# Constrained Optimization on Hierarchies and Braids of Partitions

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This work was partly funded by the French ANR-2010-BLAN-0205-03 program KIDICO

ISMM 2015
Reykjavik, Iceland
may 27-29 2015

## Constrained Optimization

$$\underset{\pi\in\Pi(E,H)}{\text{minimize}} \quad \sum_{S\in\pi} \omega_\varphi(S)$$

$$\text{subject to} \quad \sum_{S\in\pi} \omega_\partial(S) \le C$$

Lagrangian function

$$\omega(\pi,\lambda) = \omega_\varphi(\pi) + \lambda\cdot\omega_\partial(\pi)$$

Minimization by Dynamic programming

$$\pi^*(S) = \begin{cases} \{S\}, & \omega(S) \le \text{comp}(\omega(\pi^*(a))) \\ \bigsqcup_{a\in\pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$

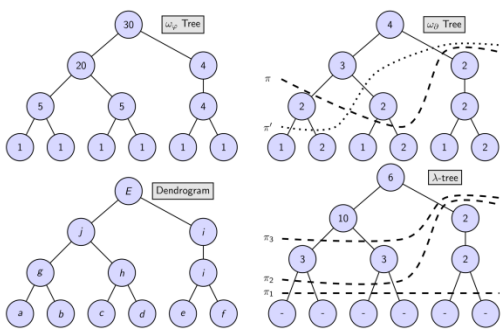
where $a \in \pi(S)$ are elements of partial partition with support $S$.

Related Problems

- Decision Tree pruning [Breiman 1984]
- BPT-pruning [Salembier-Garrido 2001]
- Scale-Sets [Guigues 2003]

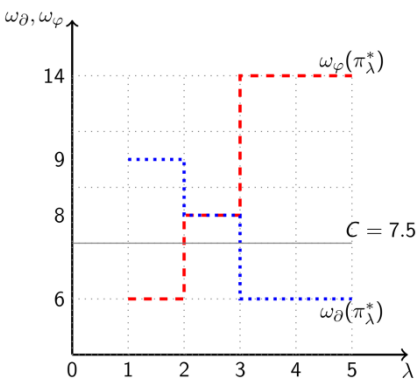
## $\lambda$-cuts are Upper bounds

$\forall$ Parents $\lambda = -\frac{\Delta\omega_\varphi}{\Delta\omega_\partial}$




$\lambda$-cuts: $\pi_1, \pi_2, \pi_3$ for $\lambda = 1, 2, 3$

Other Cuts: $\pi, \pi'$




- $\omega_\varphi(\pi') = 11$ $\omega_\partial(\pi') = 7$
- $\omega_\partial \ne C$ where $C = 7.5$ at any time!
- Set valued contraint function

## Multiplier-Constraint

- Lack of $\lambda \to C$ mapping: Given cost $\omega_\partial \le C$ one is not assured a corresponding multipler $\lambda$.
- Uniqueness is lost, even when $\omega_\varphi$ is strictly $h$-increasing.
- $\pi(\lambda^*)$ is only the upper-bound for a given $C$.
- The distance $|\ \omega_\partial(\pi^*(\lambda^*)) - C\ |$ gives no information on $|\ \omega_\varphi(\pi^*(\lambda^*)) - \omega_\varphi(\pi)\ |$ where $\pi$ is a constrained minimal cut.

## Everett's Theorem

Given the multiplier $\lambda \in \mathbb{R}$

$$\min_{\pi\in\Pi(E,H)} \left\{ \sum_\pi \omega_\varphi(S) + \lambda \sum_\pi \omega_\partial(S) \right\}$$

The solution $\bar\pi(\lambda)$ to this unconstrained minimization is also an optimal solution to perturbed primal problem:

$$\underset{\pi\in\Pi(E,H)}{\text{minimize}} \quad \sum_\pi \omega_\varphi(S)$$

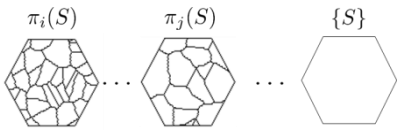
$$\text{subject to} \quad \sum_\pi \omega_\partial(S) \le \sum_{\pi(\lambda)} \omega_\partial(S)$$

## Optimal Dual Parameter

- Changing to the dual domain does not aid solving a combinatorial problem, it provides some upper-bound.
- [Salembier-Garrido 2001] searches $\lambda$-cuts to approx. constraint value $\omega_\partial(\pi) \approx C$
- [Guigues 2003] produces causal scale-set description of image using defined energy.
- By choosing multiplier $\lambda$ from possible $\lambda$'s one achieves an optimum for a constraint value is $\lambda$-dependent (Everett's Theorem [Everett 1963]).

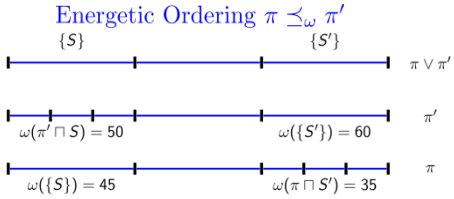
## Energetic-Lattices Recall

Singular Energies




$$\omega(\{S\}) \ne \omega(\pi_i(S)), \forall i \in [1,n]$$

Energetic Ordering $\pi \preceq_\omega \pi'$




$$\forall S \in \pi \vee \pi' \ \omega(\pi \sqcap \{S\}) \le \ \omega(\pi' \sqcap \{S\})$$

## Cut Constraint(LMCC)

**Find cut in lattice $\pi_\varphi \in \Pi_{\omega_\varphi}$ subject to $\pi_\varphi \preceq_{\omega_\partial} \pi_C$.**

Feasible set

$$\Pi_C = \{\pi \mid \pi \in \Pi,\ \pi \preceq_{\omega_\partial} \pi_C\}$$

Induces energetic lattice $\Pi_{\omega(\lambda)}$ with minimal cut $\pi^*(\lambda)$. Optimal $\lambda$:

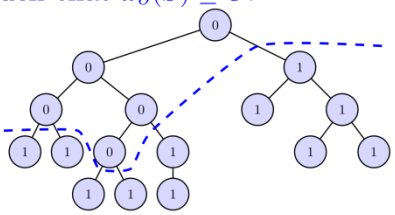
$$\lambda^* = \sup\{\lambda \mid \pi^*(\lambda) \preceq_{\omega_\partial} \pi_C\}$$

Constraint satisfaction
$\Pi_C = \{\pi \mid \pi \in \Pi, \pi \preceq_{\omega_\partial} \pi_C\} \ne \emptyset$
Positive multiplier: $\lambda \ge 0$
Lattice constraint: $\pi_{\omega_\varphi} \succeq_{\omega_\varphi} \pi^*(\lambda^*)$

The unique optimal dual parameter then is $\lambda \ge \lambda^*$, and unique optimal cut is $\pi^*(\lambda^*)$

## Class Constraint (CCM)

**Find $\pi \in \Pi(E,H)$ minimizing $\omega_\varphi(\pi)$ such that $\omega_\partial(S) \le C$.**



**Inf-Modular Energy:**
$\omega_\partial(\{S\}) \le \bigwedge\{\omega_\partial(a), a \sqsubseteq \{S\}\}$.

- 1-classes have $\omega_\partial > C$, 0-classes have $\omega_\partial \le C$.
- When for children $T_i$ of class $S$, $\omega_\partial(T_i) \le C$, then replace $S$ by $\{T_i\}$.

Feasible set
$\pi \in \mathcal{A}(C) \ \Leftrightarrow \ \{S \sqsubseteq \pi \ \Rightarrow \ \omega_\partial(S) \le C\}$

Family $\mathcal{A}(C)$ is closed under the refinement infimum, it admits a smallest element $\pi_0$:

Optimal Cut
$\pi_0 = \wedge\{\pi, \pi \in \mathcal{A}(C)\}$

Constrained cut $\pi_0$ shown in dotted blue line

## CO by Energetic-Lattices (LME)

**Cut**: Given a braid $B$, a constraint value $C$, and objective and constraint functions $\omega_\varphi$ and $\omega_\partial$, find the cut(s) $\pi \in B$ that minimize $\omega_\varphi(\pi)$, subject to the constraint $\omega_\partial(\pi) \le C$.

**Multiplier**: Given a braid $B$ on $E$ and two energies $\omega_\varphi$ and $\omega_\partial$, find the parameter $\lambda$ which maximizes $\omega(\pi^*(\lambda))$, subject to the constraint $\lambda > 0$.

Let $\lambda^* = \inf\{\lambda \mid \omega_\partial(\pi^*(\lambda)) \le C\}$. If
Feasibility: $\{\pi,\ \pi \in \Pi(E,B),\ \omega_\partial(\pi) \le C\} \ne \emptyset$
$\lambda$-$C$ map: $\omega_\partial(\pi^*(\lambda^*)) - C = 0$,

then $\pi^*(\lambda^*)$ and $\lambda^*$ are unique solutions.