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SOME GEOMETRIC PROPERTIES OF THE SUBORDINATION FUNCTION ASSOCIATED TO AN OPERATOR-VALUED FREE CONVOLUTION SEMIGROUP

SERBAN TEODOR BELINSCHI

ABSTRACT. In his article *On the free convolution with a semicircular distribution*, Biane found very useful characterizations of the boundary values of the imaginary part of the Cauchy-Stieltjes transform of the free additive convolution of a probability measure on \mathbb{R} with a Wigner (semicircular) distribution. Biane's methods were recently extended by Huang to measures which belong to the partial free convolution semigroups introduced by Nica and Speicher. The main purpose of this short note is to further extend some of Biane's methods and results to free convolution powers of operator-valued distributions and to free convolutions with operator-valued semicirculars.

1. INTRODUCTION

Free probability, introduced by Voiculescu in order to study free group factors, gained considerable importance after the discovery in [20] of the connection between freeness and the asymptotic behaviour of large random matrices. One of the most significant consequences of the main result of [20] is the fact that two independent selfadjoint random matrices $H_N, A_N - H_N$ being a gaussian matrix - are asymptotically free as $N \rightarrow \infty$. Thanks to Wigner's work, it is known since the '50s that the asymptotic distribution of H_N as $N \rightarrow \infty$ is the semicircle law. In particular, the distribution of $A_N + H_N$ is modeled by Voiculescu's free additive convolution [24] of a standard semicircular distribution with the limiting distribution of A_N . In [7], this convolution is analyzed in great detail: among others, a formula for the density of the corresponding distribution is provided, and it is shown that this density is bounded, continuous and analytic wherever positive. However, in order to study the asymptotic eigenvalue distribution of more general selfadjoint polynomials $P(A_N, H_N)$ it is necessary to consider the more general framework of free convolutions of *operator-valued distributions* [21, 16, 17, 10, 3]. In the present note, we find certain operator-valued counterparts of Biane's results from [7]; necessarily, several of the conclusion of [7] cannot hold in this more general setup.

As it is shown in [2], there exists an intimate connection between the free additive convolution with an operator-valued semicircular distribution and the free convolution powers of operator-valued distributions. In particular, it turns out that the analytic tools used in the study of free convolution powers of operator-valued distributions are a generalization of the analytic tools used in the study of the free convolution with an operator-valued semicircular distribution. Thus, we write our proofs in the more general context. This has the advantage of allowing us to draw several conclusions about more general free additive convolutions of operator-valued distributions.

The second section is dedicated to introducing the main concepts and tools we require. We state and prove our main results in the third and fourth section.

2. NONCOMMUTATIVE FUNCTIONS, DISTRIBUTIONS AND CONVOLUTIONS

2.1. Noncommutative probability spaces and distributions. Following D. Voiculescu [24, 21], by a non-commutative probability space we understand a pair (\mathcal{A}, τ) where \mathcal{A} is a unital $*$ -algebra over \mathbb{C} and $\tau: \mathcal{A} \rightarrow \mathbb{C}$ is a positive functional with $\tau(1) = 1$. Let \mathcal{B} be a unital C^* -algebra. A \mathcal{B} -valued non-commutative probability space is a triple $(\mathcal{A}, \mathbb{E}_{\mathcal{B}}, \mathcal{B})$, where \mathcal{A} is a unital $*$ -algebra containing \mathcal{B} as a $*$ -subalgebra and $\mathbb{E}_{\mathcal{B}}$ is a unit-preserving positive conditional expectation from \mathcal{A} onto \mathcal{B} (in particular, the units of \mathcal{A} and \mathcal{B} coincide). If $\mathcal{B} \subset \mathcal{A}$ is an inclusion of unital C^* -algebras, then we call $(\mathcal{A}, \mathbb{E}_{\mathcal{B}}, \mathcal{B})$ a \mathcal{B} -valued non-commutative C^* -probability space. For simplicity, we will suppress the subscript of $\mathbb{E}_{\mathcal{B}}$ whenever there is no risk of confusion, and denote our conditional expectation by \mathbb{E} . Elements $X \in \mathcal{A}$ are called *random variables* or (in the second context) *\mathcal{B} -valued* (or *operator-valued*) *random variables*.

We use the notation $\mathcal{B}\langle \mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n \rangle$ for the $*$ -algebra freely generated by \mathcal{B} and the noncommuting selfadjoint symbols $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$. If $X \in \mathcal{A}$ is a selfadjoint element, then we will also use the notation $\mathcal{B}\langle X \rangle$ for the $*$ -algebra generated by X and \mathcal{B} . Following [2] we denote set of all positive, unit preserving, conditional expectations from $\mathcal{B}\langle \mathcal{X} \rangle$ to \mathcal{B} by $\Sigma(\mathcal{B})$. Given $\mu \in \Sigma(\mathcal{B})$, its n^{th} moment is the $n - 1$ -linear map $\mu_n: \mathcal{B} \times \dots \times \mathcal{B} \rightarrow \mathcal{B}$ defined by $\mu_n(b_1, b_2, \dots, b_{n-1}) := \mu[\mathcal{X}b_1\mathcal{X}b_2 \dots \mathcal{X}b_{n-1}\mathcal{X}]$. We define the zeroth moment to be $1 \in \mathcal{B}$ and the first moment to be $\mu[\mathcal{X}] \in \mathcal{B}$. We also denote $\Sigma_0(\mathcal{B})$ the set of all $\mu \in \Sigma(\mathcal{B})$ whose moments do not grow faster than exponentially, that is there exists some $M > 0$ such that, for all positive integers m , all $b_1, \dots, b_m \in M_m(\mathcal{B})$ and $\mathcal{X}_m = \mathcal{X} \otimes 1_m$ we have that

$$\|(\mu \otimes \text{Id}_m)(\mathcal{X}_m b_1 \mathcal{X}_m b_2 \dots \mathcal{X}_m b_m \mathcal{X}_m)\| < M^{n+1} \|b_1\| \dots \|b_m\|.$$

If $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ is a \mathcal{B} -valued noncommutative probability space and $X = X^* \in \mathcal{A}$, we define its distribution with respect to \mathbb{E} to be the element $\mu_X \in \Sigma_0(\mathcal{B})$ satisfying

$$\mu_X(P(\mathcal{X})) = \mathbb{E}(P(X)) \text{ for all } P(\mathcal{X}) \in \mathcal{B}\langle \mathcal{X} \rangle.$$

If X belongs to a \mathcal{B} -valued noncommutative C^* -probability space, then $\mu_X \in \Sigma_0(\mathcal{B})$. Conversely, as shown by Voiculescu in [21], if $\mu \in \Sigma_0(\mathcal{B})$, then there exist a \mathcal{B} -valued C^* -noncommutative probability space containing an element $X = X^*$ such that $\mu_X = \mu$. In the simpler case $\mathcal{B} = \mathbb{C}$, μ_X can be identified with a Borel probability measure supported on the compact set $\sigma(X)$, the spectrum of X (see [1]).

2.2. Free independence and relevant transforms. We present next the free independence, and define the relevant analytic transforms, in a C^* -algebraic context, as this is the context that is considered most often in this paper.

Definition 2.1. *Let $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ be a \mathcal{B} -valued noncommutative C^* -probability space and $\{X_i\}_{i \in I}$ be a family of selfadjoint elements from \mathcal{A} . The family $\{X_i\}_{i \in I}$ is said to be freely independent (or just free) over \mathcal{B} with respect to \mathbb{E} if for any $n \in \mathbb{N}$, $\mathbb{E}(A_1 \dots A_n) = 0$ whenever $A_j \in \mathcal{B}\langle X_{\iota(j)} \rangle \cap \ker(\mathbb{E})$, $\iota(j) \in I$, $\iota(k) \neq \iota(k+1)$ for all $k \in \{1, \dots, n-1\}$.*

If X, Y are two freely independent \mathcal{B} -valued noncommutative random variables, then μ_{X+Y} depends only on μ_X and μ_Y and is called the free additive convolution of μ_X and μ_Y . We denote μ_{X+Y} by $\mu_X \boxplus \mu_Y$.

It is natural to denote $\underbrace{\mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}$ by $\mu^{\boxplus n}$. Obviously, $\{\mu^{\boxplus n} | n \in \mathbb{N}\}$ forms a

discrete semigroup. A remarkable result of Nica and Speicher [12] states that for any Borel probability measure μ on \mathbb{R} , there exists a partial semigroup, i.e. a family $\{\mu^{\boxplus t} : t \geq 1\}$ such that $\mu^{\boxplus 1} = \mu$ and $\mu^{\boxplus s+t} = \mu^{\boxplus s} \boxplus \mu^{\boxplus t}$, $s, t \geq 1$. This result has been extended by Curran [8] to certain operator-valued distributions. However, as it will be seen below, in the operator-valued context, analytic transforms indicate that it should be possible - or rather natural - to consider convolution powers indexed by a subset of the set of completely positive self-maps of \mathcal{B} . The main result of [2] states precisely that: given $\mu \in \Sigma_0(\mathcal{B})$, there exists a family

$$\{\mu^{\boxplus \alpha} | \alpha : \mathcal{B} \rightarrow \mathcal{B} \text{ completely positive, } \alpha - \text{Id}_{\mathcal{B}} \text{ completely positive}\} \subset \Sigma_0(\mathcal{B})$$

such that $\mu^{\boxplus \text{Id}_{\mathcal{B}}} = \mu$ and $\mu^{\boxplus \alpha + \beta} = \mu^{\boxplus \alpha} \boxplus \mu^{\boxplus \beta}$.

For the computation of free convolutions, Voiculescu [24, 21] introduced the R -transform. In order to define it, let

$$(1) \quad G_{\mu}(b) = \mu[(b - \mathcal{X})^{-1}], \quad \Im b > 0.$$

Here we denote $\Im b = (b - b^*)/2i$, $\Re b = (b + b^*)/2$, and we write $a > 0$ if $a = a^*$ and $\sigma(a) \subset (0, +\infty)$. The notation $a \geq 0$ is used when we require only that $a = a^*$ and $\sigma(a) \subset [0, +\infty)$. If $\mu \in \Sigma_0(\mathcal{B})$, then $M_{\mu}(b) = \mu[(1 - b\mathcal{X})^{-1}b] = G_{\mu}(b^{-1})$ has an analytic continuation to a neighbourhood of zero and maps 0 to itself. A simple computation shows that $M'_{\mu}(0) = \text{Id}_{\mathcal{B}}$, so that, by the inverse function theorem for Banach spaces, M_{μ} has a unique compositional inverse around zero which maps zero to itself, which we denote by $M_{\mu}^{(-1)}$. Thus, both $b^{-1}M_{\mu}^{(-1)}(b)$ and $M_{\mu}^{(-1)}(b)b^{-1}$ are analytic around zero and map zero to one. The R -transform of μ is defined via the formula $bR_{\mu}(b) = (M_{\mu}^{(-1)}(b)b^{-1})^{-1} - 1$. We prefer a slightly different form of the definition of R_{μ} , namely

$$(2) \quad R_{\mu}(b) = G_{\mu}^{(-1)}(b) - b^{-1}.$$

This formula is well-defined on an open set which has zero in its closure, and thus determines R_{μ} . The essential property of the R -transform, found by Voiculescu, is that

$$R_{\mu}(b) + R_{\nu}(b) = R_{\mu \boxplus \nu}(b)$$

on a sufficiently small neighbourhood of zero in \mathcal{B} . Clearly then, for any linear completely positive map $\alpha : \mathcal{B} \rightarrow \mathcal{B}$ such that $\alpha - \text{Id}_{\mathcal{B}}$ is still completely positive, $\mu^{\boxplus \alpha}$ will be given by

$$(3) \quad R_{\mu^{\boxplus \alpha}}(b) = \alpha(R_{\mu}(b)),$$

on a neighbourhood of zero. It has been shown in [2] that such a $\mu^{\boxplus \alpha} \in \Sigma_0(\mathcal{B})$ exists for any $\mu \in \Sigma_0(\mathcal{B})$, and that the restriction requiring $\alpha - \text{Id}_{\mathcal{B}}$ to still be completely positive is not necessary for infinitely divisible μ . A different, simpler proof of this result is given in [18], where it is also shown that the requirement that $\alpha - \text{Id}_{\mathcal{B}}$ is itself completely positive cannot be generally omitted.

It is quite obvious from (1) that G_{μ} plays a role similar to that of the Cauchy-Stieltjes transform in classical probability. However, unlike the classical Cauchy-Stieltjes transform, G_{μ} alone does not generally encode all of the distribution μ , but only its symmetric moments. It has been a crucial insight of Voiculescu that G_{μ} is just the first level of a noncommutative function that *does* encode all of μ : this will be outlined in the next subsection.

2.3. Noncommutative functions and transforms. In this subsection we largely follow [4, 15] in describing the noncommutative extensions of the analytic transforms introduced in the previous subsection, and [11] in the definition of noncommutative sets and functions. We refer to these three articles and [22, 23] for details on, and proofs of, the statements below.

First a notation: if S is a nonempty set, we denote by $M_{m \times n}(S)$ the set of all matrices with m rows and n columns having entries from S . For simplicity, we let $M_n(S) := M_{n \times n}(S)$. Given C^* -algebra \mathcal{B} , a *noncommutative set* is a family $\Omega := (\Omega_n)_{n \in \mathbb{N}}$ such that

- (a) for each $n \in \mathbb{N}$, $\Omega_n \subseteq M_n(\mathcal{B})$;
- (b) for each $m, n \in \mathbb{N}$, we have $\Omega_m \oplus \Omega_n \subseteq \Omega_{m+n}$.

The noncommutative set Ω is called *right admissible* if in addition the condition (c) below is satisfied:

- (c) for each $m, n \in \mathbb{N}$ and $a \in \Omega_m, b \in \Omega_n, w \in M_{m \times n}(\mathcal{B})$, there is an $\epsilon > 0$ such that $\begin{pmatrix} a & zw \\ 0 & b \end{pmatrix} \in \Omega_{m+n}$ for all $z \in \mathbb{C}, |z| < \epsilon$.

Given C^* -algebras \mathcal{B}, \mathcal{C} and a noncommutative set $\Omega \subseteq \prod_{n=1}^{\infty} M_n(\mathcal{B})$, a *noncommutative function* is a family $f := (f_n)_{n \in \mathbb{N}}$ such that $f_n: \Omega_n \rightarrow M_n(\mathcal{C})$ and

- (1) $f_m(a) \oplus f_n(b) = f_{m+n}(a \oplus b)$ for all $m, n \in \mathbb{N}, a \in \Omega_m, b \in \Omega_n$;
- (2) for all $n \in \mathbb{N}$, $f_n(T^{-1}aT) = T^{-1}f_n(a)T$ whenever $a \in \Omega_n$ and $T \in GL_n(\mathbb{C})$ are such that $T^{-1}aT$ belongs to the domain of definition of f_n .

A remarkable result (see [11, Theorem 7.2]) states that, under very mild conditions on Ω , local boundedness for f implies each f_n is analytic as a map between Banach spaces.

As mentioned in the previous section, the function G_μ encodes only the symmetric part of the distribution μ . It was an extremely important remark of Voiculescu that G_μ has a noncommutative extension:

$$(4) \quad G_\mu^{[n]}(b) = (\mu \otimes \text{Id}_n) [(b - \mathcal{X} \otimes 1_n)^{-1}], \quad n \in \mathbb{N}.$$

There are two noncommutative sets which are natural domains of definition for $(G_\mu^{[n]}(b))_{n \in \mathbb{N}}$ and for $(G_\mu^{[n]}(b^{-1}))_{n \in \mathbb{N}}$, respectively: the noncommutative operator upper half-plane $(\mathbb{H}^+(M_n(\mathcal{B})))_{n \in \mathbb{N}}$, where $\mathbb{H}^+(M_n(\mathcal{B})) = \{b \in M_n(\mathcal{B}) : \Im b > 0\}$, and the set of nilpotent matrices with entries from \mathcal{B} , respectively. Remarkably, as shown in [22], $G_\mu^{[n]}$ maps $\mathbb{H}^+(M_n(\mathcal{B}))$ into $\mathbb{H}^-(M_n(\mathcal{B})) := -\mathbb{H}^+(M_n(\mathcal{B}))$ and $G_\mu^{[n]}(b^*) = G_\mu^{[n]}(b)^*$. It is clear that the restriction of $(G_\mu^{[n]})_{n \in \mathbb{N}}$ to either of these two noncommutative sets determines $(G_\mu^{[n]})_{n \in \mathbb{N}}$. For a description of how to explicitly recover μ from $(G_\mu^{[n]})_{n \in \mathbb{N}}$ via its moments, we refer to [4, 15].

It follows from its definition that the R -transform has itself a noncommutative extension, which determines μ uniquely. The level-one relation (3) extends to $R_{\mu_{\boxplus \alpha}}^{[n]}(b) = (\alpha \otimes \text{Id}_n)(R_\mu^{[n]}(b))$ for $b \in M_n(\mathcal{B})$ of small enough norm. From this formula and the noncommutative extension of (2) we obtain, by adding b^{-1} , the relation $\left(G_{\mu_{\boxplus \alpha}}^{[n]}\right)^{\langle -1 \rangle}(b) = (\alpha \otimes \text{Id}_n) \left(\left(G_\mu^{[n]}\right)^{\langle -1 \rangle}(b) \right) - (\alpha \otimes \text{Id}_n - \text{Id}_{\mathcal{B}} \otimes \text{Id}_n)(b^{-1})$.

Replacing b by $G_{\mu_{\boxplus \alpha}}^{[n]}(b)$ provides $b = (\alpha \otimes \text{Id}_n) \left(G_\mu^{[n]}\right)^{\langle -1 \rangle} \left(G_{\mu_{\boxplus \alpha}}^{[n]}(b)\right) - (\alpha \otimes \text{Id}_n -$

$\text{Id}_{\mathcal{B}} \otimes \text{Id}_n \left(G_{\mu^{\boxplus \alpha}}^{[n]}(b)^{-1} \right)$. With the notations

$$(5) \quad F_{\mu}^{[n]}(b) = G_{\mu}^{[n]}(b)^{-1}, \quad b \in \mathbb{H}^+(M_n(\mathcal{B})), \quad n \in \mathbb{N},$$

$$(6) \quad h_{\mu}^{[n]}(b) = F_{\mu}^{[n]}(b) - b, \quad b \in \mathbb{H}^+(M_n(\mathcal{B})), \quad n \in \mathbb{N},$$

and

$$(7) \quad \omega_{\alpha}^{[n]}(b) = \left(G_{\mu}^{[n]} \right)^{\langle -1 \rangle} \left(G_{\mu^{\boxplus \alpha}}^{[n]}(b) \right),$$

we re-write (3) as

$$(8) \quad \omega_{\alpha}^{[n]}(b) = b + [(\alpha - \text{Id}_{\mathcal{B}}) \otimes \text{Id}_n] h_{\mu}^{[n]} \left(\omega_{\alpha}^{[n]}(b) \right), \quad b \in \mathbb{H}^+(M_n(\mathcal{B})), \quad n \in \mathbb{N},$$

with $\omega_{\alpha}^{[n]}: \mathbb{H}^+(M_n(\mathcal{B})) \rightarrow \mathbb{H}^+(M_n(\mathcal{B}))$. The above argument for the existence of $(\omega_{\alpha}^{[n]})_{n \in \mathbb{N}}$ is obviously not complete: for the rigorous proof, we refer to [2, Theorem 8.4]. This same theorem also states that for any $b \in \mathbb{H}^+(M_n(\mathcal{B}))$, $\omega_{\alpha}^{[n]}(b) \in \mathbb{H}^+(M_n(\mathcal{B}))$ is the unique attracting fixed point of the map $f_b^{[n]}: \mathbb{H}^+(M_n(\mathcal{B})) \rightarrow \mathbb{H}^+(M_n(\mathcal{B}))$, $f_b^{[n]}(w) = b + [(\alpha - \text{Id}_{\mathcal{B}}) \otimes \text{Id}_n] h_{\mu}^{[n]}(w)$, and the right inverse of the map $H^{[n]}: \mathbb{H}^+(M_n(\mathcal{B})) \rightarrow M_n(\mathcal{B})$, $H^{[n]}(w) = w - [(\alpha - \text{Id}_{\mathcal{B}}) \otimes \text{Id}_n] h_{\mu}^{[n]}(w)$.

Of importance in our analysis will be the following result of Popa and Vinnikov [15, Theorem 6.6], re-phrased in terms of the noncommutative function h :

Theorem 2.2. *Let $\mu \in \Sigma_0(\mathcal{B})$ be given. Then there exists a linear map $\eta_{\mu}: \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{B}$ and $M \in (0, +\infty)$ such that for any $k \in \mathbb{N}$, $x_1, \dots, x_k \in \mathcal{B}\langle \mathcal{X} \rangle$, we have*

$$\left(\eta_{\mu} [x_j^* x_i] \right)_{i,j=1}^k \geq 0 \quad \text{in } M_k(\mathcal{B}),$$

$$\|\eta_{\mu} [\mathcal{X} b_1 \mathcal{X} b_2 \cdots \mathcal{X} b_k \mathcal{X}]\| < M^{n+1} \|b_1\| \|b_2\| \cdots \|b_n\| \quad \text{for all } b_1, \dots, b_n \in \mathcal{B},$$

and

$$h_{\mu}^{[n]}(b) = (\eta_{\mu} \otimes \text{Id}_n) [(\mathcal{X} \otimes 1_n - b)^{-1}] - (\mu \otimes \text{Id}_n)(\mathcal{X} \otimes 1_n), \quad b \in \mathbb{H}^+(M_n(\mathcal{B})), \quad n \in \mathbb{N}.$$

In [15] it is shown that under the assumption that $\mu \in \Sigma_0(\mathcal{B})$, $\mathcal{B}\langle \mathcal{X} \rangle$ has a natural C^* -algebra completion, and then the first statement of the theorem about the norm-bounded η_{μ} becomes equivalent to its complete positivity. This will be very important in our proofs. We finally write (8) as

$$(9) \quad \omega_{\alpha}^{[n]}(b) = b - [(\alpha - \text{Id}_{\mathcal{B}}) \otimes \text{Id}_n] (\mu \otimes \text{Id}_n)(\mathcal{X} \otimes 1_n) \\ + [(\alpha - \text{Id}_{\mathcal{B}}) \otimes \text{Id}_n] (\eta_{\mu} \otimes \text{Id}_n) \left[\left(\mathcal{X} \otimes 1_n - \omega_{\alpha}^{[n]}(b) \right)^{-1} \right],$$

for $b \in \mathbb{H}^+(M_n(\mathcal{B}))$, $n \in \mathbb{N}$. This equation determines $(\omega_{\alpha}^{[n]})_{n \in \mathbb{N}}$ and thus, via the relation $G_{\mu}^{[n]} \circ \omega_{\alpha}^{[n]} = G_{\mu^{\boxplus \alpha}}^{[n]}$, equivalent to (7), determines $\mu^{\boxplus \alpha}$ in terms of μ and α .

We conclude this section with a simple remark in light of [2, 21, 19]: assume that in equation (9) above, $\mu(\mathcal{X}) = 0$ and $\nu := \eta_{\mu}$ is a conditional expectation. Denote $\beta := \alpha - \text{Id}_{\mathcal{B}}$, and assume that β is still completely positive. Then the above equation becomes

$$\omega_{\beta}^{[n]}(b) = b + (\beta \otimes \text{Id}_n)(\nu \otimes \text{Id}_n) \left[\left(\mathcal{X} \otimes 1_n - \omega_{\beta}^{[n]}(b) \right)^{-1} \right], \quad b \in \mathbb{H}^+(M_n(\mathcal{B})), \quad n \in \mathbb{N}.$$

This is precisely the subordination equation generalizing the results of [7, Lemma 4] to the operator-valued context: if γ_β is the centered operator-valued semicircular distribution of variance β , then

$$(10) \quad G_{\nu \boxplus \gamma_\beta}^{[n]} = G_\nu^{[n]} \circ \omega_\beta^{[n]}, \quad n \in \mathbb{N}.$$

There are deeper reasons for the similarity between the above formula and (9), reasons which are explored in [2].

For the purposes of our present study, we specify the object of interest: the solution in $\mathbb{H}^+(\mathcal{B})$ of the functional equation

$$(11) \quad \omega(b) = b + \mathbf{a} + \eta [(\mathcal{X} - \omega(b))^{-1}], \quad b \in \mathbb{H}^+(\mathcal{B}),$$

and its noncommutative extension to the noncommutative operator upper half-plane, where \mathcal{B} is an arbitrary unital C^* -algebra, $\mathbf{a} = \mathbf{a}^* \in \mathcal{B}$, $\mathcal{B}\langle \mathcal{X} \rangle$ has a C^* -algebra completion, and $\eta: \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{B}$ is bounded, completely positive. The function ω is necessarily the right inverse of

$$(12) \quad H(w) = w - \mathbf{a} - \eta [(\mathcal{X} - w)^{-1}], \quad w \in \mathbb{H}^+(\mathcal{B}).$$

These facts were proved in [2] and from now on we will take them for granted.

3. $(\Re\omega(\cdot + iq), \Im\omega(\cdot + iq))$ IS THE GRAPH OF A FUNCTION

Let γ_t be the semicircular law of variance $t \in (0, +\infty)$ and let μ be an arbitrary Borel probability measure on \mathbb{R} . In [7, Lemma 2] it is shown that the imaginary part of the Cauchy-Stieltjes transform of $\mu \boxplus \gamma_t$ is, up to a factor of π^{-1} , equal to the function $v_t(u)$ given as

$$v_t(u) := \inf \left\{ v \geq 0 \mid t \int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v^2} \leq 1 \right\},$$

and moreover, that this infimum is reached (i.e. $t \int_{\mathbb{R}} \frac{d\mu(x)}{(u-x)^2 + v_t(u)^2} = 1$) whenever $v_t(u) > 0$. Our next proposition establishes a slightly weaker (and necessarily so) operator-valued counterpart of this result. We denote by \mathcal{B}^{sa} the set of all selfadjoint elements of the C^* -algebra \mathcal{B} , by \mathcal{B}^+ its subset of nonnegative elements, and by \mathcal{B}^{++} the (open) subset of \mathcal{B}^{sa} of strictly positive (i.e. nonnegative and invertible in \mathcal{B}) elements.

Proposition 3.1. *Let \mathcal{B} be a C^* -algebra, η be a completely positive map on the C^* -completion of $\mathcal{B}\langle \mathcal{X} \rangle$ and \mathbf{a} be a selfadjoint element of \mathcal{B} . For any fixed $q \in \mathcal{B}$, $q > 0$, there exists a function $v_q: \mathcal{B}^{sa} \rightarrow \mathcal{B}^{++}$ such that*

$$v_q(u) = q + \eta [((\mathcal{X} - u)v_q(u)^{-1}(\mathcal{X} - u) + v_q(u))^{-1}],$$

for all $u \in \mathcal{B}^{sa}$. Moreover, the correspondence $u \mapsto v_q(u)$ is uniformly bounded (with a bound depending on q, η) and continuous in the norm topology.

Proof. It is useful to clarify first the relation between our proposition and Equation (11): taking imaginary part in this equation and recalling that (i) $\mathcal{B}\langle \mathcal{X} \rangle$ has a C^* -algebra structure, (ii) $\mathcal{X} = \mathcal{X}^*$, and (iii) η is positive, provides us with

$$\Im\omega(b) = \Im b + \eta \left[((\mathcal{X} - \Re\omega(b))(\Im\omega(b))^{-1}(\mathcal{X} - \Re\omega(b)) + \Im\omega(b))^{-1} \right].$$

We fix $\Im b = q > 0$: then our proposition states that the imaginary part of $\omega(b)$ is a continuous function of the real part of $\omega(b)$. Here, of course, $\omega(b)$ is viewed as an independent variable.

Thus, let us fix $q > 0$. Define

$$g_q: \mathcal{B}^{sa} \times \mathcal{B}^{++} \rightarrow \mathcal{B}^{++}, \quad g_q(u, v) = q + \eta [((\mathcal{X} - u)v^{-1}(\mathcal{X} - u) + v)^{-1}].$$

For any $\epsilon = \epsilon^* \in \mathcal{B}$ and $v > 0$, the relation $(v + i\epsilon)^{-1} = (v + \epsilon v^{-1}\epsilon)^{-1} - i(v + \epsilon v^{-1}\epsilon)^{-1}\epsilon v^{-1}$ implies that

$$\begin{aligned} (\mathcal{X} - u)(v + i\epsilon)^{-1}(\mathcal{X} - u) + v + i\epsilon = \\ (\mathcal{X} - u)(v + \epsilon v^{-1}\epsilon)^{-1}(\mathcal{X} - u) + v + i(\epsilon - (\mathcal{X} - u)(v + \epsilon v^{-1}\epsilon)^{-1}\epsilon v^{-1}(\mathcal{X} - u)) \end{aligned}$$

which guarantees that the real part (in the C*-algebra completion of $\mathcal{B}\langle \mathcal{X} \rangle$) of $(\mathcal{X} - u)(v + i\epsilon)^{-1}(\mathcal{X} - u) + v + i\epsilon$ is greater than v . This makes it invertible for any $\epsilon = \epsilon^* \in \mathcal{B}$, allowing the extension of g_q to $\mathcal{B}^{sa} \times (-i)\mathbb{H}^+(\mathcal{B})$, and, moreover, guarantees that $\Re g_q(u, v + i\epsilon) \geq q$ for any $(u, v + i\epsilon) \in \mathcal{B}^{sa} \times (-i)\mathbb{H}^+(\mathcal{B})$. We have thus re-written $g_q(u, \cdot)$ as a self-map of the noncommutative operator right half-plane. Precisely the same argument as the one from the proof of [2, Theorem 8.4] shows that $g_q(u, \cdot)$ maps a bounded subdomain \mathcal{D} of $\{w \in (-i)\mathbb{H}^+(\mathcal{B}) : \Re w \geq q/2\}$, depending on u and q , strictly inside itself. The Earle-Hamilton theorem [9, Section 11.1] guarantees that $g_q(u, \cdot)$ has precisely one attracting fixed point in $(-i)\mathbb{H}^+(\mathcal{B}) + q$ for any $u \in \mathcal{B}^{sa}$, point which we call $v_q(u)$. Moreover, the function $w \mapsto g_q(u, w)$ is shown in the same reference to be a *strict* contraction in the Kobayashi metric, with the contraction coefficient depending continuously on the distance from $g_q(u, \mathcal{D})$ to the complement of \mathcal{D} . Thus, the dependence of the fixed point is necessarily sequentially continuous (recall that the dependence $u \mapsto g_q(u, v)$ is smooth - in fact analytic). Since on any (norm)-bounded subset of $\mathbb{H}^+(\mathcal{B})$ which is at a strictly positive (norm)-distance from $\mathcal{B} \setminus \mathbb{H}^+(\mathcal{B})$, the topology generated by the Kobayashi metric coincides with the norm topology, this makes the correspondence $u \mapsto v_q(u)$ norm-continuous.

To conclude, observe that $g_q(u, v) > 0$ for all $u = u^*, v > 0$. Thus, the uniqueness of the attractinc fixed point of $g_q(u, \cdot)$ implies that it necessarily belongs to \mathcal{B}^{++} . \square

Remark 3.2.

- (1) It should be noted that in the above proposition's proof, the noncommutative structure of the functions involved has never come up. In particular, the requirement of complete positivity of the linear map η can be relaxed to simple positivity without affecting the conclusion of Proposition 3.1.
- (2) The existence of $\omega(r + iq)$ for any $r = r^* \in \mathcal{B}$, proved in [2, Theorem 8.4], as an attracting fixed point of $f_{r+iq}(w) = r + iq + \mathbf{a} + \eta [(\mathcal{X} - w)^{-1}]$ guarantees that there are pairs of points $(\Re \omega(r + iq), \Im \omega(r + iq)) \in \mathcal{B}^{sa} \times \mathcal{B}^{++}$ such that $g_q(\Re \omega(r + iq), \Im \omega(r + iq)) = \Im \omega(r + iq)$. The uniqueness of the fixed point of $g_q(u, \cdot)$ guarantees that whenever u is of the form $\Re \omega(r + iq)$, $v_q(\Re \omega(r + iq)) = \Im \omega(r + iq)$; in particular, the set $\{\Re \omega(r + iq), \Im \omega(r + iq) : r \in \mathcal{B}^{sa}\}$ is the graph of a function defined on \mathcal{B}^{sa} with values in \mathcal{B}^{++} .
- (3) It is remarkable in this context that g is the first level of a noncommutative map having the properties described in Proposition 3.1 at each level n . Indeed, the noncommutative extension of g is written as $g_{q \otimes 1_n}^{[n]}(u, v) = q \otimes 1_n + (\eta \otimes \text{Id}_n) [((\mathcal{X} \otimes 1_n - u)v^{-1}(\mathcal{X} \otimes 1_n - u) + v)^{-1}]$, for $u \in M_n(\mathcal{B})^{sa}$, $v \in (-i)\mathbb{H}^+(M_n(\mathcal{B}))$, $n \in \mathbb{N}$ (it should be noted that $M_n(\mathcal{B})^{sa}$, $n \in \mathbb{N}$, is a noncommutative set, but not an admissible one). This necessarily

implies that the fixed point is itself noncommutative: if $u_n = u \otimes 1_n$, then $v_{q \otimes 1_n}^{[n]}(u \otimes 1_n) = v_q(u) \otimes 1_n$.

- (4) When \mathcal{B} is finite dimensional (a C^* -algebra of matrices) - the most important case for us - it is quite easy to show that $u \mapsto v_q(u)$ is analytic in the sense that for any $u = u^*$, there exists a neighbourhood $V_{u,q}$ of u in \mathcal{B} such that v_q has an analytic extension to $V_{u,q}$. Indeed, the fact that $v_q(u)$ is an attracting fixed point for $v \mapsto g_q(u, v)$ which is in the interior of the domain of $g_q(u, \cdot)$ implies that all eigenvalues of $\partial_v g_q(u, v_q(u))$ are of absolute value strictly less than one. In order to see this, we write

$$g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v & c \\ 0 & v \end{pmatrix} \right) = \begin{pmatrix} g_q(u, v) & \partial_v g_q(u, v)(c) \\ 0 & g_q(u, v) \end{pmatrix},$$

and observe that for $c \in \mathcal{B}$ satisfying $(v^{-1/2} c v^{-1/2})(v^{-1/2} c v^{-1/2})^* < 4$ (hence for any $c \in \mathcal{B}$ of sufficiently small norm), the real part of $\begin{pmatrix} v & c \\ 0 & v \end{pmatrix}$ is strictly positive in $M_2(\mathcal{B})$. Iterating the map $\begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix} \mapsto g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix} \right)$ provides convergence in the norm of $M_2(\mathcal{B})$ to the fixed point $\begin{pmatrix} v_q(u) & 0 \\ 0 & v_q(u) \end{pmatrix}$ as $n \rightarrow \infty$. Direct computation yields the formula $\begin{pmatrix} v_q(u) & \partial_v g_q(u, v_q(u))^{on}(c) \\ 0 & v_q(u) \end{pmatrix}$ for the n^{th} iterate of the map $\begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix} \mapsto g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} v_q(u) & c \\ 0 & v_q(u) \end{pmatrix} \right)$, so we must have $\lim_{n \rightarrow \infty} \partial_v g_q(u, v_q(u))^{on}(c) = 0$ for all $c \in \mathcal{B}$. This requires the spectrum $\sigma(\partial_v g_q(u, v_q(u))) \subset \mathbb{D}$ (we have denoted by \mathbb{D} the open unit disc in the complex plane). An application of the implicit function theorem for analytic functions provides the desired result, with a formula for the derivative of v_q given by

$$\partial_u v_q(u) = [\text{Id}_{\mathcal{B}} - \partial_v g_q(u, v_q(u))]^{-1} \circ \partial_u g_q(u, v_q(u)).$$

Note that this argument only required 2-positivity for η . As a side benefit, note that $\sigma([\text{Id}_{\mathcal{B}} - \partial_v g_q(u, v_q(u))]^{-1}) \subseteq \frac{1}{2} - i\mathbb{C}^+$.

The result of Proposition 3.1 can be strengthened to the same level of strength as Remark 3.2-(4), i.e. to concluding the analyticity of $u \mapsto v_q(u)$. We sketch an argument below. This argument is rather tedious and convoluted, but we feel it is worth writing it down especially because of the way it uses noncommutative functions theory. Fix an arbitrary C^* -algebra \mathcal{B} . Direct computations show that $g_{q \otimes 1_2}^{[2]}$ can be extended to the set \mathcal{D}_2 of elements

$$\left\{ \left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} w_1 & d \\ 0 & w_2 \end{pmatrix} \right) : u_1, u_2 \in \mathcal{B}^{sa}, w_1, w_2 \in (-i)\mathbb{H}^+(\mathcal{B}), c, d \in \mathcal{B} \right\} :$$

the expressions of the (1, 1) and (2, 2) entries are $g_q(u_1, w_1)$ and $g_q(u_2, w_2)$, respectively, the (2, 1) entry is zero, and the (1, 2) entry is

$$(13) \quad \eta \left[\left((\mathcal{X} - u_1)w_1^{-1}(\mathcal{X} - u_1) + w_1 \right)^{-1} \left[(\mathcal{X} - u_1)w_1^{-1}c + cw_2^{-1}(\mathcal{X} - u_2) \right. \right. \\ \left. \left. - d + (\mathcal{X} - u_1)w_1^{-1}dw_2^{-1}(\mathcal{X} - u_2) \right] \left((\mathcal{X} - u_2)w_2^{-1}(\mathcal{X} - u_2) + w_2 \right)^{-1} \right].$$

This makes $g_{q \otimes 1_2}^{[2]}$ into a self-map of \mathcal{D}_2 . For u_1, u_2, c fixed, $\begin{pmatrix} w_1 & d \\ 0 & w_2 \end{pmatrix} \mapsto g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} w_1 & d \\ 0 & w_2 \end{pmatrix} \right)$ maps the set

$$\mathcal{D}_1 = \left\{ \begin{pmatrix} w_1 & d \\ 0 & w_2 \end{pmatrix} : w_1, w_2 \in (-i)\mathbb{H}^+(\mathcal{B}), d \in \mathcal{B} \right\}$$

into itself. We note that for fixed u_1, u_2, c , the relations $\Re w_1, \Re w_2 > q/2$ imply uniform norm boundedness for the factors $((\mathcal{X} - u_j)w_j^{-1}(\mathcal{X} - u_j) + w_j)^{-1}$, $j \in \{1, 2\}$ in the C^* -algebra completion of $\mathcal{B}(\mathcal{X})$, as well as of $(\mathcal{X} - u_j)w_j^{-1}$ etc. However, this bound might be quite large, making the estimates on (13) uniform, but useless in terms of mapping a bounded subset of \mathcal{D}_1 into itself, thus precluding another direct application of the Earle-Hamilton Theorem. We shall go around this inconvenient fact.

It is clear that if $\begin{pmatrix} w_1 & d \\ 0 & w_2 \end{pmatrix} \mapsto g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} w_1 & d \\ 0 & w_2 \end{pmatrix} \right)$ has a fixed point in \mathcal{D}_1 , then the (1, 1) and (2, 2) entries of this fixed point must be $v_q(u_1)$ and $v_q(u_2)$, respectively. This puts a very strong restriction on the (1, 2) entry of the fixed point: it must be of the form

$$\eta \left[\left((\mathcal{X} - u_1)v_q(u_1)^{-1}(\mathcal{X} - u_1) + v_q(u_1) \right)^{-1} \right. \\ \times \left[(\mathcal{X} - u_1)v_q(u_1)^{-1}c + cv_q(u_2)^{-1}(\mathcal{X} - u_2) \right. \\ \left. - d + (\mathcal{X} - u_1)v_q(u_1)^{-1}dv_q(u_2)^{-1}(\mathcal{X} - u_2) \right] \\ \left. \times \left((\mathcal{X} - u_2)v_q(u_2)^{-1}(\mathcal{X} - u_2) + v_q(u_2) \right)^{-1} \right],$$

for some $d \in \mathcal{B}$. This fixed point, if existing, must depend linearly on c . Thus, we are allowed to re-scale c as small (in norm) as we desire. However, we are still inconvenienced by the (implicit) requirement that the norm of the remaining part of the expression above (the terms not containing c) is *strictly* less than $\|d\|$. We shall complicate the problem even more in order to bypass this inconvenience: for u_1, u_2, c fixed as above, with the possible proviso that c might be re-scaled (see equation (14) below), we consider the set

$$\tilde{\mathcal{D}}_1^\epsilon = \left\{ \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} \in M_2(\mathcal{B}) : \Re \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} > \epsilon 1 \otimes 1_2 \right\}.$$

The defining inequality of $\tilde{\mathcal{D}}_1^\epsilon$ requires $\Re w_j > \epsilon 1$ and $\frac{d^* + m}{2} (\Re w_1 - \epsilon 1)^{-1} \frac{d + m^*}{2} < (\Re w_2 - \epsilon 1)$. In order to study $g_{q \otimes 1_2}^{[2]}$, we consider the expression

$$\left[\begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} + \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} \right]^{-1}$$

which appears in under $\eta \otimes \text{Id}_2$ the formula of $g_{q \otimes 1_2}^{[2]}$. Under the assumption that our argument belongs to $\tilde{\mathcal{D}}_1^\epsilon$, we determine under what conditions the element under the inverse has positive real part, and hence the whole expression above has positive real part (recall that $(-i)\mathbb{H}^+(\mathcal{B})$ is invariant under taking inverse). We write

$$\begin{aligned} & \begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{X} - u_1 & 0 \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{X} - u_1 & 0 \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{X} - u_1 & 0 \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \\ &\quad - \begin{pmatrix} \mathcal{X} - u_1 & 0 \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

for $\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} \in \tilde{\mathcal{D}}_1^\epsilon$. It is clear that the first term on the right hand side above has real part greater than or equal to zero. Since the real part of $\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}$ is greater than ϵ times the unit of $M_2(\mathcal{B})$, it follows that the norm of its inverse is no greater than ϵ^{-1} . Thus, for all $c \in \mathcal{B}$ with

$$(14) \quad \|c\| < (\min\{1/2, \epsilon\})^2 (2\|\mathcal{X}\| + \|u_1\| + \|u_2\| + 39)^{-2},$$

the norm of the sum of the real parts of the last three terms is strictly less than $\epsilon/2$. We conclude that

$$\Re \left[\begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \right] > -\frac{\epsilon}{2} 1 \otimes 1_2.$$

This guarantees that the real part of

$$\begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{X} - u_1 & -c \\ 0 & \mathcal{X} - u_2 \end{pmatrix} + \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix}$$

is strictly greater than $\frac{\epsilon}{2} 1 \otimes 1_2$. If we choose $\epsilon \in (0, 1)$ sufficiently small so that $q > 2\epsilon 1$ in \mathcal{B} , then $\begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} \mapsto g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} w_1 & d \\ m & w_2 \end{pmatrix} \right)$ maps $\tilde{\mathcal{D}}_1^\epsilon$ in a bounded subset of itself which is at strictly positive distance from the complement of $\tilde{\mathcal{D}}_1^\epsilon$. The Earle-Hamilton Theorem [9, Section 11.1] applies to provide a $d = d(u_1, u_2, c)$ such that $\begin{pmatrix} v_q(u_1) & d \\ 0 & v_q(u_2) \end{pmatrix}$ is the attracting fixed point of the correspondence given just above. As argued in the proof of Proposition 3.1, the dependence of the fixed point on the initial data (u_1, u_2, c) is norm-continuous. With the notation $d(u_1, u_2, c) = \Delta v_q(u_1, u_2)(c)$, justified by [11, Section 2], we

obtain

$$\begin{aligned}
 & \eta \left[((\mathcal{X} - u_1)v_q(u_1))^{-1}(\mathcal{X} - u_1) + v_q(u_1))^{-1} \right. \\
 & \quad \times [(\mathcal{X} - u_1)v_q(u_1))^{-1}c + cv_q(u_2))^{-1}(\mathcal{X} - u_2) \\
 & \quad - \Delta v_q(u_1, u_2)(c) + (\mathcal{X} - u_1)v_q(u_1))^{-1}\Delta v_q(u_1, u_2)(c)v_q(u_2))^{-1}(\mathcal{X} - u_2)] \\
 (15) \quad & \left. \times ((\mathcal{X} - u_2)v_q(u_2))^{-1}(\mathcal{X} - u_2) + v_q(u_2))^{-1} \right] = \Delta v_q(u_1, u_2)(c),
 \end{aligned}$$

for all $c \in \mathcal{B}$ of sufficiently small norm (estimated in (14)), and, by linearity, for all $c \in \mathcal{B}$. Moreover, this same norm estimate (14) is seen to be uniform for u_1, u_2 uniformly bounded. We conclude that the correspondence $(u_1, u_2, c) \mapsto \Delta v_q(u_1, u_2)(c)$ is not only continuous, but also locally uniformly bounded when the norm topologies are considered on $\mathcal{B}^{sa} \times \mathcal{B}^{sa} \times \mathcal{B}$ and \mathcal{B} . As shown in the same [11, Section 2], $\Delta v_q(u, u)(c) = \partial_u v_q(c)$. We conclude that the correspondence $u \mapsto v_q(u)$ is in fact C^1 in the Fréchet sense on \mathcal{B} .

We use next the property of $\Delta v_q(u_1, u_2)(c)$ to be an attracting fixed point for the map in (15). More precisely, we write the left hand side of (15) as the sum of two linear maps (one of them is applied in (15) to c , the other to $\Delta v_q(u_1, u_2)(c)$):

$$\begin{aligned}
 (16) \quad & \Delta_1 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(c) = \\
 & \eta \left[((\mathcal{X} - u_1)v_q(u_1))^{-1}(\mathcal{X} - u_1) + v_q(u_1))^{-1} \right. \\
 & \quad \times [(\mathcal{X} - u_1)v_q(u_1))^{-1}c + cv_q(u_2))^{-1}(\mathcal{X} - u_2)] \\
 & \quad \left. \times ((\mathcal{X} - u_2)v_q(u_2))^{-1}(\mathcal{X} - u_2) + v_q(u_2))^{-1} \right],
 \end{aligned}$$

and

$$\begin{aligned}
 (17) \quad & \Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(d) = \\
 & \eta \left[((\mathcal{X} - u_1)v_q(u_1))^{-1}(\mathcal{X} - u_1) + v_q(u_1))^{-1} \right. \\
 & \quad \left. [(\mathcal{X} - u_1)v_q(u_1))^{-1}dv_q(u_2))^{-1}(\mathcal{X} - u_2) - d] \right. \\
 & \quad \left. \times ((\mathcal{X} - u_2)v_q(u_2))^{-1}(\mathcal{X} - u_2) + v_q(u_2))^{-1} \right].
 \end{aligned}$$

The correspondence that we iterate is

$$(18) \quad \begin{pmatrix} v_q(u_1) & d \\ 0 & v_q(u_2) \end{pmatrix} \mapsto g_{q \otimes 1_2}^{[2]} \left(\begin{pmatrix} u_1 & c \\ 0 & u_2 \end{pmatrix}, \begin{pmatrix} v_q(u_1) & d \\ 0 & v_q(u_2) \end{pmatrix} \right).$$

The right-hand side of this correspondence is

$$\begin{pmatrix} v_q(u_1) & \Delta_1 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(c) + \Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(d) \\ 0 & v_q(u_2) \end{pmatrix}$$

(recall that $g_q(u_j, v_q(u_j)) = v_q(u_j)$ by the proof of Proposition 3.1). For reasons of space, we denote just here $S(\cdot) = \Delta_1 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(\cdot)$, $T(\cdot) = \Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))(\cdot)$. This expression makes clear that the n^{th} iteration of $g^{[2]}$ will provide in the $(1, 2)$ entry $T^n(d) + \sum_{j=0}^{n-1} T^j(S(c))$. We conclude from the existence of the limit as $n \rightarrow \infty$ of this expression for any c, d in a ball of small enough diameter (see (14)) that $\|T^n(d) + \sum_{j=0}^{n-1} T^j(S(c)) - \Delta v_q(u_1, u_2)(c)\| \rightarrow 0$ as $n \rightarrow \infty$. In particular, $\|T^n(d)\| \rightarrow 0$ as $n \rightarrow \infty$.

Now we use again the essential property of the map (18) to be a *strict* contraction in the Kobayashi metric, with the contraction coefficient uniformly bounded away from one when u_1, u_2, c vary very little in norm. This makes the above convergence

to zero uniform in c and d for c, d in small enough norm-balls around zero in \mathcal{B} . In particular, $\|T^n\| \rightarrow 0$ as $n \rightarrow \infty$, imposing $\sigma(T) \subset \mathbb{D}$. We obtain

Remark 3.3. Under the hypotheses of Proposition 3.1, the correspondence $u \mapsto v_q(u)$ is analytic on \mathcal{B}^{sa} . Moreover, the difference-differential operator associated to v_q is continuous, in the sense that

$$\mathcal{B}^{sa} \times \mathcal{B}^{sa} \times \mathcal{B} \ni (u_1, u_2, c) \mapsto \Delta v_q(u_1, u_2)(c) \in \mathcal{B}$$

is norm continuous, and the linear map $\Delta_2 g_q(u_1, u_2; v_q(u_1), v_q(u_2))$ defined above has its spectrum included in \mathbb{D} for all $u_1, u_2 \in \mathcal{B}^{sa}$.

4. THE DERIVATIVE OF ω

For the case of $\mathcal{B} = \mathbb{C}$, it is shown in [5, Theorem 4.6] that the difference quotient of ω satisfies the inequality

$$\left| \frac{\omega(z_1) - \omega(z_2)}{z_1 - z_2} \right| \leq \frac{1}{2}, \quad z_1, z_2 \in \mathbb{C}^+ \cup \mathbb{R}.$$

This is shown by proving that $\Re \omega'(\alpha) > 1/2$ for all $\alpha \in \mathbb{C}^+$. The operator-valued counterpart of this statement has the following form:

Proposition 4.1. *Let \mathcal{B} be a unital C^* -algebra and let H and ω be defined as in (12) and (11). For any $b_1, b_2 \in \mathbb{H}^+(\mathcal{B})$, the spectrum of $\Delta\omega(b_1, b_2)$ as a linear operator from \mathcal{B} to itself is included in $\{z \in \mathbb{C} : \Re z > 1/2\}$.*

Proof. The proof is very similar in spirit to the proof of [5, Theorem 4.6]. Consider $b_1, b_2 \in \mathbb{H}^+(\mathcal{B})$ and $c \in \mathcal{B}$ of sufficiently small norm so that $\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} \in \mathbb{H}^+(M_2(\mathcal{B}))$. as usual, we evaluate ω on this matrix in order to obtain

$$\begin{aligned} \begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} &= H^{[2]} \left(\omega^{[2]} \begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} \right) \\ &= \begin{pmatrix} H(\omega(b_1)) & \Delta H(\omega(b_1), \omega(b_2)) \Delta\omega(b_1, b_2)(c) \\ 0 & H(\omega(b_2)) \end{pmatrix}. \end{aligned}$$

This indicates that $\Delta H(\omega(b_1), \omega(b_2)) \circ \Delta\omega(b_1, b_2) = \text{Id}_{\mathcal{B}}$. As shown in the proof of [2, Theorem 8.4], the point $\omega^{[2]} \begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} \in \mathbb{H}^+(M_2(\mathcal{B}))$ is the unique attracting fixed point of the self-map

$$g^{[2]}: w \mapsto \begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix} + \begin{pmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{pmatrix} + (\eta \otimes \text{Id}_{M_2(\mathbb{C})}) \left[\left(\begin{pmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{X} \end{pmatrix} - w \right)^{-1} \right]$$

of $\mathbb{H}^+(M_2(\mathcal{B}))$. The methods used in the proof of Remark 3.3 apply to show that this map is a strict contraction in the Kobayashi metric. We conclude that the spectrum of $\Delta g(\omega(b_1), \omega(b_2))$ is included in the open unit disc \mathbb{D} . However, $\Delta H(\omega(b_1), \omega(b_2)) - \text{Id}_{\mathcal{B}} = \Delta g(\omega(b_1), \omega(b_2))$, which implies by the definition of the spectrum that $\sigma(\Delta H(\omega(b_1), \omega(b_2))) \subset \mathbb{D} + 1$. Analytic functional calculus rules provide

$$\sigma(\Delta\omega(b_1, b_2)) = \sigma(\Delta H(\omega(b_1), \omega(b_2))^{-1}) \subset \{z \in \mathbb{C} : \Re z > 1/2\}.$$

□

As before, the proof of the above proposition only requires η to be 2-positive.

We note a significant element: if we consider a b_0 in the *boundary* of $\mathbb{H}^+(\mathcal{B})$ and we try to apply the implicit function theorem to the function $f(b, w) = b + \mathbf{a} + \eta [(\mathcal{X} - w)^{-1}] - w$ around a point (b_0, w_0) , where w_0 is a point in $\mathbb{H}^+(\mathcal{B}) \cap C(\omega, b_0)$ (where $C(\omega, b_0)$ denotes the set of limit points of ω at b_0), it turns out that this is possible whenever $0 \notin \sigma(\Delta H(w_0, w_0))$. If \mathcal{B} is finite dimensional, the set of points w_0 with positive imaginary part that satisfy such a condition is an analytic set. It turns out that this analytic set has several properties of interest, which become quite evident when one considers rather the map $(w_1, w_2) \mapsto \Delta H(w_1, w_2)$, and which will be investigated later.

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