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Braids of Partitions

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Abstract. In obtaining a tractable solution to the problem of extracting a minimal partition from hierarchy or tree by dynamic programming, we introduce the braids of partition and h-increasing energies, the former extending the solution space from a hierarchy to a larger set, the latter describing the family of energies, for which one can obtain the solution by a dynamic programming. We also provide the singularity condition for the existence of unique solution, leading to the definition of the energetic lattice. The paper also identifies various possible braids in literature and how this structure relaxes the segmentation problem.

Keywords: Hierarchies, Dynamic Programming, Optimization, Lattice

1 Introduction

Hierarchical segmentation methods have been an important tool in providing a simplification of the image domain, following which various operations of filtering, segmentation and labeling become simpler structured problems on hierarchies of partitions (HOP), in comparison to the whole space. These problems are often formulated as optimization problems, where the space of solutions are partitions from a hierarchy. Breiman et al. [4] performed decision tree pruning to obtain a tree-classifier with least complexity to avoid overfitting, which corresponds to a pruning with minimal energy from a tree. This was first extended for the image segmentation problem, by Salembier-Garrido [13], where they calculated an optimal pruning of a binary partition tree by performing a gradient search over the Lagrange multiplier. Further on Guigues [8] introduced the scale-set descriptor, which operates on an input hierarchy of segmentations on an input image, and a parametrized energy. The scale sets are a hierarchy of minimal cuts corresponding to a given Lagrange multiplier. These methods use the Breiman Dynamic programming approach to perform pruning or extract the optimal cut. Further on Guigues provides conditions of sub-additivity of constraint function and super-additivity of objective function as conditions for finding a globally unique optimal cut, which was generalized in [9] to h-increasing energies. The λ-cut or scale-set [8] produces a descriptor based on any input hierarchy and a parametrized energy like Mumford-Shah. The attribute watersheds [6] work on the attributes of volume, area, dynamic of the component-tree hierarchy.[3]

In this paper firstly introduce a new family of partitions larger than the hierarchy over which the dynamic program is still valid, namely the braids of partitions (BOP). Further on we extend the property of h-increasingness to the braids, and as well prove the energetic lattice and ordering relation over braids.

2 Braids of partitions (BOP)

We now consider the problem of construction of other families, which no longer form chains, while they share hierarchical properties, and expand the search space for the optimization problem. We propose the braid, which on one hand provides a richer hierarchical model enabling multiple segmentations of a given region of the image domain, while on the other remains in conformance with the dynamic program substructure.

2.1 Definitions

A partition π of space E is a set of subsets of E that are pairwise disjoint and whose union reconstitutes E. A partial partition [12] of support S denoted as $\pi(S)$, is a partition of the subset $S \subset E$. The family of all partitions are denoted by $\Pi(E)$ and that of partial partitions as $\mathcal{D}(E)$. A hierarchy of partitions (HOP) is a chain of partitions $H = \{\pi_i, i \in [0, n]\}$, where $\pi_i \leq \pi_j, i < j$, where $\leq$ denotes refinement ordering. The minimal element π_0 of H is the called leaves partition which contains a finite number of elements. A cut is a partition composed of classes from a hierarchy (or more generally any family of partitions). The cuts of H are denoted by $\Pi(E, H)$.

An energy is a non-negative function on the family of partial partitions, $\omega : \mathcal{D} \to \mathbb{R}^+$. The energy of a partition or partial partition is usually obtained by the composition product, $\text{comp}(\cdot)$ of energies, by addition, supremum or other laws, over its constituent classes, e.g. $\omega(\pi(S)) = \sum_{a \sqsubset \pi(S)} \omega(a)$. Now the optimal cut in [4], [13], [8], is calculated by aggregating local optima. The local optimum at class S either choses the parent $\{S\}$, or the disjoint union of the optimums over the its children as shown in equation 1.

$$\pi^*(S) = \begin{cases} \{S\}, & \text{if } \omega(S) \leq \text{comp}(\omega(\pi^*(a))), a \in \pi(S) \\ \bigsqcup_{a \in \pi(S)} \pi^*(a), & \text{otherwise} \end{cases} \tag{1}$$

2.2 Braids

A braid is a family of partitions B, where the pairwise refinement supremum of any two elements is a cut of in some hierarchy $\Pi(E, H)$. This leads to the more formal definition:

[3] This work was partly funded by ANR-2010-BLAN-0205-03 program KIDICO.

Definition 1. *Let $\Pi(E)$ be the complete lattice of all partitions of set E; let H be a hierarchy in $\Pi(E)$. A braid B of monitor H is a family in $\Pi(E)$ where the refinement supremum of any pair of distinct partitions $\pi_1, \pi_2 \in B$ is a cut of H, other than $\{E\}$, that is in, $\Pi(E, H) \setminus \{E\}$:*

$$\forall \pi_1, \pi_2 \in B \quad \Rightarrow \pi_1 \vee \pi_2 \in \Pi(E, H) \setminus \{E\} \tag{2}$$

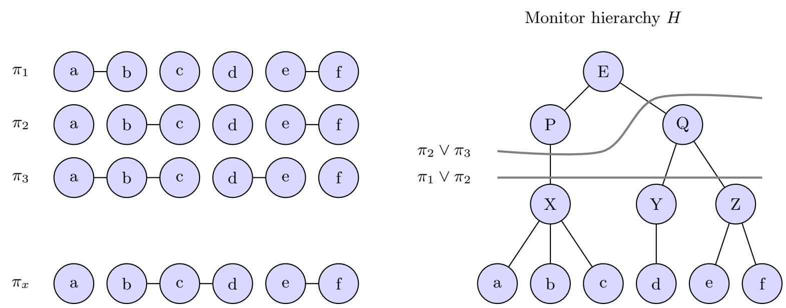

Fig. 1: Space E is partitioned into leaves $\{a, b, c, d, e, f\}$. The family $B_1 = \{\pi_1, \pi_2, \pi_3\}$ forms a braid, whose pairwise supremum is indicated on the dendrogram. Note that $\pi_1(X), \pi_2(X)$ have a common parent X, but $\pi_2(Q), \pi_3(Q)$ a common grand parent Q. However the family $\pi_x \cup B_1$ is not a braid since $\pi_3 \vee \pi_x$ gives the whole space E.

Given three partitions π_1, π_2, π_3 then the classes of suprema partitions $\pi_1 \vee \pi_2, \pi_1 \vee \pi_3$ are nested or disjoint. A braid can posses multiple monitoring hierarchies. One thus still has a scale selection to perform in the context of choosing a monitor hierarchy for a given application.

In Figure 1 we demonstrate a simple example of a braid family with the dendrogram corresponding to its monitor hierarchy. As we can see the classes of partitions π_1, π_2 are neither nested nor disjoint, and basically correspond to different segmentation hypotheses that exist in the stack of segmentations. The set of all cuts of a braid B is denoted by $\Pi(E, B)$. A braid may also contain its monitor H, though this is not necessary. On the other hand, any hierarchy is a braid with itself as monitor. A braid cannot be represented by a single saliency function, except when it reduces to a hierarchy whose classes are connected sets.

The partition with one class $\{E\}$ is not considered in Definition 2, since this would imply that any family of arbitrary partitions would form a braid, with $\{E\}$ as supremum, thus losing any useful structure. In case of a hierarchy the cone or family of classes containing a point $x \in E$ can only be nested or disjoint. While the cone of classes in the BOP, that contain a single point, are not necessarily nested, though their suprema are. This provides the local-global substructure for the dynamic program.

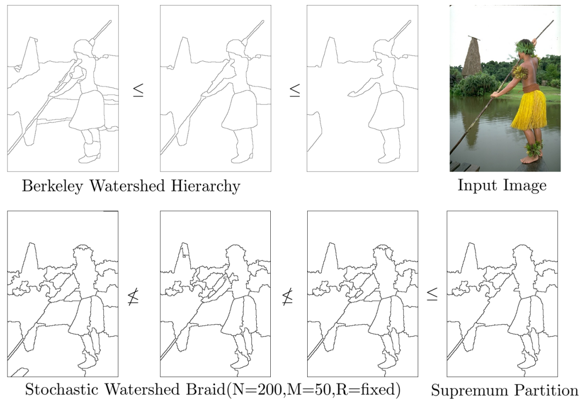

Fig. 2: HOP vs BOP: Ultrametric contour map (UCM [3]), hierarchy (top) and a braid of partitions (bottom). Braids of partitions were produced from multiple instances of random marker based stochastic watershed, with same number of regions. The supremum or monitoring partition, corresponding to these unordered family of partitions is shown. Braids help reorganize partial refinement between partitions.

2.3 Underlying questions

In the process of trying to create a structure where the dynamic program substructure holds, we are in fact posing the following sequence of questions. Given a general set of partitions $B = \{\pi_i\}, i \in \{1, 2, 3, ...n\}$: Firstly, how the partial optimum between any two partitions with a non-trivial supremum is calculated ? Over what support are partial partitions compared ? Secondly and more profoundly, given that there are cuts extractable other than these n-partitions, how does one index these different cuts. What are the types of ordering relations observable between any two partitions with a non-trivial supremum ? Furthermore it would also be useful understand the combinatorial nature by calculating the number of optimal cuts can one extract.

When B is a hierarchy, any two partitions are ordered by refinement, i.e. $\pi_i \leq \pi_j$ or $\pi_j \leq \pi_i, \forall i, j, \in \{1, 2, 3, ...n\}$. We now observe the possible ordering relations possible between pairs of partial partitions over a supports from their supremum $S \in \pi_1 \vee \pi_2$, we have: either parent-child/child parent $\pi_i \sqcap S \leq \pi_j \sqcap S$ or a braid structure $\pi_i \sqcap S \neq \pi_j \sqcap S$, though here one must maintain a nested or disjoint supremum S to ensure a local ordering to follow. Given two partitions, we can observe various local ordering between classes. This is discussed and demonstrated in an illustrative example in Figure 3.

There are two problems that are related but that very different in algorithmic complexity when dealing with braids: 1. Generating general braid of partitions and 2. Validating that a given general family of partitions is a braid. In both questions the underlying problem to evaluate is the order of refinement between the partitions. To generate braids one needs to fix some how this choice of partial

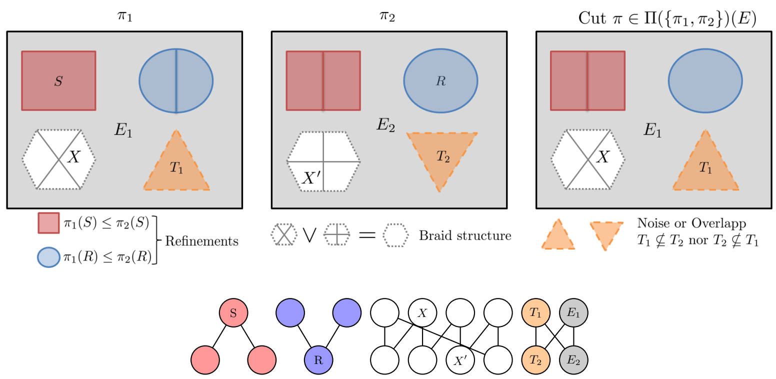

Fig. 3: We show two partitions π_1, π_2 demonstrating four ordering relations: parent→child and child→parent relation (red, blue), a p.p.→p.p. braid structure (hexagon), and finally overlapping classes that aren't inclusions (orange). One can note that once we have a refinement relation between two partitions locally, as in case of classes R, S, this implies that the remaining pairs of classes are either equal, ordered themselves or are partial partitions forming a partial braid structure since they share a common supremum. We also show the intersection graph produced by connecting regions with non-void overlaps to visualize the different ordering relations. The classes corresponding to the components of the intersection graph, gives the supremum of the two partitions. We also show a cut extracted from π_1, π_2.

order, while in case of validation one needs to verify this property of ordering of supremum as evoked in the braids definition in equation (2). We also can easily note that question (2) is a combinatorial problem since partial order across pairs of partitions need to be validated. While question (1) is simpler. We shall use the stochastic watershed model [2] here to demonstrate how one can control the partial order in generating a braid. Though the generation of braids can be done using a variety of methods. Another simple way to generate a braid would be to fragment/regroup differently an already existing hierarchy of partitions. The disadvantage is that here one fixes the monitoring hierarchy.

2.4 Motivation and finding Braids in literature

The need for such models arises in several situations. Firstly we observe that many super-pixel segmentation algorithms, and also multivariate segmentation algorithms [17], [18], operate on agglomerative clustering and region merging. In the former case we obtain a quick super-pixel segmentation by using the clustering tree, while in the latter case we compose partitions of the image domain based on different components of a vectorial image. In a paper close to our work, [5] models the image segmentation problem as the extraction of maximally weighted independent set (MWIS) on the intersection graph. This graph is built over the regions of segmentations produced using various super-pixel low level segmentations. They further associated an energy with each region or node. The algorithm of MWIS consists in calculating the MWIS by dynamic programming. There are two differences between this paper and [5]: Firstly, the segmentations used in [5] do not ensure a stable pairwise supremum, resulting in holes or overlaps. Secondly the intersection graph is blind to the the partial ordering relation between partitions. We demonstrate a counter example in figure 7, where we show different refinement orders, and how they break the dynamic program substructure. Thus following the refinement order during the DP is necessary, when one calculates the optimal cut that is at the energetic infimum.

Furthermore in optimization frameworks such as the MRF, one also notes that in forcing uniqueness, certain solution spaces are excluded. In [15], one considers the K-best solutions i.e. a local segmentation hypothesis. It is well know in segmentation evaluation that one encounters variation in partition boundaries, as a result of mainly different algorithmic parametrization and subjectivity in human expert annotations [16]. Braids enable the comparison between regions of machine segmentations and ground truth partitions (see Figure 3), which are neither purely refinements nor non-void intersections. It has already been well studied that the "segmentation soup" (family of partitions generated from across different algorithms and parameterizations) provided a better support for object detection [11].

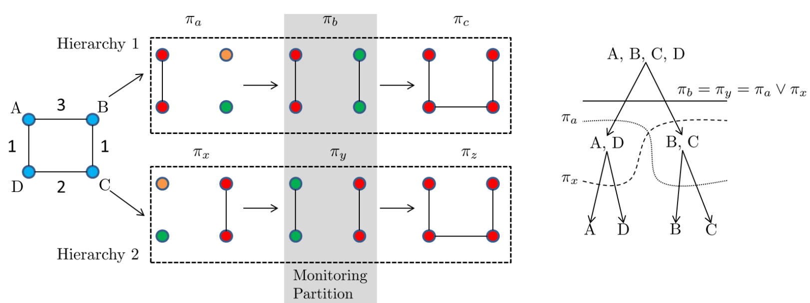

Fig. 4: There can be multiple minimum spanning trees (MSTs) for a given edge-weighted graph [15]. Figure shows a planar weighted graph with two different possible choices in selecting the lowest weighted edge in prim's algorithm. This leads to two different partitions of the nodes set as extracted by the components of the graph. This gives two different hierarchies that can be extracted. The supremum or monitoring partition is created when the second edge is added. Here partition $\{A, B, C, D\}$ monitors over partitions $\{\{A, D\}, \{B\}, \{C\}\}$ and $\{\{B, C\}, \{A\}, \{D\}\}$. We have demonstrated here how the the distinct MST enumeration can used to generated a braid.

We describe shortly the braids found in literature. Angulo et al. [2] accumulate watersheds of stochastically sampled markers chosen from the image domain. This produces an estimate the density function of gradient of the image. The set of partitions produced during the iterations of the stochastic watershed algorithm form a braid structure. This essentially corresponds to a random marker based watershed extracted from the minimum spanning tree. An example is demonstrated in Figure 2. K-Smallest Spanning Tree Segmentations [15] propose multiple distinct segmentations of the image by considering the K-smallest distinct minimum spanning trees. It can be shown easily that the degenerate set of weighted edges with equal weights when permuted over in Prim's algorithm produce different segmentations, which by definition have a common supremum, defined by the heaviest weighted edge governing the degenerate lower weighted edges. This is demonstrated in Figure 4. In a similar line, one can also demonstrate that the attribute watersheds [6] based on area, volume and dynamic, together produce a braid structure with volume hierarchies usually monitoring the other two [10]. Particular versions of braids have appeared in classification problems, for example Diday [7], demonstrates pyramids, where a child may have two parents.

3 Dynamic Programming and h-increasingness

h-increasingness is a property of energies, which preserves the optimal substructure in extracting the minimal cut so that one can use a dynamic program to solve it. It states that the ordering of energies is preserved under concatenation of partial partitions (Figure 5).

Definition 2. *(h-increasingness) Let $\pi_1(S)$, $\pi_2(S)$ be two different p.p. of the same support $S \in E$, be a family of disjoint supports over E. Let π_0 be any partial partition in $\mathcal{D}$ other than $\pi_1(S), \pi_2(S)$. A finite singular energy ω on the partial partitions $\mathcal{D}(E)$ is h-increasing when for every triplet $\{\pi_1(S), \pi_2(S), \pi_0\}$ one has:*

$$\omega(\pi_1(S)) \leq \omega(\pi_2(S)) \Rightarrow \omega(\pi_1(S) \sqcup \pi_0) \leq \omega(\pi_2(S) \sqcup \pi_0) \tag{3}$$

In implication (3) when the inequality is made strict, we have what we call strict h-increasingness. h-increasingness was first introduced in [9], which generalized the condition of *separable energies* of Guigues [8]. Separability in equation (1), is obtained by replacing comp($\cdot$) by a sum of the energies of the constituent classes of a partial partition, to calculate the energy of the partial partition. We can also perform a composition by supremum [14], [17].

Both laws are indeed particular cases of the classical Minkowski expression

$$\omega(\pi(S)) = \left[\sum_{u=1}^{q} \omega(T_u)^{\alpha} \right]^{\frac{1}{\alpha}} \tag{4}$$

which is a norm in $\mathbb{R}^n$ for $\alpha \geq 1$. Even though over partial partitions $\mathcal{D}(E)$, it is no longer a norm, it yields strictly h-increasing energies for all $\alpha \in]-\infty, +\infty[$:

Proposition 1. *Let $E \in \mathcal{P}(E)$, let $\omega : P(E) \to \mathbb{R}$ be a positive or negative energy defined on $\mathcal{P}(E)$. Then the extension of ω to the partial partitions $\mathcal{D}(E)$ by means of Relation (4) is strictly h-increasing.*

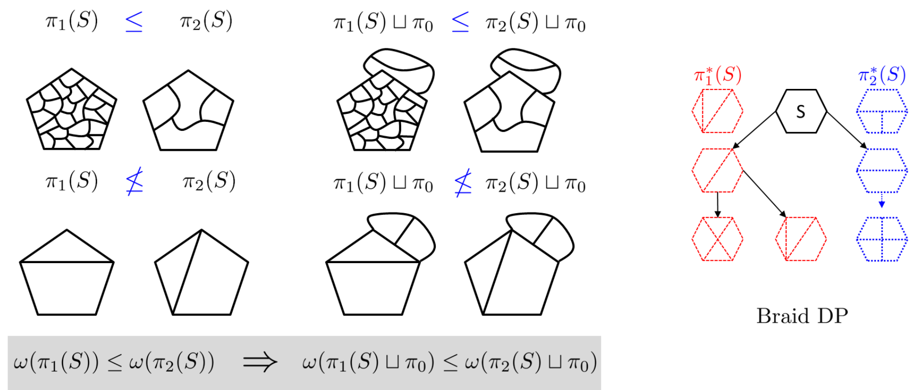

Fig. 5: Left: h-increasingness for HOP (top) versus BOP (bottom). Right: An elementary step of the dynamic program on a partial braid over a support S. The optimal partial partition is the minimum across the partial optima and S, i.e. $\omega(S), \omega(\pi_1^*(S)), \omega(\pi_2^*(S))$.

Proof. Let $\pi(S)$ $\pi'(S)$ be two p.p. of support S, with q, q' elements each, respectively. When $0 \leq \alpha < \infty$, the mapping $y = x^\alpha$ on $\mathbb{R}^+$ is strictly increasing and, according to Relation (6), the inequality $\omega(\pi(S)) < \omega(\pi'(S))$ implies

$$\sum_1^q [\omega(T_u)]^\alpha < \sum_1^{q'} [\omega(T'_u)]^\alpha \implies \sum_1^q [\omega(T_u)]^\alpha + [\omega(\pi_0)]^\alpha < \sum_1^{q'} [\omega(T'_u)]^\alpha + [\omega(\pi_0)]^\alpha \tag{5}$$

hence $\omega(\pi_1 \sqcup \pi_0) < \omega(\pi_2 \sqcup \pi_0)$. When $\alpha \leq 0$, the sense of the inequality changes on both sides of implication in (5) but changes again when applying the $(\cdot)^\alpha$. This again leads to $\omega(\pi_1 \sqcup \pi_0) < \omega(\pi_2 \sqcup \pi_0)$, and achieves the proof. □

One can easily check that the proposition remains true when $\omega : P(E) \to \mathbb{R}^-$ is a negative energy. For $\alpha = +\infty$ (resp.$-\infty$), Minkowski expression yields the supremum (resp. the infimum), which is h-increasing but not strictly. A number of other laws are compatible with h-increasingness, such as weighted sum, alternating compositions varying with level in the hierarchy [10].

Table 1: Table composition laws for different α's in equation (4)

α	**Composition laws**	**Applications**
$-\infty$	Infimum	Ground truth energies [9]
0	Number of Classes	CART classifier complexity [4]
$+1$	Addition	Salembier-Garrido, Guigues [13], [8]
$+\infty$	Supremum	Valero[17], Veganzones[18], Soille[14]

Many other α's that are left open to be explored. The parameter α in fact alike λ-cuts [8] provides a way to control the refinement of the optimal cut [10].

As demonstrated in Figure 5, the dynamic program substructure would now consist in making a choice between the parent supremum (if it is a class of the braid), and the partial partitions that it monitors. We consider in the figure a braid composed of two hierarchies (this is to be able to index the partial partitions.). Equation (6) gives the DP step for BOP shown for HOPs in equation (1). Equation (6) demonstrates a DP sub-structure very similar to the hierarchies except now they are compared over the monitoring supremum class S. When $\omega(\pi_1(S)) = \omega(\pi_2(S))$, and $\omega(\pi_1(S)) < \omega(\{S\}))$, we can pick randomly, as long as we pick one of the partial partitions, so that in a strict sense to keep the energies remain singular.

$$\pi^*(S) = \arg\min \left\{ \omega(\{S\}), \omega(\pi_1^*(S)), \omega(\pi_2^*(S)) \right\} \tag{6}$$

4 Energetic Ordering and Energetic Lattices

Given the problem of finding an optimal cut, we review separately the requirement of obtaining a unique solution. On the HOP, this has been enforced by many authors [4], [13], [8], [17], [1] as a partition which is either the largest or the smallest, amongst optimal cuts with the same energy. The classical energy based minimization associates an energy with every cut, and takes the cut which has the smallest energy[4]. A hierarchy can have multiple cuts with the same minimal energy, and to ensure a unique solution we introduce the following axiom of singularity:

Definition 3. *Let ω be an energy on the partial partitions $\mathcal{D}(E)$, and B be a braid B with a monitor hierarchy H. Energy ω is singular when*

1. *the energies $\omega(\pi(S))$ of all p.p. $\pi(S)$ of H are either strictly smaller, or strictly greater, than the energies of their supports S:*
$$\forall\ \pi(S) \in \Pi(S), \omega(\{S\}) < \omega(\pi(S))\} \text{ or } \omega(\{S\}) > \omega(\pi(S))\}, \tag{7}$$
2. *if $\forall \pi_1, \pi_2 \in B$ and $\forall S \in \pi_1 \vee \pi_2$, we have $\omega(\pi_1 \sqcap S) \neq \omega(\pi_2 \sqcap S)$.*

Consider now two partial partitions $\pi(S), \pi'(S)$ over support S, which is also their refinement supremum $S = \pi(S) \vee \pi'(S)$ (see Figure 6). Intuitively, one may assess that partition π_1 is less energetic than π_2 for an energy ω when $\omega(\pi_1 \sqcap \{S\}) \leq \omega(\pi_2 \sqcap \{S\})$ in each class of $\pi_1 \vee \pi_2$.

Theorem 1. *Given $\pi_1, \pi_2 \in \Pi(E)$ two partitions of space E, and an energy ω, the partition π_1 is said to be less energetic than π_2, i.e. $\pi_1 \preceq_\omega \pi_2$ when in each class of supremum $\pi_1 \vee \pi_2$ the energy of the partial partition of π_1 is smaller or equal to that of π_2*

$$\pi_1 \preceq_\omega \pi_2 \Leftrightarrow \{S \in \pi_1 \vee \pi_2 \Rightarrow \omega(\pi_1 \sqcap \{S\}) \leq \omega(\pi_2 \sqcap \{S\})\} \tag{8}$$

[4] A finite set E of only 25 leaves can be partitioned in 0.5×10^{18} different manners, following the Bell's number.

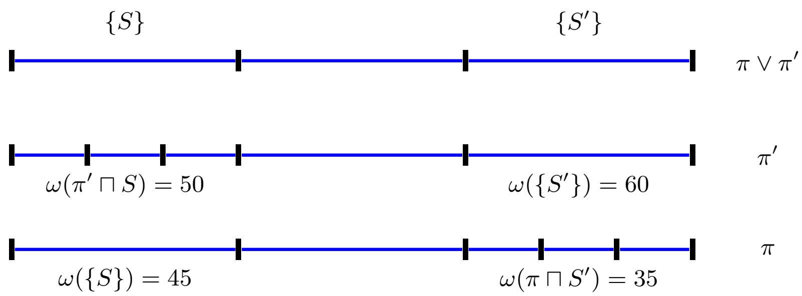

Fig. 6: An example of energetic ordering: We have $\pi \preceq_\omega \pi'$ since in each class of $\pi \vee \pi'$, the energy ω of π is less than or equal to that of π'.

The relation $\preceq_\omega$ called energetic ordering, is an ordering relation for all singular energies ω, if and only if the family Π is the set $\Pi(\omega, E, B)$ of all cuts of a braid B. Proof given in thesis [10].

To prove that the energetic order yields a complete lattice, we must remark two properties. Firstly, consider a hierarchy H reduced to the two partitions π_0 and π_1, with $\pi_0 \leq \pi_1$. Then the unique smallest partition of $\Pi(E, H)$ is obviously obtained by replacing each class S of π_1 by the corresponding p.p. of π_0 when the latter has an energy smaller than that of S. Denote the resulting minimal partition by $\pi_0 \curlywedge_\omega \pi_1$. In case of a braid, π_0 and π_1 are no longer ordered by refinement, and the energetic comparisons have to be performed in each class of $\pi_0 \vee \pi_1$. Secondly, consider now a standard hierarchy H, (i.e. with $n+1$ levels), and k cuts $\{\pi_j,\, 1 \leq j \leq k\}$ of H. The sequence 9 generates a new hierarchy H' where each two classes are ordered or disjoint, hence are classes of H.

$$\pi_1' = \wedge_1^k \pi_j;\; \pi_2' = \wedge_2^k \pi_j; ...;\; \pi_k' = \pi_k \tag{9}$$

Theorem 2. *Let B be a braid of monitor $H = \{\pi_i, 0 \leq i \leq n\}$, and ω a singular energy. The family $\Pi(\omega, H)$ of all cuts of H has a unique minimal element*

$$\pi^* = (((\pi_0 \curlywedge_\omega \pi_1) \curlywedge_\omega \pi_2)...) \curlywedge_\omega \pi_n \tag{10}$$

and a unique maximal element $\pi^{**} = ((\pi_0 \curlyvee_\omega \pi_1)...) \curlyvee_\omega \pi_n$. This property extends to braid B.

Proof. As $\pi_0 \curlywedge_\omega \pi_1$ is the less energtic cut made of classes of π_0 and π_1, the same can be stated with $(\pi_0 \curlywedge_\omega \pi_1) \curlywedge_\omega \pi_2$ for the classes of π_0, π_1, π_2. Thus, by induction, the cut (2) is the unique smallest cut of $\Pi(\omega, H)$. The dual approach leads to the largest energetic cut. Finally, if H is replaced by braid B, then each class of S may have to be compared with several sets of children partial partitions a_1, a_2, etc. but again every minimal (resp. maximal) choice is unique by singularity, which achieves the proof. $\square$

Corollary 1. *When in addition to 2 the energy ω is h-increasing, then $\Pi(\omega, H)$, and further $\Pi(\omega, B)$ turn out to be complete lattices. The infimum and supremum of family $\{\pi_j, 1 \leq j \leq k\}$ are denoted by $\curlywedge_\omega \pi_j$ and $\curlyvee_\omega \pi_j$.*

Finally, we must remark here that given a singular energy on braid, one ensures unique optimal cut, but one which cannot be obtainable by a dynamic program. While a singular and h-increasing energy yields itself to a DP producing an optimal cut, though there can exist other cuts with the same minimal energy. Finally a singular and strictly h-increasing energy is one which yields a unique optimal cut with the DP.

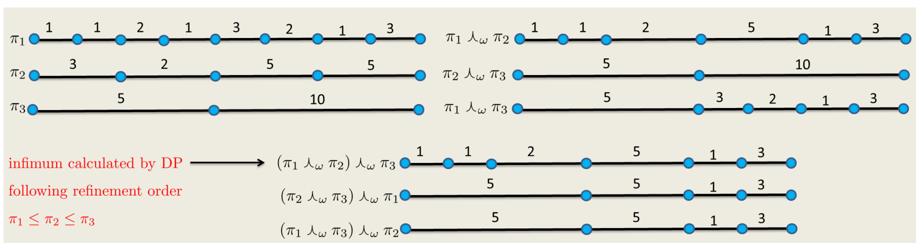

Fig. 7: A counter example showing the breakdown of DP when not following refinement ordering between partitions. Three partitions π_1, π_2, π_3 with their energies over each class. We demonstrate the different infima achievable for different orders of refinement followed across the partitions, while applying the dynamic program. We see for two orders we don't achieve the global infimum. We thus always need an algorithm that works in the order of refinement to keep the DP substructure. This is also why one uses a bottom up pruning [13] or climbing [8]. The actual infimum of the energetic lattice is obtained by following the order of refinement.

5 Conclusion

The paper introduced the new hierarchical structure of the braids of partitions, which expanded the space of hierarchies for the problem of extracting optimal cuts. Furthermore it showed that the braids are the largest family of partitions over which the energetic lattice can be defined. The DP to extract a unique minimal cut consists inherently of an ordering based optimization problem, which is expressed by the energetic lattice structure. A generalized h-increasingness condition for energies operable on braids was also demonstrated. This gives the DP that aggregates local optima to obtain the global optimum. Finally, the paper also provided a short review of braids available in literature, and provides a perspective on how the braid model can be used to become algorithm independent while organizing image domain or space into a family of partitions which preserves the dynamic program substructure. We foresee applications in the domain of multivariate optimization, machine learning and super-pixel segmentation based optimization.

References

1. Akcay, H.G., Aksoy, S.: Automatic detection of geospatial objects using multiple hierarchical segmentations. IEEE T. Geoscience & Remote Sensing 46(7), 2097–2111 (2008)
2. Angulo, J., Jeulin, D.: Stochastic watershed segmentation. In: Banon, G.J.F., Barrera, J., Braga-Neto, U.d.M., Hirata, N.S.T. (eds.) Proceedings, (ISMM), Rio de Janeiro. vol. 1, pp. 265–276. INPE (Oct 2007)
3. Arbelaez, P., Maire, M., Fowlkes, C., Malik, J.: Contour detection and hierarchical image segmentation. IEEE Trans. PAMI 33(5), 898–916 (may 2011)
4. Breiman, L., Friedman, J.H., Olshen, R.A., Stone, C.J.: Classification and Regression Trees. Wadsworth (1984)
5. Brendel, W., Todorovic, S.: Segmentation as maximum-weight independent set. In: Lafferty, J.D., Williams, C.K.I., Shawe-Taylor, J., Zemel, R.S., Culotta, A. (eds.) NIPS. pp. 307–315. Curran Associates, Inc. (2010)
6. Cousty, J., Najman, L.: Incremental algorithm for hierarchical minimum spanning forests and saliency of watershed cuts. In: Soille, P., Pesaresi, M., Ouzounis, G.K. (eds.) ISMM. LNCS, vol. 6671, pp. 272–283. Springer (2011)
7. Diday, E.: Spatial pyramidal clustering based on a tessellation. In: Banks, D., McMorris, F., Arabie, P., Gaul, W. (eds.) Classification, Clustering, and Data Mining Applications, pp. 105–120. Studies in Classification, Data Analysis, and Knowledge Organisation, Springer Berlin Heidelberg (2004)
8. Guigues, L., Cocquerez, J.P., Men, H.L.: Scale-sets image analysis. International Journal of Computer Vision 68(3), 289–317 (2006)
9. Kiran, B.R., Serra, J.: Global-local optimizations by hierarchical cuts and climbing energies. Pattern Recognition 47(1), 12 – 24 (2014)
10. Kiran, B.R.: Energetic Lattice based optimization. Ph.D. thesis, Université Paris-Est, LIGM-A3SI, ESIEE (2014)
11. Malisiewicz, T., Efros, A.A.: Improving spatial support for objects via multiple segmentations. In: British Machine Vision Conference (BMVC) (September 2007)
12. Ronse, C.: Partial partitions, partial connections and connective segmentation. J. Math. Imaging Vis. 32(2), 97–125 (Oct 2008)
13. Salembier, P., Garrido, L.: Binary partition tree as an efficient representation for image processing, segmentation, and information retrieval. IEEE Trans. on Image Processing 9(4), 561–576 (2000)
14. Soille, P.: Constrained connectivity for hierarchical image partitioning and simplification. IEEE Transactions on Pattern Analysis and Machine Intelligence 30(7), 1132–1145 (2008)
15. Straehle, C., Peter, S., Köthe, U., Hamprecht, F.: K-smallest spanning tree segmentations. In: Weickert, J., Hein, M., Schiele, B. (eds.) Pattern Recognition, LNCS, vol. 8142, pp. 375–384. Springer Berlin Heidelberg (2013)
16. Unnikrishnan, R., Pantofaru, C., Hebert, M.: Toward objective evaluation of image segmentation algorithms. IEEE Transactions on Pattern Analysis and Machine Intelligence 29(6), 929–944 (June 2007)
17. Valero, S.: Hyperspectral image representation and Processing with Binary Partition Trees. Ph.d., Universitat Politècnica de Catalunya (UPC) (2011)
18. Veganzones, M., Tochon, G., Dalla-Mura, M., Plaza, A., Chanussot, J.: Hyperspectral image segmentation using a new spectral unmixing-based binary partition tree representation. ITIP 23(8), 3574–3589 (Aug 2014)