k+\varepsilon)\eta^2_n$ at $t = 0$. These inequalities hold for all $t \ge 0$, again by the maximum principle. Reverting to the original functions we obtain $(k-\varepsilon)\eta^2 < \eta^1 < (k+\varepsilon)\eta^2$ for $t \ge t_n$ and $n \ge n_\varepsilon$, from which, letting $n \to \infty$ and $\varepsilon \to 0^+$, we eventually infer that $\eta^1 \equiv k\eta^2$ for all $t \in \mathbb{R}$. □

In the particular case $T = 0$, the inequality (23) reads

$$\min_{x\in\mathbb{R}^N} \eta_\lambda(x,t) \geq C \max_{x\in\mathbb{R}^N} \eta_\lambda(x,t). \tag{28}$$

Notice that, in contrast with standard parabolic Harnack's inequality, the two sides are evaluated at the same time. This particular instance of (23) will be sometimes used in the sequel.

Until the end of the proof of Theorem 2.1, for $\lambda > 0$, η_λ stands for the (unique up to a multiplicative constant) function given by Lemma 3.1.

3.2 The speeds of the waves

Lemma 3.2. *There is a uniformly Lipschitz-continuous function $S_\lambda : \mathbb{R} \to \mathbb{R}$ satisfying* (19).

Proof. Properties (22)-(23) yield the existence of a constant $\beta > 0$ such that

$$\forall t \in \mathbb{R},\ T \geq 0, \quad \left| \ln \|\eta_\lambda(\cdot, t+T)\|_{L^\infty(\mathbb{R}^N)} - \ln \|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \right| \leq \beta(1+\lambda^2)T.$$

For all $n \in \mathbb{N}$, we define S_λ on $[n, n+1]$ as the affine function satisfying

$$S_\lambda(n) = \frac{1}{\lambda} \ln \|\eta_\lambda(\cdot,n)\|_{L^\infty(\mathbb{R}^N)}, \qquad S_\lambda(n+1) = \frac{1}{\lambda} \ln \|\eta_\lambda(\cdot,n+1)\|_{L^\infty(\mathbb{R}^N)}.$$

Then for all $t \in (n, n+1)$,

$$|S'_\lambda(t)| = \left| \frac{1}{\lambda} \ln \|\eta_\lambda(\cdot,n+1)\|_{L^\infty(\mathbb{R}^N)} - \frac{1}{\lambda} \ln \|\eta_\lambda(\cdot,n)\|_{L^\infty(\mathbb{R}^N)} \right| \leq \beta \frac{1+\lambda^2}{\lambda}.$$

Hence, S_λ is uniformly Lipschitz-continuous over $\mathbb{R}$. Moreover, if $t \in [n, n+1]$, one has

$$\begin{aligned} \left| S_\lambda(t) - \frac{1}{\lambda} \ln \|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \right| &\leq \left| S_\lambda(t) - S_\lambda(n) \right| + \frac{1}{\lambda} \left| \ln \|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} - \ln \|\eta_\lambda(\cdot,n)\|_{L^\infty(\mathbb{R}^N)} \right| \\ &\leq 2\beta \frac{1+\lambda^2}{\lambda}. \end{aligned}$$

Hence, $t \mapsto S_\lambda(t) - \frac{1}{\lambda} \ln \|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}$ is uniformly bounded over $\mathbb{R}$. □

Owing to Lemma 3.2, the function c_λ defined for (a.e.) $t \in \mathbb{R}$ by

$$c_\lambda(t) := S'_\lambda(t), \tag{29}$$

belongs to $L^\infty(\mathbb{R})$. We will use it as a possible speed for a transition wave to be constructed.

Let us investigate the properties of the least mean of the $(c_\lambda)_{\lambda>0}$. It follows from (19) that

$$\lfloor c_\lambda \rfloor = \frac{1}{\lambda} \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \ln \frac{\|\eta_\lambda(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}. \tag{30}$$

Hence, by (22) and (23), we derive

$$\underline{\alpha}\lambda - \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\left[\min_{x\in\mathbb{R}^N} \mu(x,\cdot)\right] \le \lfloor c_\lambda \rfloor \le \overline{\alpha}\lambda + \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\left[\max_{x\in\mathbb{R}^N} \mu(x,\cdot)\right]. \tag{31}$$

Analogous bounds hold for the upper mean:

$$\underline{\alpha}\lambda - \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\left[\min_{x\in\mathbb{R}^N} \mu(x,\cdot)\right] \le \lceil c_\lambda \rceil \le \overline{\alpha}\lambda + \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\left[\max_{x\in\mathbb{R}^N} \mu(x,\cdot)\right]. \tag{32}$$

We have seen in Section 2.3 that, when the coefficients are periodic in t, one can take $S_\lambda(t) := (k(\lambda)/\lambda)t$, whence $c_\lambda \equiv k(\lambda)/\lambda$. It follows that $\lambda c_\lambda = k(\lambda)$, and we know from the arguments in the proof of Proposition 5.7 part (iii) in [3] that the function k is convex. In the general heterogeneous framework considered in the present paper, we use the same arguments as in [3] to derive the Lipschitz continuity of the function $\lambda \mapsto \lambda \lfloor c_\lambda \rfloor$. If the functions c_λ admit a uniform mean then these arguments actually imply that $\lambda \mapsto \lambda \lfloor c_\lambda \rfloor$ is convex, but we do not know if this is true in general.

Lemma 3.3. *The functions $\lambda \mapsto \lfloor c_\lambda \rfloor$ and $\lambda \mapsto \lceil c_\lambda \rceil$ are locally uniformly Lipschitz continuous on $(0,+\infty)$.*

Proof. Fix $\Lambda > 0$ and $-\Lambda \le \lambda_0 \le \Lambda$. Let λ_1 be such that $|\lambda_1 - \lambda_0| = 2\Lambda$. For $j = 0, 1$, the function $v_j(x,t) := e^{-\lambda_j x\cdot e}\eta_{\lambda_j}(x,t)$ satisfies (17). Hence, setting $v_j = e^{w_j}$, we find that

$$\partial_t w_j - \mathrm{Tr}\big(AD^2 w_j\big) + q\cdot Dw_j = \mu + \mathrm{Tr}\big(ADw_j \otimes Dw_j\big), \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}.$$

For $\tau \in (0,1)$, the function $w := (1-\tau)w_0 + \tau w_1$ satisfies, for $x \in \mathbb{R}^N$, $t \in \mathbb{R}$,

$$\begin{aligned}\partial_t w - \mathrm{Tr}(AD^2 w) + q\cdot Dw &= \mu + \mathrm{Tr}\Big(A\big((1-\tau)Dw_0 \otimes Dw_0 + \tau Dw_1 \otimes Dw_1\big)\Big)\\ &\ge \mu + \mathrm{Tr}(ADw \otimes Dw).\end{aligned}$$

As a consequence, e^w is a supersolution of (17) and then, since

$$e^{w(x,t)} = e^{-((1-\tau)\lambda_0 + \tau\lambda_1)x\cdot e}\eta_{\lambda_0}^{1-\tau}(x,t)\eta_{\lambda_1}^{\tau}(x,t),$$

the function $\eta_{\lambda_0}^{1-\tau}\eta_{\lambda_1}^{\tau}$ is a supersolution of (25) with $\lambda = \lambda_\tau := (1-\tau)\lambda_0 + \tau\lambda_1$. We can therefore apply the comparison principle between this function and η_{λ_τ} and derive, for $t \in \mathbb{R}$, $T > 0$,

$$\begin{aligned}\frac{\|\eta_{\lambda_\tau}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_\tau}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} &\le \frac{\|\eta_{\lambda_0}^{1-\tau}\eta_{\lambda_1}^{\tau}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\min_{x\in\mathbb{R}^N}\eta_{\lambda_0}^{1-\tau}\eta_{\lambda_1}^{\tau}(x,t)}\\ &\le \left(\frac{\|\eta_{\lambda_0}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\min_{x\in\mathbb{R}^N}\eta_{\lambda_0}(x,t)}\right)^{1-\tau}\left(\frac{\|\eta_{\lambda_1}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\min_{x\in\mathbb{R}^N}\eta_{\lambda_1}(x,t)}\right)^{\tau}.\end{aligned}$$

Whence, using the inequality (28) for η_{λ_0} and η_{λ_1} (with the same C depending on Λ), we obtain

$$\frac{\|\eta_{\lambda_\tau}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_\tau}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le C^{-1}\left(\frac{\|\eta_{\lambda_0}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_0}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}\right)^{1-\tau}\left(\frac{\|\eta_{\lambda_1}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_1}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}\right)^{\tau}. \tag{33}$$

Consider the function Γ defined by $\Gamma(\lambda) := \lambda \lfloor c_\lambda \rfloor$. It follows from (30) and (33) that

$$\Gamma(\lambda_\tau) \leq \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \left((1-\tau) \ln \frac{\|\eta_{\lambda_0}(\cdot, t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_0}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}} + \tau \ln \frac{\|\eta_{\lambda_1}(\cdot, t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_1}(\cdot, t)\|_{L^\infty(\mathbb{R}^N)}} \right).$$

If the $(c_\lambda)_{\lambda>0}$ admit uniform mean, the above inequality and (30) imply that Γ is convex. Otherwise, we can only infer that

$$\Gamma(\lambda_\tau) \leq (1-\tau)\Gamma(\lambda_0) + \tau\lambda_1 \lceil c_{\lambda_1} \rceil.$$

We have therefore shown that

$$\forall \tau \in (0,1), \quad \Gamma(\lambda_\tau) - \Gamma(\lambda_0) \leq \tau(\lambda_1 \lceil c_{\lambda_1} \rceil - \lambda_0 \lfloor c_{\lambda_0} \rfloor).$$

Thus, by (31) and (32) there exists a constant $K > 0$, depending on A, q, μ, such that

$$\forall \tau \in (0,1), \quad \Gamma(\lambda_\tau) - \Gamma(\lambda_0) \leq K(\Lambda^2 + 1)\tau.$$

This proves the Lipschitz continuity of Γ on $[-\Lambda, \Lambda]$, because $|\lambda_\tau - \lambda_0| = 2\Lambda\tau$, concluding the proof of the lemma.

The same arguments lead to the local Lipschitz continuity of $\lambda \mapsto \lceil c_\lambda \rceil$.

□

3.3 Definition of the critical speed

In order to define the critical speed c_*, we introduce the following set:

$$\Lambda := \{\lambda > 0 \ : \ \exists \overline{k} > 0, \ \forall 0 < k < \overline{k}, \ \lfloor c_\lambda - c_{\lambda+k} \rfloor > 0\}. \tag{34}$$

Lemma 3.4. *There exists $\lambda_* > 0$ such that $\Lambda = (0, \lambda_*)$. Moreover, the function $\lambda \mapsto \lfloor c_\lambda \rfloor$ is decreasing on Λ.*

Proof. Fix $\lambda_0, \lambda_1 > 0$. For $\tau \in (0,1)$, we set $\lambda_\tau := (1-\tau)\lambda_0 + \tau\lambda_1$. Taking the ln in (33), and recalling that $c_\lambda = S'_\lambda$ with S_λ satisfying (19) yields

$$\int_t^{t+T} [(1-\tau)\lambda_0 c_{\lambda_0} + \tau\lambda_1 c_{\lambda_1} - \lambda_\tau c_{\lambda_\tau}] \geq \ln C - 4\lambda_\tau \beta.$$

Whence,

$$\lambda_\tau \int_t^{t+T} (c_{\lambda_0} - c_{\lambda_\tau}) \geq \tau\lambda_1 \int_t^{t+T} (c_{\lambda_0} - c_{\lambda_1}) + \ln C - 4\lambda_\tau \beta.$$

Dividing both sides by T, taking the inf on $t \in \mathbb{R}$ and then the lim as $T \to +\infty$, we derive

$$\forall \tau \in (0,1), \quad \lfloor c_{\lambda_0} - c_{\lambda_\tau} \rfloor \geq \tau \frac{\lambda_1}{\lambda_\tau} \lfloor c_{\lambda_0} - c_{\lambda_1} \rfloor. \tag{35}$$

If, instead, we divide by $-T$, we get

$$\forall \tau \in (0,1), \quad \lfloor c_{\lambda_\tau} - c_{\lambda_0} \rfloor \leq \tau \frac{\lambda_1}{\lambda_\tau} \lfloor c_{\lambda_1} - c_{\lambda_0} \rfloor. \tag{36}$$

Analogous estimates hold of course for the upper mean. The characterization of Λ follows from these inequalities, by suitable choices of λ_0, λ_1 and τ. We prove it in four steps.

Step 1. $\Lambda \neq \emptyset$.

The first inequality in (31), together with (12), yield

$$\lim_{\lambda\to 0^+} \lfloor c_\lambda - c_1 \rfloor \geq \lim_{\lambda\to 0^+} \lfloor c_\lambda \rfloor - \lceil c_1 \rceil = +\infty.$$

There exists then $0 < \lambda < 1$ such that $\lfloor c_\lambda - c_1 \rfloor > 0$. Applying (35) with $\lambda_0 = \lambda$, $\lambda_1 = 1$, we eventually infer that $\lfloor c_\lambda - c_{\lambda+k} \rfloor > 0$, for all $0 < k < 1 - \lambda$, that is, $\lambda \in \Lambda$.

Step 2. Λ *is bounded from above.*

By (31) we obtain

$$\lim_{\lambda\to+\infty} \lfloor c_1 - c_\lambda \rfloor \leq \lfloor c_1 \rfloor - \lim_{\lambda\to+\infty} \lfloor c_\lambda \rfloor = -\infty.$$

There exists then $\lambda' > 1$ such that, for $\lambda > \lambda'$, $\lfloor c_1 - c_\lambda \rfloor < 0$. Hence, for $k > 0$, applying (36) with $\lambda_0 = \lambda + k$, $\lambda_1 = 1$ and $\tau = k/(k + \lambda - 1)$, we derive

$$\lfloor c_\lambda - c_{\lambda+k} \rfloor \leq \frac{k}{(k+\lambda-1)\lambda} \lfloor c_1 - c_{\lambda+k} \rfloor < 0.$$

Namely, $\lambda \notin \Lambda$ and thus Λ is bounded from above by λ'.

Step 3. *If* $\lambda \in \Lambda$ *then* $(0, \lambda] \subset \Lambda$.

Let $0 < \lambda' < \lambda$ and $k > 0$. Using first (35) and then (36) we get

$$\lfloor c_{\lambda'} - c_{\lambda'+k} \rfloor \geq \left(\frac{k}{k+\lambda-\lambda'}\right)\left(\frac{\lambda+k}{\lambda'+k}\right) \lfloor c_{\lambda'} - c_{\lambda+k} \rfloor \geq \left(\frac{\lambda+k}{\lambda'+k}\right)\frac{\lambda}{\lambda'} \lfloor c_\lambda - c_{\lambda+k} \rfloor .$$

Thus, $\lambda \in \Lambda$ implies $\lambda' \in \Lambda$.

Step 4. $\sup \Lambda \notin \Lambda$.

Let $\lambda^* := \sup \Lambda$ and $k > 0$. For all $n \in \mathbb{N}$, there exists $0 < k_n < 1/n$ such that $\lfloor c_{\lambda^*+1/n} - c_{\lambda^*+1/n+k_n} \rfloor \leq 0$. For n large enough, we have that $1/n + k_n < k$ and then, by (35),

$$0 \geq \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+1/n+k_n} \rfloor \geq \left(\frac{k_n}{k-1/n}\right)\left(\frac{\lambda^*+k}{\lambda^*+1/n+k_n}\right) \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+k} \rfloor .$$

Whence,

$$\lfloor c_{\lambda^*} - c_{\lambda^*+k} \rfloor \leq \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+k} \rfloor + \lceil c_{\lambda^*} - c_{\lambda^*+1/n} \rceil \leq \lceil c_{\lambda^*} - c_{\lambda^*+1/n} \rceil .$$

Using the the analogue of (36) for the upper mean, we can control the latter term as follows:

$$\lceil c_{\lambda^*} - c_{\lambda^*+1/n} \rceil \leq \frac{1/n}{\lambda_* + 2/n} \lceil c_{\lambda^*/2} - c_{\lambda^*+1/n} \rceil \leq \frac{1/n}{\lambda_* + 2/n} \left(\lceil c_{\lambda^*/2} \rceil - \lfloor c_{\lambda^*+1/n} \rfloor \right),$$

which goes to 0 as $n \to \infty$ (recall that $\lambda \mapsto \lfloor c_\lambda \rfloor$ is continuous by Lemma 3.3). We eventually infer that $\lfloor c_{\lambda^*} - c_{\lambda^*+k} \rfloor \leq 0$, that is, $\lambda^* \notin \Lambda$.

It remains to show that $\lambda \mapsto \lfloor c_\lambda \rfloor$ is decreasing on Λ. Assume by contradiction that there are $0 < \lambda_1 < \lambda_2 < \lambda^*$ such that $\lfloor c_{\lambda_1} \rfloor \leq \lfloor c_{\lambda_2} \rfloor$. The function $\lambda \mapsto \lfloor c_\lambda \rfloor$, being

continuous, attains its minimum on $[\lambda_1, \lambda_2]$ at some λ. Since $\lfloor c_{\lambda_1} \rfloor \leq \lfloor c_{\lambda_2} \rfloor$, we can assume that $\lambda \in [\lambda_1, \lambda_2)$. The definition of Λ implies that there exists $\lambda' \in (\lambda, \lambda_2)$ such that $\lfloor c_\lambda - c_{\lambda'} \rfloor > 0$. Whence

$$\lfloor c_{\lambda'} \rfloor \leq \lfloor c_\lambda \rfloor + \lceil c_{\lambda'} - c_\lambda \rceil = \lfloor c_\lambda \rfloor - \lfloor c_\lambda - c_{\lambda'} \rfloor < \lfloor c_\lambda \rfloor .$$

This is impossible. □

We are now in position to define the critical speed c_*:

$$c_* := \lfloor c_{\lambda_*} \rfloor , \tag{37}$$

where λ_* is given in Lemma 3.4.

Remark 1. When the terms in (1) are periodic in time, resuming from Section 2.3, we know that the speeds $(c_\lambda)_{\lambda>0}$ are constant and satisfy $c_\lambda \equiv k(\lambda)/\lambda$, where $k(\lambda)$ is the principal eigenvalue of problem (20). Hence,

$$\lfloor c_\lambda - c_{\lambda+\kappa} \rfloor = \frac{k(\lambda)}{\lambda} - \frac{k(\lambda+\kappa)}{\lambda+\kappa}.$$

As $\lambda \mapsto k(\lambda)$ is strictly convex (see [16]) and, by (31),

$$\lim_{\lambda\to+\infty} \frac{k(\lambda)}{\lambda} = +\infty, \qquad \lim_{\lambda\to 0^+} k(\lambda) = \lim_{\lambda\to 0^+} \lambda c_\lambda \geq \left\lfloor \min_{x\in\mathbb{R}^N} \mu(x,\cdot) \right\rfloor > 0,$$

straightforward convexity arguments yield that λ_* given by Lemma 3.4 is the unique minimizer of $\lambda \mapsto k(\lambda)/\lambda$. Therefore, $c_* = \min_{\lambda>0} k(\lambda)/\lambda$, which is known to be the minimal speed for pulsating travelling fronts (see [16]).

3.4 Construction of a subsolution and conclusion of the proof

In order to prove Theorem 2.1, we introduce a family of functions $(\varphi_\lambda)_{\lambda>0}$ which play the role of the spatial periodic principal eigenfunctions in the time-independent case. For $\lambda > 0$, let η_λ be the function given by Lemma 3.1, normalized by $\|\eta_\lambda(\cdot, 0)\|_{L^\infty(\mathbb{R}^N)} = 1$. We define

$$\varphi_\lambda(x,t) := e^{-\lambda S_\lambda(t)} \eta_\lambda(x,t).$$

By (19) and (28), there exist two positive constants C_λ, β such that

$$\forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \quad C_\lambda \leq \varphi_\lambda(x,t) \leq e^{\lambda\beta}. \tag{38}$$

We will make use of the following key property of the least mean, provided by Lemma 3.2 of [18]:

$$\forall g \in L^\infty(\mathbb{R}), \quad \lfloor g \rfloor = \sup_{\sigma\in W^{1,\infty}(\mathbb{R})} \inf_{t\in\mathbb{R}} (\sigma' + g)(t). \tag{39}$$

Proof of Theorem 2.1. Fix $\gamma > c_*$. Since the function $\lambda \mapsto \lfloor c_\lambda \rfloor$ is continuous by Lemma 3.3 and tends to $+\infty$ as $\lambda \to 0^+$ by (31), and $\Lambda = (0, \lambda_*)$ by Lemma 3.4, there exists $\lambda \in \Lambda$ such that $\lfloor c_\lambda \rfloor = \gamma$. The function w defined by

$$w(x,t) := \min\left(e^{-\lambda x\cdot e}\eta_\lambda(x,t)\,,\ 1\right)$$

is a generalized supersolution of (1).

In order to construct a subsolution, consider the constant ν in (10). By the definition of Λ, there exists $\lambda < \lambda' < (1+\nu)\lambda$ such that $\lfloor c_\lambda - c_{\lambda'} \rfloor > 0$. We then set $\psi(x,t) := e^{\sigma(t)-\lambda'(x\cdot e - S_\lambda(t)+S_{\lambda'}(t))}\eta_{\lambda'}(x,t)$, where $\sigma \in W^{1,\infty}(\mathbb{R})$ will be chosen later. We have that

$$\partial_t\psi - \mathrm{Tr}(A(x,t)D^2\psi) + q(x,t)\cdot D\psi - \mu(x,t)\psi = \left[\sigma'(t) + \lambda'(c_\lambda(t) - c_{\lambda'}(t))\right]\psi.$$

Since $\lfloor \lambda'(c_\lambda - c_{\lambda'}) \rfloor = \lambda' \lfloor c_\lambda - c_{\lambda'} \rfloor > 0$, by (39) we can choose $\sigma \in W^{1,\infty}(\mathbb{R})$ in such a way that $K := \inf_{\mathbb{R}}(\sigma' + \lambda'(c_\lambda - c_{\lambda'})) > 0$. Hence,

$$\partial_t\psi - \mathrm{Tr}(A(x,t)D^2\psi) + q(x,t)\cdot D\psi \geq (\mu(x,t) + K)\psi, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

We define:

$$v(x,t) := e^{-\lambda x\cdot e}\eta_\lambda(x,t) - m\psi(x,t),$$

where m is a positive constant to be chosen. By computation,

$$e^{-\lambda x\cdot e}\eta_\lambda(x,t) - m\psi(x,t) = e^{-\lambda(x\cdot e - S_\lambda(t))}\left(\varphi_\lambda(x,t) - m\varphi_{\lambda'}(x,t)e^{\sigma(t)-(\lambda'-\lambda)(x\cdot e - S_\lambda(t))}\right).$$

Since φ_λ, $\varphi_{\lambda'}$ satisfy (38) and $\sigma \in L^\infty(\mathbb{R})$, it follows that, choosing m large enough, $v(x,t) \leq 0$ if $x\cdot e - S_\lambda(t) \leq 0$, and that v is less than $\delta \in (0,1]$ from (10) everywhere. If $v(x,t) > 0$, and then $x\cdot e - S_\lambda(t) > 0$, we see that

$$\begin{aligned}
\partial_t v - \mathrm{Tr}(A(x,t)D^2 v) + q(x,t)\cdot Dv - \mu(x,t)v &\leq -mK\psi \\
&\leq -mK\psi\frac{v^{1+\nu}}{e^{-(1+\nu)\lambda x\cdot e}\eta_\lambda^{1+\nu}} \\
&= -mKv^{1+\nu}\frac{\varphi_{\lambda'}}{\varphi_\lambda^{1+\nu}}e^{\sigma(t)-(\lambda'-(1+\nu)\lambda)(x\cdot e - S_\lambda(t))} \\
&\leq -mKv^{1+\nu}C_{\lambda'}e^{-(1+\nu)\lambda\beta}\inf_{s\in\mathbb{R}} e^{\sigma(s)},
\end{aligned}$$

where, for the last inequality, we have used (38) and the fact that $\lambda' < (1+\nu)\lambda$. As a consequence, by hypothesis (10), for m sufficiently large, v is a subsolution of (1) in the set where it is positive.

Using again (38), one computes:

$$\begin{aligned}
v(x + S_\lambda(t)e, t) &= e^{-\lambda x\cdot e}\left(\varphi_\lambda(x + S_\lambda(t)e, t) - m\varphi_{\lambda'}(x + S_\lambda(t)e, t)e^{\sigma(t)-(\lambda'-\lambda)x\cdot e}\right) \\
&\geq e^{-\lambda x\cdot e}\left(C_\lambda - mC_{\lambda'}e^{\lambda'\beta + \|\sigma\|_\infty - (\lambda'-\lambda)x\cdot e}\right).
\end{aligned}$$

Hence, taking R large enough, one has:

$$\inf_{\substack{x\cdot e=R\\ t\in\mathbb{R}}} v(x+S_\lambda(t)e,t) \geq e^{-\lambda R}\Big(C_\lambda - mC_{\lambda'}e^{\lambda'\beta+\|\sigma\|_\infty-(\lambda'-\lambda)R}\Big) =: \omega \in (0,1).$$

Consequently, the function $\underline{v}$ defined by

$$\underline{v}(x,t) := \begin{cases} v(x,t) & \text{if} \quad x\cdot e \geq S_\lambda(t)+R, \\ \max\big(\omega, v(x,t)\big) & \text{if} \quad x\cdot e < S_\lambda(t)+R, \end{cases}$$

is continuous and, because of (8), it is a generalized subsolution of (1). Moreover, since $v \leq w$ and $w(x+S_\lambda(t)e,t) \geq e^{-\lambda R}C_\lambda > \omega$ if $x\cdot e < R$, one sees that $\underline{v} \leq w$. A solution $\underline{v} \leq u \leq w$ can therefore be obtained as the limit of (a subsequence of) the solutions $(u_n)_{n\in\mathbb{N}}$ of the problems

$$\begin{cases} \partial_t u_n - \mathrm{Tr}(A(x,t)D^2u_n) + q(x,t)\cdot Du_n = f(x,t,u_n), \quad x\in\mathbb{R}^N,\ t>-n \\ u_n(x,-n) = w(x,-n), \quad x\in\mathbb{R}^N. \end{cases}$$

The strong maximum principle yields $u>0$. One further sees that

$$\lim_{x\cdot e\to+\infty} u(x+e\int_0^t c_\lambda(s)ds,t) \leq \lim_{x\cdot e\to+\infty} w(x+e\int_0^t c_\lambda(s)ds,t) \leq \lim_{x\cdot e\to+\infty} e^{-\lambda x\cdot e}\varphi_\lambda(x,t) = 0,$$

uniformly with respect to $t\in\mathbb{R}$. It remains to prove that

$$\lim_{x\cdot e\to-\infty} u(x+e\int_0^t c_\lambda(s)ds,t) = 1$$

holds uniformly with respect to $t\in\mathbb{R}$. Set

$$\vartheta := \lim_{r\to-\infty} \inf_{\substack{x\cdot e\leq r\\ t\in\mathbb{R}}} u(x+e\int_0^t c_\lambda(s)ds,t).$$

Our aim is to show that $\vartheta=1$. We know that $\vartheta\geq\omega>0$, because $u(x,t)\geq\underline{v}(x,t)\geq\omega$ if $x\cdot e<S_\lambda(t)+R$. Let $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ and $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ be such that

$$\lim_{n\to\infty} x_n\cdot e = -\infty, \qquad \lim_{n\to\infty} u\big(x_n+e\int_0^{t_n} c_\lambda(s)ds,t_n\big) = \vartheta.$$

For $n\in\mathbb{N}$, let $k_n\in\prod_{j=1}^N l_j\mathbb{Z}$ be such that $y_n := x_n + e\int_0^{t_n} c_\lambda(s)ds - k_n \in \prod_{j=1}^N[0,l_j)$ and define $v_n(x,t) := u(x+k_n,t+t_n)$. The functions $(v_n)_{n\in\mathbb{N}}$ are solutions of

$$\partial_t v_n - \mathrm{Tr}(A(x,t+t_n)D^2v_n) + q(x,t+t_n)\cdot Dv_n = f(x,t+t_n,v_n), \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}.$$

By parabolic estimates, one can show using the same types of arguments as in the proof of Lemma 3.1 that $(v_n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly to some function v satisfying

$$\partial_t v - \mathrm{Tr}(\tilde{A}(x,t)D^2v) + \tilde{q}(x,t)\cdot Dv = g(x,t) \geq 0, \quad x\in\mathbb{R}^N,\ t\in\mathbb{R},$$

where $\tilde{A}$ and $\tilde{q}$ are the strong limits in $L^\infty_{loc}(\mathbb{R}^{N+1})$ and g is the weak limit in $L^p_{loc}(\mathbb{R}^N\times\mathbb{R})$ of (a subsequence of) $A(x,t+t_n)$, $q(x,t+t_n)$ and $f(x,t+t_n,v_n(x,t))$ respectively, the inequality $g\geq 0$ coming from hypothesis (8). Furthermore, letting y be the limit of (a converging subsequence of) $(y_n)_{n\in\mathbb{N}}$, we find that $v(y,0)=\vartheta$ and

$$\forall x\in\mathbb{R}^N,\ t\in\mathbb{R},\quad v(x,t)=\lim_{n\to\infty}u(x+x_n+e\int_0^{t_n}c_\lambda(s)ds-y_n,t+t_n)\geq\vartheta.$$

As a consequence, the strong maximum principle yields $v=\vartheta$ in $\mathbb{R}^N\times(-\infty,0]$. In particular, $g=0$ in $\mathbb{R}^N\times(-\infty,0)$. Using the Lipschitz continuity of $f(x,t,\cdot)$, we then derive for all $(x,t)\in\mathbb{R}^N\times(-\infty,0)$:

$$\forall T>0,\quad 0=\lim_{n\to+\infty}f(x,t+t_n,v_n(x,t))=\lim_{n\to+\infty}f(x,t+t_n,\vartheta)\geq\inf_{(x,t)\in\mathbb{R}^{N+1}}f(x,t,\vartheta).$$

This, by (8), implies that either $\vartheta=0$ or $\vartheta=1$, whence $\vartheta=1$ because $\vartheta\geq\omega>0$. □

3.5 A criterion for the existence of generalized transition waves in space-time general heterogeneous media

As already emphasized above, our proof holds in more general media, without assuming that the coefficients satisfy (11), that is, without the space periodicity assumption. We then need to assume that the linearized equation admits a family of solutions satisfying some global Harnack inequality. We conclude the existence part of the paper by stating such a result. We omit its proof since one only needs to check that the previous arguments still work.

Theorem 3.5. *In addition to* (3)-(10)*, assume that there exists* $\overline{\lambda}>0$ *such that for all* $\lambda\in(0,\overline{\lambda})$*, there exists a Lipschitz-continuous time-global solution* η_λ *of*

$$\partial_t\eta_\lambda=\mathrm{Tr}(AD^2\eta_\lambda)-(q+2\lambda Ae)D\eta_\lambda+(\mu+\lambda^2eAe+\lambda q\cdot e)\eta_\lambda,\quad x\in\mathbb{R}^N,\ t\in\mathbb{R}$$

satisfying

$$\frac{1}{C}\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}e^{-CT}\leq\eta_\lambda(x,t+T)\leq C\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}e^{CT},$$

for some $C=C(\lambda)>0$ *and for all* $T>0$, $(x,t)\in\mathbb{R}^{N+1}$.

Then there exists $\lambda_*\in(0,\overline{\lambda})$ *such that for all* $\gamma>c_*:=\lfloor S'_{\lambda_*}\rfloor$*, where* S_λ *is a Lipschitz continuous function satisfying* (19)*, there exists a generalized transition wave with speed* $c_\lambda=S'_\lambda$ *such that* $\lfloor c_\lambda\rfloor=\gamma$.

4 Non-existence result

Our aim is to find bounded subsolutions to the linearized problem

$$\partial_t u-Tr\big(A(x,t)D^2u\big)+q(x,t)\cdot Du=\mu(x,t)u,\quad x\in\mathbb{R}^N,\ t\in\mathbb{R},\tag{40}$$

in order to get a lower bound for the speed of travelling wave solutions. We recall that no spatial-periodic condition is now assumed. Looking for solutions of (40) in the form $u(x,t) = e^{-\lambda(x\cdot e - ct)}\phi(x,t)$, with λ and c constant, leads to the equation

$$(\mathcal{P}_\lambda + c\lambda)\phi = 0, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{41}$$

where $\mathcal{P}_\lambda$ is the linear parabolic operator defined by

$$\begin{aligned}\mathcal{P}_\lambda w := \ & \partial_t w - Tr\big(A(x,t)D^2 w\big) + \big(q(x,t) - 2\lambda A(x,t)e\big)\cdot Dw \\ & -\big(\lambda^2 e A(x,t)e + \lambda q(x,t)\cdot e + \mu(x,t)\big)w.\end{aligned}$$

We consider the generalized principal eigenvalue introduced in [8]:

$$\kappa(\lambda) := \inf\Big\{k \in \mathbb{R}\ :\ \exists\varphi,\ \inf\varphi > 0,\ \sup\varphi < \infty,\ \sup|D\varphi| < \infty,\ \mathcal{P}_\lambda\varphi \le k\varphi,\ \text{in } \mathbb{R}^N \times \mathbb{R}\Big\}, \tag{42}$$

where the functions φ belong to $L^{N+1}_{loc}(\mathbb{R}^{N+1})$, together with their derivatives ∂_t, D, D^2 (and then the differential inequalities are understood to hold a.e.). This is the minimal regularity required for the maximum principle to apply, see, e.g. [13].

Taking $\varphi \equiv 1$ in the above definition we get, for $\lambda \in \mathbb{R}$,

$$\kappa(\lambda) \le -\underline{\alpha}\lambda^2 + \sup_{\mathbb{R}^{N+1}}|q||\lambda| - \inf_{\mathbb{R}^{N+1}}\mu. \tag{43}$$

We now derive a lower bound for $\kappa(\lambda)$. Assume by way of contradiction that there exists a function φ as in the definition of $\kappa(\lambda)$, associated with some k satisfying

$$k < -\overline{\alpha}\lambda^2 - \sup_{\mathbb{R}^{N+1}}|q||\lambda| - \sup_{\mathbb{R}^{N+1}}\mu.$$

For $\beta > 0$, the function $\psi(x,t) := e^{-\beta t}$ satisfies

$$\frac{\mathcal{P}_\lambda\psi}{\psi} \ge -\beta - \overline{\alpha}\lambda^2 - \sup_{\mathbb{R}^{N+1}}|q||\lambda| - \sup_{\mathbb{R}^{N+1}}\mu.$$

Hence, β can be chosen small enough in such a way that the latter term is larger than k, that is, $\mathcal{P}_\lambda\psi \ge k\psi$. The function ψ is larger than φ for t less than some t_0, whence $\psi \ge \varphi$ for all t by the comparison principle. It follows that $\varphi \to 0$ as $t \to +\infty$, which is impossible since φ is bounded from below away from 0. This shows that $\kappa(\lambda) > -\infty$.

We can now define c^* by setting

$$c^* := -\max_{\lambda>0}\frac{\kappa(\lambda)}{\lambda}. \tag{44}$$

This definition is well posed if $\kappa(0) < 0$ because $\kappa(\lambda)/\lambda \to -\infty$ as $\lambda \to +\infty$ by (43), and we know from [8] that $\lambda \mapsto \kappa(\lambda)$ is Lipschitz-continuous [3]. Let us show that (12) implies that

[3] The coefficients are assumed to be Hölder continuous in [8], but one can check that it does not matter in the proof of the continuity.

$\kappa(0) < 0$ and then that c^* is well defined and finite. Writing a positive function φ in the form $\varphi(t) := e^{-\sigma(t)}$, we see that

$$\mathcal{P}_0\varphi = -(\sigma'(t) + \mu(x,t))\varphi \leq -\left(\sigma'(t) + \inf_{x\in\mathbb{R}} \mu(x,t)\right)\varphi.$$

Thus, (39) implies that, for given $\varepsilon > 0$, there exists $\sigma \in W^{1,\infty}(\mathbb{R})$ such that

$$\mathcal{P}_0\varphi \leq -\left(\left\lfloor \inf_{x\in\mathbb{R}^N} \mu(x,\cdot)\right\rfloor - \varepsilon\right)\varphi.$$

Therefore, if (12) holds, taking $\varepsilon < \lfloor \min_{x\in\mathbb{R}^N} \mu(x,\cdot)\rfloor$ we derive $\kappa(0) < 0$.

The proof of Theorem 2.2 proceeds in two steps. In the following section we show that the average between 0 and $+\infty$ of the speed of a wave cannot be smaller than c^*. More precisely, we derive

Proposition 4.1. *Assume that* (3)-(6) *hold and that* $\kappa(0) < 0$. *Then, for any nonnegative supersolution* u *of* (1) *such that there is* $c \in L^\infty(\mathbb{R})$ *satisfying* (2), *there holds*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t c(s)ds \geq c^* := -\max_{\lambda>0} \frac{\kappa(\lambda)}{\lambda}.$$

In this statement, the notion of (sub, super) solution is understood as in the definition of $\kappa(\lambda)$: $u, \partial_t u, Du, D^2u \in L^{N+1}_{loc}(\mathbb{R}^{N+1})$. Notice that the least mean of a function is in general smaller than the average between 0 and $+\infty$. In the last section, we establish a general property of the least mean that allows us to deduce Theorem 2.2 by applying Proposition 4.1 to suitable time translations of the original problem.

4.1 Lower bound on the mean speed for positive times

We start with constructing subsolutions with a slightly varying exponential behaviour as $x \cdot e \to \pm\infty$. These will be then used to build a generalized subsolution with an arbitrary modulation of the exponential behaviour. The term "generalized subsolution" refers to a function that, in a neighbourhood of each point, is obtained as the supremum of some family of subsolutions. Then, using the fact that the generalized subsolutions satisfy the maximum principle, we will be able to prove Proposition 4.1.

Lemma 4.2. *Let* $c, \lambda \in \mathbb{R}$ *be such that* $\kappa(\lambda) + c\lambda < 0$. *There exists then* $\varepsilon > 0$ *and* $M > 1$ *such that, for any* $z \in \mathbb{R}$, (40) *admits a subsolution* $\underline{v}$ *satisfying*

$$\text{if } x\cdot e - ct \geq z, \qquad \frac{1}{M}e^{-(\lambda+\varepsilon)(x\cdot e-ct)} \leq \underline{v}(x,t) \leq Me^{-(\lambda+\varepsilon)(x\cdot e-ct)},$$

$$\inf_{z-1<x\cdot e-ct<z} \underline{v}(x,t) > 0,$$

$$\text{if } x\cdot e - ct \leq z-1, \qquad \frac{1}{M}e^{-(\lambda-\varepsilon)(x\cdot e-ct)} \leq \underline{v}(x,t) \leq Me^{-(\lambda-\varepsilon)(x\cdot e-ct)}.$$

Proof. By the definition of $\kappa(\lambda)$, there is a bounded function φ with positive infimum satisfying

$$\mathcal{P}_\lambda \varphi \leq k\varphi, \quad x \in \mathbb{R}^N,\ t > T,$$

for some $k < -c\lambda$. It follows that $v(x,t) := e^{-\lambda(x\cdot e - ct)}\varphi(x,t)$ is a subsolution of (40). Fix $z \in \mathbb{R}$ and consider a smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\zeta = \lambda - \varepsilon \text{ in } (-\infty, z-1], \qquad \zeta = \lambda + \varepsilon \text{ in } [z, +\infty), \qquad 0 \leq \zeta' \leq 3\varepsilon, \qquad |\zeta''| \leq h\varepsilon,$$

where $\varepsilon > 0$ has to be chosen and h is a universal constant. We define the function $\underline{v}$ by setting $\underline{v}(x,t) := e^{-(x\cdot e - ct)\zeta(x\cdot e - ct)}\varphi(x,t)$. Calling $\rho := x \cdot e - ct$, we find that

$$\begin{aligned}[\partial_t \underline{v} - a_{ij}(x,t)\partial_{ij}\underline{v} + q_i(x,t)\partial_i \underline{v} - \mu(x,t)\underline{v}]e^\rho \leq &(\mathcal{P}_\zeta + c\zeta)\varphi \\ &+ C[(1 + \rho + \rho|\zeta| + \rho^2|\zeta'|)|\zeta'| + \rho|\zeta''|],\end{aligned}$$

where ζ, ζ', ζ'' are evaluated at ρ and C is a constant depending on N, c and the L^∞ norms of $a_{ij}, q, \mu, \varphi, D\varphi$. The second term of the above right-hand side is bounded by $H(\varepsilon)$, for some continuous function H vanishing at 0. The first term satisfies

$$(\mathcal{P}_\zeta + c\zeta)\varphi \leq (\mathcal{P}_\lambda + c\lambda)\varphi + C((\zeta - \lambda) + |\zeta^2 - \lambda^2|) \leq (k + c\lambda)\varphi + C(\varepsilon + 2|\lambda|\varepsilon + \varepsilon^2).$$

We thus derive

$$\partial_t \underline{v} - a_{ij}(x,t)\partial_{ij}\underline{v} + q_i(x,t)\partial_i \underline{v} - \mu(x,t)\underline{v} \leq e^\rho[(k + c\lambda)\varphi + C\varepsilon(1 + 2|\lambda| + \varepsilon^2) + H(\varepsilon)].$$

Since $k < -c\lambda$ and $\inf \varphi > 0$, ε can be chosen small enough in such a way that $\underline{v}$ is a subsolution of (40). □

Lemma 4.3. *Let $\underline{\lambda}, \overline{\lambda}, c \in \mathbb{R}$ satisfy $\underline{\lambda} < \overline{\lambda}$ and*

$$\max_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} (\kappa(\lambda) + c\lambda) < 0.$$

Then there exists a generalized, bounded subsolution $\underline{v}$ of (40) satisfying

$$\lim_{r \to -\infty} \sup_{x\cdot e - ct < r} \underline{v}(x,t)e^{\underline{\lambda}(x\cdot e - ct)} = 0, \qquad \lim_{r \to +\infty} \sup_{x\cdot e - ct > r} \underline{v}(x,t)e^{\overline{\lambda}(x\cdot e - ct)} = 0.$$

$$\forall r_1 < r_2, \qquad \inf_{r_1 < x\cdot e - ct < r_2} \underline{v}(x,t) > 0. \tag{45}$$

Proof. For $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, let ε_λ, M_λ be the constants given by Lemma 4.2 associated with c and λ. Call I_λ the interval $(\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$. The family $(I_\lambda)_{\underline{\lambda} \leq \lambda \leq \overline{\lambda}}$ is an open covering of $[\underline{\lambda}, \overline{\lambda}]$. Let $(I_{\lambda_i})_{i=1,\dots,n}$ be a finite subcovering and set for short $\varepsilon_i := \varepsilon_{\lambda_i}$, $M_i := M_{\lambda_i}$. Up to rearranging the indices and extracting another subcovering if need be, we can assume that

$$\forall i = 1, \dots, n-1, \quad \lambda_{i+1} - \varepsilon_{i+1} < \lambda_i - \varepsilon_i < \lambda_{i+1} + \varepsilon_{i+1} < \lambda_i + \varepsilon_i.$$

Let v_1 be the subsolution of (40) given by Lemma 4.2 associated with $\lambda = \lambda_1$ and $z = 0$. Set $z_1 := 0$, $k_1 := 1$ and

$$k_2 := \frac{e^{(\lambda_2 + \varepsilon_2 - (\lambda_1 - \varepsilon_1))(z_1 - 1)}}{M_1 M_2}.$$

Consider then the subsolution v_2 associated with $\lambda = \lambda_2$ and z equal to some value $z_2 < z_1 - 1$ to be chosen. We have that

$$\text{if } x \cdot e - ct = z_1 - 1, \qquad \frac{v_1(x,t)}{v_2(x,t)} \geq \frac{k_2}{k_1},$$

$$\text{if } x \cdot e - ct = z_2, \qquad \frac{k_1 v_1(x,t)}{k_2 v_2(x,t)} \leq (M_1 M_2)^2 e^{(\lambda_2 + \varepsilon_2 - (\lambda_1 - \varepsilon_1))(z_2 - z_1 + 1)}.$$

Since $\lambda_2 + \varepsilon_2 > \lambda_1 - \varepsilon_1$, we can choose z_2 in such a way that the latter term is less than 1. By a recursive argument, we find some constants $(z_i)_{i=1,\dots,n}$ satisfying $z_n < z_{n-1} - 1 < \cdots < z_1 - 1 = -1$, such that the family of subsolutions $(v_i)_{i=1,\dots,n}$ given by Lemma 4.2 associated with the $(\lambda_i)_{i=1,\dots,n}$ and $(z_i)_{i=1,\dots,n}$ satisfies, for some positive $(k_i)_{i=1,\dots,n}$,

$$\forall i = 1, \dots, n-1, \quad k_{i+1} v_{i+1} \leq k_i v_i \ \text{ if } x \cdot e - ct = z_i - 1, \quad k_{i+1} v_{i+1} \geq k_i v_i \ \text{ if } x \cdot e - ct = z_{i+1}.$$

The function $\underline{v}$, defined by

$$\underline{v}(x,t) := \begin{cases} v_1(x,t) & \text{if } x \cdot e - ct \geq z_1 \\ \max(k_i v_i(x,t), k_{i+1} v_{i+1}(x,t)) & \text{if } z_{i+1} \leq x \cdot e - ct < z_i \\ k_n v_n(x,t) & \text{if } x \cdot e - ct < z_n, \end{cases}$$

is a generalized subsolution of (40) satisfying the desired properties. □

Proof of Proposition 4.1. Let u, c be as in the statement of the proposition, and call $\phi(x,t) := u(x + e\int_0^t c(s)ds, t)$. Since $\phi(x,t) \to 1$ as $x \cdot e \to -\infty$, uniformly with respect to $t \in \mathbb{R}$, one can find $\rho \in \mathbb{R}$ such that

$$\inf_{\substack{x \cdot e < \rho \\ t \in \mathbb{R}}} \phi(x,t) > 0.$$

We make now use of Lemma 3.1 in [23], which, under the above condition, establishes a lower bound for the exponential decay of an entire supersolution ϕ of a linear parabolic equation (notice that the differential inequality for ϕ can be written in linear form with a bounded zero order term: $f(x,t,\phi) = [f(x,t,\phi)/\phi]\phi$). The result of [23] implies the existence of a positive constant λ_0 such that

$$\inf_{\substack{x \cdot e > \rho - 1 \\ t \in \mathbb{R}}} \phi(x,t) e^{\lambda_0 x \cdot e} > 0.$$

By the definition of c^*, the hypotheses of Lemma 4.3 are fulfilled with $\underline{\lambda} = 0$, $\overline{\lambda} = \lambda_0$ and $c = c^* - \varepsilon$, for any given $\varepsilon > 0$. This is also true if one penalizes the nonlinear term $f(x,t,u)$ by subtracting δu, with δ small enough, since this just raises the principal eigenvalues $\kappa(\lambda)$ by δ. Therefore, Lemma 4.3 provides a function $\underline{v}$ such that, for $h > 0$ small enough, $h\underline{v}$ is a subsolution of (1). We choose h in such a way that, together with the above property, $h\underline{v}(x,0) < u(x,0)$. This can be done, due to the lower bounds of $u(x,0) = \phi(x,0)$, because $\underline{v}$ is bounded and decays faster than $e^{-\lambda_0 x \cdot e}$ as $x \cdot e \to +\infty$. Applying the parabolic comparison

principle we eventually infer that $h\underline{v} < u$ for all $x \in \mathbb{R}^N$, $t \geq 0$. It follows that u satisfies (45) with $c = c^* - \varepsilon$ for $t > 0$. We derive, in particular,

$$0 < \inf_{t>0} u((c^* - \varepsilon)te, t) = \inf_{t>0} u\Big(\Big[(c^* - \varepsilon)t - \int_0^t c(s)ds\Big]e + e\int_0^t c(s)ds, t\Big),$$

which, in virtue of the second condition in (2), implies that

$$\limsup_{t\to+\infty} \Big((c^* - \varepsilon)t - \int_0^t c(s)ds\Big) < +\infty.$$

This concludes the proof due to the arbitrariness of ε. □

4.2 Property of the least mean and proof of Theorem 2.2

Roughly speaking, the least mean of a function is the infimum of its averages in sufficiently large intervals. We show that, in some sense, this infimum is achieved up to replacing the function with an element of its *ω-limit* set. The ω-limit (in the L^∞ weak-$\star$ topology) of a function g, denoted by ω_g, is the set of functions obtained as L^∞ weak-$\star$ limits of translations of g.

Proposition 4.4. *Let $g \in L^\infty(\mathbb{R})$ and let ω_g denote its ω-limit set (in the L^∞ weak-$\star$ topology). Then*

$$\lfloor g \rfloor = \min_{\tilde{g}\in\omega_g} \left(\lim_{t\to+\infty} \frac{1}{t}\int_0^t \tilde{g}(s)\,ds \right).$$

Proof. We can assume without loss of generality that $\lfloor g \rfloor = 0$. Clearly, any $\tilde{g} \in \omega_g$ satisfies $\lfloor \tilde{g} \rfloor \geq \lfloor g \rfloor$, whence

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \tilde{g}(s)\,ds \geq \lfloor \tilde{g} \rfloor \geq \lfloor g \rfloor = 0.$$

Our aim is to find a function $\tilde{g} \in \omega_g$ satisfying

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \tilde{g}(s)\,ds \leq 0. \tag{46}$$

We claim that, for any $n \in \mathbb{N}$, there exists $t_n \in \mathbb{N}$ such that

$$\forall j = 1, \dots, n, \quad n\int_{t_n}^{t_n+j} g(s)\,ds \leq j.$$

Assume by way of contradiction that this property fails for some $n \in \mathbb{N}$. By the definition of least mean, for $K \in \mathbb{N}$ large enough, there is $\tau \in \mathbb{R}$ such that

$$\frac{1}{K}\int_\tau^{\tau+Kn} g(s)\,ds < \frac{1}{2}.$$

On the other hand, there is $j \in \{1, \dots, n\}$ such that $n\int_\tau^{\tau+j} g(s)\,ds > j$. Then, there is $h \in \{1, \dots, n\}$ such that $n\int_{\tau+j}^{\tau+j+h} g(s)\,ds > h$, hence $n\int_\tau^{\tau+j+h} g(s)\,ds > j + h$. We repeat

this argument until we find $k \in \{1, \dots, n\}$ such that $n \int_\tau^{\tau+Kn+k} g(s)\,ds > Kn + k$. From this we deduce that

$$\int_\tau^{\tau+Kn} g(s)\,ds > K + \frac{k}{n} - \int_{\tau+Kn}^{\tau+Kn+k} g(s)\,ds > K - n\|g\|_{L^\infty(\mathbb{R})}.$$

A contradiction follows taking $K > 2n\|g\|_{L^\infty(\mathbb{R})}$. The claim is proved. The L^∞ weak-$\star$ limit $\tilde{g}$ as $n \to \infty$ of (a subsequence of) $g(\cdot + t_n)$ satisfies the desired property. Indeed,

$$\forall j \in \mathbb{N}, \quad \int_0^j \tilde{g}(s)\,ds = \lim_{n\to\infty} \int_{t_n}^{t_n+j} g(s)\,ds = 0,$$

from which (46) follows since $\tilde{g}$ is bounded. □

Proof of Theorem 2.2. Let u be a generalized transition wave with speed c. Proposition 4.4 yields that there exists $\tilde{c} \in \omega_c$ such that

$$\lfloor c \rfloor = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \tilde{c}(s)ds. \tag{47}$$

The definition of ω_c gives a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $c(\cdot + t_n) \rightharpoonup \tilde{c}$ as $n \to +\infty$ for the L^∞ weak-$\star$ topology. For $n \in \mathbb{N}$, consider the functions

$$A_n(x,t) := A\Big(x + e\int_0^{t_n} c(s)ds, t + t_n\Big), \qquad q_n(x,t) := q\Big(x + e\int_0^{t_n} c(s)ds, t + t_n\Big),$$

$$\mu_n(x,t) := \mu\Big(x + e\int_0^{t_n} c(s)ds, t + t_n\Big), \qquad u_n(x,t) := u\Big(x + e\int_0^{t_n} c(s)ds, t + t_n\Big).$$

For any $\varepsilon \in (0,1)$ there exists $m \in (0,1)$ such that

$$\forall (x,t) \in \mathbb{R}^{N+1},\ u \in [0,1], \quad f(x,t,u) \geq (\mu(x,t) - \varepsilon)u(m-u).$$

It follows that the u_n satisfy

$$\partial_t u_n - \mathrm{Tr}(A_n(x,t)D^2u_n) + q_n(x,t)Du_n \geq (\mu_n(x,t) - \varepsilon)u_n(m - u_n), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

On the other hand, the L^p parabolic interior estimates ensure that the sequences $(\partial_t u_n)_{n\in\mathbb{N}}$, $(Du_n)_{n\in\mathbb{N}}$, $(D^2u_n)_{n\in\mathbb{N}}$ are bounded in $L^p(Q)$ for all $p \in (1,\infty)$ and $Q \Subset \mathbb{R}^{N+1}$. Hence, by the embedding theorem, $(u_n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly in $\mathbb{R}^{N+1}$ to some function $\tilde{u}$, and the derivatives ∂_t, D, D^2 of the $(u_n)_{n\in\mathbb{N}}$ weakly converge to those of $\tilde{u}$ in $L^p_{loc}(\mathbb{R}^{N+1})$. Therefore, letting $\tilde{A}$, $\tilde{q}$ be the locally uniform limits of (subsequences of) $(A_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ and $\tilde{\mu}$ be the L^∞ weak-$\star$ limit of (a subsequence of) $(\mu_n)_{n\in\mathbb{N}}$, we infer that

$$\partial_t \tilde{u} - \mathrm{Tr}(\tilde{A}(x,t)D^2\tilde{u}) + \tilde{q}(x,t)D\tilde{u} \geq (\tilde{\mu}(x,t) - \varepsilon)\tilde{u}(m - \tilde{u}), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

Hence, $\tilde{u}$ is a supersolution of an equation of the type (1) whose terms satisfy (3)-(5) and (6). Moreover, it is easily derived from the definition of the speed c and the L^∞ weak-$\star$ convergence to $\tilde{c}$, that $\tilde{u}$ satisfies (2) with c replaced by $\tilde{c}$, uniformly with respect to $t \in \mathbb{R}$.

In order to apply Proposition 4.1 to the function $\tilde{u}$, we need to show that $\tilde{\kappa}(0) < 0$, where $\lambda \mapsto \tilde{\kappa}(\lambda)$ is defined like $\lambda \mapsto \kappa(\lambda)$, but with $\tilde{A}$, $\tilde{q}$, $\tilde{\mu} - \varepsilon$ in place of A, q, μ respectively. Namely, the $\kappa(\lambda)$ are the principal eigenvalues in the sense of (42) for the operators $\tilde{\mathcal{P}}_\lambda$ defined as follows:

$$\begin{aligned}\tilde{\mathcal{P}}_\lambda w := \;& \partial_t w - Tr\big(\tilde{A}(x,t)D^2 w\big) + \big(\tilde{q}(x,t) - 2\lambda\tilde{A}(x,t)e\big)\cdot Dw \\ & -\big(\lambda^2 e\tilde{A}(x,t)e + \lambda\tilde{q}(x,t)\cdot e + \tilde{\mu}(x,t) - \varepsilon\big)w.\end{aligned}$$

This will be achieved by showing that

$$\forall \lambda > 0, \quad \tilde{\kappa}(\lambda) \le \kappa(\lambda) + \varepsilon, \tag{48}$$

whence $\tilde{\kappa}(0) < 0$ as soon as $\varepsilon < -\kappa(0)$ (recall that $\kappa(0) < 0$ by (12)). Let us postpone for a moment the proof of (48). Applying Proposition 4.1 to $\tilde{u}$ yields

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t \tilde{c}(s)ds \ge -\max_{\lambda>0}\frac{\tilde{\kappa}(\lambda)}{\lambda} = -\frac{\tilde{\kappa}(\hat{\lambda})}{\hat{\lambda}},$$

for some $\hat{\lambda} > 0$. In virtue of (47) and (48), from this inequality we deduce

$$\lfloor c \rfloor \ge -\frac{\kappa(\hat{\lambda}) + \varepsilon}{\hat{\lambda}},$$

from which $\lfloor c \rfloor \ge c^*$ follows by the arbitrariness of ε.

It remains to prove (48). Let $k > \kappa(\lambda)$. By definition (43) there exists φ such that $\inf\varphi > 0$, $\varphi, D\varphi \in L^\infty(\mathbb{R}^N \times \mathbb{R})$ and $\mathcal{P}_\lambda\varphi \le k\varphi$ in $\mathbb{R}^N \times \mathbb{R}$. We would like to perform on φ the same limit of translations as done before to obtain $\tilde{u}$ from u. This would yield a function $\tilde{\varphi}$ satisfying $\tilde{\mathcal{P}}_\lambda\tilde{\varphi} \le (k+\varepsilon)\tilde{\varphi}$. But this argument requires the L^p_{loc} estimates of the derivatives ∂_t, D, D^2 of the translated of φ, which are not available because φ is a subsolution and not a solution of an equation. However, it is possible to replace φ with a solution of a semilinear equation of the type $\mathcal{P}_\lambda w = g(w)$ in $\mathbb{R}^N \times \mathbb{R}$, with g smooth and such that $g(w) \le (k+\varepsilon)w$, which satisfies the same properties as φ, as well as the desired additional regularity properties. This is done in the proof of Theorem A.1 of [23], whose arguments can be exactly repeated here. We can therefore apply the translation argument that provides a function $\tilde{\varphi}$ such that $\tilde{\mathcal{P}}_\lambda\tilde{\varphi} \le (k+\varepsilon)\tilde{\varphi}$. Moreover, $\inf\tilde{\varphi} > 0$ and $\sup\tilde{\varphi} < \infty$. In order to be able to use $\tilde{\varphi}$ in the definition of $\tilde{\kappa}(\lambda)$ and derive $\tilde{\kappa}(\lambda) \le k+\varepsilon$, we only need to have that $\sup|D\tilde{\varphi}| < \infty$. This property does not follow automatically from the L^p estimates and the embedding theorem as in the elliptic case treated in [23]. This is the reason why we need the extra assumption (13) on A. Indeed, we use Theorem 1.4 of [22], with, using the same notations as in [22], F the nonlinear operator associated with equation $\mathcal{P}_\lambda w = g(w)$, Hypothesis 1.2 of [22] being satisfied since A satisfies (13), q is bounded and $f = f(x,t,u)$ is bounded with respect to $(x,t,u) \in \mathbb{R}^N \times \mathbb{R} \times [0,1]$, and Hypothesis 1.3 being satisfied with $\varphi(x,t) := e^{Mt}(1+|x|^2)$ and M large enough. Hence, we get a uniform L^∞ bound on Dw, where w is the solution of $\mathcal{P}_\lambda w = g(w)$. Using w instead of φ, we get that this bound is inherited by $\tilde{\varphi}$ and we therefore deduce $\tilde{\kappa}(\lambda) \le k+\varepsilon$. As $k > \kappa(\lambda)$ is arbitrary, we eventually get (48).

□

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