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An extremal eigenvalue problem arising in heat conduction^{*†}

Grégoire Nadin[‡] Yannick Privat[§]

Abstract

This article is devoted to the study of two extremal problems arising naturally in heat conduction processes. We look for optimal configurations of thermal axisymmetric fins and model this problem as the issue of (i) minimizing (for the worst shape) or (ii) maximizing (for the best shape) the first eigenvalue of a selfadjoint operator having a compact inverse. We impose a pointwise lower bound on the radius of the fin, as well as a lateral surface constraint. Using particular perturbations and under a smallness assumption on the pointwise lower bound, one shows that the only solution is the cylinder in the first case whereas there is no solution in the second case. We moreover construct a maximizing sequence and provide the optimal value of the eigenvalue in this case. As a byproduct of this result, and to propose a remedy to the non-existence in the second case, we also investigate the well-posedness character of another optimal design problem set in a class enjoying good compactness properties.

Keywords: extremal problem, calculus of variation, Sturm-Liouville eigenvalue, Lebesgue density theorem, lateral surface constraint, parabolic equation.

AMS classification: 49J15, 49K15, 34E05.

1 Introduction

The current work is inspired and motivated by [1], where the authors considered the problem of maximizing, with respect to the cross sectional area, the rate of heat transfer through a bar of given mass. For the sake of clarity, we first state the extremal problem we will investigate in Section 1.1 and we will thus provide several explanations on the physical frame of our study in Section 1.2.

1.1 Setting of the extremal problems

Let us introduce the extremal problems we will deal with. Let $a_0 > 0$. For the reasons evoked in Section 1.2, the admissible set will consist of radii $a(\cdot)$ belonging to $W^{1,\infty}(0, \ell)$ such that

(H₁) Pointwise constraint. There holds $a(x) \geq a_0$ for every $x \in [0, \ell]$;

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(H₂) Lateral surface constraint. There holds

$$\int_0^\ell a(x)\sqrt{1+a'(x)^2} dx \leq S_0.$$

Let us thus introduce the class of admissible functions defined by

$$\mathcal{S}_{a_0,\ell,S_0} = \{a \in W^{1,\infty}(0,\ell) \text{ satisfying } \mathbf{(H_1)} \text{ and } \mathbf{(H_2)}\},$$

where $S_0 > a_0\ell$ is given, so that the class $\mathcal{S}_{a_0,\ell,S_0}$ be non-empty.

According to Section 1.2, the functional we aim at optimizing is $a \mapsto \lambda_1(a)$, where $\lambda_1(a)$ stands for the first eigenvalue the operator \mathcal{L}_a , defined by

$$\lambda_1(a) = \min_{\substack{\varphi \in H^1(0,\ell) \\ \varphi \neq 0}} \mathfrak{R}[a, \varphi] \quad (1)$$

with

$$\mathfrak{R}[a, \varphi] = \frac{\alpha \int_0^\ell a(x)^2 \varphi'(x)^2 dx + \beta \int_0^\ell a(x) \sqrt{1+a'(x)^2} \varphi(x)^2 dx + \sigma a(\ell)^2 \varphi(\ell)^2}{\int_0^\ell a(x)^2 \varphi(x)^2 dx + \delta \varphi(0)^2}, \quad (2)$$

where $\alpha, \beta, \delta, \sigma$ denote positive real numbers.

Note that, according to the Sturm-Liouville eigenfunctions theory, it is standard that $\lambda_1(a)$ is simple (see e.g. [5, 6]) and that its associated normalized eigenfunction denoted $\varphi_{1,a}$ solves the ordinary differential system

$$\begin{aligned} -\alpha (a(x)^2 \varphi'_{1,a}(x))' + \beta a(x) \sqrt{1+a'(x)^2} \varphi_{1,a}(x) &= \lambda_1(a) a(x)^2 \varphi_{1,a}(x), \quad x \in (0, \ell) \\ \gamma a(0)^2 \varphi'_{1,a}(0) &= -\lambda_1(a) \varphi_{1,a}(0) \\ \varphi'_{1,a}(\ell) &= -\frac{\sigma}{\alpha} \varphi_{1,a}(\ell), \end{aligned} \quad (3)$$

with $\gamma = \alpha/\delta$.

This quantity $\lambda_1(a)$ can be viewed as the exponential cooling rate of decay of the fin. This will be explained in Section 1.2 below. We are thus led to investigate the two following extremal problems:

- **Minimization of $\lambda_1(a)$** (worst shape of a fin).

$$\boxed{\inf \{\lambda_1(a), a \in \mathcal{S}_{a_0,\ell,S_0}\}}. \quad (4)$$

- **Maximization of $\lambda_1(a)$** (best shape of a fin).

$$\boxed{\sup \{\lambda_1(a), a \in \mathcal{S}_{a_0,\ell,S_0}\}}. \quad (5)$$

Remark 1. The issue of optimizing eigenvalues of Sturm-Liouville operators is a long story. For a survey of such problems, one refers for instance to [9]. Moreover, one also mentions [10, 15] where the authors deal with a “lateral surface” constraint similar to the one considered in this article. Nevertheless, to the opinion of the authors, the technics, based on the standard change of variable for Sturm-Liouville equation $y = \int_0^x \frac{dt}{a(t)^2}$ with the notations of the paper, cannot be adapted in a simple way to solve the problems investigated in this article. Indeed, this change of variable was used to introduce an auxiliary problem for which one showed that the optimal value coincided with the one of the initial optimal design problem. It also allowed to construct minimizing/maximizing sequences. Unfortunately, such technic does not provide a sharp estimate of the optimal value, and we have to use another approach.

1.2 Motivations in convection-conduction theory

State of the art about shape optimization in convection-conduction theory. Among many applications of the optimal design problem we will investigate, let us mention the strong importance in the computer industry of finding cooling fins in microprocessors having good performances. In [1], the authors used a simplified one-dimensional model of conduction within an axisymmetric geometry, where heat transfers from the side of the fin are neglected, in order to get analytical results for optimal configurations of thermal fins.

In [12], the authors dealt with a simplified one-dimensional stationary model of axisymmetric fin taking into account the lateral heat transfers of the fin. They analyzed the optimal design problem and in particular the existence issues as well as the determination of maximizing sequences.

Many engineering works focused on modeling the direct problem in order to assess the efficiency of different fin shapes. Notice that these studies are mainly numerical and no mathematical approach is used to determine the optimal profiles of fins (see for instance [2, 3, 8, 14]).

In a more general context, let us mention several study dealing essentially with numerical aspects of conduction/convection problems in shape optimization. The model used combines a Fluid Mechanics partial differential equation with a parabolic equation involving a transport term (see e.g. [7, 13]).

In this work, we will consider a more accurate model of one-dimensional thermal bar in non-stationary regime, where convective phenomena from the side of the fin are considered. According to our main theorems (see Theorems 2 and 3 in Section 3), we show in this article that this term plays a crucial role for determining the optimal shape of the fin.

Modeling of the problem Let us consider an axisymmetric fin represented by a domain Ω_a of length $\ell > 0$ and radius $a(x)$ at abscissa x , as displayed in Figure 1, defined in a Cartesian coordinate system by

$$\Omega_a = \{(r \cos \theta, r \sin \theta, x) \mid r \in [0, a(x)), \theta \in \mathbb{S}^1, x \in (0, \ell)\}, \quad (6)$$

where $a \in W^{1,\infty}(0, \ell)$ is such that $a(x) \geq a_0$ for every $x \in [0, \ell]$ with a_0 a positive constant.

Figure 1 sums-up the situation and the notations we will use throughout this article. According to the approach and the model described in [1, Sections 1, 2 and 6], we make the two following assumptions:

- (i) the convective coefficient h , modeling the heat transfer between the fin surface and the fluid flow, does not depend on the variable x and θ . This hypothesis allows to reduce the three-dimensional problem to an axisymmetric one, which justifies that the temperature T along the fin can be considered as a function of t (the time), r and x only.
- (ii) for most of its operating conditions, the fin can be viewed as thermally thin along the r -axis. As a consequence, its radial thermal resistance is low enough in comparison with the convective heat transfer h and it is relevant to claim that $\partial T / \partial r \simeq 0$ almost everywhere in Ω_a . This is why we will impose from now on that the temperature T is a function of the variables t (the time) and x only.

The inlet of the fin, as well as the fluid surrounding the fin are assumed to be at a constant temperatures, denoted respectively T_d and T_∞ . Considering processes where the fin aims at cooling a thermal system, *i.e.* where the heat flows from its basis towards the fluid, we will assume that $0 < T_\infty < T_d(\cdot)$ almost everywhere in $(0, \ell)$. Moreover, we will consider a base mass M_0 attached at the end point $x = 0$.

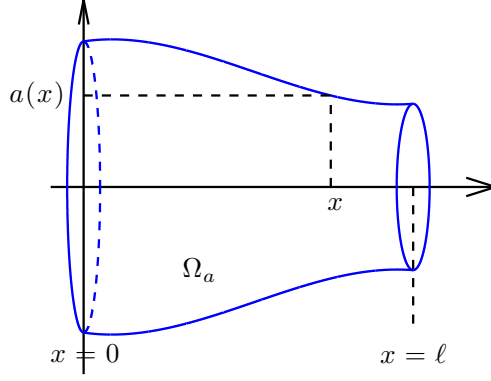


Figure 1: Scheme of the axisymmetric fin

Let $T_d \in L^2(0, \ell)$. The temperature T is then solution of the following parabolic partial differential equation

$$\begin{aligned}
 a(x)^2 \frac{\partial T}{\partial t} &= \frac{k}{\rho c} \frac{\partial}{\partial x} \left(a(x)^2 \frac{\partial T}{\partial x} \right) - \frac{h}{\rho c} a(x) \sqrt{1 + a'(x)^2} (T - T_\infty) & t > 0, \quad x \in (0, \ell) \\
 cM_0 \frac{\partial T}{\partial t}(0, t) &= ka(0)^2 \frac{\partial T}{\partial x}(0, t) & t > 0 \\
 \frac{\partial T}{\partial x}(\ell, t) &= -\frac{h_r}{k} (T(\ell, t) - T_\infty) & t > 0 \\
 T(x, 0) &= T_d(x) & x \in (0, \ell)
 \end{aligned} \tag{7}$$

where k denotes the thermal conductivity of the fin, ρ its density, c its specific heat capacity, h the convective heat transfer coefficient modeling the heat transfer between the fin surface and the fluid flow, and h_r the convective coefficient characterizing the heat transfer over the tip. We will assume in the sequel that the real numbers k , M_0 , ρ , c , h and h_r are positive. Some physical explanations about the derivation of the temperature model may be found in [2, 17].

From now on, we will rather use the notations

$$\alpha = \frac{k}{\rho c}, \quad \beta = \frac{h}{\rho c}, \quad \gamma = \frac{k}{cM_0} \quad \text{and} \quad \sigma = \frac{h_r}{\rho c}, \tag{8}$$

for the sake of readability. It can be proved using standard semigroups arguments and since $a \geq a_0$ on $[0, \ell]$, that the solution T of the partial differential equation (7) belongs to $L^2(0, T, H^1(0, \ell))$.

As did the authors of [1], it is convenient to represent the solution T in terms of series of eigenfunctions. For that purpose, let us introduce the operator

$$\begin{aligned}
 \mathcal{L}_a : C^0([0, \ell]) &\longrightarrow H^1(0, \ell), \\
 f &\longmapsto \varphi_a
 \end{aligned} \tag{9}$$

where φ_a denotes the unique solution of the o.d.e.

$$\begin{aligned}
 \alpha (a(x)^2 \varphi_a'(x))' - \beta a(x) \sqrt{1 + a'(x)^2} \varphi_a(x) &= a(x)^2 f(x), \quad x \in (0, \ell) \\
 \gamma a(0)^2 \varphi_a'(0) &= f(0) \\
 \varphi_a'(\ell) &= -\frac{\sigma}{\alpha} \varphi_a(\ell)
 \end{aligned} \tag{10}$$

According to Lax-Milgram's theorem and since $a \in W^{1,\infty}(0, \ell)$ and $a \geq a_0$ on $[0, \ell]$, this system has a unique solution that belongs to $H^1(0, \ell)$. Let us introduce the inner-product $\langle \cdot, \cdot \rangle_a$ in the space $C^0([0, \ell])$ defined by

$$\langle f, g \rangle_a = \int_0^\ell a(x)^2 f(x)g(x) dx + \frac{\alpha}{\gamma} f(0)g(0),$$

for every $(f, g) \in (C^0([0, \ell]))^2$. The completion of $C^0([0, \ell])$ for the topology inherited from the inner-product $\langle \cdot, \cdot \rangle_a$ is a Hilbert space, and the definition of $\langle \cdot, \cdot \rangle_a$ extends clearly to elements of that space. We denote it by \mathcal{C}_a . We also define the norm $\| \cdot \|_a$ induced by the inner product $\langle \cdot, \cdot \rangle_a$.

With a slight abuse of notation, let us still denote by \mathcal{L}_a this extension. One has thus the following result, whose proof is postponed to Appendix A.

Lemma 1. *The operator \mathcal{L}_a is selfadjoint and compact in \mathcal{C}_a .*

As a consequence of Lemma 1, the operator \mathcal{L}_a is diagonalizable in \mathcal{C}_a and there exist a sequence of positive real numbers $(\lambda_n)_{n \in \mathbb{N}^*}$ diverging to $+\infty$ and a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of elements of \mathcal{C}_a such that $\mathcal{L}_a \varphi_n = \frac{1}{\lambda_n} \varphi_n$ for every $n \in \mathbb{N}^*$, which rewrites

$$\begin{aligned} -\alpha (a(x)^2 \varphi_n'(x))' + \beta a(x) \sqrt{1 + a'(x)^2} \varphi_n(x) &= \lambda_n(a) a(x)^2 \varphi_n(x), \quad x \in (0, \ell) \\ \gamma a(0)^2 \varphi_n'(0) &= -\lambda_n(a) \varphi_n(0) \\ \varphi_n'(\ell) &= -\frac{\sigma}{\alpha} \varphi_n(\ell). \end{aligned} \tag{11}$$

Moreover, according to the so-called min-max principle by Courant-Fisher (see e.g. [5]), there holds

$$\lambda_n(a) = \max_{\substack{E \subset H^1(0, \ell) \\ \text{s.t. } \dim E = n}} \min_{\substack{\varphi \in E \\ \varphi \neq 0}} \Re[a, \varphi],$$

where $\Re[a, \varphi]$ is defined by (2).

These considerations allow us to decompose the solution T of (7) as

$$T(t, x) - T_\infty = \sum_{n=1}^{+\infty} \langle T_d - T_\infty, \varphi_n \rangle_a e^{-\lambda_n t} \varphi_n(x), \tag{12}$$

Toward an extremal problem According to (12), one has the following asymptotic for the solution of (7)

$$T(t, \cdot) - T_\infty \underset{t \rightarrow +\infty}{\sim} \langle T_d(\cdot) - T_\infty, \varphi_1 \rangle_a e^{-\lambda_1(a)t} \varphi_1,$$

provided that $T_d(\cdot) - T_\infty$ be non-orthogonal to φ_1 for the inner-product $\langle \cdot, \cdot \rangle_a$, which is non restrictive and will be assumed from now.

Since we are looking for the shape of a fin optimizing its cooling properties, it is then natural to consider:

- **the problem of finding the best shape of a thermal fin**, by maximizing the first eigenvalue $\lambda_1(a)$ with respect to the function a , so that the temperature of the material to cool will become close to the fluid temperature T_∞ as quick as possible.
- **the problem of finding the worst shape of a thermal fin**, by minimizing $\lambda_1(a)$ with respect to the function a .

Finally, let us briefly comment on the choice of the admissible set of radii a . We will impose:

- (i) a regularity assumption, namely $a \in W^{1,\infty}(0, \ell)$, to guarantee that the surface element be defined almost everywhere.
- (ii) a pointwise lower bound assumption that prevent the fin to collapse: there exists $a_0 > 0$ such that $a(x) \geq a_0$ for every $x \in [0, \ell]$. Moreover, to consider a class of shapes as large as possible, we will choose a_0 suitably small (the precise sense of the word “small” will be made explicit in the statement of the main theorems of this paper).
- (iii) a global lateral surface assumption, to model a limitation on the manufacturing cost. More precisely, we assume an upper bound on the lateral surface of Ω_a , that is given by

$$\text{lateral surface of } \Omega_a = 2\pi \int_0^\ell a(x) \sqrt{1 + a'(x)^2} dx.$$

In the next section, we sum-up the previous considerations and state the extremal problems we will solve.

2 Solving of Problem (4) (looking for the worst shape)

2.1 Main results

This section is devoted to the investigation of Problem (4).

As highlighted in [12, Lemma 3.1], the class $\mathcal{S}_{a_0, \ell, S_0}$ does not share nice compactness properties. In particular, it is not closed nor bounded in $W^{1,\infty}(0, \ell)$ (endowed with the strong topology), whereas it is bounded in $L^\infty(0, \ell)$. This drives us to introduce a new optimal design problem in a subclass of $W^{1,\infty}(0, \ell)$ enjoying good topological properties.

To this aim, let $M > a_0$ and let us define the *truncated* class

$$\mathcal{S}_{a_0, \ell, S_0}^M = \left\{ a \in W^{1,\infty}(0, \ell) \text{ satisfying } (\mathbf{H}_1), (\mathbf{H}_2) \text{ and } a\sqrt{1 + a'^2} \leq M \text{ a.e. in } (0, \ell) \right\}. \quad (13)$$

Since $\mathcal{S}_{a_0, \ell, S_0}$ is a bounded set of $L^\infty(0, \ell)$, it is easy to see that $\mathcal{S}_{a_0, \ell, S_0}^M$ is bounded and closed in $W^{1,\infty}(0, \ell)$. In particular, it inherits useful compactness properties in a weak sense that will be made precise in the sequel. In the following theorem, we investigate the minimization of $a \mapsto \lambda_1(a)$ over the class $\mathcal{S}_{a_0, \ell, S_0}^M$.

Theorem 1. *Let $\alpha, \beta, \delta, a_0, \ell$ and S_0 be positive real numbers such that $S_0 > a_0 \ell$ and $\sigma \geq 0$. The constant function $a(\cdot) = a_0$ minimizes the functional λ_1 over the class $\mathcal{S}_{a_0, \ell, S_0}^M$.*

As a consequence, we infer the following result for the original problem (4).

Corollary 1. *For every positive real numbers α, β, δ , every positive numbers a_0, ℓ and S_0 such that $S_0 > a_0 \ell$, and $\sigma \geq 0$, the constant function $a(\cdot) = a_0$ is the unique solution of the extremal problem (4).*

The approach used to prove the results above rests upon the use of a particular perturbation that we will introduce in Section 2.2. The proofs of Theorem 1 and Corollary 1 are then gathered in Section 2.3.

Remark 2. The optimal value of the function $\lambda_1(\cdot)$ can be explicitly computed. For example, assume that $a_0 < \left(\frac{\beta\delta}{\sigma}\right)^{1/3}$. Recall that, for modeling reasons, such a smallness assumption on a_0 is

of particular interest in the framework of our study, as underlined in Section 1.2. Hence, according to (1) and considering $\varphi \equiv 1$ as a test function, we claim that

$$a_0 \lambda_1(a_0) - \beta = \min_{\substack{\varphi \in H^1(0,\ell) \\ \varphi \neq 0}} \frac{\alpha a_0^3 \int_0^\ell \varphi'(x)^2 dx + \sigma a_0^3 \varphi(\ell)^2 - \beta \delta \varphi(0)^2}{a_0^2 \int_0^\ell \varphi(x)^2 dx + \delta \varphi(0)^2} \leq \frac{\sigma a_0^3 - \beta \delta}{a_0^2 \ell + \delta} < 0.$$

A straightforward computation leads to the following expression of the associated eigenfunction

$$\varphi_{1,a_0}(x) = A \left(\cosh(\omega_1(a_0)x) - \frac{\lambda_1(a_0)}{\gamma a_0^2 \omega_1(a_0)} \sinh(\omega_1(a_0)x) \right),$$

with $\omega_1(a_0)^2 = \frac{\beta - \lambda_1(a_0)a_0}{\alpha a_0}$, where A denotes the normalization constant for the norm $\|\cdot\|_{a_0}$. The boundary condition at $x = \ell$ yields that, $\lambda = \lambda_1(a_0)$ is the first positive root of the transcendental equation

$$\frac{\lambda}{\gamma a_0^2 \omega} = \frac{\alpha \omega \sinh(\omega \ell) + \sigma \cosh(\omega \ell)}{\alpha \omega \cosh(\omega \ell) + \sigma \sinh(\omega \ell)} \quad \text{with} \quad \omega^2 = \frac{\beta - \lambda_1(a_0)a_0}{\alpha a_0}.$$

The construction of $\lambda_1(a_0)$ is illustrated on Figure 2.

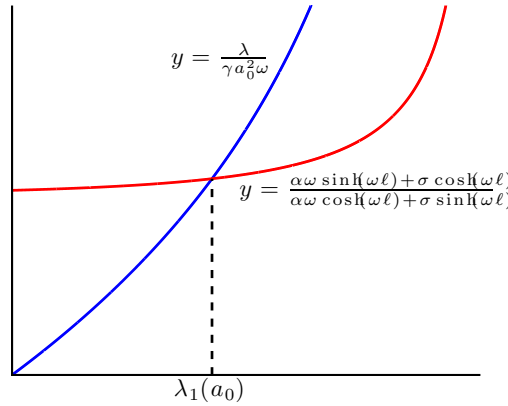


Figure 2: Construction of $\lambda_1(a_0)$

2.2 A key technical lemma

This section is devoted to the description of particular perturbations that we will use to solve at the same time the problems of minimizing and maximizing λ_1 over the class $\mathcal{S}_{a_0, \ell, S_0}^M$.

Let $\varepsilon > 0$, $S_0 \geq a_0 \ell$, $a \in \mathcal{S}_{a_0, \ell, S_0}^M$ such that $a(\cdot) \neq a_0$, and set $b = a\sqrt{1 + a'^2}$. Let us introduce the two families of perturbations we will use in the sequel. Their construction is based on the straightforward claim holding, up to a null (Lebesgue) measure set:

$$(0, \ell) = \{b = M\} \cup \{a = a_0\} \cup \{a_0 < b < M\} \quad \text{since} \quad \{b = a_0\} = \{a = a_0\}.$$

Perturbation of type I (worst shape). Assume that the function a is not identically equal to a_0 . It follows that the set $\{a_0 < b \leq M\}$ is of positive measure. We will then consider the particular perturbation of b of the form

$$b_\varepsilon = b - c\chi_{\mathcal{V}_0(\varepsilon)},$$

where x_0 denotes a Lebesgue point of the set $\{a_0 < b \leq M\}$ and $\mathcal{V}_0(\varepsilon) = \{a_0 < b \leq M\} \cap (x_0 - \varepsilon, x_0 + \varepsilon)$, and $c > 0$ is chosen so that there holds at the same time $a_0 \leq b_\varepsilon \leq M$ a.e. in $(0, \ell)$ and $\int_0^\ell b_\varepsilon(x) dx \leq S_0$.

Perturbation of type II (best shape). Assume that M is large, more precisely that $M\ell > S_0$. Therefore, one has necessarily $b \not\equiv M$ in the sense that the measure of the set $\{b = M\}$ is strictly lower than ℓ .

Starting from there, we will consider the particular perturbations of b of the form

$$b_\varepsilon = b + c \left(\chi_{\mathcal{V}_{x_0}(\varepsilon)} - r_\varepsilon \chi_{\mathcal{V}_{y_0}(\varepsilon)} \right),$$

where $c > 0$, x_0 is a Lebesgue point of $\{a_0 \leq b < M\}$, $y_0 \neq x_0$ is a Lebesgue point of $\{a_0 < b \leq M\}$, the measurable sets $\mathcal{V}_{x_0}(\varepsilon)$ and $\mathcal{V}_{y_0}(\varepsilon)$ are defined by

$$\mathcal{V}_{x_0}(\varepsilon) = \{a_0 \leq b < M\} \cap (x_0 - \varepsilon, x_0 + \varepsilon), \quad \mathcal{V}_{y_0}(\varepsilon) = \{a_0 < b \leq M\} \cap (y_0 - \varepsilon, y_0 + \varepsilon),$$

with ε is small enough so that these two sets do not intersect, and $r_\varepsilon = \frac{|\mathcal{V}_{x_0}(\varepsilon)|}{|\mathcal{V}_{y_0}(\varepsilon)|}$. The Lebesgue density theorem yields that $\lim_{\varepsilon \rightarrow 0} r_\varepsilon = 1$. Obviously, since $a(\cdot) \not\equiv a_0$, there holds $a_0 \leq b_\varepsilon \leq M$ almost everywhere in $(0, \ell)$ since the set $\{b = M\} \cup \{a_0 < b < M\}$ is measurable of positive measure, provided that ε and c be small enough.

Using a choice of function b_ε as above, we will now construct a perturbation a_ε of a within the class $\mathcal{S}_{a_0, \ell, S_0}^M$. This is the content of the following lemma.

Lemma 2. *Let $b_\varepsilon \in L^\infty(0, \ell)$ be a perturbation of b either of type I, or II. Then, there exists a family $(a_\varepsilon)_{\varepsilon > 0}$ such that*

- $a_\varepsilon \in \mathcal{S}_{a_0, \ell, S_0}^M$ for every $\varepsilon > 0$,
- $a_\varepsilon \sqrt{1 + a_\varepsilon'^2} = b_\varepsilon$ almost everywhere in $(0, \ell)$ and for every $\varepsilon > 0$,
- one has the remainder estimate: $\|a_\varepsilon - a\|_{L^\infty(0, \ell)} \leq C\varepsilon^2$, where the constant C only depends on M , c and the constants α , β , δ , a_0 , ℓ , S_0 and σ .

The statement of this lemma is close to [12, Lemma 3.4]. Nevertheless, a notable difference lies in the fact that we have to deal with perturbations of b that are the sums of characteristic functions of measurable sets, instead of open sets.

Proof. The constructions of the function a_ε satisfying the aforementioned assertions when b_ε is either of type I, or II are quite close. Nevertheless, since the case of a perturbation of type II requires a little bit more technicity, we focus on it in this proof. The content of the proof can then be easily adapted (and even simplified) to deduce the construction of a_ε for perturbations of type I.

Let us now describe the construction of a_ε . First, one imposes that

$$a_\varepsilon = a \quad \text{on} \quad [0, \ell] \setminus ((x_0 - \varepsilon, x_0 + \varepsilon) \cup (y_0 - \varepsilon, y_0 + \varepsilon)).$$

Without loss of generality, we will focus within this proof on the characterization of the perturbation $a_\varepsilon - a$ on $\mathcal{V}_{x_0}(\varepsilon)$, the definition of $a_\varepsilon - a$ on $\mathcal{V}_{y_0}(\varepsilon)$ being similar (in absolute value).

Hence, let us define a_ε on $\mathcal{V}_{x_0}(\varepsilon)$. The difficulty lies in controlling the L^∞ distance between a and a_ε . The algorithmic procedure to define a_ε writes as follows:

- i one considers a regular subdivision of $[x_0 - \varepsilon, x_0 + \varepsilon]$ into $k_\varepsilon = \lceil 1/\varepsilon^2 \rceil + 1$ intervals of length $\eta = 2\varepsilon/k_\varepsilon$.
- ii on every subinterval $[\bar{x}, \bar{x} + \eta]$ of this subdivision, one creates one oscillation by setting

$$a_\varepsilon = a_{\eta,2} \quad \text{on } (\bar{x}, \xi) \quad \text{and} \quad a_\varepsilon = a_{\eta,1} \quad \text{on } (\xi, \bar{x} + \eta),$$

where the functions $a_{\eta,1}$ and $a_{\eta,2}$ satisfy in particular $a_{\eta,i} \sqrt{1 + a_{\eta,i}'^2} = b_\varepsilon$ for $i \in \{1, 2\}$, $a_{\eta,1}$ is decreasing and $a_{\eta,2}$ is increasing.

More precisely, we define the function $a_{\eta,2}$ as a solution of the Cauchy problem

$$\begin{aligned} a_{\eta,2}'(x) &= \frac{\sqrt{b_\varepsilon(x)^2 - a_{\eta,2}(x)^2}}{a_{\eta,2}(x)} & x \in (\bar{x}, \bar{x} + \eta) \\ a_{\eta,2}(\bar{x}) &= a(\bar{x}), \end{aligned}$$

and the function $a_{\eta,1}$ is a solution of the Cauchy problem

$$\begin{aligned} a_{\eta,1}'(x) &= -\frac{\sqrt{b_\varepsilon(x)^2 - a_{\eta,1}(x)^2}}{a_{\eta,1}(x)} & x \in (\bar{x}, \bar{x} + \eta) \\ a_{\eta,1}(\bar{x} + \eta) &= a(\bar{x} + \eta), \end{aligned}$$

where $\xi \in (\bar{x}, \bar{x} + \eta)$ is chosen so that $a_{\eta,1}(\xi) = a_{\eta,2}(\xi)$.

It remains to verify that such a construction is possible. Clearly, it rests upon the fact that the graphs of $a_{\eta,1}$ and $a_{\eta,2}$ intersect at a point whose abscissa belongs to $[\bar{x}, \bar{x} + \eta]$, which comes to

$$a_{\eta,2}(\bar{x} + \eta) \geq a(\bar{x} + \eta) \quad \text{and} \quad a_{\eta,1}(\bar{x}) \geq a(\bar{x}).$$

according to the intermediate value theorem.

Let us show that $a_{\eta,i} \geq a$ everywhere in $(\bar{x}, \bar{x} + \eta]$ for $i \in \{1, 2\}$. Because of the symmetrical definitions of $a_{\eta,1}$ and $a_{\eta,2}$, it suffices to prove this fact for $i = 2$. Assume by contradiction and by continuity of a and $a_{\eta,2}$ the existence of $x_\eta \in (\bar{x}, \bar{x} + \eta]$ and $\delta_\eta > 0$ such that $a(x) > a_{\eta,2}(x)$ on $(x_\eta, x_\eta + \delta_\eta)$. Then, since $b_\varepsilon \geq b$ almost everywhere on this interval and since $\mathcal{V}_{x_0}(\varepsilon)$ and $\mathcal{V}_{y_0}(\varepsilon)$ are disjoint, we claim that

$$a_{\eta,2}'(x) = \sqrt{\frac{b_\varepsilon(x)^2}{a_{\eta,2}(x)^2} - 1} > \sqrt{\frac{b(x)^2}{a(x)^2} - 1} = a'(x),$$

for almost every $x \in (x_\eta, x_\eta + \delta_\eta)$. Integrating both sides of this inequality leads to

$$\int_{x_\eta}^{x_\eta + \delta_x} a_{\eta,2}'(x) dx > \int_{x_\eta}^{x_\eta + \delta_x} a'(x) dx,$$

for every $\delta_x \in (0, \delta_\eta)$, which rewrites $a_{\eta,2}(x_\eta + \delta_x) > a(x_\eta + \delta_x)$ since $a_{\eta,2}(x_\eta) = a(x_\eta)$. This is in contradiction with the assumption above and proves that for every $x \in (\bar{x}, \bar{x} + \eta]$, there holds $a_{\eta,2}(x) \geq a(x)$. This justifies the definition above. The construction of a_ε is illustrated on Figure 3.

Moreover, the lateral surface constraint remains satisfied by a_ε since

$$\int_0^\ell b_\varepsilon(x) dx = \int_0^\ell b(x) dx - c|\mathcal{V}_{x_0}(\varepsilon)| + c|\mathcal{V}_{x_1}(\varepsilon)| \leq S_0,$$

and obviously $b_\varepsilon \leq b \leq M$ on $(0, \ell)$. Thus, $a_\varepsilon \in \mathcal{S}_{a_0, \ell, S_0}^M$.

At this step, the function a_ε defined as previously satisfies the two first assertions of the lemma. It remains now to estimate the L^∞ -norm of $a_\varepsilon - a$. Since

$$\begin{aligned} \frac{1}{2}(a_\varepsilon(x)^2 - a_\varepsilon(\bar{x})^2) &= \int_{\bar{x}}^x a_\varepsilon(s) a'_\varepsilon(s) ds \leq \int_{\bar{x}}^x b_\varepsilon(x) dx \\ \text{and } \frac{1}{2}(a(x)^2 - a(\bar{x})^2) &= \int_{\bar{x}}^x a(s) a'(s) ds \leq \|a'\|_{L^\infty(0, \ell)} \|a\|_{L^\infty(0, \ell)} \end{aligned}$$

for every $x \in (\bar{x}, \bar{x} + \eta)$, there holds

$$\begin{aligned} a(x)^2 \leq a_\varepsilon(x)^2 &\leq 2 \int_{\bar{x}}^{\bar{x} + \eta} b_\varepsilon(x) dx + a(\bar{x})^2 \\ &\leq 2\eta(\|b + c\|_{L^\infty(0, \ell)} + 2\|a'\|_{L^\infty(0, \ell)} \|a\|_{L^\infty(0, \ell)}) + a(\bar{x})^2, \end{aligned}$$

for every $x \in [\bar{x}, \bar{x} + \eta]$. Moreover, since $a \in W^{1, \infty}(0, \ell)$ and according to **(H₁)**, one has

$$0 \leq a_\varepsilon(x) - a(x) \leq \frac{(\|b + c\|_{L^\infty(0, \ell)} + \|a'\|_{L^\infty(0, \ell)} \|a\|_{L^\infty(0, \ell)})}{a_0([1/\varepsilon^2] + 1)} 2\varepsilon$$

for every $x \in [\bar{x}, \bar{x} + \eta]$. As $\|a'\|_{L^\infty(0, \ell)} \leq M/a_0$ and, according to Lemma 2 in [12], $\|a\|_{L^\infty(0, \ell)}$ is bounded by a constant which only depends on S_0 and ℓ , it follows that $\|a_\varepsilon - a\|_{L^\infty(\bar{x}, \bar{x} + \eta)} \leq C\varepsilon^2$, for a constant C only depending on M , c and the constants of our problem.

The proof of the lemma is then complete.

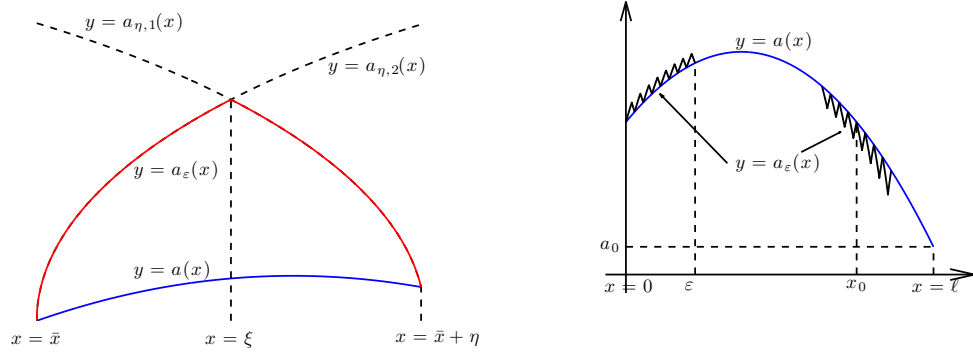


Figure 3: Left: Zoom on one oscillation. Right: the perturbation a_ε

□

2.3 Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. We first investigate the existence of a minimizer within the class $\mathcal{S}_{a_0, \ell, S_0}^M$. In view of that, we will need the following straightforward lemma.

Lemma 3. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of $L^2(0, \ell)$ converging to some function u weakly in $L^2(0, \ell)$. Assume moreover that $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, \ell)$. Then, $\sqrt{1 + u_n^2}$ belongs to $L^2(0, \ell)$ for every $n \in \mathbb{N}$ and the sequence $(\sqrt{1 + u_n^2})_{n \in \mathbb{N}}$ converges weakly in $L^2(0, \ell)$ to a function U satisfying $\sqrt{1 + u^2} \leq U$ a.e. in $(0, \ell)$.*

Proof. Even if this result is straightforward, we nevertheless provide elements of proof for the sake of completeness. Since $\sqrt{1 + u_n^2} \leq 1 + |u_n|$ a.e. in $(0, \ell)$, the first claim follows. By assumption, there exists $\bar{u} > 0$ such that $|u_n| \leq \bar{u}$ a.e. in $(0, \ell)$, for every $n \in \mathbb{N}$. Notice moreover that the functional $v \mapsto \sqrt{1 + v^2}$ is convex and continuous for the strong L^2 -topology on the set $\mathcal{U}_{\bar{u}} = \{v \in L^2(0, \ell) \mid \|v\|_\infty \leq \bar{u}\}$. Indeed, the convexity is obvious and the continuity is obtained by considering a sequence of $\mathcal{U}_{\bar{u}}$ denoted $(v_n)_{n \in \mathbb{N}}$ that converges strongly in L^2 to a function v , and by writing

$$\int_0^\ell (\sqrt{1 + v_n^2} - \sqrt{1 + v^2})^2 dx = \int_0^\ell \frac{(v_n - v)^2 (v_n + v)^2}{(\sqrt{1 + v_n^2} + \sqrt{1 + v^2})^2} dx \leq \bar{u}^2 \|v_n - v\|_{L^2}^2.$$

We thus infer that $v \mapsto \sqrt{1 + v^2}$ is also lower semi-continuous for the weak-topology of $L^2(0, \ell)$ (see e.g. [4, 16]).

By weak-compactness of the bounded sets of $L^2(0, \ell)$, there exists a function U such that, up to a subsequence, $(\sqrt{1 + u_n^2})_{n \in \mathbb{N}}$ converges weakly in $L^2(0, \ell)$ to U , and the expected pointwise inequality follows by weak-semicontinuity of $v \mapsto \sqrt{1 + v^2}$ on $\mathcal{U}_{\bar{u}}$. \square

Lemma 4. *The problem of minimizing λ_1 over $\mathcal{S}_{a_0, \ell, S_0}^M$ has a solution.*

Proof. Let us consider a minimizing sequence $(a_n)_{n \in \mathbb{N}}$ for this problem. Denote by $\lambda_{1,n}$ the associated eigenvalue $\lambda_1(a_n)$ and by $\varphi_{1,n}$ the associated eigenfunction such that $\|\varphi_{1,n}\|_{a_n} = 1$ and $\varphi_{1,n}(0) > 0$. The function $\varphi_{1,n}$ solves the system

$$\begin{aligned} -\alpha (a_n(x)^2 \varphi'_{1,n}(x))' + \beta a_n(x) \sqrt{1 + a_n'(x)^2} \varphi_{1,n}(x) &= \lambda_{1,n} a_n(x)^2 \varphi_{1,n}(x), \quad x \in (0, \ell) \\ \gamma a_n(0)^2 \varphi'_{1,n}(0) &= -\lambda_{1,n} \varphi_{1,n}(0) \\ \varphi'_{1,n}(\ell) &= -\frac{\sigma}{\alpha} \varphi_{1,n}(\ell). \end{aligned} \tag{14}$$

Multiplying this system by $\varphi_{1,n}$ and integrating by parts leads to

$$\begin{aligned} \alpha \int_0^\ell a_n(x)^2 \varphi'_{1,n}(x)^2 dx + \beta \int_0^\ell a_n(x) \sqrt{1 + a_n'(x)^2} \varphi_{1,n}(x)^2 dx + \sigma a_n(\ell)^2 \varphi_{1,n}(\ell)^2 \\ = \lambda_{1,n} \left(\int_0^\ell a_n(x)^2 \varphi_{1,n}(x)^2 dx + \delta \varphi_{1,n}(0)^2 \right). \end{aligned} \tag{15}$$

Since the sequence $(\lambda_{1,n})_{n \in \mathbb{N}}$ is bounded and since $\min\{a_n, b_n\} \geq a_0$ a.e. in $(0, \ell)$ with $b_n = a_n \sqrt{1 + a_n'^2}$, the sequence $(\varphi_{1,n})_{n \in \mathbb{N}}$ is bounded in $H^1(0, \ell)$. Then, using the compact Sobolev embeddings $H^1(0, \ell) \hookrightarrow L^2(0, \ell)$ and $H^1(0, \ell) \hookrightarrow C^0([0, \ell])$, we infer that, up to a subsequence, $(\varphi_{1,n})_{n \in \mathbb{N}}$ converges weakly in $H^1(0, \ell)$ and strongly in $L^2(0, \ell)$ and in $C^0([0, \ell])$ to some function $\underline{\varphi}$.

Moreover, one has $a_0 |a_n'(x)| \leq M$ for every $n \in \mathbb{N}$ and almost every $x \in (0, \ell)$. Hence $(a_n)_{n \in \mathbb{N}}$ is uniformly Lipschitz-continuous and bounded. According to the Arzelà-Ascoli theorem, the

sequence $(a_n)_{n \in \mathbb{N}}$ converges, up to a subsequence, to some Lipschitz-continuous limit \underline{a}_M , satisfying $a_0 \leq \underline{b}_M \leq M$ where $\underline{b}_M = \underline{a}_M \sqrt{1 + \underline{a}_M'^2}$. Since $\|a_n'\|_\infty \leq M$ and according to Lemma 3, one shows furthermore that the sequence $\left(a_n \sqrt{1 + a_n'^2}\right)_{n \in \mathbb{N}}$ converges, up to a subsequence, weakly in $L^2(0, \ell)$ to a function \bar{b}_M such that $a_0 \leq \underline{b}_M \leq \bar{b}_M \leq M$ a.e. in $(0, \ell)$. All these considerations allow to make n go to $+\infty$ in (15) and one gets that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \lambda_{1,n} &= \frac{\alpha \int_0^\ell \underline{a}_M(x)^2 \underline{\varphi}'(x)^2 dx + \beta \int_0^\ell \bar{b}_M \underline{\varphi}(x)^2 dx + \sigma \underline{a}_M(\ell)^2 \underline{\varphi}(\ell)^2}{\int_0^\ell \underline{a}_M(x)^2 \underline{\varphi}(x)^2 dx + \delta \underline{\varphi}(0)^2} \\ &\geq \frac{\alpha \int_0^\ell \underline{a}_M(x)^2 \underline{\varphi}'(x)^2 dx + \beta \int_0^\ell \underline{b}_M \underline{\varphi}(x)^2 dx + \sigma \underline{a}_M(\ell)^2 \underline{\varphi}(\ell)^2}{\int_0^\ell \underline{a}_M(x)^2 \underline{\varphi}(x)^2 dx + \delta \underline{\varphi}(0)^2} \\ &\geq \min_{\substack{\varphi \in H^1(0, \ell) \\ \varphi \neq 0}} \mathfrak{R}[\underline{a}_M, \varphi] = \lambda_1(\underline{a}_M). \end{aligned}$$

By minimality, we thus infer that the previous inequalities are in fact equalities and it follows that necessarily, $\underline{b}_M = \bar{b}_M$. As a result, the sequence $(\lambda_{1,n})_{n \in \mathbb{N}}$ converges to $\lambda_1(\underline{a}_M)$ as $n \rightarrow +\infty$. and the problem of minimizing λ_1 over the class $\mathcal{S}_{a_0, \ell, S_0}^M$ has (at least) one solution. \square

As previously, denote by \underline{a}_M a solution of the problem $\inf_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a)$. Let us now show that necessarily, $\underline{a}_M = a_0$. Denoting by $\varphi_{1,M}$ the eigenfunction associated to the eigenvalue $\lambda_1(\underline{a}_M)$, one writes

$$\min_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a) = \lambda_1(\underline{a}_M) = \min_{\substack{\varphi \in H^1(0, \ell) \\ \varphi \neq 0}} \mathfrak{R}[\underline{a}_M, \varphi] = \mathfrak{R}[\underline{a}_M, \varphi_{1,M}]$$

where $\mathfrak{R}[a, \varphi]$ is defined by (2).

To show that $\underline{b}_M = a_0$, assume by contradiction that the set $\{a_0 < \underline{b}_M \leq M\}$ has a nonzero Lebesgue measure. Then according both to the continuity of \underline{a}_M and the assumption (\mathbf{H}_1) , there exists a Lebesgue point x_0 of the set $\{a_0 < \underline{b}_M \leq M\}$ such that $(x_0 - \varepsilon/2, x_0 + \varepsilon/2) \cap \{a_0 < \underline{b}_M \leq M\}$ is of positive Lebesgue measure.

Using the notations of Section 2.2, let $\mathcal{V}_0(\varepsilon) = \{a_0 < b \leq M\} \cap (x_0 - \varepsilon, x_0 + \varepsilon)$, where $c > 0$ is chosen so that there holds at the same time $a_0 \leq b_\varepsilon \leq M$ a.e. in $(0, \ell)$ and $\int_0^\ell b_\varepsilon(x) dx \leq S_0$.

Denote then by b_ε the function $b - c\chi_{\mathcal{V}_0(\varepsilon)}$. In other words and according to Section 2.2, b_ε is a perturbation of type (I). Hence, Lemma 2 yields the existence of $a_\varepsilon \in \mathcal{S}_{a_0, \ell, S_0}^M$ such that $a_\varepsilon \sqrt{1 + a_\varepsilon'^2} = b_\varepsilon$ and $\|\underline{a}_M - a_\varepsilon\|_\infty = O(\varepsilon^2)$.

Furthermore, since $\frac{b_\varepsilon - b}{\varepsilon}$ converges in the sense of measures to the Dirac measure $-c\delta_{x_0}$ as ε goes to zero, we then infer that

$$\begin{aligned} \lambda_1(a_\varepsilon) &\leq \mathfrak{R}[a_\varepsilon, \varphi_{1,M}] \underset{\varepsilon \rightarrow 0}{\sim} \mathfrak{R}[\underline{a}_M, \varphi_{1,M}] - \varepsilon \frac{\beta c \varphi_{1,M}(x_0)^2}{\int_0^\ell \underline{a}_M(x)^2 \varphi_{1,M}(x)^2 dx + \delta \varphi_{1,M}(0)^2} + O(\varepsilon^2) \\ &< \lambda_1(\underline{a}_M), \end{aligned}$$

provided that ε be small enough.

This is in contradiction with the minimality of \underline{a}_M and we conclude that $|\{a_0 < \underline{b}_M \leq M\}| = 0$.

Proof of Corollary 1. The sequence $(\mathcal{S}_{a_0, \ell, S_0}^M)_{M > a_0}$ is obviously decreasing for the inclusion, and there holds

$$\mathcal{S}_{a_0, \ell, S_0} = \bigcup_{M > a_0} \mathcal{S}_{a_0, \ell, S_0}^M.$$

It thus follows that

$$\inf_{a \in \mathcal{S}_{a_0, \ell, S_0}} \lambda_1(a) = \inf_{M > a_0} \inf_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a) = \lim_{M \rightarrow +\infty} \min_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a) = \lambda_1(a_0),$$

according to Theorem 1. The conclusion follows. \square

3 Solving of Problem (5) (looking for the best shape)

3.1 Functional setting

Define the class of admissible designs

$$\mathcal{A}_{a_0, \ell} = \{a \in W^{1, \infty}(0, \ell), a \geq a_0 \text{ a.e. in } (0, \ell)\},$$

and the product space $\widehat{\mathcal{A}}_{a_0, \ell}$ defined by

$$\widehat{\mathcal{A}}_{a_0, \ell} = \{(a, b), a \in \mathcal{A}_{a_0, \ell} \text{ and } b = a\sqrt{1 + a'^2}\}.$$

Introduce the functional \widehat{F} defined on $\widehat{\mathcal{A}}_{a_0, \ell}$ by

$$\widehat{\lambda}_1(a, a\sqrt{1 + a'^2}) = \lambda_1(a), \quad (16)$$

for every $a \in \mathcal{A}_{a_0, \ell}$. Here and in the sequel, the notation $\mathcal{M}(0, \ell)$ stands for the space of Radon measures on $(0, \ell)$.

Definition 1. Let $(a_n, b_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\widehat{\mathcal{A}}_{a_0, \ell}$. We will say that $(a_n, b_n)_{n \in \mathbb{N}}$ τ -converges to $(a, b) \in C^0([0, \ell]) \times \mathcal{M}(0, \ell)$ if

- $(a_n)_{n \in \mathbb{N}}$ converges to a , locally uniformly in $(0, \ell)$;
- $(b_n)_{n \in \mathbb{N}}$ converges to b in the sense of measures.

We endow $\widehat{\mathcal{A}}_{a_0, \ell}$ with the topology inherited from the τ -convergence. Moreover, one has the following continuity result

Proposition 1. Assume that the sequence $(a_n, a_n\sqrt{1 + a_n'^2})_{n \in \mathbb{N}}$ τ -converges to (a_M, μ) . Then the sequence $(\lambda_1(a_n))_{n \in \mathbb{N}}$ converges to $\widehat{\lambda}_1(a_M, \mu)$, where

$$\widehat{\lambda}_1(a_M, \mu) = \min_{\substack{\varphi \in H^1(0, \ell) \\ \varphi \neq 0}} \frac{\alpha \int_0^\ell a(x)^2 \varphi'(x)^2 dx + \beta \langle \mu, \varphi^2 \rangle_{\mathcal{M}(0, \ell)} + \sigma a(\ell)^2 \varphi(\ell)^2}{\int_0^\ell a(x)^2 \varphi(x)^2 dx + \delta \varphi(0)^2}, \quad (17)$$

and the eigenfunction $\varphi_{1, n}$ associated with $\lambda_1(a_n)$ converges to a minimizer φ of (17) strongly in $H^1(0, \ell)$.

Proof. Denote by $\varphi_{1, n}$ the first eigenfunction solution of (11) with $a = a_n$ normalized by $\int_0^\ell a_n^2 \varphi_{1, n}^2 = 1$. Multiplying the main equation of (11) by $\varphi_{1, n}$ and integrating then by parts leads to

$$\alpha \int_0^\ell a_n(x)^2 \varphi_{1, n}'(x)^2 dx + \beta \int_0^\ell a_n(x) \sqrt{1 + a_n'(x)^2} \varphi_{1, n}(x)^2 dx + \sigma a_n(\ell)^2 \varphi_{1, n}(\ell)^2 = \lambda_1(a_n). \quad (18)$$

Since the sequence $(\lambda_1(a_n))_{n \in \mathbb{N}}$ is bounded and since $\min\{a_n, b_n\} \geq a_0$ a.e. in $(0, \ell)$ with $b_n = a_n \sqrt{1 + a_n'^2}$, the sequence $(\varphi_{1,n})_{n \in \mathbb{N}}$ is bounded in $H^1(0, \ell)$. Then, using the compact Sobolev embeddings $H^1(0, \ell) \hookrightarrow L^2(0, \ell)$ and $H^1(0, \ell) \hookrightarrow C^0([0, \ell])$, we infer that, up to a subsequence, $(\varphi_{1,n})_{n \in \mathbb{N}}$ converges weakly in $H^1(0, \ell)$ and strongly in $L^2(0, \ell)$ and in $C^0([0, \ell])$ to some function φ . We conclude by using the classical arguments recalled in detail in the proof of Lemma 4, by making each term converging in the variational formulation of the equation solved by $\varphi_{1,n}$. The strong convergence in $H^1(0, \ell)$ follows from classical arguments.

Finally, the fact that φ is associated to the first eigenvalue $\widehat{\lambda}_1(a_M, \mu)$ is due to the fact that each element of the sequence $(\varphi_{1,n})_{n \in \mathbb{N}}$ is positive on $(0, \ell)$, and that according to the compact embedding $H^1(0, \ell) \hookrightarrow C^0([0, \ell])$, the limit φ is nonnegative on $(0, \ell)$. \square

3.2 Main results

The following theorem constitutes the main result of this section.

Theorem 2. *Let $\alpha, \beta, \delta, a_0, \ell$ and S_0 be positive real numbers such that $S_0 > a_0 \ell$, $\sigma \geq 0$ and*

$$a_0 \left(\beta + 4\sigma + \sigma \frac{S_0}{\ell^2} \right) < \frac{\delta \beta}{S_0}. \quad (19)$$

Problem (5) has no solution. Moreover, every sequence $(a_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{S}_{a_0, \ell, S_0}$ such that $(a_n, a_n \sqrt{1 + a_n'^2})_{n \in \mathbb{N}}$ τ -converges to $(a_0, a_0 + (S_0 - a_0 \ell) \delta_0)$, where δ_0 denotes the Dirac measure at $x = 0$, is a maximizing sequence for Problem (5).

Remark 3. Note that assumption (19) is satisfied provided that a_0 be small enough, the other parameters being fixed. This is satisfying since $a \geq a_0$ is mainly a technical assumption ensuring the ellipticity and thus the well-posedness of the equation.

Remark 4 (Example of maximizing sequence). Introduce $M_m = a_0 + (S - a_0 \ell)m$ for every $m \in \mathbb{N}^*$. Let us construct the sequence $(a_{S,m})_{m \in \mathbb{N}^*}$ oscillating m times on $[0, 1/m]$. More precisely, we define $a_{S,m}$ by

$$a_{S,m}(x) = \begin{cases} \sqrt{M_m^2 - (\sqrt{M_m^2 - a_0^2} - x)^2} & \text{on } \left[0, \frac{1}{2m^2}\right]; \\ a_{S,m} \left(\frac{1}{m^2} - x\right) & \text{on } \left[\frac{1}{2m^2}, \frac{1}{m^2}\right]; \\ a_{S,m} \left(x - \frac{i}{m^2}\right) & \text{on } \left[\frac{i}{m^2}, \frac{i+1}{m^2}\right], \quad i \in \{1, \dots, m-1\}; \\ a_0 & \text{on } \left[\frac{1}{m}, \ell\right]. \end{cases} \quad (20)$$

Then, one shows easily that

$$\|a_{S,m} - a_0\|_\infty = a_{S,m} \left(\frac{1}{2m^2} \right) - a_0 = o \left(\frac{1}{m^2} \right) \quad \text{as } m \rightarrow +\infty,$$

and that

$$a_{S,m}(x) \sqrt{1 + a_{S,m}'^2(x)} = \begin{cases} a_0 + (S - a_0 \ell)m & \text{if } x \in [0, 1/m] \\ a_0 & \text{if } x \in (1/m, \ell]. \end{cases}$$

Hence, the sequence $(a_{S,m}, a_{S,m} \sqrt{1 + a_{S,m}'^2})_{m \in \mathbb{N}}$ τ -converges to $(a_0, a_0 + (S - a_0 \ell) \delta_0)$ as m diverges.

Remark 5. The optimal value of the function $\lambda_1(\cdot)$ can be explicitly computed. Introduce the notation $\bar{\lambda} = \widehat{\lambda}_1(a_0, a_0 + (S_0 - a_0\ell)\delta_0)$. In other words,

$$\bar{\lambda} = \min_{\substack{\varphi \in H^1(0,\ell) \\ \varphi \neq 0}} \frac{\alpha a_0^2 \int_0^\ell \varphi'(x)^2 dx + \beta a_0 \int_0^\ell \varphi(x)^2 dx + \beta(S_0 - a_0\ell)\varphi(0)^2 + \sigma a_0^2 \varphi(\ell)^2}{a_0^2 \int_0^\ell \varphi(x)^2 dx + \delta \varphi(0)^2} \quad (21)$$

Denote then by φ_0 the associated normalized eigenfunction (for the norm $\|\cdot\|_{a_0}$, see Section 1.2 for its definition) of the spectral problem in the right-hand-side of (21), in other word, the first eigenfunction of the system

$$\begin{cases} -\alpha a_0 \varphi_0'' = (\bar{\lambda} a_0 - \beta) \varphi_0 & \text{on } (0, \ell), \\ -\alpha a_0^2 \varphi_0'(0) = (\bar{\lambda} \delta - \beta(S_0 - a_0\ell)) \varphi_0(0), \\ \varphi_0'(\ell) = -\frac{\sigma}{\alpha} \varphi_0(\ell). \end{cases} \quad (22)$$

Since different cases may arise, depending on the sign of $\bar{\lambda} a_0 - \beta$, let us assume for example that $S_0 > a_0\ell + \delta/a_0$. Hence, according to (21), one has

$$a_0 \bar{\lambda} - \beta = \min_{\substack{\varphi \in H^1(0,\ell) \\ \varphi \neq 0}} \frac{\alpha a_0^3 \int_0^\ell \varphi'(x)^2 dx + \beta(a_0 S_0 - a_0^2 \ell - \delta) \varphi(0)^2 + \sigma a_0^3 \varphi(\ell)^2}{a_0^2 \int_0^\ell \varphi(x)^2 dx + \delta \varphi(0)^2} > 0.$$

A straightforward computation leads to the following expression of the associated eigenfunction

$$\varphi_{1,a_0}(x) = A \left(\cos(\bar{\omega}x) - \frac{\bar{\lambda} \delta - \beta(S_0 - a_0\ell)}{\alpha a_0^2 \bar{\omega}} \sin(\bar{\omega}x) \right),$$

with $\bar{\omega}^2 = \frac{\bar{\lambda} a_0 - \beta}{\alpha a_0}$, where A denotes the normalization constant for the norm $\|\cdot\|_{a_0}$. Moreover, tedious computations show that $\bar{\lambda}$ is the first positive root of the transcendental equation

$$\frac{\delta - \beta(S_0 - a_0\ell)}{\alpha a_0^2 \bar{\omega}} = \frac{\sigma \cos(\bar{\omega}\ell) - \alpha \bar{\omega} \sin(\bar{\omega}\ell)}{\sigma \sin(\bar{\omega}\ell) + \alpha \bar{\omega} \cos(\bar{\omega}\ell)}.$$

As in Section 2.1, this gives a practical way of determining $\bar{\lambda}$.

Let $M > a_0\ell$. The proof of Theorem 2 relies on the investigation of the maximization problem

$$\sup_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a), \quad (23)$$

settled on a smaller class of admissible functions $a(\cdot)$. The real number M can be interpreted as a penalization parameter used to obtain compactness properties of the maximizing sequence. The precise study of this auxiliary problem will allow to make M go to $+\infty$ and deduce the expected non-existence result.

Theorem 3. *Let $\alpha, \beta, \delta, a_0, \ell$ and S_0 be such that $S_0 > a_0\ell, \sigma \geq 0$ and*

$$a_0 \left(\beta + 4\sigma + \sigma \frac{S_0}{\ell^2} \right) < \frac{\delta \beta}{S_0}. \quad (24)$$

Assume that M is chosen such that

$$M > \max\{\bar{M}, S_0/\ell\} \quad \text{where} \quad \bar{M} = (S_0^2/\ell^2 + 4S_0) \frac{\beta S_0 + \sigma(S_0^2/\ell^2 + 4S_0)}{\beta(a_0^2 + \delta)}. \quad (25)$$

Then, Problem (23) has a solution a_M . Moreover, the function $b_M = a_M \sqrt{1 + (a'_M)^2}$ satisfies

$$b_M(x) = \begin{cases} M & \text{on } (0, x_M) \\ a_0 & \text{on } (x_M, \ell), \end{cases}$$

with $x_M = \frac{S_0 - a_0 \ell}{M - a_0}$.

Remark 6. Notice that the assumption “ $M > S_0/\ell$ ” is imposed to guarantee that the function $a_M(x) \sqrt{1 + a'_M(x)^2}$ does not coincide with the constant function equal to M . Such an assumption is necessary to use Lemma 2 for perturbations of type (II).

The result stated in Theorem 3 is intrinsically interesting since it allows to consider Problem (23) as a remedy to the non-existence result stated in Theorem 2.

3.3 Proof of Theorem 3

In what follows and for $f \in L^\infty(0, \ell)$, we will respectively denote by f_+ and f_- its positive and negative part.

Consider a maximizing sequence $(a_n)_{n \in \mathbb{N}}$ and let $b_n = a_n \sqrt{1 + (a'_n)^2}$. Since $(b_n)_{n \in \mathbb{N}}$ is uniformly bounded by M in $L^\infty(0, \ell)$, we can assume that it converges to a limit b_M for the L^∞ weak- \star topology. Similarly, $(a_n)_{n \in \mathbb{N}}$ is uniformly Lipschitz-continuous and bounded and we can assume that it converges to a function a_M for the $W^{1, \infty}((0, \ell))$ weak- \star topology and uniformly in $(0, \ell)$. Lastly, if $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of eigenfunctions associated with $\lambda_1(a_n)$, then it follows from the same arguments as in the proof of Proposition 1 that it converges in $H^1(0, \ell)$, up to extraction, to an eigenfunction φ_M associated to the first eigenvalue of the problem

$$\begin{aligned} -\alpha (a_M(x)^2 \varphi'_M(x))' + \beta b_M(x) \varphi_M(x) &= \lambda_M a_M(x)^2 \varphi_M(x), & x \in (0, \ell) \\ \gamma a_M(0)^2 \varphi'_M(0) &= -\lambda_M \varphi_M(0) \\ \alpha \varphi'_M(\ell) &= -\delta \varphi_M(\ell). \end{aligned} \tag{26}$$

Proposition 2. For almost every $x_0 \in \{b_M < M\}$ and $y_0 \in \{b_M > a_0\}$, one has $\varphi_M(x_0) \leq \varphi_M(y_0)$.

Proof of Proposition 2. Let

$$A_c = \{x \in (0, \ell), b_M(x) \leq M - c\} \quad \text{and} \quad B_c = \{x \in (0, \ell), a_0 + c \leq b_M(x)\}^1.$$

As b_M is not uniformly equal to a_0 according to Theorem 1 nor to M since $M > a_0 \ell$, one has $|A_c| > 0$ and $|B_c| > 0$ for all $c > 0$ small enough. Consider two Lebesgue points x_0 of A_c and y_0 of B_c . We claim that

$$\begin{aligned} \liminf_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{|B(x_0, \eta/2) \cap \{b_n < M - c/2\}|}{\eta} &> 0 \\ \text{and} \quad \liminf_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{|B(y_0, \eta/2) \cap \{b_n > a_0 + c/2\}|}{\eta} &> 0. \end{aligned} \tag{27}$$

Let us prove the first claim, the second one being showed in a similar way. Assume by contradiction that

$$\liminf_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{|B(x_0, \eta/2) \cap \{b_n < M - c/2\}|}{\eta} = 0.$$

¹Here the definition must be understood up to some subset of zero Lebesgue measure.

This yields the existence of a sequence $(\eta_k)_k$ converging to 0 and of $N_0 \in \mathbb{N}$ such that for $n \geq N_0$ and for all k , there holds

$$\int_{B(x_0, \eta_k/2)} \left(b_n - M + \frac{c}{2}\right)_- dx \leq \frac{c}{2} |B(x_0, \eta_k/2) \cap \{b_n < M - c/2\}| < \frac{\eta_k c}{10}. \quad (28)$$

As x_0 is a Lebesgue point of A_c , one can take k large enough such that

$$\frac{1}{\eta_k} \int_{B(0, \eta_k/2)} \left(b_M - M + \frac{c}{2}\right) dx < -\frac{c}{4}. \quad (29)$$

Lastly, using the L^∞ weak- \star convergence of $(b_n)_{n \in \mathbb{N}}$ to b_M , there exists $N_k \in \mathbb{N}$ such that

$$\forall n \geq N_k, \quad \int_{B(x_0, \eta_k/2)} (b_n - b_M) dx < \frac{\eta_k c}{10}. \quad (30)$$

Let us choose k large enough and $n \geq \max(N_0, N_1)$. According to (29), (28) and (30), we get

$$\begin{aligned} 0 &\leq \frac{1}{\eta_k} \int_{B(x_0, \eta_k/2)} \left(b_n - M + \frac{c}{2}\right)_+ dx \\ &\leq \frac{1}{\eta_k} \int_{B(x_0, \eta_k/2)} (b_n - b_M) dx + \frac{1}{\eta_k} \int_{B(x_0, \eta_k/2)} \left(b_M - M + \frac{c}{2}\right) dx \\ &\quad - \frac{1}{\eta_k} \int_{B(x_0, \eta_k/2)} \left(b_n - M + \frac{c}{2}\right)_- dx \\ &\leq \frac{c}{10} + \frac{c}{10} - \frac{c}{4} < 0 \end{aligned}$$

leading to a contradiction. Therefore, the assertion (27) is proved.

Let us introduce $\nu = \min\{\nu_0, \nu_1\}$, where

$$\begin{aligned} \nu_0 &= \liminf_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{|B(x_0, \eta/2) \cap \{b_n < M - c/2\}|}{\eta}, \\ \text{and } \nu_1 &= \liminf_{\eta \rightarrow 0} \limsup_{n \rightarrow +\infty} \frac{|B(y_0, \eta/2) \cap \{b_n > a_0 + c/2\}|}{\eta}. \end{aligned}$$

According to the previous analysis, there holds $\nu > 0$.

This result allows us to consider subsequences $(\eta_k)_{k \in \mathbb{N}}$, and $(n_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, one has

$$|B(x_0, \eta_k/2) \cap \{b_{n_k} < M - c/2\}| \geq \frac{\nu \eta_k}{2} \quad \text{and} \quad |B(y_0, \eta_k/2) \cap \{b_{n_k} > a_0 + c/2\}| \geq \frac{\nu \eta_k}{2}.$$

Without loss of generality, we also assume that $B(x_0, \eta_k/2) \cap B(y_0, \eta_k/2) = \emptyset$ for every $k \in \mathbb{N}$.

Assume by contradiction that $\varphi_M(x_0) > \varphi(y_0)$. Then one could assume (extracting n_k large enough):

$$\lambda_1(a_{n_k}) > \sup_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a) - \eta_k \frac{c\nu}{8} (\varphi_M(x_0)^2 - \varphi_M(y_0)^2). \quad (31)$$

Let $k \in \mathbb{N}$, $\mathcal{V}_{x_0}(\eta_k)$ be any measurable subset of $B(x_0, \eta_k/2) \cap \{b_{n_k} < M - c/2\}$ of Lebesgue measure $\nu \eta_k/2$ and $\mathcal{V}_{y_0}(\eta_k)$ be any measurable subset of $B(y_0, \eta_k/2) \cap \{b_{n_k} > a_0 + c/2\}$ of Lebesgue measure $\nu \eta_k/2$. Introduce $r_k = \frac{|\mathcal{V}_{x_0}(\eta_k)|}{|\mathcal{V}_{y_0}(\eta_k)|}$. Recall that, according to the Lebesgue density theorem, there holds $\lim_{k \rightarrow +\infty} r_k = 1$

Now, introduce the perturbation b_{n_k, η_k} defined for every $k \in \mathbb{N}$ by

$$b_{n_k, \eta_k} = b_{n_k} + \frac{c}{4}(\chi_{\mathcal{V}_{x_0}(\eta_k)} - r_k \chi_{\mathcal{V}_{y_0}(\eta_k)}).$$

According to Section 2.2 and Lemma 2, b_{n_k, η_k} defines a perturbation of type (II) and there exists a sequence $(a_{n_k, \eta_k})_{k \in \mathbb{N}}$ such that

- $a_{n_k, \eta_k} \in \mathcal{S}_{a_0, \ell, S_0}^M$ for every $k \in \mathbb{N}$,
- $a_{n_k, \eta_k} \sqrt{1 + a_{n_k, \eta_k}'^2} = b_{n_k, \eta_k}$ almost every in $(0, \ell)$ and for every $k \in \mathbb{N}$,
- $\|a_{n_k, \eta_k} - a_{n_k}\|_{L^\infty(0, \ell)} \leq C \eta_k^2$ for every $k \in \mathbb{N}$, where the constant C only depends on M, c and the constants $\alpha, \beta, \delta, a_0, \ell, S_0$ and σ .

Let us investigate the L^∞ weak- \star convergence of the sequence $(b_{n_k, \eta_k})_{k \in \mathbb{N}}$. Since $\frac{|\mathcal{V}_{x_0}(\eta_k)|}{\eta_k} = \frac{\nu}{2}$, according to the Banach-Alaoglu-Bourbaki theorem, there exists $\rho_0 \in L^\infty(0, \ell; [0, 1])$ such that the positive measure $\frac{1}{\eta_k} \chi_{\mathcal{V}_{x_0}(\eta_k)}$ converges weakly- \star to ρ_0 in L^∞ . Moreover, since the support of $\chi_{\mathcal{V}_{x_0}(\eta_k)}$ shrinks to $\{x_0\}$, one easily infers that $\rho_0 = \frac{\nu}{2} \delta_{x_0}$ as k goes to $+\infty$. Similarly, one shows that the measure $\frac{1}{\eta_k} \chi_{\mathcal{V}_{y_0}(\eta_k)}$ converges weakly- \star to $\frac{\nu}{2} \delta_{y_0}$ as k goes to $+\infty$. Let φ_k the principal eigenfunction associated with $\lambda_1(a_{n_k, \eta_k})$ and normalized by $\int_0^\ell a_{n_k, \eta_k}(x)^2 \varphi_k(x)^2 dx + \delta \varphi_k(0)^2 = 1$. We know from Proposition 1 that it converges in $H^1(0, \ell)$ to a minimizer φ_M of:

$$\lambda_M := \min_{\substack{\varphi \in H^1(0, \ell) \\ \varphi \neq 0}} \frac{\alpha \int_0^\ell a_M(x)^2 \varphi'(x)^2 dx + \beta \int_0^\ell b_M(x) \varphi(x)^2 dx + \sigma a_M(\ell)^2 \varphi(\ell)^2}{\int_0^\ell a_M(x)^2 \varphi(x)^2 dx + \delta \varphi(0)^2}. \quad (32)$$

As a consequence, one computes

$$\begin{aligned} \lambda_1(a_{n_k, \eta_k}) &= \alpha \int_0^\ell a_{n_k, \eta_k}(x)^2 \varphi_k'(x)^2 dx + \beta \int_0^\ell b_{n_k, \eta_k}(x) \varphi_k(x)^2 dx + \sigma a_{n_k, \eta_k}(\ell)^2 \varphi_k(\ell)^2 \\ &\geq \alpha \int_0^\ell a_{n_k}(x)^2 \varphi_k'(x)^2 dx + \beta \int_0^\ell b_{n_k, \eta_k}(x) \varphi_k(x)^2 dx + \sigma a_{n_k}(\ell)^2 \varphi_k(\ell)^2 - C' \eta_k^2 \\ &= \alpha \int_0^\ell a_{n_k}(x)^2 \varphi_k'(x)^2 dx + \beta \int_0^\ell b_{n_k}(x) \varphi_k(x)^2 dx + \sigma a_{n_k}(\ell)^2 \varphi_k(\ell)^2 \\ &\quad + \frac{c}{4} \int_0^\ell (\chi_{\mathcal{V}_{x_0}(\eta_k)} - r_k \chi_{\mathcal{V}_{y_0}(\eta_k)}) \varphi_k^2 - C' \eta_k^2 \\ &\geq \lambda_1(a_{n_k}) + \frac{c}{4} \int_0^\ell (\chi_{\mathcal{V}_{x_0}(\eta_k)} - r_k \chi_{\mathcal{V}_{y_0}(\eta_k)}) \varphi_k^2 - C' \eta_k^2 \end{aligned}$$

where the constant C' does not depend on k . Moreover, according to the considerations above, one has

$$\lim_{k \rightarrow +\infty} \frac{1}{\eta_k} \int_0^\ell (\chi_{\mathcal{V}_{x_0}(\eta_k)} - r_k \chi_{\mathcal{V}_{y_0}(\eta_k)}) \varphi_k^2 = \frac{\nu}{2} (\varphi_M(x_0)^2 - \varphi_M(y_0)^2).$$

Take k large enough so that

$$\frac{c\nu}{16} (\varphi_M(x_0)^2 - \varphi_M(y_0)^2) > C' \eta_k.$$

Combining with (31), one gets $\lambda_1(a_{n_k, \varepsilon}) > \sup_{a \in \mathcal{S}_{a_0, \ell, S_0}^M} \lambda_1(a)$ provided that k be large enough. This is a contradiction, and it follows that $\varphi_M(x_0) \leq \varphi_M(y_0)$. The Lebesgue density theorem yields that this inequality holds for almost every $x_0 \in A_c$ and $y_0 \in B_c$ and we conclude by taking the union of these sets over $c \in \mathbb{Q} \cap (0, \infty)$. \square

Define

$$\xi_M := \operatorname{esssup}\{\varphi_M(x), b_M(x) < M\} \quad \text{and} \quad \xi'_M := \operatorname{essinf}\{\varphi_M(x), b_M(x) > a_0\}.$$

These two sets both have a positive measure since $b_M \equiv a_0$ and $b_M \equiv M$ are excluded. Proposition 2 yields $\xi_M \leq \xi'_M$. Moreover, if $\xi_M < \xi'_M$, then as φ_M is continuous, there exists $x \in (0, \ell)$ such that $\xi_M < \varphi_M(x) < \xi'_M$, which would imply $b_M(x) = M$ since $\varphi_M(x) > \xi_M$ and $b_M(x) = a_0$ since $\varphi_M(x) < \xi'_M$. Hence, there holds $\xi_M = \xi'_M$ or in other words:

$$\operatorname{esssup}\{\varphi_M(x), b_M(x) < M\} = \operatorname{essinf}\{\varphi_M(x), b_M(x) > a_0\}.$$

Lemma 5. *One cannot have $\varphi_M \leq \xi_M$ everywhere in $(0, \ell)$.*

Proof of Lemma 5. Assume by contradiction that $\varphi_M \leq \xi_M$ on $(0, \ell)$. As $\varphi'_M(0) = -\lambda_M \varphi_M(0) / \gamma a(0)^2 < 0$, there exists $x_0 \in (0, \ell]$ such that $\varphi_M(x) < \xi_M$ in $(0, x_0)$. Taking x_0 the largest, we could assume that $\varphi_M(x_0) = \xi_M$.

The definition of ξ_M yields that for almost every $x \in (0, \ell)$ such that $\varphi_M(x) < \xi_M$, one has $b_M(x) = a_0$. Hence $x_0 < \ell$ since we know from Theorem 1 that $b_M \not\equiv a_0$. Moreover, as $b_M \geq a_M \sqrt{1 + (a'_M)^2} \geq a_0$ a.e. on $(0, \ell)$, the identity $b_M(x) = a_0$ is equivalent to $a_M(x) = a_0$ for almost every x . Hence $a_M = a_0$ in $(0, x_0)$.

We now distinguish between two cases:

- First case: assume that $\lambda_M a_0 \geq \beta$. Then φ_M satisfies

$$-\alpha a_0^2 \varphi_M'' = (\lambda_M a_0^2 - \beta a_0) \varphi_M \geq 0 \quad \text{in } (0, x_0)$$

and thus φ_M is concave in $(0, x_0)$. But as $\varphi'_M(0) < 0$ and φ_M reaches its maximum on $(0, x_0)$ at x_0 , this is a contradiction.

- Second case: assume that $\lambda_M a_0 \leq \beta$. Then φ_M is convex in $(0, x_0)$. As $\varphi_M(x) < \varphi_M(x_0)$ for all $x < x_0$, one gets $\varphi'_M(x_0) > 0$ and thus $\varphi_M(x) > \xi_M$ on a right neighborhood of x_0 , a contradiction since $\varphi_M \leq \xi_M$ on $(0, \ell)$.

We have thus reached a contradiction in all cases. \square

Lemma 6. *For $M > \overline{M}$ where \overline{M} is defined in the statement of Theorem 3, there exists $x_M > 0$ such that $\{\varphi_M > \xi_M\} = (0, x_M)$.*

Proof of Lemma 6. The open set $\{\varphi_M > \xi_M\}$ is not empty by Lemma 5. The definition of ξ_M yields that $\{\varphi_M > \xi_M\} \subset \{b_M \equiv M\}$. Consider any connected component (x_0, x_1) of the open set $\{\varphi_M > \xi_M\}$. Clearly $\varphi_M(x_0) = \xi_M$ if $x_0 > 0$ and $\varphi_M(x_1) = \xi_M$ if $x_1 < \ell$. Moreover, the function φ_M satisfies

$$\alpha (a_M(x)^2 \varphi'_M(x))' = (\beta M - \lambda_M a_M(x)^2) \varphi_M(x) \geq 0 \quad \text{in } (x_0, x_1)$$

since $\beta M \geq \lambda_M a_M(x)^2$. Indeed, we know from Lemma 2 in [12] that

$$a(x) \leq \sqrt{S_0^2 / \ell^2 + 4S_0} \quad \text{for all } x \in (0, \ell) \quad (33)$$

and, taking $\varphi \equiv 1$ as a test-function in the definition of $\lambda_1(a)$, one has

$$\lambda_1(a) \leq \frac{\beta S_0 + \sigma a(\ell)^2}{a_0^2 + \delta}, \quad (34)$$

leading to $\beta M \geq \lambda_M a_M(x)^2$ for all $x \in (0, \ell)$.

Hence, φ_M does not admit any interior maximum in (x_0, x_1) , which is a contradiction if $x_0 > 0$ and $x_1 < \ell$ since $\varphi_M(x) > \xi_M$ in (x_0, x_1) . Similarly, if $x_1 = \ell$, then $x_0 > 0$ otherwise one would get $b_M = M$ in $(x_0, x_1) = (0, \ell)$, and thus $\varphi_M(x_0) = \xi_M$. It follows that the maximum of φ_M on $[x_0, x_1]$ could only be reached at x_1 . According to the Hopf lemma, there holds $\varphi'_M(x_1) > 0$, and on the contrary, one has $\varphi'_M(x_1) \leq 0$ since (26) and $\sigma \geq 0$. We have reached a contradiction. Hence, the only possible connected component of $\{\varphi_M > \xi_M\}$ is an interval of the form $(0, x_M)$. \square

Lemma 7. *If $\sigma > 0$ and $M > \overline{M}$, then the set $\{\varphi_M < \xi_M\}$ is not empty and there exists $y_M \in (0, \ell)$ such that $\{\varphi_M < \xi_M\} = (y_M, \ell)$.*

Proof of Lemma 7. If $\varphi_M \geq \xi_M$ on $(0, \ell)$, then as $\sigma > 0$, one has $\varphi_M(x) > \xi_M$ on some set (y_0, ℓ) , which implies $b_M(x) = M$ on this interval. Hence $y_0 > 0$ and we could assume that $\varphi_M(y_0) = \xi_M$. But as $M \geq \overline{M}$, one has $(a_M^2 \varphi'_M)' \geq 0$ on (y_0, ℓ) as in the proof of Lemma 6 and thus φ_M does not admit any interior local maximum on this interval, a contradiction.

Next, we know that $\{\varphi_M < \xi_M\} \subset \{b_M = a_M = a_0\}$ and we consider a connected component (y_0, y_1) of $\{\varphi_M < \xi_M\}$.

$$\alpha a_0^2 \varphi''_M = (\beta a_0 - \lambda_M a_0^2) \varphi_M \quad \text{in } (y_0, y_1).$$

Distinguishing between the cases $\lambda_M a_0 \geq \beta$ and $\lambda_M a_0 < \beta$ as in the proof of Lemma 5 and using the same types of arguments on minimas as in proof of Lemma 6, we get the conclusion. \square

Lemma 8. *Let $\sigma \geq 0$ and assume (24) and (25) (so that, in particular, the conclusion of Lemma 6 and 7 hold). One has $x_M = y_M$. In other words, there exists $x_M \in (0, \ell)$ such that*

- $\varphi_M(x) > \xi_M$ and $b_M(x) = M$ if $x \in (0, x_M)$,
- $\varphi_M(x) < \xi_M$ and $b_M(x) = a_0$ if $x \in (x_M, \ell)$,

where x_M is uniquely given by $x_M M + (\ell - x_M) a_0 = S_0$. In particular, b_M is bang-bang.

Proof of Lemma 8. Assume in a first time that $\sigma > 0$. Then, one has clearly $\varphi'_M(\ell) \neq 0$ and thus $x_M \leq y_M < \ell$. Lemma 6 and 7 will yield that the conclusion follows provided that we prove that $\{a_0 < b_M < M\} \subset [x_M, y_M]$ has zero measure.

According to Proposition 2 and Lemma 6 and 7, the following situation occurs:

- on $\{\varphi_M > \xi_M\} = (0, x_M)$, one has $b_M \equiv M$,
- on $\{\varphi_M(x) < \xi_M\} = (y_M, \ell)$, one has $b_M \equiv a_0$,
- on $\{\varphi_M(x) = \xi_M\} = [x_M, y_M]$, one has $\beta b_M = \lambda_M a_M^2$ on (x_M, y_M) since φ_M solves Equation (26).

Assume now by contradiction that $x_M \neq y_M$. Since $b_M \geq a_M \sqrt{1 + (a'_M)^2}$ a.e. in $(0, \ell)$, there holds $\lambda_M a_M^2 \geq \beta a_M \sqrt{1 + (a'_M)^2}$ on (x_M, y_M) and therefore

$$a_M \geq \frac{\beta}{\lambda_M} \geq \frac{\beta(a_0^2 + \delta)}{\beta S_0 + \sigma a_M(\ell)^2} \geq \frac{\beta \delta}{\beta S_0 + \sigma(S_0^2/\ell^2 + 4S_0)} \quad \text{on } (x_M, y_M),$$

by using the estimates (33) and (34). Hence, according to (24), one has $\lim_{\substack{x \rightarrow y_M \\ x < y_M}} a_M(x) > a_0$ which is in contradiction with the continuity of a_M and the fact that $a_M = a_0$ on (y_M, ℓ) . It follows that necessarily, $x_M = y_M$ yielding the expected conclusion. The computation of x_M follows from the identity $\int_0^\ell b_M = S_0$.

It remains to show that the same conclusion holds in the case where $\sigma = 0$. Let us roughly provide the argument. Consider a sequence $(\sigma_p)_{p \in \mathbb{N}}$ of positive real numbers converging to zero. Since the parameter σ varies, we change temporarily our notations, denoting by $\lambda_{1,\sigma}$ the eigenvalue given by (1), by $(a_{M,\sigma}, b_{M,\sigma})$ any τ -limit of a maximizing sequence for the optimal design problem (23), and by $\lambda_{M,\sigma}$ the eigenvalue given by (32). Denote also by $\varphi_{M,\sigma}$ the minimizer of the associated Rayleigh quotient. Thus, one has

$$\lambda_{1,\sigma_p}(a) \leq \lambda_1(a_{M,\sigma_p}) = \lambda_{M,\sigma_p} \quad \forall a \in \mathcal{S}_{a_0,\ell,S_0}^M, \quad \forall p \in \mathbb{N}.$$

Making p go to $+\infty$ and passing to the limit yields that

$$\lambda_{1,0}(a) \leq \lambda_1(a_{M,0}) = \lambda_{M,0} \quad \forall a \in \mathcal{S}_{a_0,\ell,S_0}^M,$$

by noting that \overline{M} is an increasing function of σ , that the graph of b_{M,σ_p} does not depend on $\sigma_p > 0$ whenever $M > \overline{M}$ and by using the same (standard) techniques as in the proof of Proposition 1. \square

It remains now to show that necessarily, $b_M = a_M \sqrt{1 + (a'_M)^2}$. Let $d_n = M - a_n \sqrt{1 + (a'_n)^2}$ and $I_M = \{b_M = M\}$. Then $0 \leq d_n \leq M - a_0$ in $(0, \ell)$ and thus

$$\int_{I_M} d_n^2 dx \leq (M - a_0) \int_{I_M} d_n dx \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

by definition of I_M (indeed, $a_n \sqrt{1 + (a'_n)^2} \chi_{I_M} \rightharpoonup M \chi_{I_M}$ weakly- \star in $L^\infty(0, \ell)$ as $n \rightarrow +\infty$). Therefore, the sequence $(a_n \sqrt{1 + (a'_n)^2} \chi_{I_M})_{n \in \mathbb{N}}$ converges strongly to $M \chi_{I_M}$ in $L^2(0, \ell)$. Hence, up to extraction, one can assume that $a_n \sqrt{1 + (a'_n)^2} \rightarrow M$ almost everywhere in I_M . Since $(a_n)_{n \in \mathbb{N}}$ converges locally uniformly to a_M , it easily follows that $(a'_n)_{n \in \mathbb{N}}$ converges almost everywhere to a'_M in I_M and thus $a_M \sqrt{1 + a_M'^2} = M$ in I_M . Moreover, according to Lemma 3, there holds $a_M \sqrt{1 + a_M'^2} \leq b_M$ and thus

$$a_M \sqrt{1 + a_M'^2} = a_M = a_0 = b_M \quad \text{in } \{b_M = a_0\}.$$

We have thus proved that $a_M \sqrt{1 + a_M'^2} = b_M$ almost everywhere in $(0, \ell)$, yielding that $\lim_{n \rightarrow +\infty} \lambda_1(a_n) = \lambda_1(a_M)$. As a consequence, Problem (23) has a solution a_M that is *bang-bang*.

3.4 Proof of Theorem 2

Notice that the sequence of sets $(\mathcal{S}_{a_0,\ell,S_0}^M)_{M > a_0}$ is increasing for the inclusion, and there holds

$$\mathcal{S}_{a_0,\ell,S_0} = \bigcup_{M > a_0} \mathcal{S}_{a_0,\ell,S_0}^M.$$

It thus follows that

$$\begin{aligned} \sup_{a \in \mathcal{S}_{a_0,\ell,S_0}} \lambda_1(a) &\leq \lim_{M \rightarrow +\infty} \sup_{a \in \mathcal{S}_{a_0,\ell,S_0}^M} \lambda_1(a) \\ &= \lim_{M \rightarrow +\infty} \lambda_1(a_M). \end{aligned}$$

Finally, since

$$a_M(x)\sqrt{1+a'_M(x)^2} = \begin{cases} M & \text{if } 0 < x < x_M \\ a_0 & \text{if } x_M < x < \ell, \end{cases}$$

with $x_M = \frac{S_0 - a_0\ell}{M - a_0}$, the family $(a_M, a_M(x)\sqrt{1+a'_M(x)^2})_{M>a_0}$ τ -converges to $(a_0, a_0 + (S - a_0\ell)\delta_0)$. Therefore, using the same arguments as in Proposition 1, there holds

$$\sup_{a \in \mathcal{S}_{a_0, \ell, S_0}} \lambda_1(a) \leq \frac{\alpha a_0^2 \int_0^\ell \varphi_0'(x)^2 dx + \beta a_0 \int_0^\ell \varphi_0(x)^2 dx + \beta(S_0 - a_0\ell)\varphi_0(0)^2 + \sigma a_0^2 \varphi_0(\ell)^2}{a_0^2 \int_0^\ell \varphi_0(x)^2 dx + \delta \varphi_0^2(0)} = \bar{\lambda}. \quad (35)$$

Conversely, considering any sequence $(a_n, a_n\sqrt{1+a_n'^2})_{n \in \mathbb{N}}$ that τ -converges to $(a_0, a_0 + (S_0 - a_0\ell)\delta_0)$ (for instance, the one exhibited in Remark 4) yields

$$\sup_{a \in \mathcal{S}_{a_0, \ell, S_0}} \lambda_1(a) \geq \bar{\lambda},$$

by using Proposition 1. The conclusion follows.

4 Perspectives

In this article, we have addressed the issue of finding the optimal shape of a fin, by using simplifying assumptions about the model: axisymmetric shape, one-dimensional parabolic equation governing the temperature inside the fin, etc. Even if such assumptions are often considered in the engineering literature (see e.g. [2, 17]), it would be natural to investigate a more realistic model of thermal conduction.

The simplified model considered in this article shows that a relaxation type phenomenon arises when investigating the existence of a fin optimizing cooling properties. We propose the following general formulation for a general thermal fin: the domain Ω occupied by the fin is assumed to be simply connected and bounded. Introduce Γ_i , the inlet of the fin, Γ_{lat} its lateral surface and Γ_o the outlet of the fin, so that $\partial\Omega = \Gamma_i \cup \Gamma_{lat} \cup \Gamma_o$. A possible temperature model for this fin writes

$$\begin{aligned} \frac{\partial T}{\partial t} - \Delta T &= 0 && \text{in } \Omega \\ T(0, \cdot) &= T_d && \text{in } \Omega \\ -k \frac{\partial T}{\partial \nu} &= h(T - T_\infty) && \text{on } \Gamma_{lat} \\ \frac{\partial T}{\partial t} &= \gamma \frac{\partial T}{\partial \nu} && \text{on } \Gamma_i \\ -k \frac{\partial T}{\partial \nu} &= h_r(T - T_\infty) && \text{on } \Gamma_o \end{aligned} \quad (36)$$

where all the constant are positive and can be defined according to [1, 2, 17], $T_d \in H^{1/2}(\Gamma_i)$, $\frac{\partial}{\partial \nu}$ is the outward normal derivative on the boundary $\partial\Omega$.

Performing the same asymptotic analysis as the one in Section 1.2, it appears relevant to investigate the shape optimization problem

$$\inf_{\substack{\Omega \text{ open} \\ \text{Per}(\Omega) = P_0}} \mu_1(\Omega), \quad (37)$$

where P_0 is a positive real number, and $\mu_1(\Omega)$ is the first eigenvalue associated to the eigenvalue problem

$$\begin{aligned} -\Delta \varphi &= \mu(\Omega)\varphi && \text{in } \Omega \\ -k \frac{\partial \varphi}{\partial \nu} &= h\varphi && \text{on } \Gamma_{lat} \\ \gamma \frac{\partial \varphi}{\partial \nu} &= -\mu(\Omega)\varphi && \text{on } \Gamma_i \\ -k \frac{\partial \varphi}{\partial \nu} &= h_r\varphi && \text{on } \Gamma_o \end{aligned} \quad (38)$$

We foresee to analyze this shape optimization problem in a future study.

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A Proof of Lemma 1

Note first that $H^1(0, \ell) \hookrightarrow \mathcal{C}_a$ continuously. Moreover, for every f and g in \mathcal{C}_a , there holds using an integration by parts and introducing $\varphi_a = \mathcal{L}_a f$ and $\psi_a = \mathcal{L}_a g$,

$$\begin{aligned} \langle \mathcal{L}_a f, g \rangle_a &= \langle \varphi_a, g \rangle_a \\ &= \int_0^\ell a(x)^2 \varphi_a(x) g(x) dx + \frac{\alpha}{\gamma} \varphi_a(0) g(0) \\ &= -\alpha \int_0^\ell a(x)^2 \varphi'_a(x) \psi'_a(x) dx - \sigma a(\ell)^2 \varphi_a(\ell) \psi_a(\ell) - \beta \int_0^\ell a(x) \sqrt{1 + a'(x)^2} \varphi_a(x) \psi_a(x) dx \\ &= \langle f, \psi_a \rangle_a = \langle f, \mathcal{L}_a g \rangle_a, \end{aligned}$$

since φ_a solves Equation (10). We thus infer that \mathcal{L}_a is selfadjoint in \mathcal{C}_a .

Let us show that \mathcal{L}_a is compact. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{C}_a such that $\|f_n\|_a \leq 1$ for every $n \in \mathbb{N}$, where $\|\cdot\|_a$ stands for the norm induced by the inner-product $\langle \cdot, \cdot \rangle_a$. We will prove that, up to a subsequence, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ where $\varphi_n = \mathcal{L}_a f_n$ converges in \mathcal{C}_a . Let $n \in \mathbb{N}$. Multiplying the main equation of (10) (where φ and f have been respectively replaced by φ_n and f_n) by φ_n , one gets by using an integration by parts,

$$\alpha \int_0^\ell a(x)^2 \varphi'_n(x)^2 dx + \sigma a(\ell)^2 \varphi_n(\ell)^2 + \beta \int_0^\ell a(x) \sqrt{1 + a'(x)^2} \varphi_n(x)^2 dx = -\langle \varphi_n, f_n \rangle_a.$$

On the one hand,

$$\alpha \int_0^\ell a(x)^2 \varphi'_n(x)^2 dx + \sigma a(\ell)^2 \varphi_n(\ell)^2 + \beta \int_0^\ell a(x) \sqrt{1 + a'(x)^2} \varphi_n(x)^2 dx \geq \min\{\alpha a_0^2, \beta a_0\} \|\varphi_n\|_{H^1(0, \ell)}^2,$$

and on the other hand, combining the Cauchy-Schwarz inequality with the continuity of the mapping $\text{Tr} : H^1(0, \ell) \ni \varphi \mapsto \varphi(0)$, one gets the existence of a positive constant C such that

$$|\langle \varphi_n, f_n \rangle_a| \leq \|f_n\|_a \|\varphi_n\|_a \leq \|\varphi_n\|_a \leq C \|\varphi_n\|_{H^1(0, \ell)}.$$

Therefore, the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is bounded in $H^1(0, \ell)$ and converges thus up to a subsequence, weakly in $H^1(0, \ell)$ and strongly in $L^2(0, \ell)$ to some element $\varphi \in H^1(0, \ell)$ according to Rellich compactness embedding theorem. Finally, by using the compactness of the mapping Tr , one gets that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges to φ in \mathcal{C}_a .

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