

Braids Of Partitions

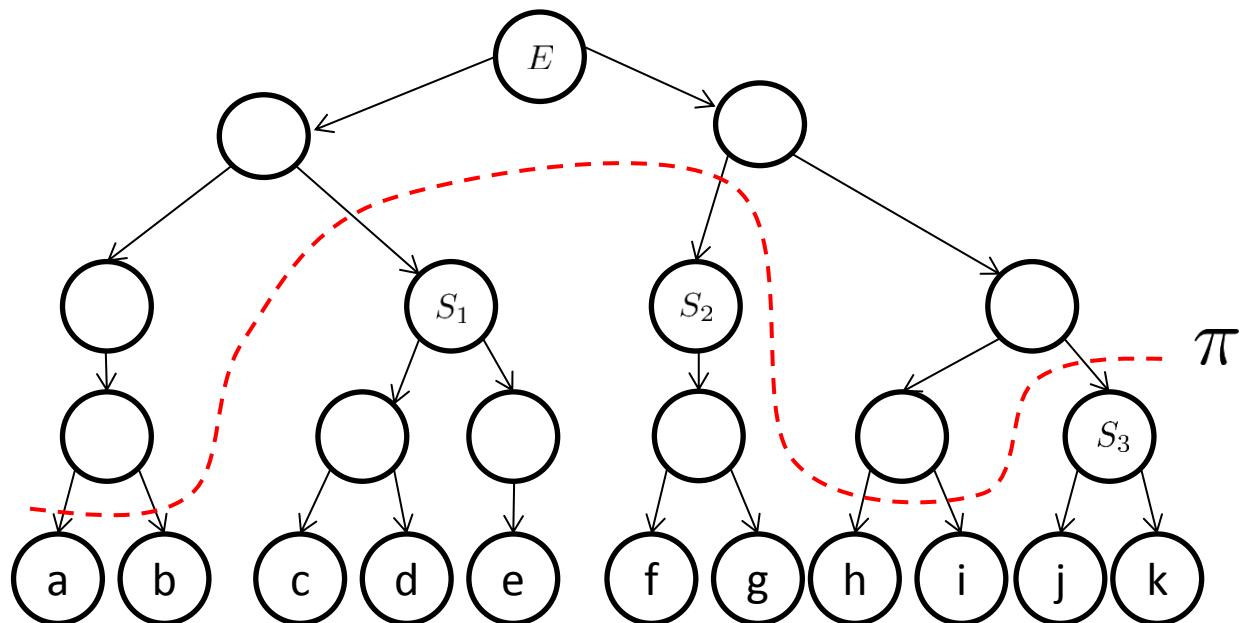
B Ravi Kiran*, Jean Serra**

*Centre de robotique, MINES ParisTech, PSL-Research University

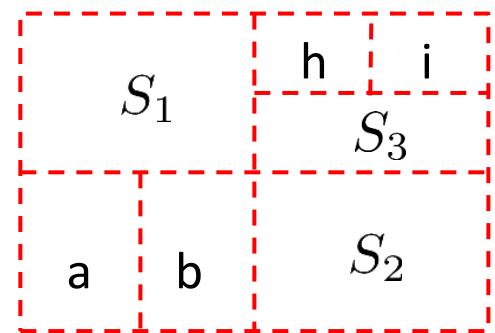
**Université Paris-Est, A3SI-ESIEE LIGM

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Problem Formulation

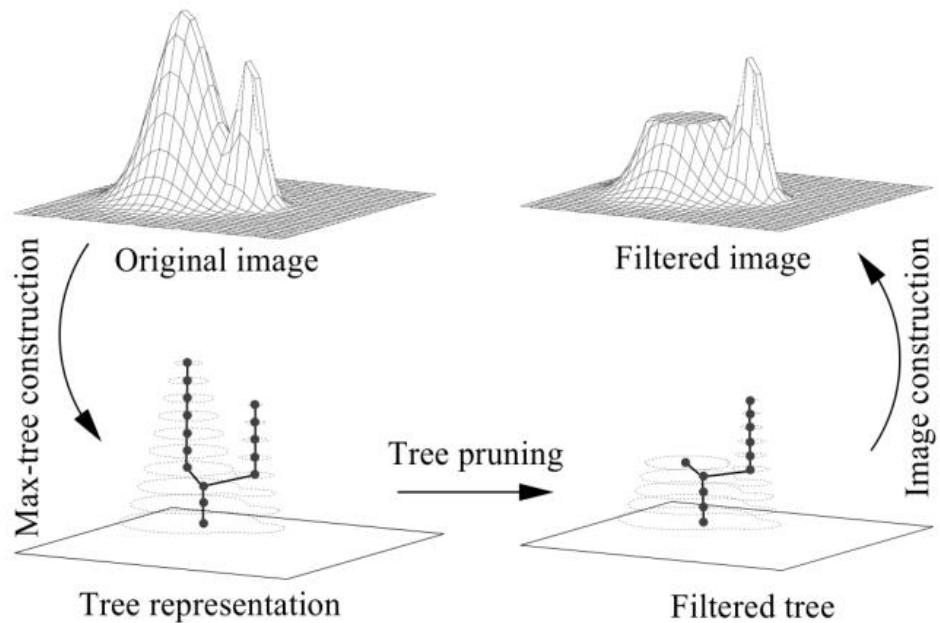
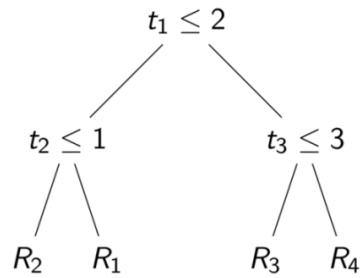
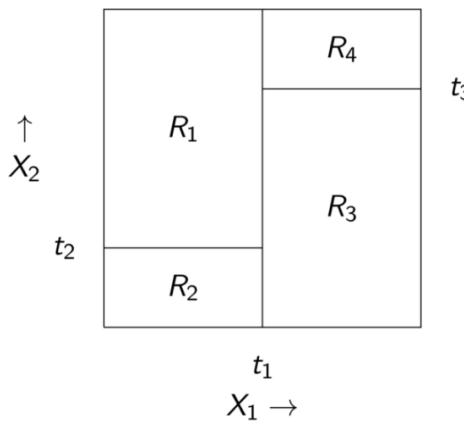


d		h	i
c	e	j	k
a	b	f	
		g	



$$\underset{\pi \in \Pi(E, H)}{\text{minimize}} \quad \sum_{S \in \pi} \omega(S)$$

Constrained Optimization on Trees



Decision Tree Cost-Complexity Post-pruning

[Breiman et al. 1984]

λ -dependent rate satisfaction.
[Salembeier-Garrido 2000]

Extraction sequence of λ -cuts given:

$$H, \omega(\mathcal{S}, \lambda), \lambda_i \implies H^*(\lambda)$$

λ -cuts form a hierarchy.

[Guigues 2003]

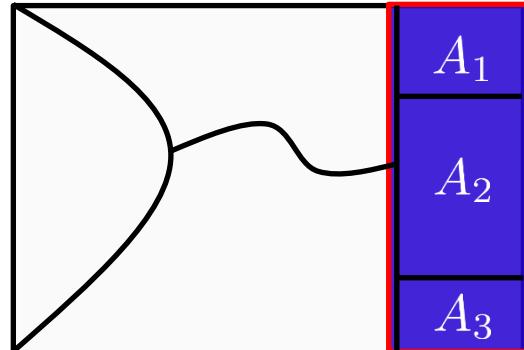
Partial Partitions

A partial partition of a subset $S \subseteq E$ is defined as:

$$\pi(S) = \{A_i \mid A_i \subseteq S, A_i \cap A_j = \emptyset\}$$

$S = \cup A_i$ is called the support of $\pi(S)$

$$\begin{aligned} S &= A_1 \cup A_2 \cup A_3 \\ E &:= \mathbb{R}^2 \text{ or } \mathbb{Z}^2 \end{aligned}$$



Energies on P.P.

Energy/Function on partial partitions:

$$\omega : \mathbf{PPs} \rightarrow \mathbb{R}$$

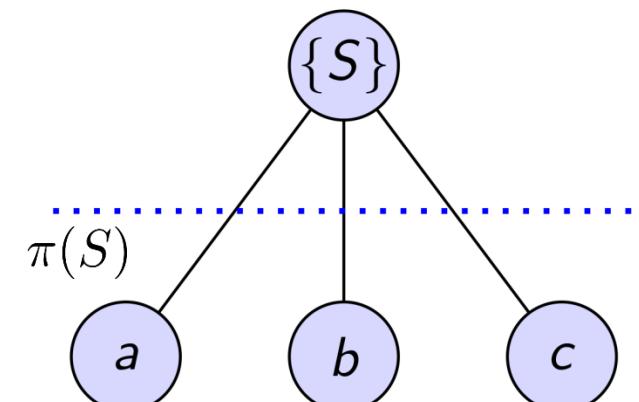
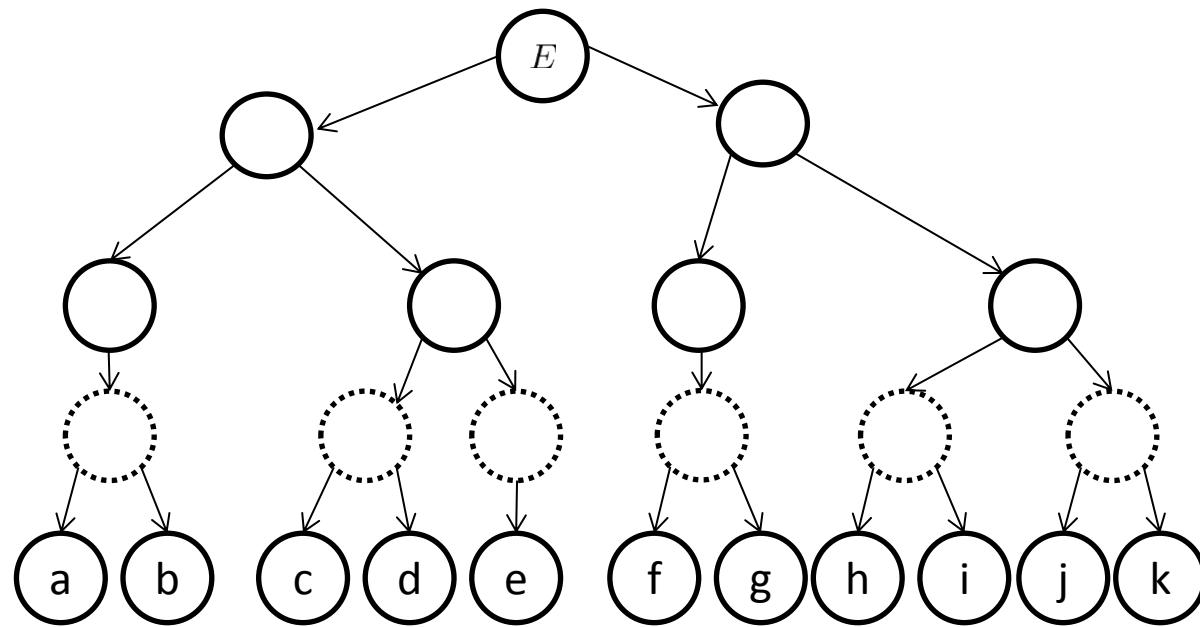
Energy of a partial partition can be written by composing energies of its classes

$$\omega(\pi(S)) = \sum_{A_i \in \pi(S)} \omega(A_i)$$

Note! Energy is defined on Partial partitions and not conventionally where it was defined sets

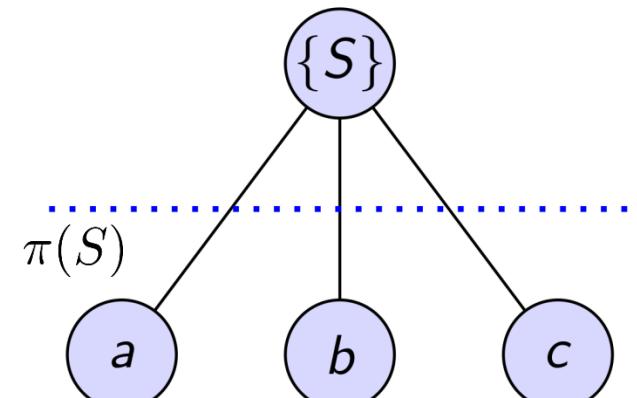
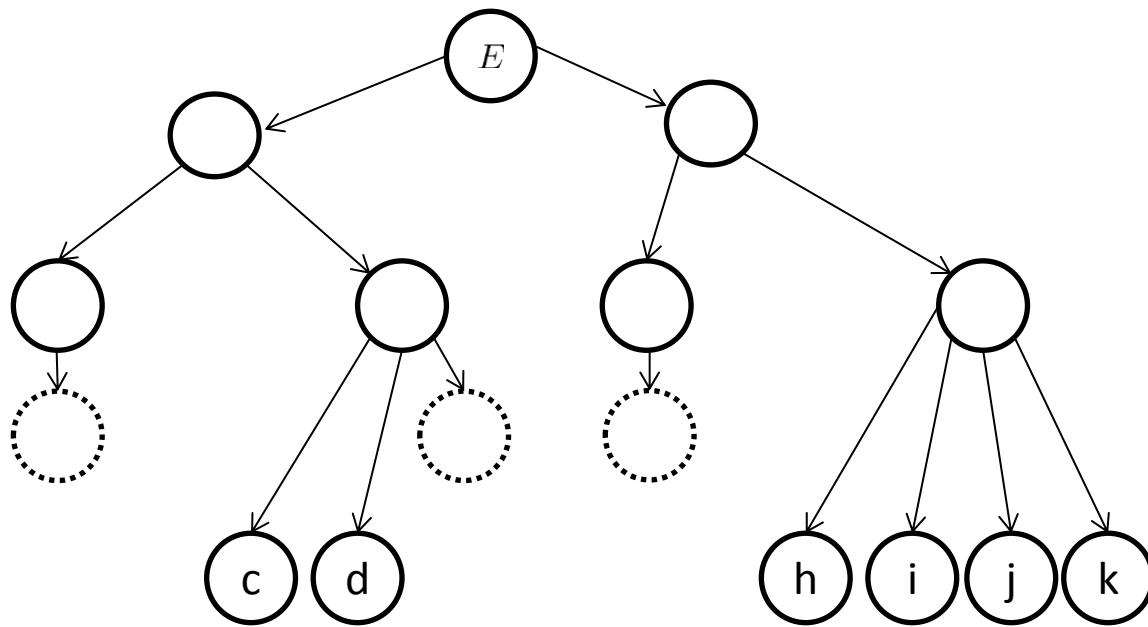
Dynamic Program

$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$



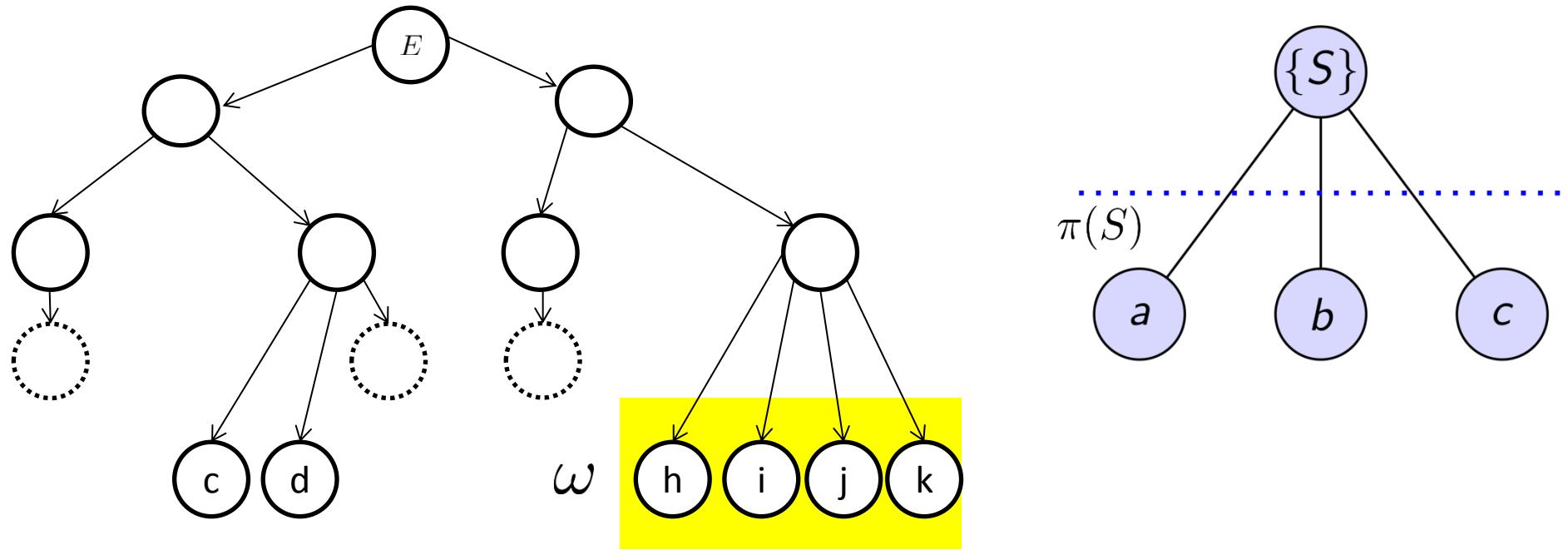
Dynamic Program

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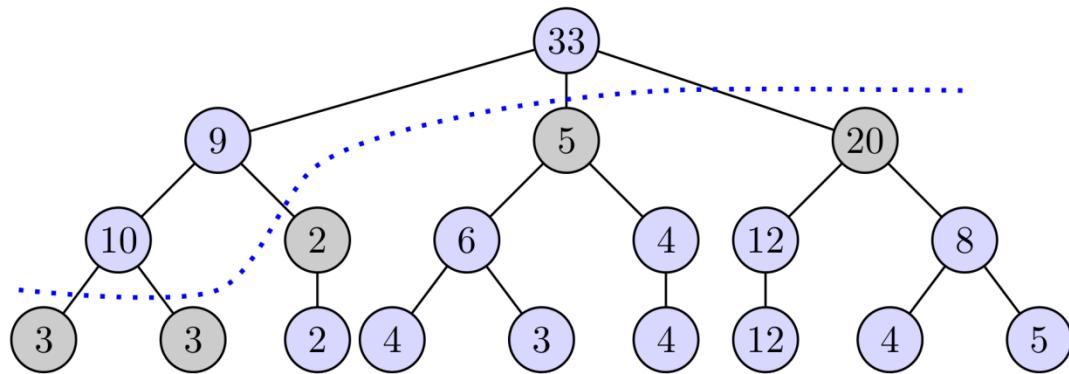
Dynamic Program

$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \text{comp}(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases}$$



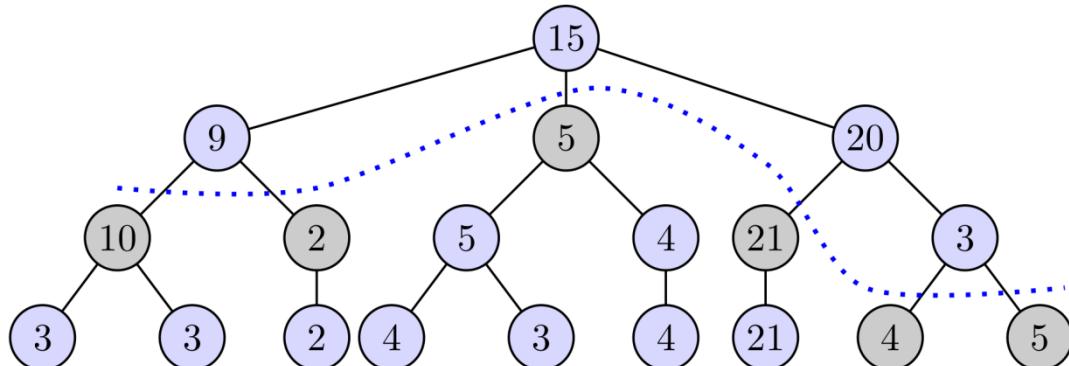
What are the valid comp() ?
Braids of Partitions

Composition Laws



$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \sum_{a \in \pi(S)} \omega(a) \\ \pi(S), & \text{otherwise} \end{cases}$$

[Salembier Garrido 2000, Guigues 2003]



S more energetic than all its descendants.

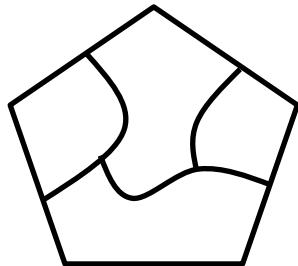
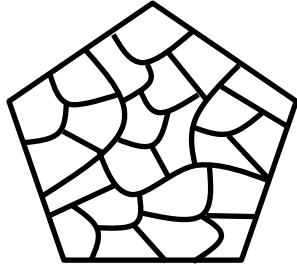
$$\omega(S^*) \geq \bigvee_{\pi(S)} \omega(T_i)$$

[Akçay-Akçoy 2008]

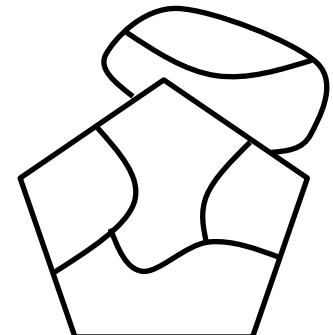
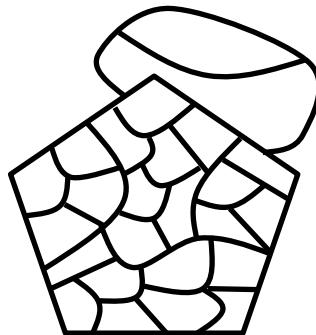
h -increasingness on HOP

[Serra DGCI 2011, Kiran-Serra PR 2013]

$$\pi_1(S) \leq \pi_2(S)$$



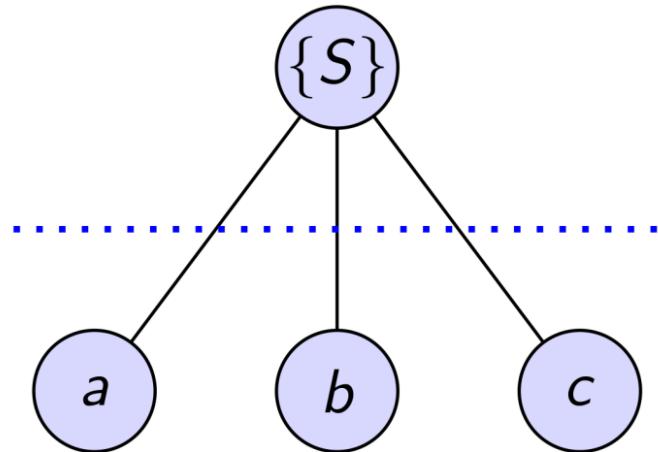
$$\pi_1(S) \sqcup \pi_0 \leq \pi_2(S) \sqcup \pi_0$$



$$\omega(\pi_1(S)) \leq \omega(\pi_2(S)) \Rightarrow \omega(\pi_1(S) \sqcup \pi_0) \leq \omega(\pi_2(S) \sqcup \pi_0)$$

\sqcup : disjoint union to concatenate partial partitions during DP

Generalized Minkowski composition



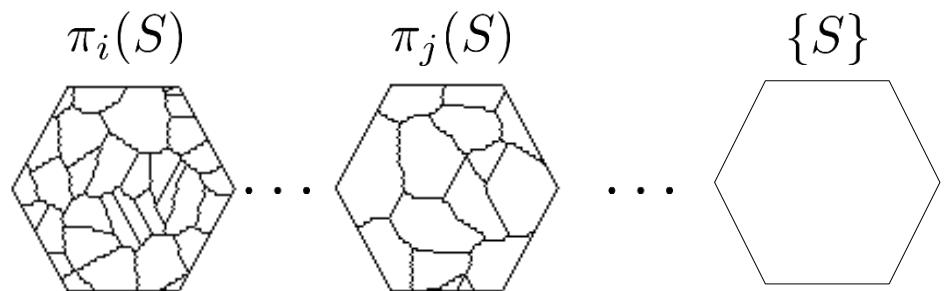
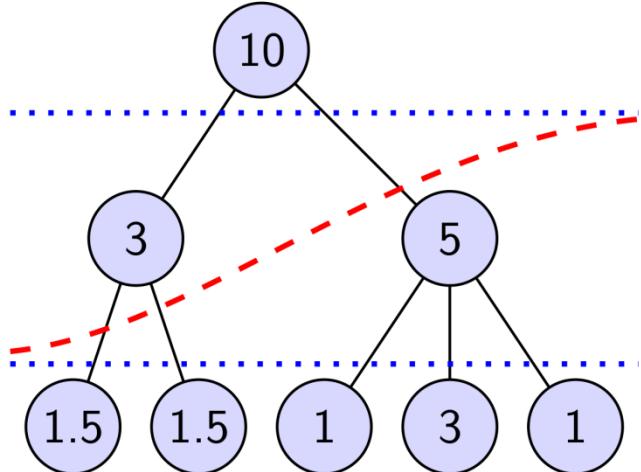
$$\omega(\pi(S), \alpha) = \left[\sum_{\text{children}} \omega(u)^\alpha \right]^{\frac{1}{\alpha}}$$

α	$\omega(T_i)$ Composition Law
$-\infty$	infimum
-1	harmonic sum
0	number of classes
$+1$	sum
$+2$	quadratic sum
$+\infty$	supremum

- Additive: [Breiman et al. 1984, Salembier-Garrido 2000, Guigues 2003]
- Supremum & Dominant Ancestor: [Akcay-Akcoy 2008, Soille 2008, Valero 2011, Veganzones-Chanussot 2014]
- Minkowski norm generalization: [Salembier 2015]
- Max-pooling type, alternating compositions

Uniqueness and Singular Energy

[Kiran-Serra PR 2013]

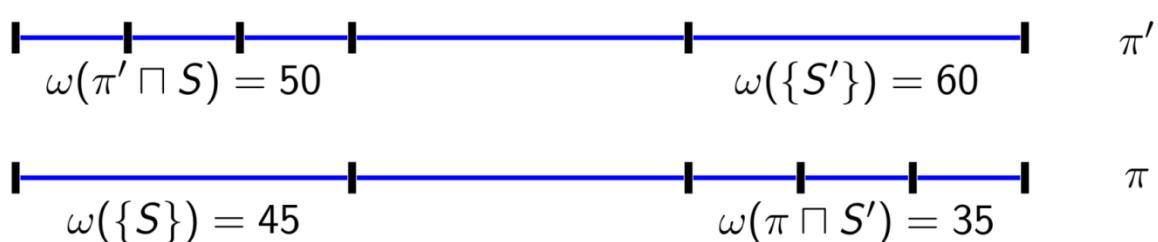


$$\omega(\{S\}) \neq \omega(\pi_i(S)), \forall i \in [1, n]$$

Various authors indirectly use the singularity condition for a unique solution.



$\pi \vee \pi'$ Energetic Lattice



$$\pi \preceq_{\omega} \pi' \Leftrightarrow \forall S \in \pi \vee \pi' \text{ we have } \omega(\pi \cap \{S\}) \leq \omega(\pi' \cap \{S\})$$

- The energetic lattice ($\preceq_{\omega}, \vee_{\omega}$) derives from the energetic order.
- Existence of unique solution when ω singular.
- local minimum \implies global minimum.
- Given H, ω we have an energetic lattice iff ω is singular. h -increasingess

Optimal λ -Cut Example

luminance fidelity term

$$\omega_\varphi(T) = \int_T \|l(x) - \mu(T)\|^2 dx$$

chrominance fidelity term

$$\omega_\varphi(T) = \sum_i \int_T \|c_i(x) - \mu_i(T)\|^2 dx$$

Contour length

$$\omega_\partial(T_k) = \partial T_k$$

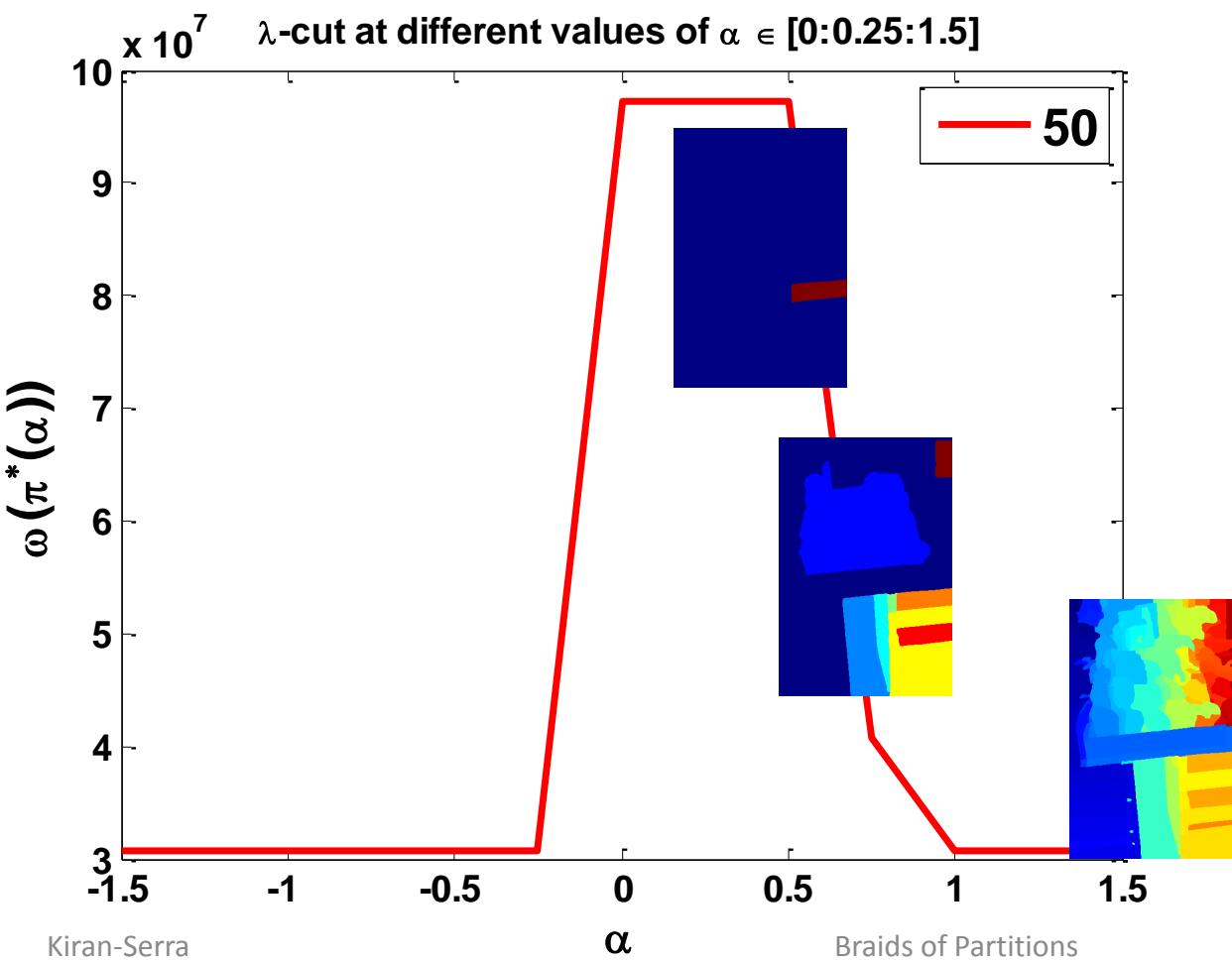


$$\omega(\pi(S), \lambda) = \sum_k \omega_\varphi(T_k) + \lambda \sum_k \omega_\partial(T_k)$$

λ fixed to have partition with same coding cost



Optimal α -Cuts

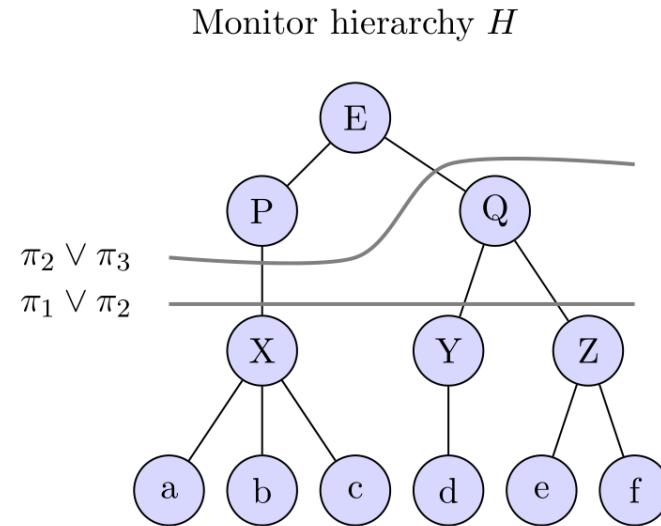
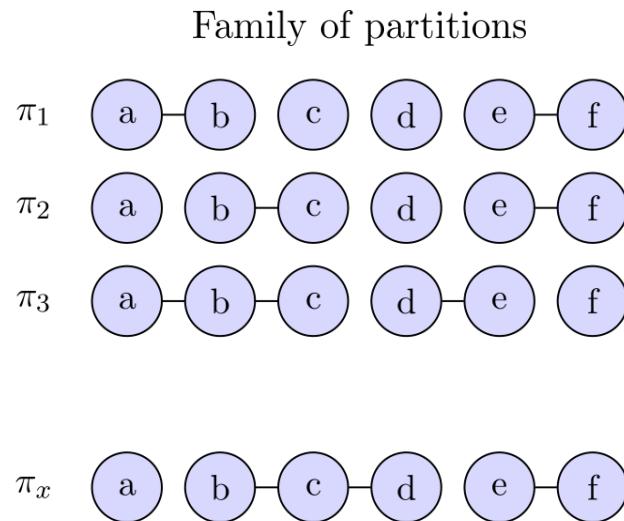


30-level UCM hierarchy

Braids of Partitions

$$\forall \pi_1, \pi_2 \in B \Rightarrow \pi_1 \vee \pi_2 \in \Pi(H, E) \setminus \{E\}$$

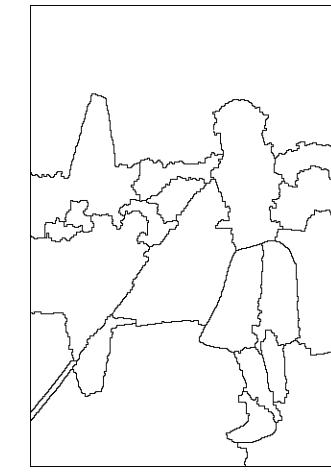
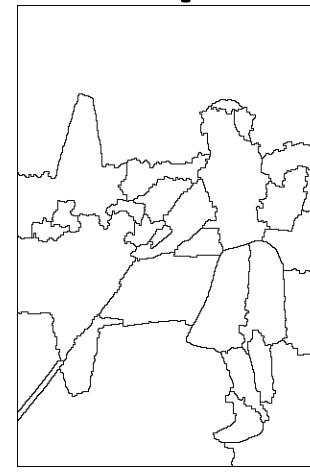
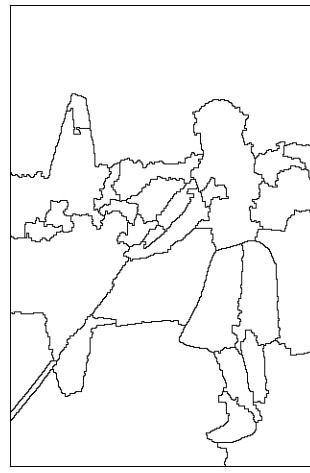
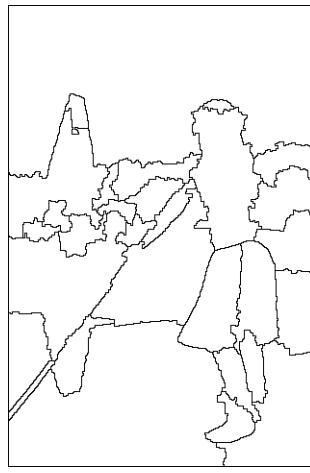
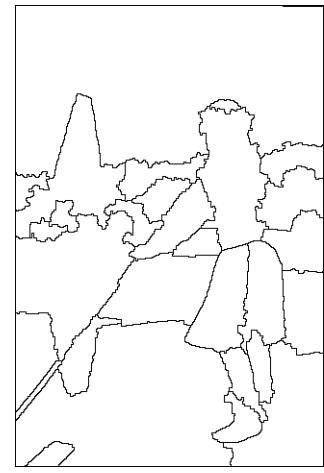
Pair-wise Refinement supremum are hierarchical



$$B_1 = \{\pi_1, \pi_2, \pi_3\} \checkmark \quad B_2 = B_1 \cup \pi_x \times$$

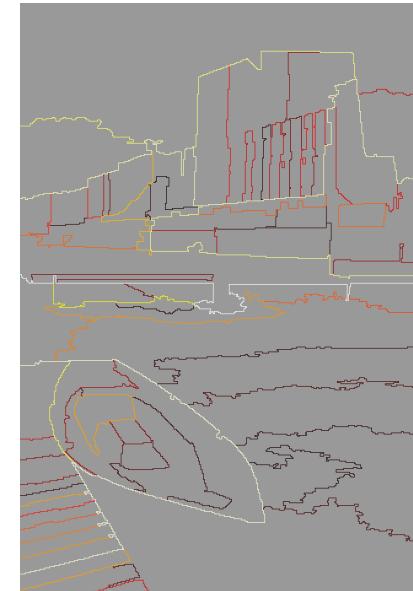
H is called the monitoring hierarchy

Braids Examples



Stochastic Watershed [Angulo et al. 2007] forms a Braid

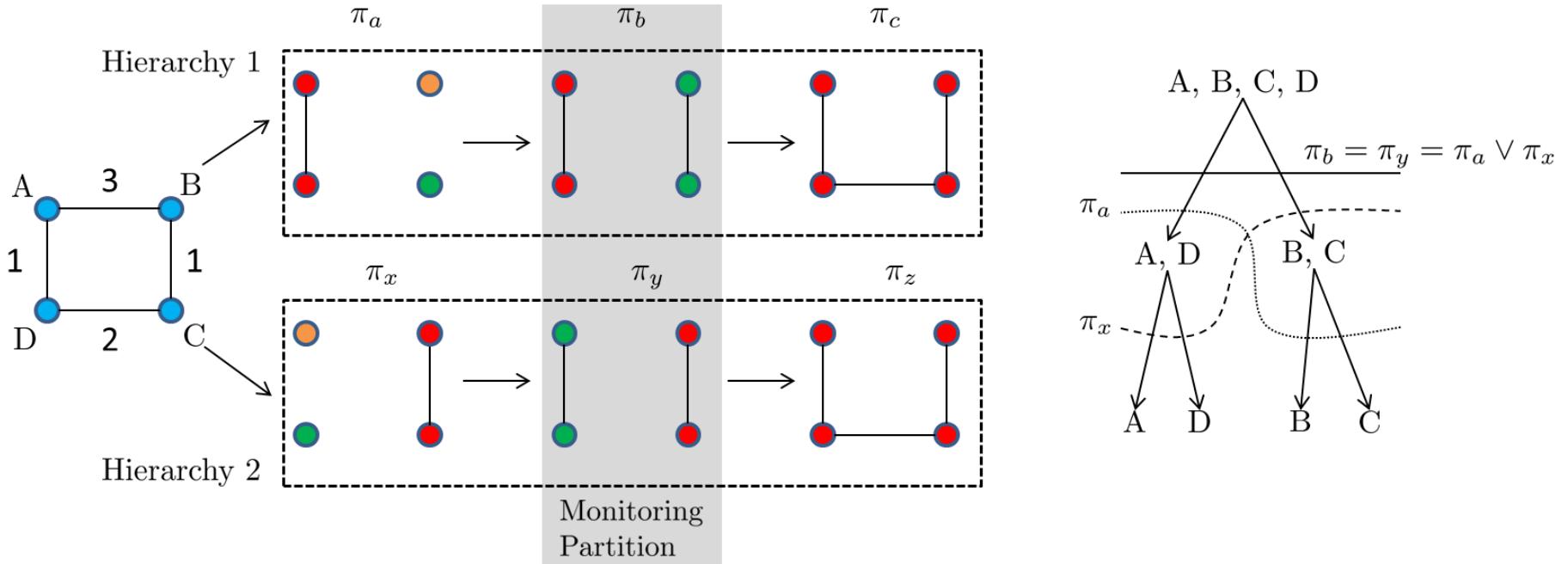
Monitoring Partition Input Image



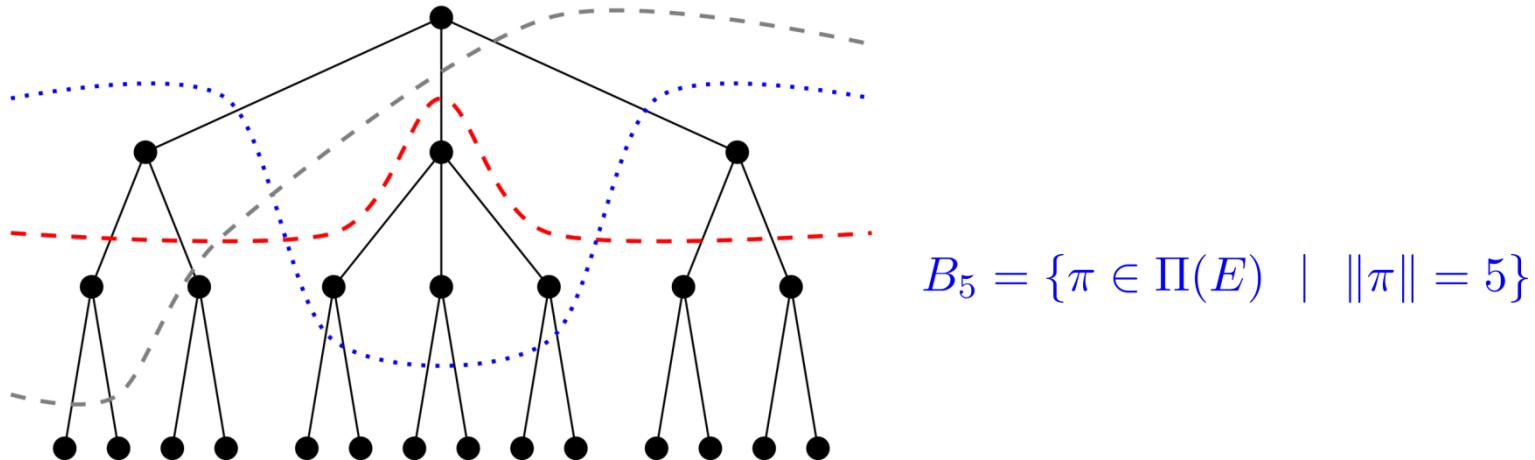
Kiran-Serra
Input Image

Area & Volume Attribute Hierarchies
[Cousby et al. 2011]

Braids of Partitions
Monitor Hierarchy



Minimum Spanning Tree(MST) braid



Kiran-Serra
Feasible family of partitions in Constrained Optimization problem
Braids of Partitions

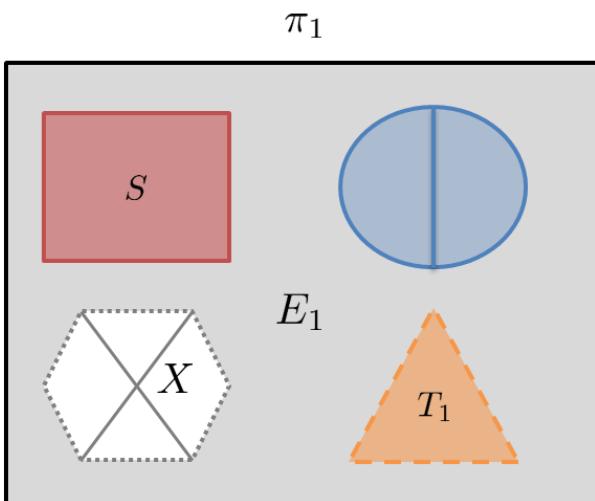
Why Braids

Uncertain boundaries \implies multiple partial partitions.

Multivariate energy minimization.

Accomodates variability in Human & Machine segmentations

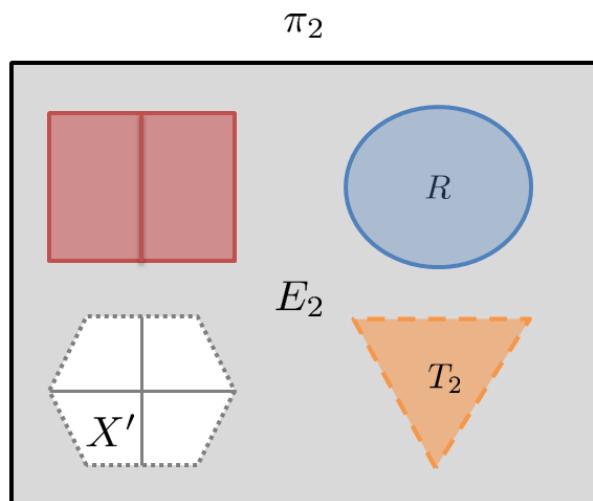
Better DP infimum & compatible with Energetic Lattice.



$\boxed{\text{Red}} \quad \pi_1(S) \leq \pi_2(S)$

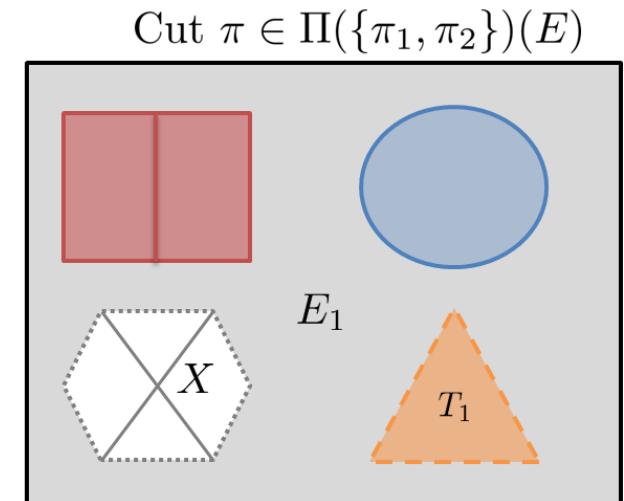
$\boxed{\text{Blue}} \quad \pi_1(R) \leq \pi_2(R)$

Refinements



$$\otimes \vee \oplus = \circlearrowleft$$

Braid structure



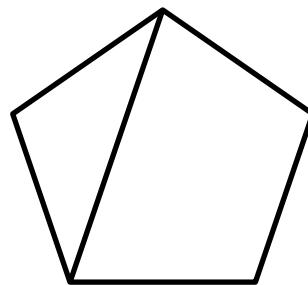
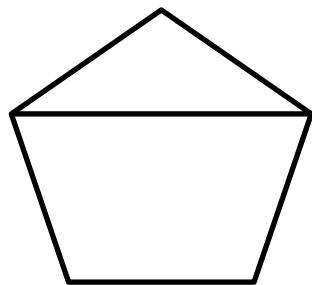
Noise or Overlap
 $T_1 \not\subseteq T_2$ nor $T_2 \not\subseteq T_1$

h -increasingness on BOP

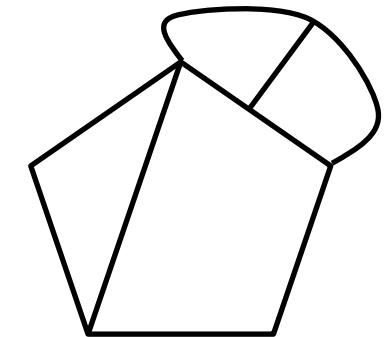
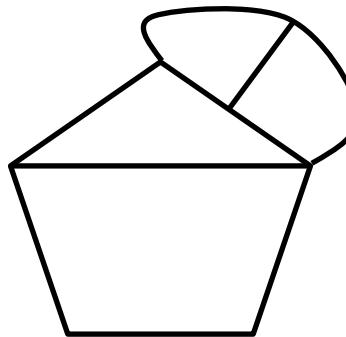
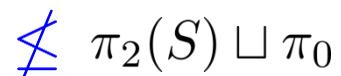
$$\pi_1(S)$$



$$\pi_2(S)$$



$$\pi_1(S) \sqcup \pi_0$$



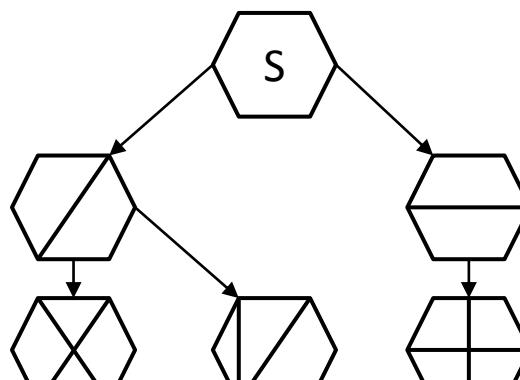
$$\omega(\pi_1(S)) \leq \omega(\pi_2(S))$$



$$\omega(\pi_1(S) \sqcup \pi_0) \leq \omega(\pi_2(S) \sqcup \pi_0)$$



$$\pi_1^*(S)$$



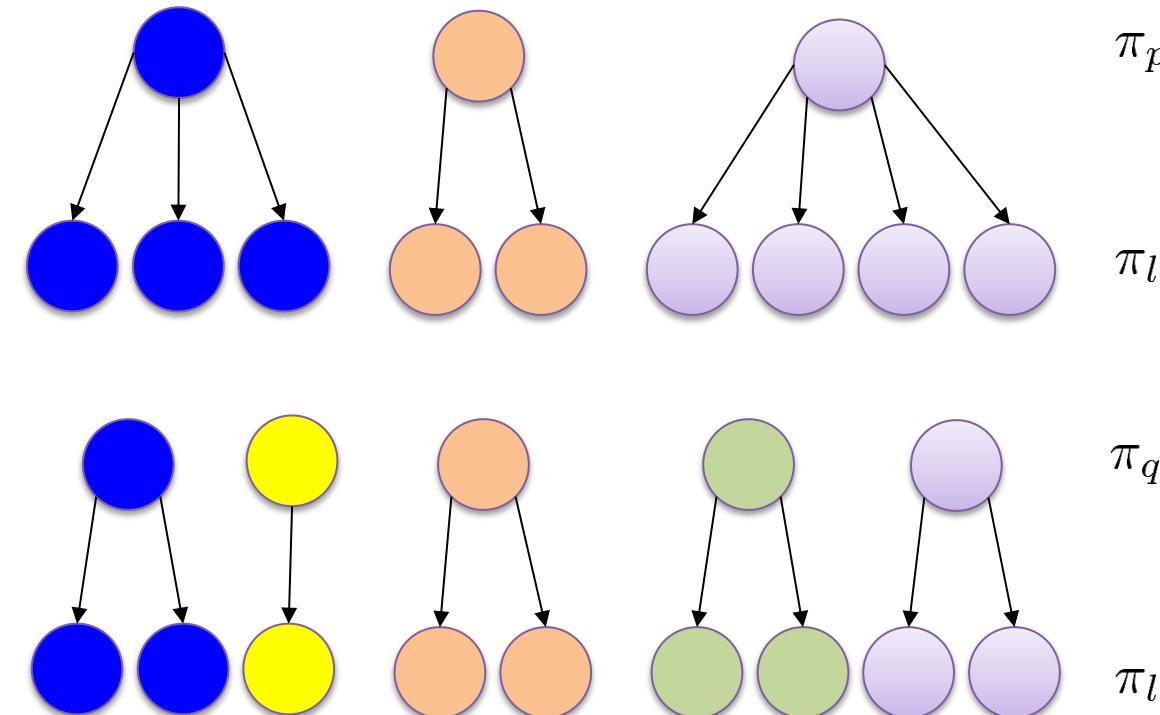
$$\pi_2^*(S)$$

Braids of Partitions

[Ravi Kiran PhD Thesis 2014]
20

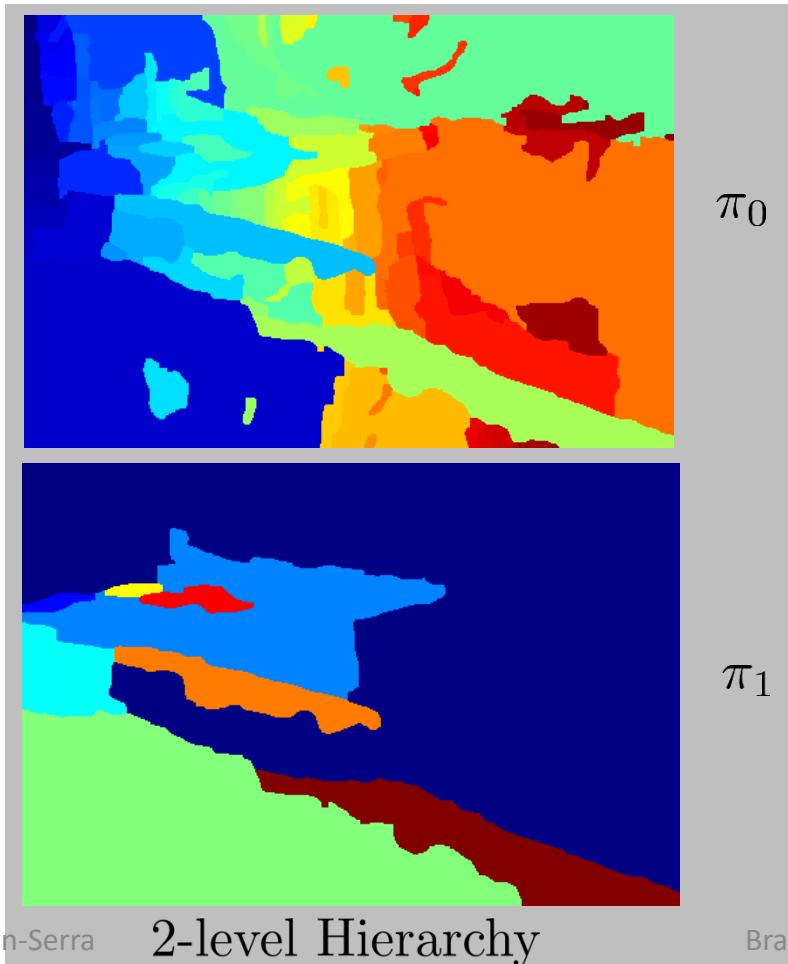
Searching better optimum

Random re-compositions on hierarchies

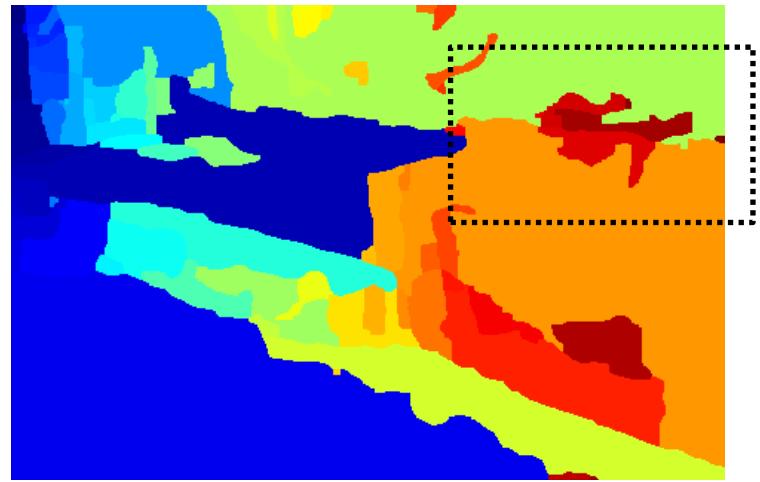


```
For  $\pi_i \in H$ 
  For  $S \in \pi_i$ 
    if( $\omega(\pi_q(S)) < \pi_p^*(S)$ )
       $\pi^*(S) = \pi_q(S)$ 
    else
      iterate
```

For every class of parent partition π_p we regroup children in π_l at random to search possible finer parent level π_q



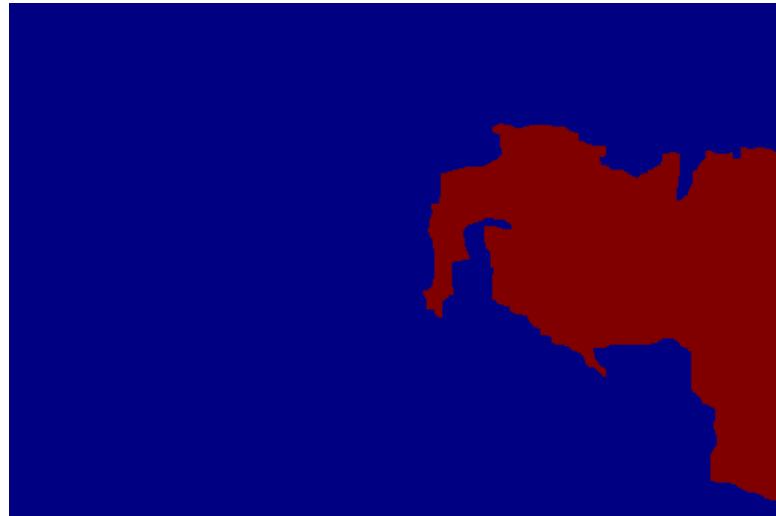
Optimal cut from hierarchy $\pi^*(\lambda = 550)$
84 classes $\omega(\pi^*(\lambda = 550)) = 20313749.3404$



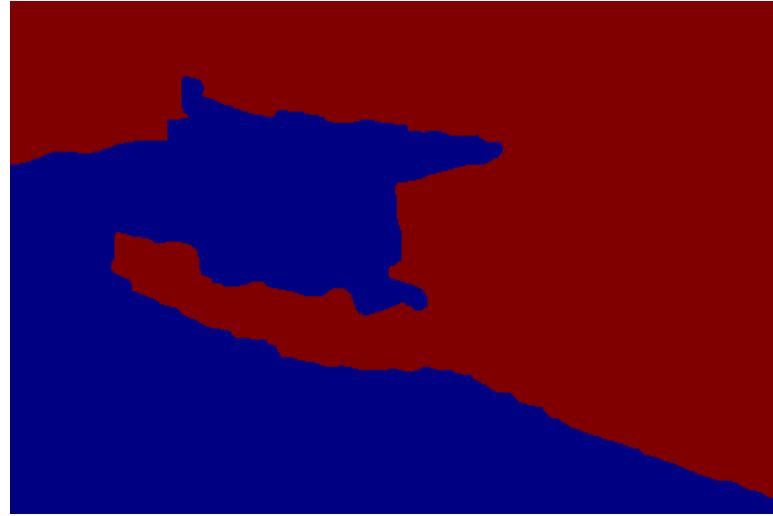
(Perturbed Hierarchy forms a braid)
Optimal Cut from Braid $\pi_p^*(\lambda = 550)$
74 classes $\omega(\pi_p^*(\lambda = 550)) = 19672933.8262$



Class in partition $\pi_0 \in H$

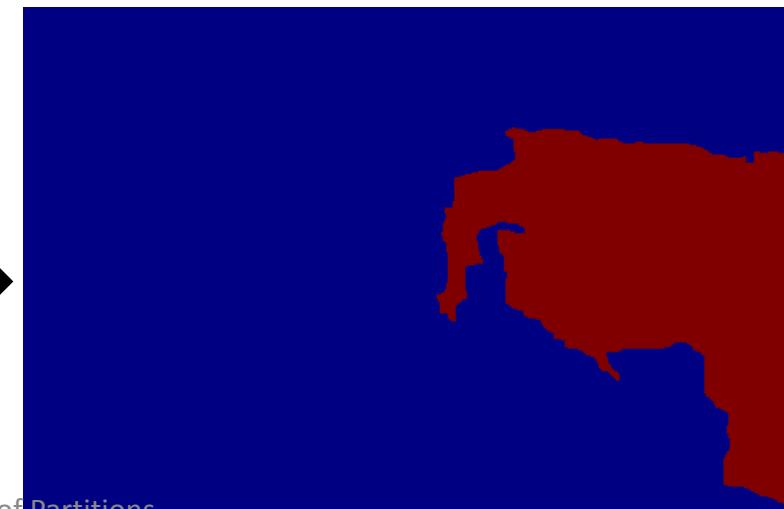
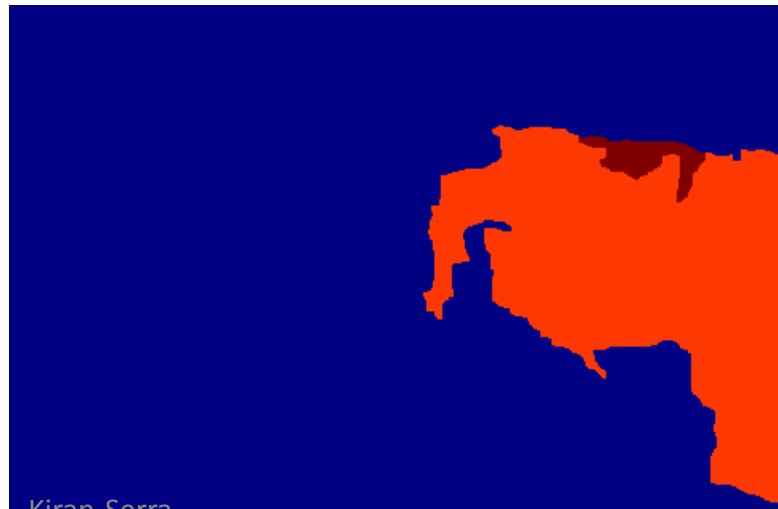


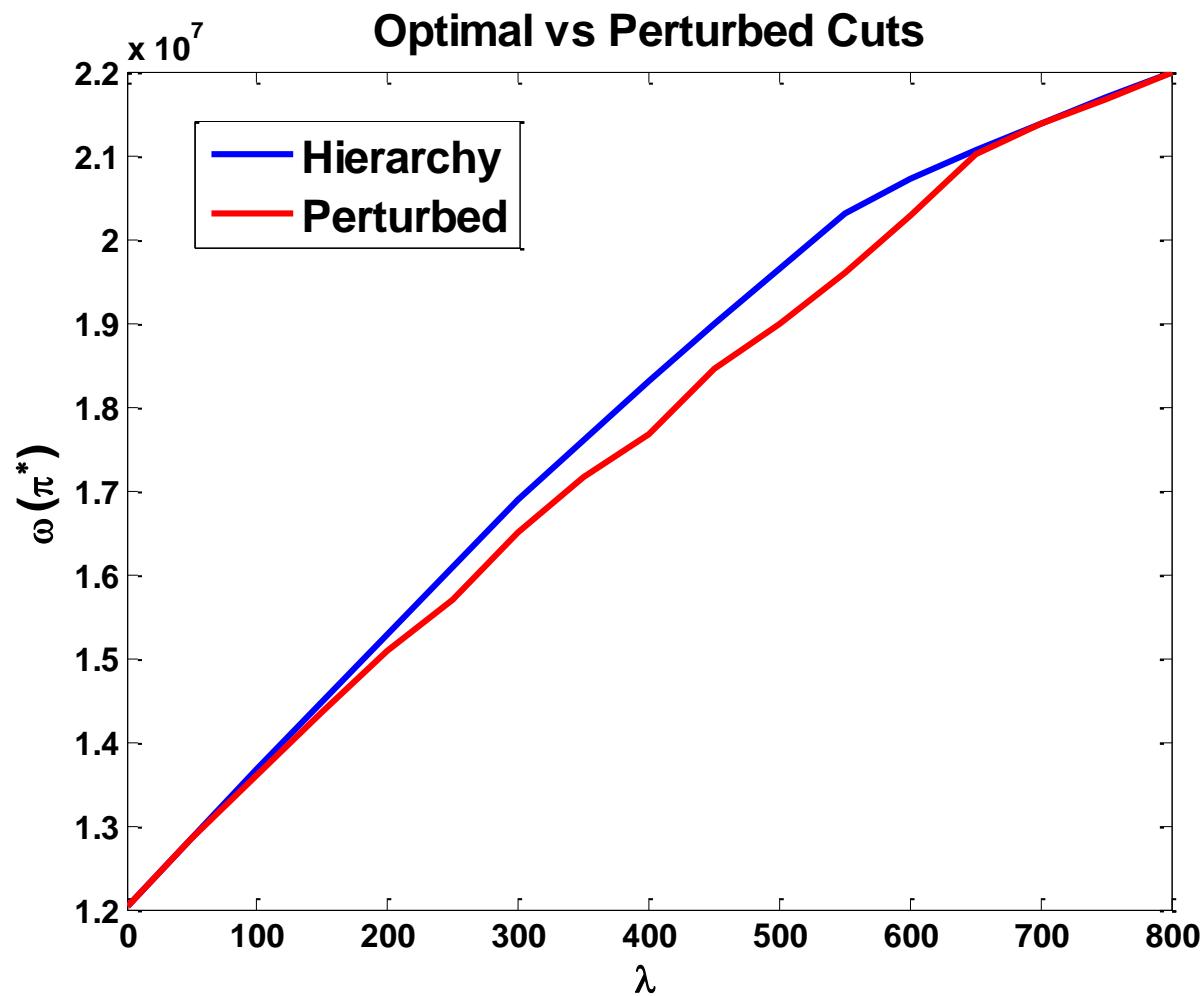
Class in partition $\pi_1 \in H$



Two classes chosen at random during perturbation step
not grouped into single parent in H

Produces better partial partition and lower energy





Observations

Observation

Perturbation and search is λ -dependent.

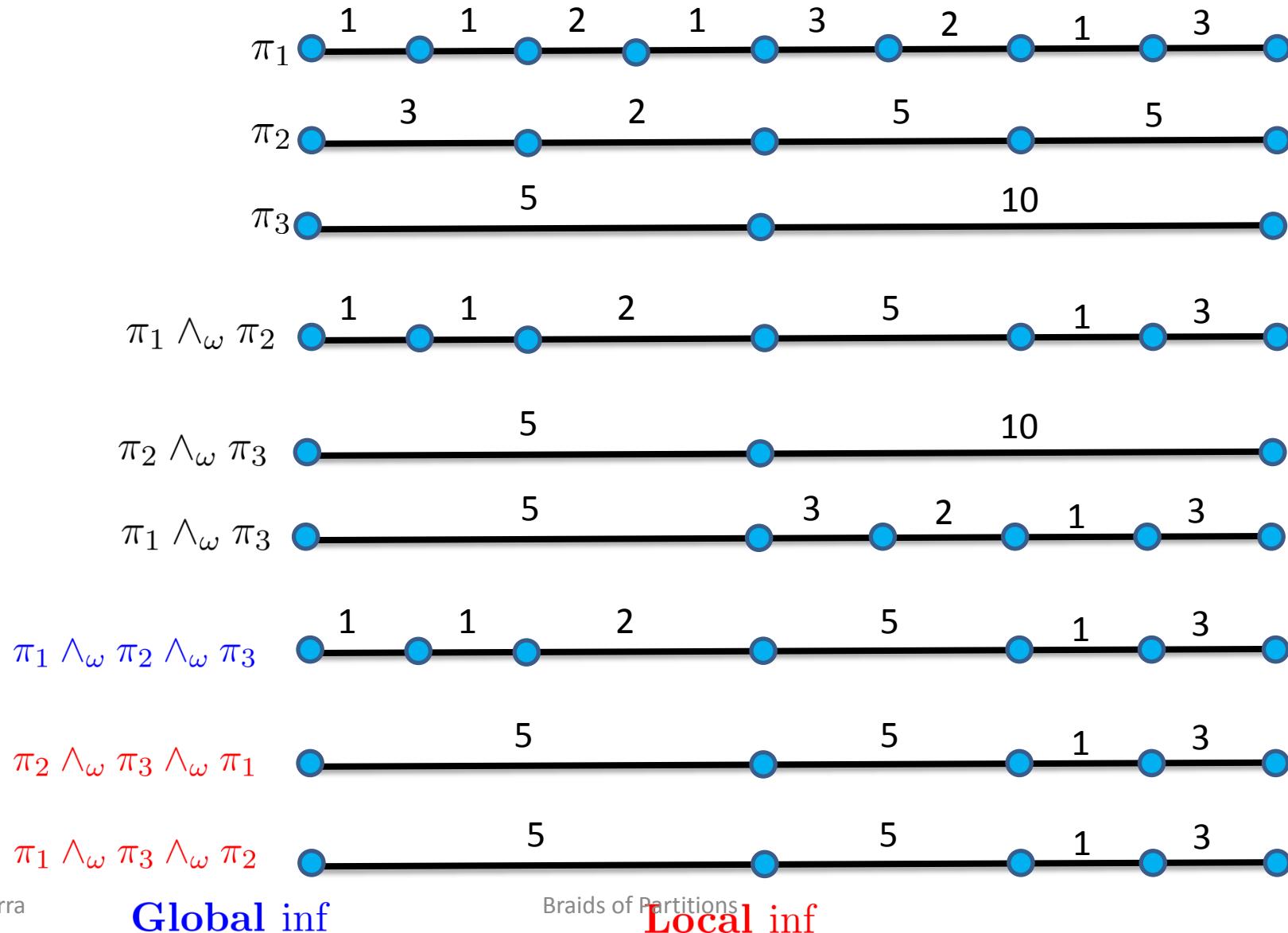
Locally random perturbations. (More evolved versions possible)

Refinement order respected.

Conclusion

- DP-substructure preserved by h -increasing energies and braids.
- Braids are the largest family for which the energetic lattice structure holds.
- Flexibility to incorporate multiple hypothesis in segmentation.
- Many ways a braid can be built, as for a hierarchy.
- Global infimum requires partial refinement order information
- Improvement of minimum by regrouping children randomly.

Local vs global Infimum



Other Plots

