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# Smoothing effect in $BV_\Phi$ for entropy solutions of scalar conservation laws

PIERRE CASTELLI\*, STÉPHANE JUNCA†

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## Abstract

This paper deals with a sharp smoothing effect for entropy solutions of one-dimensional scalar conservation laws with a degenerate convex flux. We briefly explain why degenerate fluxes are related with the optimal smoothing effect conjectured by Lions, Perthame, Tadmor for entropy solutions of multidimensional conservation laws. It turns out that generalized spaces of bounded variation  $BV_\Phi$  are particularly suitable -better than Sobolev spaces- to quantify the regularizing effect and to obtain traces as in  $BV$ . The function  $\Phi$  in question is linked to the degeneracy of the flux. Up to the present, the Lax-Oleïnik formula has provided optimal results for an uniformly convex flux. In this paper we first need to validate this formula for a more general flux, i.e. a  $C^1$  strictly convex flux. We also give a complete proof that for a nonlinear degenerate convex flux the Lax-Oleïnik formula provides the unique entropy solution, namely the Kruzkov solution.

**Keywords:** scalar conservation laws, entropy solution, smoothing effect, traces, Lax-Oleïnik formula, generalized  $BV$  spaces,  $C^1$  degenerate strictly convex flux, modulus of degeneracy.

**Mathematics Subject Classification:** 35L65 ; 35B65 ; 35C05.

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## 1 Introduction

This paper focuses on a smoothing effect for the entropy solution of the nonlinear scalar conservation law:

$$\partial_t u + \partial_x f(u) = 0, \quad u(x, t = 0) = u_0(x) \in L^\infty. \quad (1.1)$$

This regularizing effect is linked to the nonlinearity of the flux  $f$ . Indeed, if  $f(u) = cu$  is a linear flux, then the solution is  $u(x, t) = u_0(x - ct)$ , so that the regularity of the initial data is not improved. Lax and Oleřnik proved in the 1950s ([La2, O]) that, if  $f$  is an uniformly convex flux, then the solution becomes immediately more regular. More precisely, for all  $t > 0$ ,  $x \mapsto u(x, t)$  is locally in  $BV$ , the space of functions of bounded variation. In particular, the solution admits traces everywhere -right traces and left traces-, like shock waves. For more general convex fluxes like  $f(u) = |u|^3$  or  $f(u) = u^4$ , K.S. Cheng ([Cheng1]) showed that there is no more regularization in  $BV$ . There are only few results quantifying in some Banach spaces the improved regularity of the entropy solutions. After Lax-Oleřnik in the 1950s it took until the 1990s ([LPT]) to get a smoothing effect in Sobolev spaces for a general multidimensional nonlinear flux. Furthermore Lions, Perthame, Tadmor conjectured the optimal smoothing effect.

We first point out a link between the multidimensional regularizing effect and the one-dimensional one for degenerate fluxes. So consider the scalar conservation law

$$\partial_t v + \operatorname{div}_X F(v) = 0, \quad v(X, 0) = v_0(X) \in L^\infty(\mathbb{R}^d, \mathbb{R}),$$

where  $X \in \mathbb{R}^d$ . Assume for example that  $d = 3$  and  $F(v) = (f(v), g(v), h(v))$ , then the equation

becomes

$$\partial_t v + \partial_x f(v) + \partial_y g(v) + \partial_z h(v) = 0, \quad v(x, y, z, 0) = v_0(x, y, z) \in L^\infty(\mathbb{R}^3, \mathbb{R}).$$

Consider the nonlinear most degenerate scalar flux among  $f, g, h$  (assume that it is  $h$ ) and then the one-dimensional corresponding equation

$$\partial_t u + \partial_z h(u) = 0, \quad u(0, z) = u_0(z).$$

If we choose  $v_0(x, y, z) = u_0(z)$ , then the entropy solution is  $v(x, y, z, t) = u(z, t)$ , thus the multidimensional smoothing effect cannot exceed the one-dimensional one associated to the less nonlinear scalar flux. This is a key point in [Ju] to bound the maximal regularizing effect conjectured by Lions, Perthame, Tadmor in [LPT] and also to enable the propagation of high frequency waves in [CJR]. For instance, the simplest genuinely nonlinear multidimensional flux ([CJR, COW]) generating a smoothing effect is not  $f(v) = v^2, g(v) = v^2, h(v) = v^2$  (since  $v(x, y, z, t) = U(x - y)$  is a stationary solution) but

$$f(v) = v^2, g(v) = v^3, h(v) = v^4.$$

This vectorial flux involves the nonlinear degenerate cubic and quartic fluxes, for which there is no  $BV$  smoothing effect [Cheng1]. This is the reason why we are interested in degenerate nonlinear fluxes.

Another regularizing effect was obtained by De Lellis, Otto, Westdickenberg in [DOW1]: with only a  $L^\infty$  initial data, entropy solutions have got traces like  $BV$  functions. Lions, Perthame, Tadmor did not recover this traces regularity, since their work involves fractional Sobolev spaces  $W^{s,p}$  with too small regularity. More precisely, the exponents  $s$  and  $p$  satisfy  $sp < 1$  in the one-dimensional case, which does not enable traces. However if  $sp > 1$ , then the regularity is too large -functions are continuous-, which does not enable shocks. The suitable Sobolev space could only be  $W^{s,p}(\mathbb{R})$  with  $p = \frac{1}{s}$ , but it does not work too. So our idea was to look for a space which would give the smoothing effect and the traces properties simultaneously. The generalized space of bounded variation  $BV_\Phi$  provides the satisfying framework. Shortly speaking, the function  $u$  is in  $BV_\Phi$  if the total  $\Phi$ -variation of  $u$ :

$$TV^\Phi u = \sup_{n \in \mathbb{N}^*, x_0 < x_1 < \dots < x_n} \sum_{i=1}^n \Phi(|u(x_i) - u(x_{i-1})|)$$

is finite. We precisely recall in Section 5 the properties of generalized  $BV$  spaces related to a positive convex function  $\Phi$ . The function  $\Phi$  quantifies the regularity of the solutions and is linked to the nonlinearity of the flux  $f$ . If the degeneracy of the flux is like a power law, then  $\Phi$  is simply a power law and we get the fractional  $BV$  spaces  $BV^s$ . In this case the optimality of the smoothing effect is obtained for a smooth general convex flux in [CJ] and [BGJ]. It yields the optimal smoothing effect conjectured by [LPT] in  $W^{s,p}(\mathbb{R}_x, \mathbb{R})$  with the optimal  $s$  and the optimal

$p = \frac{1}{s}$ . We extend the smoothing effect proved in [BGJ] for any  $C^1$  strictly convex flux, more general than a flux with a power law behavior.

The first tool which has provided optimal regularity results is the Lax-Oleïnik formula for an uniformly convex flux. In order to get our smoothing effect -in the end part of this article- we first need to rigorously validate the well-known Lax-Oleïnik formula for a nonlinear degenerate convex flux. Then we will give an uniformly  $BV_\Phi$  regularizing effect for entropy solutions of a one-dimensional nonlinear scalar conservation law with only a  $L^\infty$  data and a  $C^1$  strictly convex flux. Usually authors consider an uniformly convex flux  $f$ , i.e. a  $C^2$  flux such that  $\inf f'' > 0$ . The main example is the Burgers' flux:  $f(u) = u^2$ . Subsequently we consider the more general case of a  $C^1$  strictly convex flux:

**Definition 1.** [ $C^1$  strictly convex flux]  $f$  is a  $C^1$  strictly convex flux if its derivative  $f'$  is increasing.

Many papers -for instance [ADGV, AMV, AV, Gh, JVG, Le1]- use the Lax-Oleïnik formula in this context. However, up to our knowledge, the direct link with the Kruzkov entropy solution for this larger class of fluxes was never written. An important part of the paper (Sections 3, 4 and Appendix) is devoted to the Lax-Oleïnik formula for a  $C^1$  strictly convex flux with a  $L^\infty$  initial data. In particular we show that the Lax-Oleïnik formula provides traces. Regulated functions are strongly related to the generalized  $BV$  spaces. Indeed, for every regulated function  $u$ , there exists a function  $\Phi$  such that  $u \in BV_\Phi$  ([GMW]). In the end of our paper we show that the function  $\Phi$  is the same for all  $t > 0$  and for all solutions with the same bound  $|u| \leq M$ .

There are also other approaches of the regularizing effect for entropy solutions and also for a larger class of solutions, the solutions with bounded entropy production (BEP solutions). For entropy solutions of one-dimensional scalar conservation laws, a generalized one-sided Oleïnik condition and a  $BV$  regularity for  $a(u)$  are obtained by Dafermos, Cheng, Jenssen, Sinestrari ([D3, Cheng2, JS]). This regularity does not provide immediately regularity for  $u$ . For instance, set  $f(u) = \frac{u^3}{3}$  and  $a(u) = u^2$ , then a function taking only the values 1 and  $-1$  will not be in  $BV$  but satisfies  $a(u) = 1$  everywhere ! Nevertheless, some authors use the regularity of  $a(u)$  coupled with the kinetic formulation to obtain traces ([DR]) and also an optimal smoothing effect in  $W^{s,1}$  ([Ja]). The bounds of the optimality are established in [DW] and [CJ]. The Hamilton-Jacobi approach ([E]) provides the entropy solution but the regularity is not easy to obtain since we have to differentiate the viscosity solution to study the entropy solution. Notice also that the compactness result given by Panov ([Pa2, Pa3]) for a continuous or discontinuous flux can be interpreted as a regularizing effect. In the multidimensional case, a  $BV$  regularity only for some averagings of  $u$  on some hyperplanes is obtained in [Chev].

BEP solutions are not studied in our paper. This larger class of solutions is the natural framework to use the kinetic formulation of scalar conservation laws ([LPT]). In the one-dimensional case and for an uniformly convex flux, the regularity is bounded from above ([DW]) and its optimality is proved ([Go, GP]) using quantitative estimates throu compensated compactness.

In Section 2 we state the main results namely the  $BV_\Phi$  smoothing effect and the validity of the Lax-Oleřnik formula. Section 3 recalls basics on the Lax-Oleřnik formula and we give then some stability results and traces properties, which will be used in the following section. In Section 4 we prove that the Lax-Oleřnik formula provides the Kruzkov (entropy) solution for a larger class of degenerate convex fluxes. Finally, we recall in Section 5 the definitions and the main properties of the  $BV_\Phi$  spaces, we quantify the degeneracy of a  $C^1$  strictly convex flux and we prove the smoothing effect in this class of fluxes.

## 2 Main results

Our main result is the uniform  $BV_\Phi$  regularizing effect for entropy solutions of (1.1) with only a  $L^\infty$  initial data and a  $C^1$  strictly convex flux (see Definition 1). In order to prove this, we will first validate the Lax-Oleřnik formula for this larger class of fluxes, which contains for instance the convex power law fluxes:  $f(u) = |u|^{1+\alpha}$ ,  $\alpha > 0$ .

The regularizing effect depends on the nonlinearity of the flux. We quantify this nonlinearity by introducing the modulus of degeneracy of a  $C^1$  strictly convex flux  $f$ . Suppose that  $u_0(\mathbb{R}) \subset [-M, M]$ ,  $M > 0$ . The modulus of degeneracy of  $f$  for  $h \in [0, 2M]$  is defined with its derivative  $a = f'$ :

$$\varphi(h) = \min_{-M \leq x \leq M-h} |a(x+h) - a(x)|. \quad (2.1)$$

We denote by  $\Phi$  the greatest convex function such that  $0 \leq \Phi \leq \varphi$  on  $[0, 2M]$ .

**Theorem 1.** *Let the initial data  $u_0$  belong to  $L^\infty(\mathbb{R}, \mathbb{R})$ ,  $M \geq \|u_0\|_\infty$  and  $f$  be a  $C^1([-M, M], \mathbb{R})$  strictly convex flux. Then the Kruzkov entropy solution  $x \mapsto u(x, t)$  belongs to  $BV_{\Phi, loc}(\mathbb{R}_x, \mathbb{R})$  for all  $t > 0$ .*

*Remark 1.*

1. The strictly convexity of  $f$  on  $[\inf u_0, \sup u_0]$  is enough since the entropy solution satisfies the maximum principle.
2. The bound of the total  $\Phi$ -variation in  $BV_{\Phi, loc}$  depends only on  $M$ , that is to say that it is uniform for the  $L^\infty$  ball of initial data  $\{u_0, \|u_0\|_\infty \leq M\}$ .
3. For a convex power law flux  $f(u) = \frac{|u|^{1+\alpha}}{1+\alpha}$ ,  $\alpha > 0$ , we have  $\Phi(u) = \varphi(u) = |u|^s$ ,  $s = \max(1, \alpha)$  and then  $BV_\Phi = BV^s$  ([BGJ]).
4. Our results handle a more general degeneracy than the power law degeneracy. Take for instance the very flat flux  $f(u) = \exp(-2/u^2)$ ,  $|u| \leq 1$  or a "near power law" flux:  $f(u) = -\frac{|u|^{1+\alpha}}{\ln(|u|)}$ ,  $|u| < 1$ ,  $\alpha > 1$ .

5. For every positive time the entropy solution is a regulated function. Thus, the Lax entropy condition is then well defined and enough to single out the unique Kruzkov entropy solution. It is well known for an uniformly convex flux that one entropy is enough to characterize the Kruzkov solution ([D4, DOW2, Pa1]).

We now recall the Lax-Oleïnik formula in Definition 2 below. Historically it was established for an uniformly convex and superlinear flux  $f$ . We claim that we can generalize this formula for a  $C^1$  strictly convex flux on  $[\inf u_0, \sup u_0]$ , without assuming  $f$  to be superlinear. The precise arguments are given in Section 3, in which we also give some useful properties about the Lax-Oleïnik formula : stability, traces, Lax's entropy condition.

**Definition 2. [Lax-Oleïnik solution]**

Let  $f$  be a  $C^1$  strictly convex flux and  $u_0 \in L^\infty$ . If necessary, we modify  $f$  outside  $[\inf u_0, \sup u_0]$  so that  $f$  becomes superlinear. We denote  $a = f'$  the velocity,  $b = a^{-1}$  its inverse function,  $g$  the Legendre transform of  $f$ :  $g' = b$ , and  $U'_0 = u_0$  an antiderivative of  $u_0$ . The following function

$$h_{(x,t)}(y) = U_0(y) + tg \left( \frac{x-y}{t} \right)$$

admits at least one minimizer  $y = y(x, t)$  for  $t > 0$  and an unique minimizer for almost all  $x$ . The Lax-Oleïnik solution denoted by  $\mathcal{LO}[f, u_0]$  is:

$$\mathcal{LO}[f, u_0](x, t) = u(x, t) = b \left( \frac{x - y(x, t)}{t} \right). \tag{2.2}$$

**Fact 1.** Notice that this formula does not depend on the extension of  $f$  (see Proposition 8).

According to [La1, La2, E] the function  $u(x, t)$  is uniquely defined almost everywhere. Notice that the Legendre transform  $g$  is strictly convex and superlinear. Lax and Oleïnik used their formula in the 1950s. Twenty years later, Kruzkov [K] stated his general existence and uniqueness theorem related to entropy condition and for all  $C^1$  fluxes without any convex assumptions. Let us recall the definition of a Kruzkov entropy solution.

**Definition 3. [Kruzkov entropy solution]**

A solution is said to be a Kruzkov entropy solution if for every convex function  $\eta$ ,

$$\frac{\partial}{\partial t}(\eta(u)) + \frac{\partial}{\partial x}(q(u)) \leq 0 \text{ in the sense of distributions on } (0, T) \times \mathbb{R}, \text{ where } q' = \eta' f'; \tag{2.3}$$

and if the initial data is recovered in  $L^1_{loc}(\mathbb{R}, \mathbb{R})$ :

$$\text{ess lim}_{t \rightarrow 0} u(x, t) = u_0(x). \tag{2.4}$$

Kruzkov [K] showed that there exists then a unique solution  $u(x, t)$  of (1.1) satisfying both conditions (2.3) and (2.4) above. The solution given by Kruzkov's theorem will be denoted by  $\mathcal{K}[f, u_0]$ .

Notice that the Kruzkov solution is a weak solution of (1.1) with the convex (degenerate) entropies  $\eta(u) = \pm u$ .

A natural question, already asked by Lax himself in 1954 ([La1] first conjecture p.6), is the link between  $\mathcal{LO}[f, u_0]$  and the solution given by the viscosity method, which will be twenty years later known as  $\mathcal{K}[f, u_0]$ .

**Theorem 2.** *If  $u_0 \in L^\infty(\mathbb{R})$  and  $f \in C^1$  is strictly convex on  $[\inf u_0, \sup u_0]$ , then*

$$\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0].$$

In other terms the Kruzkov entropy solution is represented by the Lax-Oleřnik formula.

*Remark 2.* We need the continuity of the velocity  $a = f'$  to generalize the Lax-Oleřnik formula and also to define the function  $\Phi$ . An open question is what occurs with a nonlinear convex but not differentiable flux, for instance with a Lipschitz nonlinear convex flux ?

To prove Theorem 2 in Section 4, we first prove in Subsection 3.2 the stability of the Lax-Oleřnik formula with respect to the flux and the initial data. Then we use the same well-known stability for the Kruzkov solution.

### 3 Lax-Oleřnik formula for a $C^1$ strictly convex flux

In this section we give some useful results on the Lax-Oleřnik formula before proving in the next section that the Lax-Oleřnik solution is the Kruzkov solution.

First, we recall shortly that the Lax-Oleřnik formula is well defined for a  $C^1$ -strictly convex and superlinear flux, see also [ADGV, AMV, AV]. Second, we obtain stability of the Lax-Oleřnik formula with respect to the flux and the initial data. Third, the traces property is simply derived from the Lax-Oleřnik formula, without any  $BV$  regularity and before the proof of the  $BV_\Phi$  regularity. Finally, we explain why the superlinearity of the flux is not a fundamental assumption.

#### 3.1 Lax-Oleřnik formula revisited

We first emphasize on the convexity involved in our paper :  $f \in C^1$  is strictly convex if  $a(u) = f'(u)$  is an increasing function. [La1] considers the uniformly convex case:  $f''(u) > \delta > 0$  on  $\mathbb{R}$ . Various authors [E, H, La1, La2] consider the convex case  $f''(u) > 0$  everywhere. Indeed, on any compact set this condition is equivalent to the uniform convexity. Definition 1 allows the second derivative  $f''$  to vanish, when  $f$  is smooth (for instance power law). In order to use the Lax-Oleřnik formula, we first need to suppose  $f$  to be superlinear, i.e.  $\lim_{|u| \rightarrow +\infty} \frac{f(u)}{|u|} = +\infty$ . In fact, this assumption is not essential since the behavior of  $f$  is important only on the segment  $[\inf u_0, \sup u_0]$ .

The Lax-Oleřnik formula is related to the Legendre transform  $g$  of the flux  $f$  ([E]) and the inverse  $b$  of the function velocity  $a$ :



$$\forall v \in \mathbb{R}, g(v) = \max_u (vu - f(u)) = vb(v) - f(b(v)). \quad (3.1)$$

**Lemma 1.** 1)  $a : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism.

2) Let  $b$  be the inverse of  $a$  and let  $g$  be the antiderivative of  $b$  such that  $g(a(0)) = 0$ . Then  $g$  is strictly convex and superlinear.

The second part of the lemma is well known for the Legendre transformation and useful to have a well posed minimization problem. The first part is simple but new. Indeed, notice that the velocity  $a(\cdot)$  is not a diffeomorphism as in [La2, O]. It is the reason why  $BV$  regularity is lost ([BGJ, CJ]).

*Proof.* 1) Since  $f$  is strictly convex,  $a(u) = f'(u)$  is increasing. If  $a \leq C$ , then for  $u \geq 0$ ,  $f(u) - f(0) \leq Cu$ , in contradiction with the superlinearity of  $f$ . In the same way, we have  $\inf a = -\infty$ . So  $a$  is not bounded and is a homeomorphism.

2) Since  $b = a^{-1}$ ,  $b$  is also an increasing homeomorphism and then  $g$  is strictly convex. Let  $A > 0$ , for  $u$  large enough ( $v \geq v_A$ ),  $b(v) \geq A$ , so  $g(v) - g(v_A) \geq A(v - v_A)$ , which proves  $\lim_{v \rightarrow +\infty} \frac{g(v)}{v} = +\infty$ . We prove similarly that  $\lim_{v \rightarrow -\infty} \frac{g(v)}{v} = -\infty$ . Then  $g$  is superlinear.  $\square$

We define  $U_0(y) = \int_0^y u_0(z) dz$ . The two following results are already proved in [H, La2] (see also for instance [E] for the case  $f$  uniformly convex). The proof is valid for  $f$  strictly convex.

**Proposition 1.** [Minimizer  $y(x, t)$ ]

1) For all  $(x, t)$  with  $t > 0$ , there exists at least a real  $y = y(x, t)$  which minimizes

$$h_{(x,t)}(y) = U_0(y) + tg \left( \frac{x - y}{t} \right).$$

2) Let  $t > 0$ . For all  $x$ , except on a set at most countable, there exists only one real  $y = y(x, t)$  which minimizes  $h_{(x,t)}(y)$ .

Notice that  $U_0$  has at most a linear growth at infinity, so the superlinearity and the convexity of the function  $g$  is enough to get a well posed minimization problem with at least one solution. Again, the convexity of  $g$  yields to monotonicity of the minimizer [E, H, La2].

**Lemma 2.** Let  $t > 0$ . If for all  $x$ ,  $y(x, t)$  denotes a minimizer related to  $(x, t)$ , then for all  $x_1, x_2$  such that  $x_1 < x_2$ ,  $y(x_1, t) \leq y(x_2, t)$  (we say that  $x \mapsto y(x, t)$  is non-decreasing).

For convenience, we recall here that the Lax-Oleïnik solution is given by

$$\mathcal{LO}[f, u_0](x, t) = b \left( \frac{x - y(x, t)}{t} \right)$$

uniquely a. e. and also everywhere but not uniquely since it can depend on the choice of  $y(x, t)$ . For fixed  $(x, t)$ , denote by  $y^+(x, t)$  and  $y^-(x, t)$  the largest and smallest of  $y$  for which the function

$h_{(x,t)}(y)$  assumes its minimum. The Lax-Oleïnik formula can be defined uniquely everywhere by  $\mathcal{LO}^+$  or  $\mathcal{LO}^-$  :

$$\mathcal{LO}^\pm[f, u_0](x, t) = b \left( \frac{x - y^\pm(x, t)}{t} \right). \quad (3.2)$$

There are obvious consequences of this representation formula, first when  $u_0$  is continuous and then for only  $u_0$  in  $L^\infty$  in Proposition 5.

**Proposition 2.** *Let  $u_0$  be a continuous bounded function.*

**1) [Method of characteristics]** *If  $(x, t)$  is a point of continuity of  $y(x, t)$  and if  $u_0$  is continuous at  $y(x, t)$ , then:*

$$\mathcal{LO}[f, u_0](x, t) = u_0(y(x, t)). \quad (3.3)$$

*In particular, if  $u_0$  is continuous, then Equality (3.3) is valid almost everywhere.*

**2) [Maximum principle]** *For almost all  $(x, t)$ :  $\inf_{y \in \mathbb{R}} u_0(y) \leq u(x, t) \leq \sup_{y \in \mathbb{R}} u_0(y)$ . In particular:*

$$|u(x, t)| \leq \|u_0\|_\infty.$$

**3) [Finite speed of propagation]** *For almost all  $(x, t)$ :  $|y(x, t) - x| \leq t \|a(u_0)\|_\infty$ .*

*Proof.* According to the definition of  $y(x, t)$  we have indeed:

$$0 = \frac{\partial h_{(x,t)}}{\partial y}(y(x, t)) = u_0(y(x, t)) - b \left( \frac{x - y(x, t)}{t} \right),$$

So for almost all  $(x, t)$ :

$$u_0(y(x, t)) = b \left( \frac{x - y(x, t)}{t} \right) \text{ and } y(x, t) = x - t a(u_0(y(x, t))).$$

The three statements follow from these last equalities. □

## 3.2 Stability

We start this subsection by proving two lemmas, which we use later to get stability properties.

**Lemma 3.** *For  $n \in \mathbb{N}$ , let  $\alpha_n : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous, bijective and increasing function. Assume that  $(\alpha_n)$  converges pointwise to a function  $\alpha$  which is continuous, bijective and increasing. Then  $(\beta_n)$  converges uniformly to  $\beta$  on each segment of  $\mathbb{R}$ , where  $\beta_n$  (respectively  $\beta$ ) is the inverse of  $\alpha_n$  (respectively  $\alpha$ ).*

*Proof.* Let  $J = [l, r]$  be a segment of  $\mathbb{R}$ . Let  $I = \beta(J) = [\beta(l), \beta(r)]$  and for  $n \in \mathbb{N}$ ,  $I_n = \beta_n(J) = [\beta_n(l), \beta_n(r)]$ .

i) We first prove the pointwise convergence of  $(\beta_n)$  towards  $\beta$ . Let  $\varepsilon > 0$  and  $x \in \mathbb{R}$ . Since  $\alpha_n(\beta(x) + \varepsilon) \rightarrow \alpha(\beta(x) + \varepsilon)$  and  $\alpha(\beta(x) + \varepsilon) > x$ , then for  $n$  large enough,  $\alpha_n(\beta(x) + \varepsilon) \geq x$ .

It follows that  $\beta(x) + \varepsilon \geq \beta_n(x)$ . Similarly for  $n$  large enough, the following inequality holds:  $\beta(x) - \varepsilon \leq \beta_n(x)$ . Then  $\beta_n(x) \rightarrow \beta(x)$ .

ii) It follows in particular from i) that there exists a segment  $K$  such that  $I \subset K$  and for  $n$  large enough ( $n \geq n_0$ ),  $I_n \subset K$ .

iii) It follows from Dini's second theorem that the convergence of  $(\alpha_n)$  to  $\alpha$  is uniform on  $K$ . Let  $\varepsilon > 0$ . Since  $\alpha$  is a homeomorphism,  $\beta$  is continuous, so  $\beta$  is uniformly continuous on the segment  $J$ : there exists  $\eta > 0$  such that for all  $y, y' \in J$ ,  $|y - y'| \leq \eta \implies |\beta(y) - \beta(y')| \leq \varepsilon$ . Since  $(\alpha_n)$  converges uniformly to  $\alpha$  on  $K$ , there exists  $n_1 \geq n_0$  such that for all  $n \geq n_1$  and for all  $x \in K$ ,  $|\alpha_n(x) - \alpha(x)| \leq \eta$ . For all  $n \geq n_1$  and  $y \in J$ ,  $|\alpha_n(\beta_n(y)) - \alpha(\beta_n(y))| \leq \eta$ , i.e.  $|y - \alpha(\beta_n(y))| \leq \eta$ , therefore:  $|\beta(y) - \beta_n(y)| \leq \varepsilon$ . It follows that  $(\beta_n)$  converges uniformly to  $\beta$  on  $J$ .  $\square$

The second lemma is a simple "gamma-convergence" result.

**Lemma 4.** *Let  $K$  be a segment of  $\mathbb{R}$ . Let  $\varphi$  and  $\varphi_n, n \in \mathbb{N}$  be functions defined on  $K$  such that:*

- 1)  $\varphi$  has a unique minimizer  $x$  in  $K$  ;
- 2)  $\varphi_n$  has a minimizer  $x_n$  in  $K$  ;
- 3)  $(\varphi_n)$  converges uniformly to  $\varphi$  on  $K$ .

*Then  $(x_n)$  converges to  $x$ .*

*Proof.* Let  $r > 0$ . By the uniqueness of the minimizer  $x$ , there exists  $\gamma > 0$  such that for all  $y \in K$ ,  $|y - x| > r \implies \varphi(y) > \varphi(x) + 2\gamma$ . For  $n$  large enough,  $\|\varphi_n - \varphi\|_\infty \leq \gamma$ , so if  $|y - x| > r$ , then  $\varphi_n(y) \geq \varphi(y) - \gamma > \varphi(x) + \gamma$  whereas  $\varphi_n(x) \leq \varphi(x) + \gamma$ . It follows that for  $n$  large enough,  $|x_n - x| \leq r$ , so  $(x_n)$  converges to  $x$ .  $\square$

We are now in a position to prove stability properties in the next two propositions.

**Proposition 3. [Stability with regard to the flux]** *Let  $u_0 \in L^\infty$  and for  $n \in \mathbb{N}$ ,  $f_n$  strictly convex and superlinear such that  $(f_n)$  converges to  $f$  in  $C^1_{loc}$ , where  $f$  is strictly convex and superlinear. Then  $(\mathcal{LO}[f_n, u_0])_n$  converges to  $\mathcal{LO}[f, u_0]$  in  $L^1_{loc}$  and also pointwise a.e..*

*Proof.* We choose  $(x, t)$  such that  $y(x, t)$  is uniquely defined so the minimization problem has a unique minimizer. According to (2.2), we write:  $\mathcal{LO}[f, u_0](x, t) = b \left( \frac{x - y(x, t)}{t} \right)$  (respectively  $\mathcal{LO}[f_n, u_0](x, t) = b_n \left( \frac{x - y_n(x, t)}{t} \right)$ ), where  $y(x, t)$  (respectively  $y_n(x, t)$ ) minimizes  $h_{(x,t)}(y) = U_0(y) + tg \left( \frac{x - y}{t} \right)$  (respectively  $h^n_{(x,t)}(y) = U_0(y) + tg_n \left( \frac{x - y}{t} \right)$ ). First note that  $(g_n)$  converges towards  $g$  in  $C^1_{loc}$ , since from Lemma 3  $(b_n)$  converges uniformly to  $b$ . Since  $g$  is convex and superlinear, we can restrict the minimization on a fixed compact set  $K$ . Moreover  $(g_n)$  converges to  $g$  in  $C^1(K)$  so, for  $n$  large enough,  $g_n$  admits its global minimizer in  $K$ . Indeed, we choose  $K = [c, d]$  large enough such that  $g$  on the boundary is greater than  $|g(0)| + 1$  and  $g'(c) < 0 < g'(d)$ . We can choose  $\varepsilon > 0$  small enough and  $n$  large enough such that  $g'(c) + \varepsilon < 0 < g'(d) - \varepsilon$ ,  $\|g_n - g\|_{C^1(K)} < \varepsilon$ , thus the minimizers of  $g_n$  are still in the same compact  $K$ . It follows from

Lemma 4 that  $(y_n(x, t))_n$  converges to  $y(x, t)$ . We conclude then that  $(\mathcal{LO}[f_n, u_0])_n$  converges pointwise to  $\mathcal{LO}[f, u_0]$ . Furthermore, the inequality 2) from Proposition 2 yields the convergence in  $L^1_{loc}$ .  $\square$

**Proposition 4. [Stability with regard to the initial data]** *Let  $f \in C^1$  be strictly convex and superlinear and for  $n \in \mathbb{N}$ ,  $u_0^n \in C^0 \cap BV$  such that  $(u_0^n)$  converges to  $u_0$  in  $L^1_{loc}$  and for all  $n$ ,  $\|u_0^n\|_\infty \leq \|u_0\|_\infty$ , where  $u_0 \in L^\infty$ . Then  $(\mathcal{LO}[f, u_0^n])_n$  converges to  $\mathcal{LO}[f, u_0]$  in  $L^1_{loc}$ .*

This result is already written in [La2] for an uniformly convex flux, also with respect to the weak convergence of the initial data. We gives a proof to be self-contained.

*Proof.* We choose  $(x, t)$  such that  $y(x, t)$  is uniquely defined so the minimization problem has a unique minimizer. According to (2.2), we write:  $\mathcal{LO}[f, u_0](x, t) = b\left(\frac{x - y(x, t)}{t}\right)$  (respectively  $\mathcal{LO}[f, u_0^n](x, t) = b\left(\frac{x - y_n(x, t)}{t}\right)$ ), where  $y(x, t)$  (respectively  $y_n(x, t)$ ) minimizes  $h_{(x, t)}(y) = U_0(y) + tg\left(\frac{x - y}{t}\right)$  (respectively  $h^n_{(x, t)}(y) = U_0^n(y) + tg\left(\frac{x - y}{t}\right)$ ). It follows from Lemma 4 that  $(y_n(x, t))_n$  converges to  $y(x, t)$ . Since  $b$  is continuous, we conclude that  $(\mathcal{LO}[f, u_0^n])_n$  converges pointwise a.e. to  $\mathcal{LO}[f, u_0]$ . Furthermore, the inequality 2) from Proposition 2 and the assumption  $\|u_0^n\|_\infty \leq \|u_0\|_\infty$  yield the convergence in  $L^1_{loc}$ .  $\square$

Thanks to Proposition 4 and Lemma 4 we can extend two results given above in Proposition 2: the maximum principle and the finite speed of propagation.

**Proposition 5.** *The points 2) and 3) of Proposition 2 are still valid for  $u_0 \in L^\infty$ .*

*Proof.* i) According to Proposition 2, we have for almost all  $(x, t)$ :  $\inf_{y \in \mathbb{R}} u_0(y) \leq u(x, t) \leq \sup_{y \in \mathbb{R}} u_0(y)$  for  $u_0 \in C^0 \cap L^\infty$ . Then by stability with respect to the initial data we keep the same result for  $u_0 \in L^\infty$ .

ii) According to Proposition 2, we have for almost all  $(x, t)$ :  $|y(x, t) - x| \leq t \|a(u_0)\|_\infty$  for  $u_0 \in C^0 \cap L^\infty$ . Suppose now that  $u_0 \in L^\infty$ . There exists a sequence  $(u_0^n)$  of  $C^0$  which converges pointwise to  $u_0$ . For all  $n$ , we get:  $|y_n(x, t) - x| \leq t \|a(u_0^n)\|_\infty$ . The inequality for  $u_0$  follows then from Lemma 4.  $\square$

### 3.3 Traces and Lax-entropy condition

The traces are a direct consequence of the Lax-Oleïnik formula. We find these traces again in Section 5 thanks to the  $BV_\Phi$  regularizing effect.

**Proposition 6.** *For all  $t > 0$ ,  $x \mapsto \mathcal{LO}[f, u_0](x, t)$  is a regulated function (it admits a left limit and a right limit at each point).*

This result implies that for each time  $t > 0$ , the entropy solution belongs to a space  $BV_\Phi$ . Indeed, for every regulated function  $u$ , there exists a convex function  $\Phi$  such that  $u \in BV_\Phi$  ([GMW]). In the end of our paper we show that the function  $\Phi$  is the same for all  $t > 0$  and for all solutions with the same bound  $\|u_0\|_\infty \leq M$ . Moreover, for each  $t > 0$ , the total  $\Phi$ -variation is also locally uniformly bounded.

*Proof.* Let  $t > 0$ . Since  $x \mapsto x$  and  $x \mapsto y(x, t)$  are non-decreasing,  $x \mapsto \frac{x - y(x, t)}{t}$  is of bounded variation and is then a regulated function. Since  $b$  is continuous,  $x \mapsto \mathcal{LO}[f, u_0](x, t) = b\left(\frac{x - y(x, t)}{t}\right)$  is also a regulated function.  $\square$

**Proposition 7.** *The function  $u = \mathcal{LO}[f, u_0]$  satisfies the Lax-entropy condition, i.e.: for each discontinuity at  $(x, t)$ ,  $a(u_r) < \frac{f(u_r) - f(u_l)}{u_r - u_l} < a(u_l)$ , where  $u_r$  (respectively  $u_l$ ) denotes the right (respectively left) limit of  $u(\cdot, t)$ .*

*Proof.* Let  $t > 0$  and  $x_1, x_2$  such that  $x_1 < x_2$ . Let  $y_1$  (respectively  $y_2$ ) be a minimizer related to  $(x_1, t)$  (respectively  $(x_2, t)$ ). According to Lemma 2,  $y_1 \leq y_2$ . Moreover:  $u(x_1, t) = b\left(\frac{x_1 - y_1}{t}\right)$  and  $u(x_2, t) = b\left(\frac{x_2 - y_2}{t}\right)$ . Since  $b$  is increasing, it follows that:  $b\left(\frac{x_1 - y_1}{t}\right) \geq b\left(\frac{x_1 - y_2}{t}\right)$ , so we get the inequality:  $u(x_2, t) - u(x_1, t) \leq b\left(\frac{x_2 - y_2}{t}\right) - b\left(\frac{x_1 - y_2}{t}\right)$ . According to Proposition 6,  $u(\cdot, t)$  is a regulated function. Since  $b$  is continuous, it follows from the previous inequality that  $u_r < u_l$ , so we deduce:  $a(u_r) < a(u_l)$ . Finally, since  $f$  is convex, the Lax-entropy condition is satisfied.  $\square$

### 3.4 About the flux superlinearity

To conclude this section we show that the superlinearity can be removed, simply, by modifying the flux outside  $[\inf u_0, \sup u_0]$ .

**Proposition 8.** *Let  $f, \tilde{f} \in C^1$  be superlinear strictly convex fluxes and  $u_0 \in C^0 \cap L^\infty$  ( $u_0(\mathbb{R}) \subset K := [-M, M]$ ,  $M > 0$ ). If  $f = \tilde{f}$  on  $K$ , then  $\mathcal{LO}[f, u_0] = \mathcal{LO}[\tilde{f}, u_0]$ .*

*Proof.* We start with  $u_0 \in C^0 \cap L^\infty$  and then by the stability with respect to the initial data we keep the same result for  $u_0 \in L^\infty$ .

Since  $f = \tilde{f}$  on  $K$ , it follows that  $a = \tilde{a}$  on  $K$ ,  $b = \tilde{b}$  on  $a(K)$  and  $g = \tilde{g}$  on  $a(K)$ . Let  $t > 0$ . For all  $x$ , except on a set at most countable, there exists only one real  $y = y(x, t)$  which minimizes  $h_{(x,t)}(y)$  and one real  $\tilde{y} = \tilde{y}(x, t)$  which minimizes  $\tilde{h}_{(x,t)}(y)$ . For almost all  $x$ ,  $(x, t)$  is a point of continuity of both  $y(x, t)$  and  $\tilde{y}(x, t)$ , therefore  $b\left(\frac{x - y}{t}\right) = \mathcal{LO}[f, u_0](x, t) = u_0(y)$  and  $\tilde{b}\left(\frac{x - \tilde{y}}{t}\right) = \mathcal{LO}[\tilde{f}, u_0](x, t) = u_0(\tilde{y})$ . In particular,  $\frac{x - y}{t} \in a(K)$  and  $\frac{x - \tilde{y}}{t} \in a(K)$ . But  $g = \tilde{g}$  on  $a(K)$ , so  $h_{(x,t)}(\tilde{y}) = \tilde{h}_{(x,t)}(\tilde{y}) \leq \tilde{h}_{(x,t)}(y) = h_{(x,t)}(y)$ , which means that  $\tilde{y}$  minimizes  $h_{(x,t)}$ . By uniqueness of the minimizer, it follows that  $\tilde{y} = y$ , and then  $\mathcal{LO}[f, u_0](x, t) = \mathcal{LO}[\tilde{f}, u_0](x, t)$ .  $\square$

**Fact 2.** *Proposition 8 above allows us to assume for instance (which we will afterwards) that  $f(u)$  is quadratic for  $u$  large enough:  $f(u) = \alpha u^2$ ,  $\alpha > 0$ . Then for  $u$  large enough,  $a$  and  $b = a^{-1}$  will be linear and  $g(u) = \beta u^2 + \gamma$ ,  $\beta > 0$ .*

We can now define the Lax-Oleïnik formula for a not superlinear flux.

**Definition 4. [Lax-Oleïnik formula for a general strictly convex flux]** Let  $f$  be a strictly convex flux on  $[\inf u_0, \sup u_0]$  and  $\tilde{f}$  be a superlinear strictly convex flux on  $\mathbb{R}$  such that  $\tilde{f} = f$  on  $[\inf u_0, \sup u_0]$ , then we define  $\mathcal{LO}[f, u_0] := \mathcal{LO}[\tilde{f}, u_0]$ .

According to Proposition 8 this definition does not depend on the extension  $\tilde{f}$ , but it depends on the initial data  $u_0$ .

We will prove in the next section that  $\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0]$ .

## 4 Lax-Oleïnik solution and Kruzkov solution

We prove in this section that the Lax-Oleïnik solution is the Kruzkov entropy solution for a general strictly convex flux and a bounded initial data. This result is well known for an uniformly convex flux. In the simpler case where the initial data is moreover, smooth, we detail completely Lax's proof in the Appendix. We derive the general case from the smooth case by using the stability arguments with respect to the flux and to the initial data (Propositions 3 and 4). We can assume without loss of generality that the flux is superlinear (see Definition 4).

We first prove the result for a general flux and a smooth initial data and then for a  $L^\infty$  initial data.

**Proposition 9.** *Assume that  $f \in C^1$  is strictly convex and superlinear and that  $u_0 \in C_c^0 \cap BV$ . Then  $\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0]$ .*

*Proof.* Consider a sequence  $(f_n)$  of fluxes of class  $C^2$  and uniformly convex such that  $(f_n)$  converges to  $f$  in  $C_{loc}^1$ . It follows from Proposition 3 that  $(\mathcal{LO}[f_n, u_0])_n$  converges to  $\mathcal{LO}[f, u_0]$  in  $L_{loc}^1$  and also pointwise a.e. and from Proposition 11 that  $\mathcal{LO}[f_n, u_0] = \mathcal{K}[f_n, u_0]$ , for all  $n$ . According to [Se], we have for all  $t > 0$  and for all  $A > 0$ :

$$\int_{-A}^A |\mathcal{K}[f_n, u_0](x, t) - \mathcal{K}[f, u_0](x, t)| dx \leq t Lip(f_n - f) TV u_0[-A - t \|a(u_0)\|_\infty, A + t \|a(u_0)\|_\infty],$$
 so we deduce that  $(\mathcal{K}[f_n, u_0])_n$  converges to  $\mathcal{K}[f, u_0]$  in  $L_{loc}^1$ . The equality  $\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0]$  follows then from the uniqueness of the limit in  $L_{loc}^1$ .  $\square$

We are finally in a position to achieve the proof of Theorem 2.

*Proof.* Consider a sequence  $(u_0^n)$  of initial data in  $C_c^0 \cap BV$  such that  $(u_0^n)$  converges to  $u_0$  in  $L_{loc}^1$  and is uniformly bounded. It follows from Proposition 4 that  $(\mathcal{LO}[f, u_0^n])_n$  converges to  $\mathcal{LO}[f, u_0]$  in  $L_{loc}^1$  and also pointwise a.e. and from Proposition 9 that  $\mathcal{LO}[f, u_0^n] = \mathcal{K}[f, u_0^n]$ , for all  $n$ . According to the  $L^1$ -contraction inequality of Kruzkov, we have for all  $t > 0$ :

$\int_{-A}^A |\mathcal{K}[f, u_0^n](x, t) - \mathcal{K}[f, u_0](x, t)| dx \leq \int_{-A-t\|a(u_0)\|_\infty}^{A+t\|a(u_0)\|_\infty} |u_0^n(x) - u_0(x)| dx$  for all  $A > 0$ , so we deduce that  $(\mathcal{K}[f_n, u_0])_n$  converges to  $\mathcal{K}[f, u_0]$  in  $L_{loc}^1$ . The equality  $\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0]$  follows then from the uniqueness of the limit in  $L_{loc}^1$ .  $\square$

## 5 $BV_\Phi$ uniform regularity

We show a smoothing effect in generalized spaces of bounded variation [MO]. A  $BV^s$  smoothing effect has already been proved in [BGJ] for any non flat  $C^\infty$  convex flux (more generally any flux with a power law degeneracy as stated in Definition 6 below). The optimality is proved in [CJ]. For a more general convex and nonlinear flux ( $C^1$  strictly convex flux) we obtain a  $BV_\Phi$  smoothing effect. We recall briefly the definition of these generalized  $BV$  spaces. The interest is that  $BV_\Phi$  keeps the same features as  $BV$ : left and right traces everywhere and compactness in  $L_{loc}^1$ , but with less smoothness. The function  $\Phi$  which measures the regularity of functions in  $BV_\Phi$  is related to the flux. The key tool to quantify the nonlinearity of  $f$  is the modulus of degeneracy defined by (2.1).

### 5.1 Generalized $BV$ spaces

We recall briefly the definitions of these generalized  $BV$  spaces. We refer the reader to [MO] for the first extensive study of  $BV_\Phi$  spaces.

**Definition 5. [ $BV_\Phi$  spaces]** Let  $I$  be an non-empty interval of  $\mathbb{R}$  and let  $\mathcal{S}(I)$  be the set of subdivisions of  $I$  :  $\{(x_0, x_1, \dots, x_n), n \geq 1, x_i \in I, x_0 < x_1 < \dots < x_n\}$ .

Let  $M > 0$  and  $\Phi$  a positive convex function on  $]0, 2M]$  such that  $\Phi(0) = 0$ .

i) If  $u$  is a function defined on  $I$ , such that  $|u| \leq M$  the total  $\Phi$ -variation of  $u$  on  $I$  is:

$$TV^\Phi u[I] = \sup_{\mathcal{S}(I)} \sum_{i=1}^n \Phi(|u(x_i) - u(x_{i-1})|)$$

where the supremum is taken on all subdivisions of the interval  $I$ .

ii) If  $\Phi$  satisfies the condition

$$(\Delta_2) \quad \exists h_0 > 0, k > 0, \Phi(2h) \leq k\phi(h) \quad \text{for } 0 \leq h \leq h_0,$$

then the space  $BV_\Phi(I)$  is the set of functions  $u$  defined on  $I$  such that  $TV^\Phi u[I] < +\infty$  and in this case  $BV_\Phi(I)$  is a linear space. Else  $BV_\Phi(I) = \{u : I \mapsto \mathbb{R}, \exists \lambda > 0, TV^\Phi(\lambda u)[I] < +\infty\}$  is a metric space.

Notice that [MO] consider the case  $\Phi(u) = o(|u|)$  near 0, which leads to a less regular space than  $BV$ :  $BV \not\subset BV_\Phi$ . The case where  $\Phi(u) = u$  or  $\Phi(u) \sim u$  near 0 yields  $BV = BV_\Phi$ . For degenerate fluxes, we are in the context of [MO]:  $\Phi(u) = o(|u|)$  near 0.

*Remark 3.* In the particular case where  $\Phi$  is a power function:  $\Phi(u) = |u|^\alpha$ ,  $\alpha > 1$ , with  $p = \frac{1}{s}$ , we get a space known as  $BV^s(I)$ . For  $s = 1$ , we get the space of functions of bounded variation.

**Example 1.** 1) Let  $\Phi(u) = \exp\left(-\frac{2}{u^2}\right)$ ,  $|u| \leq 1$ . Since  $\Phi(u) = o(|u|^\alpha)$  for all  $\alpha \geq 1$ , it follows that for all  $s \in ]0, 1]$ ,  $BV^s \subset BV_\Phi$ . In particular, it follows that for all  $s \in ]0, 1]$ ,  $BV^s \neq BV_\Phi$ .

2) Let  $\Phi(u) = -\frac{|u|^\alpha}{|\ln u|}$ ,  $|u| < 1$ ,  $\alpha \geq 1$ . The following inclusions hold for all  $\varepsilon > 0$  and for  $s = \frac{1}{\alpha}$ ,  $BV^s \subset BV_\Phi \subset BV^{s-\varepsilon}$ .

We recall the compact embedding theorem in  $L^1_{loc}$ :

**Theorem 3. [Helly's extracting theorem [MO]]** Every sequence  $(u_n) \in BV_\Phi(I)$  bounded in total  $\Phi$ -variation includes a subsequence convergent to a function  $u$  of the class  $BV_\Phi(I)$  pointwise in  $I$ .

The  $L^1_{loc}(I)$  convergence of the subsequence follows from the inclusion  $BV_\Phi(I) \subset L^\infty(I)$ .

*Remark 4.* The total  $\Phi$ -variation can be extended to the class of measurable functions defined almost everywhere by setting:  $TV^\Phi u[I] = \inf_{v=u \text{ a.e.}} TV^\Phi v[I]$ . For a function  $u$  defined a.e. we can also estimate  $TV^\Phi u[I]$  by  $TV^\Phi u[I] \leq \sup_{\tilde{S}(I)} \sum_{i=1}^n \Phi(|u(x_i) - u(x_{i-1})|)$ , where  $\tilde{S}(I)$  is the set of subdivisions of  $I \setminus D$  and  $D$  is a measure-zero set where  $u$  is not defined.

## 5.2 Modulus of degeneracy

In [BGJ] we can find the following definition of the degeneracy for nonlinear convex fluxes:

**Definition 6.** Let  $f \in C^1(K, \mathbb{R})$ , where  $K$  is a compact interval of  $\mathbb{R}$ . We say that the degeneracy of  $f$  on  $K$  is at least  $q > 0$  if the continuous derivative  $a(u) = f'(u)$  satisfies:

$$\inf_{(u,v) \in (K \times K) \setminus \mathcal{D}_K} \frac{|a(u) - a(v)|}{|u - v|^q} > 0, \quad (5.1)$$

where  $\mathcal{D}_K = \{(u, v) \in (K \times K) \mid u \neq v\}$ . The lowest real number  $q$ , if there exists, is called the degeneracy of  $f$  on  $K$  and denoted  $p$ .

However, this definition is not enough general to consider all  $C^1$  strictly convex fluxes such as flat fluxes. So we introduce for the monotonic function  $a(\cdot)$  the modulus of degeneracy, which is the key function to obtain new sharp generalized  $BV$  estimates. Suppose that  $u_0(\mathbb{R}) \subset [-M, M]$ ,  $M > 0$ . We recall formula (2.1) for convenience: for  $h \in [0, 2M]$ ,

$$\varphi(h) = \min_{-M \leq x \leq M-h} |a(x+h) - a(x)|$$

and for  $h < 0$ ,  $\varphi(h) = \varphi(-h)$ . Note that  $\varphi(0) = 0$  and for all  $x, y$ :

$$\varphi(|x - y|) \leq |a(x) - a(y)| \leq \omega(|x - y|),$$



where  $\omega(h) = \sup_{|x-y|\leq h} |a(x) - a(y)|$  is the continuity modulus.

*Remark 5.* We can assume that  $a$  is increasing ; else, if  $a$  is not and is only non-decreasing, then  $\varphi$  is identically zero for  $h$  small enough.

Let us give some examples of modulus of degeneracy.

**Example 2.** If  $a = f'$  is convex, then  $\varphi(h) = a(h - M) - a(-M)$  and if  $a = f'$  is concave, then  $\varphi(h) = a(M) - a(M - h)$ .

**Lemma 5.** If  $a(u)$  is odd, increasing and convex for  $u \geq 0$ , then  $\varphi(h) = 2a\left(\frac{|h|}{2}\right)$ . In particular,  $\varphi(h)$  is convex for  $h \geq 0$ .

*Proof.* Let  $h > 0$ . Since  $a(u)$  is convex for  $u \geq 0$ , the slope function  $x \mapsto \frac{a(x+h) - a(x)}{h}$  is increasing for  $x > 0$ , so  $\min_{x \geq 0} (a(x+h) - a(x)) = a(h) - a(0) = a(h)$ . In the same way, since  $a(u)$  is odd,  $a(u)$  is concave for  $u \leq 0$  and  $\min_{x \leq -h} (a(x+h) - a(x)) = a(0) - a(-h) = a(h)$ . Suppose now that  $-h < x < 0$ : it follows from the convexity of  $a(u)$  for  $u \geq 0$  that  $a(x+h) - a(x) = a(x+h) + a(-x) \geq 2a\left(\frac{x+h}{2} + \frac{-x}{2}\right) = 2a\left(\frac{h}{2}\right)$ . Finally, since  $2a\left(\frac{h}{2}\right) < a(h)$ , we deduce that  $\varphi(h) = \min_{-M \leq x \leq M-h} (a(x+h) - a(x)) = 2a\left(\frac{h}{2}\right)$ .  $\square$

**Example 3.** It follows from Lemma 5 that a convex power law flux has got a convex modulus of degeneracy: if  $f(u) = \frac{|u|^{1+\alpha}}{1+\alpha}$ ,  $\alpha > 1$ , then  $a(u) = \text{sgn}(u) |u|^\alpha$  and  $\varphi(h) = 2^{1-\alpha} |h|^\alpha$ .

Notice that  $\varphi$  is not necessary convex:

**Example 4.** If  $a(u) = u - s(u)$ , where  $s(u) = \frac{\sin(2\pi u)}{2\pi}$  on  $[-2, 2]$ , then  $\varphi(h) = 2(h - |s(h)|)$ , so that  $\varphi$  is not convex for  $h > 1$ .

**Lemma 6.** Assume that  $f$  is convex. Then the function  $\varphi$  satisfies:

- 1)  $\varphi$  is increasing on  $[0, 2M]$ .
- 2)  $\varphi(h) > 0$  for  $h \neq 0$ .
- 3)  $\varphi$  is continuous.

*Proof.* 1) Let  $h_1 < h_2$ . Let  $x_1 \in [-M, M - h_1]$  and  $x_2 \in [-M, M - h_2] \subset [-M, M - h_1]$  such that  $\varphi(h_1) = a(x_1 + h_1) - a(x_1)$  and  $\varphi(h_2) = a(x_2 + h_2) - a(x_2)$ . We have:  $\varphi(h_1) \leq a(x_2 + h_1) - a(x_2) < a(x_2 + h_2) - a(x_2) = \varphi(h_2)$ , so  $\varphi$  is increasing.

2)  $\varphi(0) = 0$  and  $\varphi$  is increasing for  $h > 0$  yields 2).

3) It is a slight generalization of the classical following result: let  $D(x) = \sup_{y \in [\alpha, \beta]} d(x, y)$ , where  $d$  is continuous on  $\mathbb{R} \times [\alpha, \beta]$ . Then  $D$  is continuous on  $\mathbb{R}$ .  $\square$

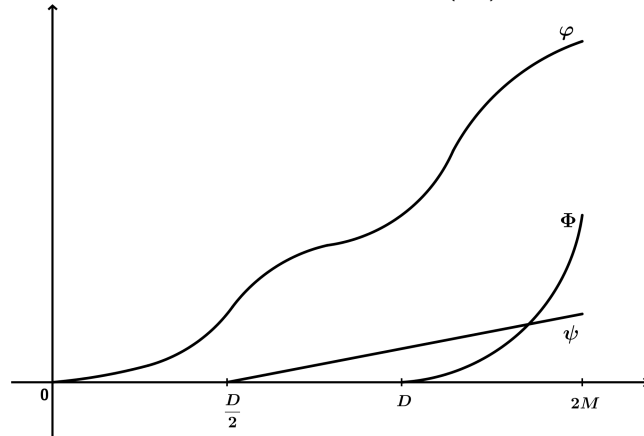
### 5.3 $BV_{\Phi}$ estimate

As we notice in Example 4 above, we cannot expect in general the modulus of degeneracy  $\varphi$  to be convex. However, the convexity is necessary to define the space  $BV_{\Phi}$ . So we define in the next subsection the closest convex function  $\Phi$  related to  $\varphi$ .

**Proposition 10.** *[The convex function  $\Phi$ ] We denote by  $\Phi$  the greatest convex, even function such that  $0 \leq \Phi \leq \varphi$  on  $[0, 2M]$ . This function  $\Phi$  is increasing and satisfies:  $\Phi(0) = 0$  and for all  $u, v \in [-M, M]$ ,  $u \neq v$ ,*

$$0 < \Phi(|u - v|) \leq \varphi(|u - v|) \leq |a(u) - a(v)|.$$

*Proof.* We show that  $\Phi$  is well-defined. Let  $\mathcal{C} = \{\psi \mid \psi \text{ is convex, even and } 0 \leq \psi \leq \varphi \text{ on } [0, 2M]\}$ . Since  $0 \in \mathcal{C}$ ,  $\mathcal{C} \neq \emptyset$ . We set:  $\Phi(x) = \sup_{\psi \in \mathcal{C}} \psi(x)$ . Then  $\Phi$  is convex, even and such that  $0 \leq \Phi \leq \varphi$  on  $[0, 2M]$ . We prove that  $\Phi \not\equiv 0$ . Let  $D = \sup\{d \in [0, 2M] \mid \Phi(d) = 0\}$  and assume that  $D \neq 0$ . We define then the piecewise linear function  $\psi$  by  $\psi(0) = 0$ ,  $\psi\left(\frac{D}{2}\right) = 0$  and  $\psi(M) = \varphi\left(\frac{D}{2}\right) > 0$ .



Setting  $\Psi = \max(\Phi, \psi)$ , we get a convex, even function such that  $0 \leq \Psi \leq \varphi$  on  $[0, 2M]$  and  $\Psi > \Phi$  on  $\left[\frac{D}{2}, D\right]$ , which is a contradiction. Since  $\Phi$  is convex,  $\Phi$  is at least continuous on  $]0, 2M[$ . Moreover,  $\Phi(0) = 0$  and  $\Phi > 0$  on  $]0, 2M[$ , so that  $\Phi$  is increasing on  $[0, 2M]$ . Finally, note that for all  $u, v$ ,  $u \neq v$ :  $0 < \Phi(|u - v|) \leq \varphi(|u - v|) \leq |a(u) - a(v)|$ .  $\square$

We are now able to prove here Theorem 1.

*Proof.* Let  $(x_i)_{1 \leq i \leq n}$  be a partition of an interval  $[A, B]$  and  $\ell := B - A$ . Then it follows from both

Propositions 10 and 5:

$$\begin{aligned}
\sum_{i=0}^{n-1} \Phi(|u(x_{i+1}, t) - u(x_i, t)|) &= \sum_{i=0}^{n-1} \Phi \left( \left| b \left( \frac{x_{i+1} - y(x_{i+1}, t)}{t} \right) - b \left( \frac{x_i - y(x_i, t)}{t} \right) \right| \right) \\
&\leq \sum_{i=0}^{n-1} \left| a \left( b \left( \frac{x_{i+1} - y(x_{i+1}, t)}{t} \right) \right) - a \left( b \left( \frac{x_i - y(x_i, t)}{t} \right) \right) \right| \\
&\leq \frac{1}{t} \sum_{i=0}^{n-1} |x_{i+1} - x_i - (y(x_{i+1}, t) - y(x_i, t))| \\
&\leq \frac{1}{t} \sum_{i=0}^{n-1} |x_{i+1} - x_i - (y(x_{i+1}, t) - x_{i+1} + x_{i+1} - x_i - (y(x_i, t) - x_i))| \\
&\leq \frac{1}{t} \sum_{i=0}^{n-1} (2|x_{i+1} - x_i| + |y(x_{i+1}, t) - x_{i+1}| + |y(x_i, t) - x_i|) \\
&\leq \frac{2}{t} (\ell + t \|a(u_0)\|_\infty).
\end{aligned}$$

Notice that the Lax-Oleĭnik formula is not defined everywhere, so that the previous inequalities do not consider all the subdivisions of  $[A, B]$ . We can use Remark 4 or Formula (3.2) to bound the total  $\Phi$ -variation on  $[A, B]$ . Moreover, this bound depends only on  $t$ ,  $M \geq \|u_0\|_\infty$ , and the length of the interval  $[A, B]$ :

$$TV^\Phi u(\cdot, t)[A, B] \leq 2 \left( \frac{B - A}{t} + \sup_{[-M, M]} |a| \right).$$

□

## Appendix : $\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0]$ (smooth case)

In this whole appendix, we assume that  $f \in C^2$  is uniformly convex ( $f'' \geq \delta > 0$ , [La2, La3]) and also that  $u_0 \in C_c^0 \cap BV$ . We detail completely Lax's proof to obtain the well known following result: for an uniformly convex flux, the Lax-Oleïnik solution is the Kruzkov entropy solution. There are three steps in our proof. First, we prove that the Lax-Oleïnik solution is a weak solution of the conservation law. Second, we show that the Lax-Oleïnik solution satisfies the Lax-entropy condition and then the Kruzkov entropy condition (2.3). Third, we focus on the  $L^1$  strong continuity in time (2.4).

**Proposition 11.** *If  $f \in C^2$  is uniformly convex and  $u_0 \in C_c^0 \cap BV$ , then  $\mathcal{LO}[f, u_0]$  is a weak solution of (1.1).*

To prove that  $\mathcal{LO}[f, u_0]$  is a weak solution of (1.1), we will use following lemma, related to Laplace's method:

**Lemma 7.** *Let  $h$  be a continuous function on  $\mathbb{R}$  such that*

$$\lim_{|y| \rightarrow +\infty} \frac{h(y)}{|y|} = +\infty \quad (5.2)$$

and let  $p$  be a continuous function on  $\mathbb{R}$  such that

$$\int_{\mathbb{R}} |p(y)| e^{-h(y)} dy < +\infty. \quad (5.3)$$

If there exists a real  $y_0$  such that for all  $y \neq y_0$ ,  $h(y) > h(y_0)$ , then:

$$\lim_{n \rightarrow +\infty} \frac{\int_{\mathbb{R}} p(y) e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} = p(y_0). \quad (5.4)$$

*Proof.* Let  $n \in \mathbb{N}^*$ . According to (5.2), the integral  $\int_{\mathbb{R}} e^{-nh(y)} dy$  is convergent and according to

both (5.2) and (5.3), the integral  $\int_{\mathbb{R}} p(y) e^{-nh(y)} dy$  is also convergent.

Considering  $\tilde{h}(y) = h(y + y_0)$  and  $\tilde{p}(y) = p(y + y_0)$  if necessary, we can assume that  $y_0 = 0$ . Moreover,

$$\frac{\int_{\mathbb{R}} p(y) e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} - p(y_0) = \frac{\int_{\mathbb{R}} (p(y) - p(y_0)) e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy},$$

so we can suppose that  $p(0) = 0$ .

Let  $\varepsilon > 0$ . Since  $p$  is continuous at 0, there exists  $\eta > 0$  such that for all  $y$ ,  $|y| \leq \eta \Rightarrow |p(y)| \leq \varepsilon$ . The continuous function  $h$  is bounded on the compact set  $[0, 1]$ : there exists  $C > 0$  such that for

all  $y \in [0, 1]$ ,  $|h(y)| \leq C$ . According to (5.2) there exists  $A > 0$  (we can assume that  $A > \eta$ ) such that for all  $y$ ,  $|y| \geq A \Rightarrow h(y) \geq 2C$ . We set  $K = [-A, A] \setminus ]-\eta, \eta[$ . The continuous function  $h$  achieves its minimum  $m > h(0)$  on the compact set  $K$ . We set:  $\delta = \frac{m - h(0)}{2} > 0$ . The continuous function  $p$  is bounded on the compact set  $[-A, A]$ : there exists  $C' > 0$  such that for all  $y$ ,  $|y| \leq A \Rightarrow |p(y)| \leq C'$ . Since  $h$  is continuous at 0, there exists  $\nu > 0$  (we may suppose that  $\nu < \eta$ ) such that for all  $y$ ,  $|y| \leq \nu \Rightarrow |h(y) - h(0)| \leq \delta$ .

We write now:

$$\frac{\int_{\mathbb{R}} p(y)e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} = \underbrace{\frac{\int_{|y| \leq \eta} p(y)e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy}}_{I_1(n)} + \underbrace{\frac{\int_{y \in K} p(y)e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy}}_{I_2(n)} + \underbrace{\frac{\int_{|y| \geq A} p(y)e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy}}_{I_3(n)}.$$

The integral  $I_1(n)$  satisfies:

$$|I_1(n)| \leq \frac{\int_{|y| \leq \eta} |p(y)| e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} \leq \frac{\int_{|y| \leq \eta} \varepsilon e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} \leq \varepsilon \frac{\int_{\mathbb{R}} e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} = \varepsilon.$$

The integral  $I_2(n)$  satisfies:

$$|I_2(n)| \leq \frac{\int_{y \in K} |p(y)| e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} \leq \frac{\int_{y \in K} C' e^{-nm} dy}{\int_{|y| \leq \nu} e^{-n(h(0)+\delta)} dy} \leq \frac{2AC' e^{-nm}}{2\nu e^{-n(h(0)+\delta)}} = \frac{AC'}{\nu} e^{-n\delta},$$

so that  $\lim_{n \rightarrow +\infty} I_2(n) = 0$ .

Next:

$$|I_3(n)| \leq \frac{\int_{|y| \geq A} |p(y)| e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} \leq \frac{\int_{|y| \geq A} |p(y)| e^{-nh(y)} dy}{\int_{y \in [0,1]} e^{-nC} dy} = \int_{|y| \geq A} |p(y)| e^{-n(h(y)-C)} dy.$$

Defining  $s_n(y) = |p(y)| e^{-n(h(y)-C)}$ , we get for  $|y| \geq A$ :  $\lim_{n \rightarrow +\infty} s_n(y) = 0$  and  $0 \leq s_n(y) \leq |p(y)| e^{-(h(y)-C)} = s(y)$ . Since from (5.3)  $s$  is integrable, we deduce from the dominated convergence theorem that  $\lim_{n \rightarrow +\infty} \int_{|y| \geq A} |p(y)| e^{-n(h(y)-C)} dy = 0$ , and then  $\lim_{n \rightarrow +\infty} I_3(n) = 0$ . Finally:

$$\lim_{n \rightarrow +\infty} \frac{\int_{\mathbb{R}} p(y) e^{-nh(y)} dy}{\int_{\mathbb{R}} e^{-nh(y)} dy} = 0.$$

□

We now turn to the proof of Proposition 11.

*Proof.* (Proposition 11) That formula (2.2) defines a function  $u$  almost everywhere follows from both Proposition 1 and Lemma 1. We will now prove that  $u$  is a weak solution of (1.1) on  $]0, +\infty[$ . In order to do so, we divide the proof into five parts.

We set for all  $(x, t)$ :  $h_{(x,t)}(y) = U_0(y) + tg \left( \frac{x-y}{t} \right)$  and  $p_{(x,t)}(y) = b \left( \frac{x-y}{t} \right)$ , and for all positive integers  $n$ :

$$u_n(x, t) = \frac{\int_{\mathbb{R}} p_{(x,t)}(y) e^{-nh_{(x,t)}(y)} dy}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy} \quad \text{and} \quad f_n(x, t) = \frac{\int_{\mathbb{R}} f(p_{(x,t)}(y)) e^{-nh_{(x,t)}(y)} dy}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy}.$$

1) We will prove the pointwise convergence of  $(u_n(x, t))_n$  (respectively  $(f_n(x, t))_n$ ) to  $u(x, t)$  (respectively  $f(u(x, t))$ ) for all  $t > 0$  and for almost all  $x$ .

From definitions of  $U_0$  and  $g$  and the continuity of  $b$ , we state that the functions  $h_{(x,t)}$  and  $p_{(x,t)}$  are continuous on  $\mathbb{R}$ . Since  $u_0$  is compactly supported,  $U_0$  is bounded. Furthermore,  $g$  is superlinear. Then:

$$\lim_{|y| \rightarrow +\infty} \frac{h_{(x,t)}(y)}{|y|} = +\infty. \quad (5.5)$$

i) According to Fact 2,  $b(u)$  is linear for  $u$  large enough. Considering limit (5.5), we claim then that  $\int_{\mathbb{R}} |p_{(x,t)}(y)| e^{-h_{(x,t)}(y)} dy < +\infty$ . Moreover, we have for all  $t > 0$  and for almost all  $x$ : for all  $y$ ,  $h_{(x,t)}(y) > h_{(x,t)}(y(x, t))$ . Hypothesis of Lemma 7 being satisfied, it follows from (5.4) that:

$\lim_{n \rightarrow +\infty} u_n(x, t) = p_{(x,t)}(y(x, t))$ , i.e.:

$$\lim_{n \rightarrow +\infty} u_n(x, t) = b \left( \frac{x - y(x, t)}{t} \right) = u(x, t). \quad (5.6)$$

ii) As  $f(u)$  is quadratic for  $u$  large enough, it follows that:  $\int_{\mathbb{R}} |f(p_{(x,t)}(y))| e^{-h_{(x,t)}(y)} dy < +\infty$ .

So we deduce as above that:

$$\lim_{n \rightarrow +\infty} f_n(x, t) = f(u(x, t)). \quad (5.7)$$

2) We bound  $u_n(x, t)$  and  $f_n(x, t)$  regardless of  $n$ .

i) Let us begin with  $u_n(x, t)$ . For almost all  $y$ :

$$\frac{\partial}{\partial y}(h_{(x,t)}(y)) = u_0(y) - b\left(\frac{x-y}{t}\right) = u_0(y) - p_{(x,t)}(y),$$

so:

$$u_n(x, t) = \frac{\int_{\mathbb{R}} p_{(x,t)}(y) e^{-nh_{(x,t)}(y)} dy}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy} = \frac{\int_{\mathbb{R}} \left(u_0(y) - \frac{\partial}{\partial y}(h_{(x,t)}(y))\right) e^{-nh_{(x,t)}(y)} dy}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy} =$$

$$\frac{\int_{\mathbb{R}} u_0(y) e^{-nh_{(x,t)}(y)} dy}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy},$$

where the last equality follows from the superlinearity of  $h_{(x,t)}$ . Then:

$$|u_n(x, t)| \leq \frac{\int_{\mathbb{R}} |u_0(y)| e^{-nh_{(x,t)}(y)} dy}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy} \leq \|u_0\|_{\infty}.$$

Note that the previous inequality yields: for almost all  $(x, t)$ ,  $|u(x, t)| \leq \|u_0\|_{\infty}$ .

ii) We bound now  $f_n(x, t)$  for  $(x, t) \in [x_0, x_1] \times [t_0, t_1] \subset \mathbb{R} \times ]0, +\infty[$ . With the substitution  $z = \frac{x-y}{t}$ , we get:

$$f_n(x, t) = \frac{\int_{\mathbb{R}} f(b(z)) e^{-nh_{(x,t)}(x-tz)} dz}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(x-tz)} dz},$$

where:  $h_{(x,t)}(x-tz) = U_0(x-tz) + tg(z)$ .

According to Fact 2 we can write for  $z$  large enough ( $|z| \geq A > 0$ ):  $f(b(z)) = \alpha z^2$  and  $g(z) = \beta z^2 + \gamma$  (with  $\alpha > 0$  and  $\beta > 0$ ). Moreover, since  $u_0$  is compactly supported,  $U_0(x-tz)$  is constant ( $= V$ ) for  $z$  large enough ( $|z| \geq B = B(x_0, x_1, t_0, t_1) > 0$ ). Let  $M = \max(A, B) > 0$  and  $D = \sup_{|z| \leq M} |f(b(z))|$ .

We write then:

$$f_n(x, t) = \frac{\int_{|z| \leq M} f(b(z)) e^{-nh_{(x,t)}(x-tz)} dz}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(x-tz)} dz} + \frac{\int_{|z| \geq M} f(b(z)) e^{-nh_{(x,t)}(x-tz)} dz}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(x-tz)} dz},$$

so that:

$$|f_n(x, t)| \leq \frac{\int_{\mathbb{R}} D e^{-nh_{(x,t)}(x-tz)} dz}{\int_{\mathbb{R}} e^{-nh_{(x,t)}(x-tz)} dz} + \frac{\int_{|z| \geq M} \alpha z^2 e^{-n(V+t(\beta z^2 + \gamma))} dz}{\int_{|z| \geq M} e^{-n(V+t(\beta z^2 + \gamma))} dz} = D + \alpha \frac{\int_{z \geq M} z^2 e^{-nt\beta z^2} dz}{\int_{z \geq M} e^{-nt\beta z^2} dz}.$$

But the function  $t \mapsto \frac{\int_{z \geq M} z^2 e^{-nt\beta z^2} dz}{\int_{z \geq M} e^{-nt\beta z^2} dz}$  is decreasing, since its derivative

$$n\beta \frac{\left( \int_{z \geq M} z^2 e^{-nt\beta z^2} dz \right)^2 - \int_{z \geq M} z^4 e^{-nt\beta z^2} dz \int_{z \geq M} e^{-nt\beta z^2} dz}{\left( \int_{z \geq M} e^{-nt\beta z^2} dz \right)^2}$$

is non-positive (according to the Cauchy-Schwarz inequality in  $L^2$ ). Moreover, integrating by parts the numerator and considering the infinitesimal behavior of the complementary error function, we claim that

$$\lim_{n \rightarrow +\infty} \frac{\int_{z \geq M} z^2 e^{-nt_0\beta z^2} dz}{\int_{z \geq M} e^{-nt_0\beta z^2} dz} = \frac{M^2}{2},$$

which enables us to conclude.

**3)** We prove that for all positive integers  $n$ :

$$u_n(x, t) = -\frac{1}{n} \frac{\partial}{\partial x} (v_n(x, t)) \quad (5.8)$$

and

$$f_n(x, t) = \frac{1}{n} \frac{\partial}{\partial t} (v_n(x, t)), \quad (5.9)$$

where  $v_n = \ln(w_n)$  and  $w_n(x, t) = \int_{\mathbb{R}} e^{-nh_{(x,t)}(y)} dy = \int_{\mathbb{R}} t e^{-n(U_0(x-tz) + tg(z))} dz$ .

i) Let  $t > 0$ . For all  $x$ :  $\frac{\partial}{\partial x} (h_{(x,t)}(x - tz)) = u_0(x - tz)$ . Moreover  $u_0$  and  $U_0$  are bounded. By differentiation under the integral sign, we get then:  $\frac{\partial}{\partial x} w_n = \int_{\mathbb{R}} -ntu_0(x - tz) e^{-nh_{(x,t)}(x-tz)} dz$ .

But:

$$\int_{\mathbb{R}} u_0(x - tz) e^{-nh_{(x,t)}(x-tz)} dz = \int_{\mathbb{R}} b(z) e^{-nh_{(x,t)}(x-tz)} dz,$$

since

$$\int_{\mathbb{R}} (u_0(x - tz) - b(z)) e^{-nh_{(x,t)}(x-tz)} dz = \int_{\mathbb{R}} \frac{1}{nt} \frac{\partial}{\partial z} (e^{-nh_{(x,t)}(x-tz)}) dz = \frac{1}{nt} [e^{-nh_{(x,t)}(x-tz)}]_{-\infty}^{+\infty} = 0.$$

It follows that:  $\frac{\partial}{\partial x} w_n = -n \int_{\mathbb{R}} tb(z) e^{-nh_{(x,t)}(x-tz)} dz = -n \int_{\mathbb{R}} b\left(\frac{x-y}{t}\right) e^{-nh_{(x,t)}(y)} dy = -nw_n u_n$ , so that (5.8) is satisfied.

ii) Let  $x \in \mathbb{R}$ . For all  $t > 0$ :  $\frac{\partial}{\partial t} (h_{(x,t)}(x - tz)) = -zu_0(x - tz) + g(z)$ . Similarly as above, we get



by differentiation under the integral sign:  $\frac{\partial}{\partial t} w_n = \int_{\mathbb{R}} (1 - nt(-zu_0(x-tz) + g(z))) e^{-nh_{(x,t)}(x-tz)} dz$ .

But:

$$\int_{\mathbb{R}} (1 + ntzu_0(x-tz)) e^{-nh_{(x,t)}(x-tz)} dz = \int_{\mathbb{R}} ntzb(z) e^{-nh_{(x,t)}(x-tz)} dz,$$

since

$$\int_{\mathbb{R}} (1 + ntzu_0(x-tz) - ntzb(z)) e^{-nh_{(x,t)}(x-tz)} dz = \int_{\mathbb{R}} \frac{\partial}{\partial z} (ze^{-nh_{(x,t)}(x-tz)}) dz = [ze^{-nh_{(x,t)}(x-tz)}]_{-\infty}^{+\infty} = 0.$$

Therefore:

$$\frac{\partial}{\partial t} w_n = nt \int_{\mathbb{R}} (zb(z) - g(z)) e^{-nh_{(x,t)}(x-tz)} dz. \quad (5.10)$$

Finally, it follows from the relation (3.1) that:  $\frac{\partial}{\partial t} w_n = nt \int_{\mathbb{R}} f(b(z)) e^{-nh_{(x,t)}(x-tz)} dz$  and then:

$$\frac{\partial}{\partial t} w_n = n \int_{\mathbb{R}} f\left(b\left(\frac{x-y}{t}\right)\right) e^{-nh_{(x,t)}(y)} dy = nw_n f_n, \text{ so that (5.9) is satisfied.}$$

4) We deduce from both (5.8) and (5.9) and from Schwarz theorem that the equation  $\frac{\partial u_n}{\partial t} + \frac{\partial f_n}{\partial x} = 0$  is satisfied in the sense of distributions.

5) Considering limits (5.6) and (5.7) and the bounds obtained at the step 2), we deduce from the dominated convergence theorem that the equation  $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$  is satisfied in the sense of distributions. In other words,  $u$  is a weak solution of (1.1) on  $]0, +\infty[$ .  $\square$

In this section, we briefly explain why the Lax-Oleïnik solution satisfies the Kruzkov entropy inequalities. For a bounded initial data and an uniformly convex flux, this result is well known [D4]. We recall that a Kruzkov entropy inequality is

$$\frac{\partial \eta(u)}{\partial t} + \frac{\partial q(u)}{\partial x} \leq 0 \quad (5.11)$$

where  $\eta$  is a convex function called the entropy and  $q$  is the associated entropy-flux, defined by  $q' = \eta' f'$ . In Kruzkov's theorem the previous inequalities have to be satisfied for all convex entropies  $\eta$ .

**Proposition 12.** *The Lax-Oleïnik solution satisfies the condition (5.11).*

*Proof.* We recall the arguments to be self-contained (see [D4]). Since  $u_0 \in BV$  and is smooth,  $y(\cdot, t)$  is increasing, it follows from the expression  $u(x, t) = u_0(y(x, t))$  (since  $u_0$  is continuous) that  $u(\cdot, t) \in BV$ . The Lax-Oleïnik solution is a weak solution of the conservation law. Thus,  $u \in BV([0, T] \times \mathbb{R})$ . The structure of  $BV$  space is used in [D4] for instance to show that it is sufficient to check the following inequality almost everywhere on the shock curves:

$$s[\eta(u)] + [q(u)] \leq 0, \quad (5.12)$$

where  $s = \frac{[f(u)]}{[u]}$  is the slope of the shock curve and  $[u] = u_+ - u_-$ . The Lax-Oleïnik solution satisfies  $u_+ < u_-$ , i.e.  $[u] < 0$ . This condition is equivalent to (5.12). To see that, it suffices to consider only the Kruzkov entropy  $\eta(u) = |u - k|$  with the entropy flux  $q(u) = \text{sign}(u - k)(f(u) - f(k))$  ([D4] second edition : p. 78 (4.5.5) and p. 219 (8.4.3)). Then the inequality (5.12) becomes simply for all  $k$  between  $u_+$  and  $u_-$ :

$$\frac{f(k) - f(u_-)}{k - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \geq \frac{f(u_+) - f(k)}{u_+ - k}$$

By convexity of the flux  $f$ , we get inequality (5.12). □

**Proposition 13.** [*L<sup>1</sup> strong continuity in time*] *The Lax-Oleïnik solution satisfies the following condition strongly in  $L^1_{loc}$ :*

$$\text{ess lim}_{t \rightarrow 0} u(x, t) = u_0(x).$$

We give a simple direct proof. Notice that the nonlinearity of the flux, the entropy conditions and the weak trace for the initial data are sufficient to recover strongly the initial data ([CR, V]).

*Proof.* Since  $u(x, t) = u_0(y(x, t))$ , we deduce from Proposition 5 that  $\lim_{t \rightarrow 0} u(x, t) = u_0(x)$ . Moreover, for almost all  $(x, t)$ ,  $|u(x, t)| \leq \|u_0\|_\infty$ . Then  $\lim_{t \rightarrow 0} u(x, t) = u_0(x)$  in  $L^1_{loc}(\mathbb{R}, \mathbb{R})$ , so that (2.4) is satisfied. □

Finally, we have proved :

**Proposition 14.** *If  $f \in C^2$  is uniformly convex and  $u_0 \in C^0_c \cap BV$ , then  $\mathcal{LO}[f, u_0] = \mathcal{K}[f, u_0]$ .*

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