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INTRODUCTION TO MODULI SPACES OF CONNECTIONS: SOME EXPLICIT CONSTRUCTIONS

FRANK LORAY

ABSTRACT. We give some concrete examples of moduli spaces of connections. Precisely, we explain how to explicitly construct the moduli spaces of rank 2 fuchsian systems and logarithmic connections on the Riemann sphere with 4 poles. The former ones are affine cubic surfaces; the latter ones are analytically isomorphic to affine surfaces but not algebraically: they do not carry non constant regular functions. We end with some remark on arbitrary number of poles.

1. INTRODUCTION: VARIATION ON AN EXAMPLE OF SERRE

Given a complex smooth projective curve X , a rank 1 (flat) connection, or equivalently a rank 1 local system on X , is an element of the cohomology group $H^1(X, \mathbb{C}^*)$. This is the easiest example of a moduli space of connections. From the exact sequence of sheaves

$$0 \rightarrow \mathbb{C}^* \rightarrow \mathcal{O}^* \rightarrow \Omega^1 \rightarrow 0$$

(where $\mathcal{O}^* \rightarrow \Omega^1$ is given by $f \mapsto \frac{df}{f}$), we can derive the associate long exact sequence of cohomology groups and extract the following

$$(1) \quad 0 \rightarrow H^0(X, \Omega^1) \rightarrow H^1(X, \mathbb{C}^*) \rightarrow \text{Jac}(X) \rightarrow 0$$

making $H^1(X, \mathbb{C}^*)$ into a principal \mathbb{C}^g -bundle over the Jacobian of the curve.

Even simpler is the case when X is an elliptic curve: we get a \mathbb{C} -bundle over the curve $X \simeq \text{Jac}(X)$ itself. One may compactify it by adding a section at infinity in order to get a \mathbb{P}^1 -bundle $P \rightarrow X$: it is one of the two indecomposable \mathbb{P}^1 -bundles found by Atiyah in [2]. The section at infinity $\sigma : X \rightarrow P$ is the unique section whose image $\Sigma := \sigma(X)$ has zero self-intersection in the total space $S := \text{Tot}(P)$ of the bundle. The group $H^1(X, \mathbb{C}^*)$ identifies, as a variety, with the complement $S \setminus \Sigma$.

Viewing X as a Riemann surface, we get the Riemann-Hilbert correspondence

$$RH : H^1(X, \mathbb{C}^*) \longrightarrow \text{Hom}(\pi_1(X), \mathbb{C}^*)$$

which associates to a local system its monodromy representation. Once we choose a system of generators for the fundamental group, we get an identification between the space of representations and the group $\mathbb{C}^* \times \mathbb{C}^*$. We thus get a complex analytic group isomorphism

$$(2) \quad H^1(X, \mathbb{C}^*) \longrightarrow \mathbb{C}^* \times \mathbb{C}^*.$$

However, the two underlying varieties are not isomorphic from the algebraic point of view: while $\mathbb{C}^* \times \mathbb{C}^*$ is an affine variety, there is no non constant regular functions on S . Indeed, such a function would extend as a rational function on S with polar divisor $n\Sigma$; one easily check that the zero divisor of such function should define a complete curve not intersecting Σ . But there is no complete curve in S since it is analytically isomorphic to an affine variety. This famous example is due to Serre.

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There is a natural non degenerate 2-form ω on S constructed from the exact sequence (1) by choosing non zero 1-forms on the base X and on the fiber; it is well-defined up to a scalar constant. Isomorphism (2) sends it to the 2-form $\frac{du}{u} \wedge \frac{dv}{v}$ (up to a scalar). The 2-form ω extends on S as a rational 2-form with divisor 2Σ .

Let us briefly mention two other facts. The maximal compact subgroup given by unitary connections/representation is an analytic real torus providing a real section of $P \rightarrow X$, and turns to be real algebraic on representation-side, namely $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^* \times \mathbb{C}^*$. The fibration $P \rightarrow X$ is sent to the foliation defined by $u\partial_u + \tau v\partial_v$ where $(1 : \tau)$ is commensurable to periods of X .

One can modify Serre's example as follows. The elliptic involution $\iota : X \rightarrow X$ lifts-up as a biregular involution $\varphi : S \rightarrow S$. This follows from the unicity of the undecomposable bundle P , or equivalently from the natural action of the involution ι on local systems. The quotient surface \underline{S} fibers over $X/\iota \simeq \mathbb{P}^1$. The involution φ has 2 fixed points over each of the 4 Weierstrass points: one of them lie on the section at infinity Σ (which is invariant) and the other one is corresponding to a 2-torsion point of $H^1(X, \mathbb{C}^*)$. We thus get 8 conic singular points on the quotient that we have to blow-up. The resulting surface $\hat{S} \rightarrow \underline{S}$ is equipped with the quotient 2-form $\underline{\omega}$ having polar divisor $2\underline{\Sigma} + E_1 + E_2 + E_3 + E_4$ where E_i are exceptional divisors of conic points along $\underline{\Sigma}$ (the image of Σ); $\underline{\omega}$ is non degenerate on the open set $M := \hat{S} \setminus (\hat{\Sigma} \cup E_1 \cup E_2 \cup E_3 \cup E_4)$. This open set is no more a group, but share all other nice properties with Serre example. For instance, the divisor $2\underline{\Sigma} + E_1 + E_2 + E_3 + E_4$ at infinity is (from the numerical point of view) like a degenerate elliptic fiber of type I_0^* in Kodaira list, there is no non constant regular function on M and it is analytically conjugated to an affine surface, namely the Cayley cubic, quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution $(u, v) \mapsto (\frac{1}{u}, \frac{1}{v})$. This second example is actually a particular case of moduli space of connections arising as space of initial conditions of one of the Painlevé VI equations, namely that one found by Picard.

These nice properties are common to general moduli spaces of connections (except structure of group, particular for the rank one) and this is a good motivation to study them. On the other hand, they naturally appear as spaces of initial conditions for isomonodromy equations, but we skip this point of view from our discussion.

2. MODULI SPACE OF FUCHSIAN SYSTEMS

Consider a \mathfrak{sl}_2 -fuchsian system on the Riemann sphere with 4 poles

$$(3) \quad \frac{dY}{dx} = \frac{A_1}{x - t_1} + \frac{A_2}{x - t_2} + \frac{A_3}{x - t_3} + \frac{A_4}{x - t_4}$$

with constant matrices A_i , $i = 1, 2, 3, 4$, satisfying

$$(4) \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \quad \text{and} \quad A_1 + A_2 + A_3 + A_4 = 0$$

(here and after we assume all 4 poles t_i 's in the affine part for simplicity). The group $\text{SL}_2(\mathbb{C})$ acts on the Y -variable and thus on residues A_i 's by simultaneous conjugacy. Indeed, change of variable $Y = MY'$ induces change of residues $A'_i = M^{-1}A_iM$. The spectrum of each matrix A_i is preserved by this action. Let us fix the spectral data

$$(5) \quad -\det(A_i) = a_i^2 + b_i c_i = \theta_i^2$$

for some $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ (eigenvalues are $\pm\theta_i$). We want to describe the quotient space $\mathcal{Fuchs}^\theta(X, D)$ of those systems by this action:

$$(6) \quad \left\{ (A_1, A_2, A_3, A_4) \in (\mathfrak{sl}_2(\mathbb{C}))^4 ; \sum_i A_i = 0, \det(A_i) = -\theta_i^2 \right\} /_{\text{SL}_2(\mathbb{C})}.$$

A straightforward computation shows that $\mathcal{Fuchs}^\theta(X, D)$ is expected to be a surface (depending on 4 parameters θ_i 's). However, it is non Hausdorff (as a topological space) in general. For instance, the orbit of the triangular system

$$(7) \quad A_i = \begin{pmatrix} \theta_i & b_i \\ 0 & -\theta_i \end{pmatrix}$$

under diagonal conjugacy is not closed; its closure contains the diagonal system (with all b_i 's vanishing) so that they define infinitesimally closed points in the quotient. Note however that, in this case, we have $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$.

Introduce the following functions on $\mathcal{Fuchs}^\theta(X, D)$:

$$(8) \quad \begin{aligned} X_1 &:= \det(A_2 + A_3), \quad X_2 := \det(A_1 + A_3), \quad X_3 := \det(A_1 + A_2) \\ \text{and } Y &= \text{tr}(A_1[A_2, A_4]) \end{aligned}$$

where $[A, B] = AB - BA$ is the Lie bracket. These functions are clearly invariant under $\text{SL}_2(\mathbb{C})$ -action.

Proposition 1. *Assume $\theta_4 \neq 0$. Then, the map*

$$(9) \quad (X_1, X_2, X_3, Y) : \mathcal{Fuchs}^\theta(X, D) \longrightarrow \mathbb{C}^4$$

sends the moduli space onto the affine cubic surface S^θ defined by

$$(10) \quad \begin{aligned} X_1 + X_2 + X_3 &= \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 \quad \text{and} \\ \frac{Y^2}{4} + X_1 X_2 X_3 + (\theta_1^2 - \theta_3^2)(\theta_2^2 - \theta_4^2) X_1 + (\theta_2^2 - \theta_3^2)(\theta_1^2 - \theta_4^2) X_2 \\ &= (\theta_1^2 + \theta_2^2 - \theta_3^2 - \theta_4^2)(\theta_1^2 \theta_2^2 - \theta_3^2 \theta_4^2) \end{aligned}$$

The map (9) above is one-to-one over the smooth part of S^θ . Singularities arise when

- $\theta_i = 0$ and $A_i = 0$ for $i = 1, 2, 3$;
- $\pm\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4 = 0$ and all A_i 's are simultaneously triangular up to conjugacy.

In particular, apart from special values of θ listed just before, the quotient $\mathcal{Fuchs}^\theta(X, D)$ is Hausdorff and is isomorphic to the smooth cubic surface S^θ .

Proof. Since $\theta_4 \neq 0$, we can assume $A_4 = \begin{pmatrix} \theta_4 & 0 \\ 0 & -\theta_4 \end{pmatrix}$. This normalization is well-defined up to diagonal conjugacy so that the monomials a_i and $b_i c_j$ are invariant. Using $A_1 + A_2 + A_3 + A_4 = 0$, we can now express A_3 in function of A_1 and A_2 :

$$\begin{cases} a_3 &= -a_1 - a_2 - \theta_4 \\ b_3 &= -b_1 - b_2 \\ c_3 &= -c_1 - c_2 \end{cases}$$

Spectral data (5) gives the following conditions

$$(11) \quad \begin{aligned} a_1^2 + b_1 c_1 &= \theta_1^2, \quad a_2^2 + b_2 c_2 = \theta_2^2 \quad \text{and} \\ 2a_1 a_2 + 2\theta_4(a_1 + a_2) + b_1 c_2 + b_2 c_1 + \theta_1^2 + \theta_2^2 - \theta_3^2 + \theta_4^2 &= 0. \end{aligned}$$

On the other hand, we get

$$(12) \quad \begin{aligned} X_1 &= 2\theta_4 a_1 + \theta_4^2 + \theta_1^2 \\ X_2 &= 2\theta_4 a_2 + \theta_4^2 + \theta_2^2 \quad \text{and} \quad Y = 4\theta_4(b_1 c_2 - b_2 c_1) \\ X_3 &= 2\theta_4 a_3 + \theta_4^2 + \theta_3^2 \end{aligned}$$

Once we know X_1, X_2, X_3, Y , we get the invariants a_1, a_2 and $b_1c_2 - b_2c_1$. From relations (11), we promptly deduce b_1c_1, b_2c_2 and $b_1c_2 + b_2c_1$, and therefore b_1c_2 and b_2c_1 . The cubic equation (10) just says that these invariants satisfy the obvious relation $(b_1c_1)(b_2c_2) = (b_1c_2)(b_2c_1)$. We can thus recover (b_1, b_2, c_1, c_2) uniquely up to the diagonal action, except when either $b_1 = b_2 = 0$, or $c_1 = c_2 = 0$; in these latter cases, there are several possible solutions (A_1, A_2, A_3) up to diagonal action, but all of them are triangular. \square

3. MODULI SPACE OF CONNECTIONS

Fix $D = t_1 + \dots + t_n$ a reduced divisor (of poles) on the Riemann sphere $X := \mathbb{P}^1$ with $n \geq 4$. Fix eigenvalues $\{\theta_i^+, \theta_i^-\} \subset \mathbb{C}$ for each $i = 1, \dots, n$ with integral sum $\sum_{i=1}^n \theta_i^+ + \theta_i^- = -d \in \mathbb{Z}$. For simplicity, assume generic condition $\theta_1^\pm + \dots + \theta_n^\pm \notin \mathbb{Z}$ for any choice of signs \pm to avoid reducible connections. Denote by $\theta = (\theta_1^\pm, \dots, \theta_n^\pm)$ the spectral data.

Consider the triples (E, ∇, \mathbf{l}) called “parabolic connections” where

- E is a rank 2 vector bundle of degree d over \mathbb{P}^1 ,
- ∇ is a logarithmic connection $\nabla : E \rightarrow E \otimes \Omega^1(D)$ with polar divisor D , having residual eigenvalues θ_i^+ and θ_i^- over the pole t_i ,
- $\mathbf{l} = (l_1, \dots, l_n)$ is a parabolic structure on E such that $l_i \subset E|_{t_i}$ belongs to the eigenspace generated by θ_i^+ .

When $\theta_i^+ \neq \theta_i^-$, we note that the parabolic l_i is **the** eigendirection of θ_i^+ ; the parabolic structure is therefore relevant only in case of equality $\theta_i^+ = \theta_i^-$. Note also that Fuchs relation says that $\sum_{i=1}^n \theta_i^+ + \theta_i^- + \deg(E) = 0$, which explains the constraints given by d in the above definitions.

We say that two parabolic connections (E, ∇, \mathbf{l}) and $(E', \nabla', \mathbf{l}')$ are equivalent when there is a bundle isomorphism $\phi : E \rightarrow E'$ conjugating connections and parabolic structures. We denote by $\text{Con}^\theta(X, D)$ the moduli space of parabolic connections for this equivalence. This can be viewed as a stack (see [1]), but it can actually be constructed by GIT method (see [3]); under our generic assumptions on θ , we simply get:

Theorem 2 (Inaba-Iwasaki-Saito). *The moduli space $\text{Con}^\theta(X, D)$ is a smooth irreducible quasi-projective variety of dimension $2(n-3)$ equipped with a regular symplectic 2-form ω .*

The construction needs a choice of weights to impose stability condition; all parabolic connections are stable under our assumptions and the resulting quotient does not actually depends on this choice. However, if we allow non generic eigenvalues, then some reducible connections become unstable, and the quotient of semi-stable points depends on the choice of weights. Note also that we could have avoid parabolic structure in the discussion by assuming also $\theta_i^+ \neq \theta_i^-$, but we will use parabolics later; this is why we already introduce this notion.

3.1. Connection matrix. Let us now be more explicit for readers that are not familiar to connections. Following Birkhoff, any rank vector bundle E on \mathbb{P}^1 splits as a direct sum of line bundles. In particular, for the rank 2 case, we have

$$E = \mathcal{O}(d_1) \oplus \mathcal{O}(d_2) \quad \text{with} \quad d_1 \leq d_2 \quad \text{and} \quad d_1 + d_2 = d.$$

Choose sections e_i of $\mathcal{O}(d_i)$ whose divisor (zero or pole) is supported by $\infty \in \mathbb{P}^1$ so that, over the affine part $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$, the vector bundle is trivial: $E|_{\mathbb{C}} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. We can describe the connection ∇ as follows:

$$\nabla|_{\mathbb{C}} : Y \mapsto dY + \Omega \cdot Y$$

where Y is a section of $E|_{\mathbb{C}}$ and Ω is a 2×2 -matrix of meromorphic 1-forms:

$$(13) \quad \Omega = \sum_{i=1}^n \frac{A_i}{x-t_i} dx + B(x) dx$$

with B holomorphic (and x the affine variable of $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$). Let e'_i be sections of $\mathcal{O}(d_i)$ defined by $e_i = x^{d_i} e'_i$. At infinity, we must express the connection in the basis (e'_1, e'_2) and, setting

$$Y = MY' \quad \text{with} \quad M = \begin{pmatrix} x^{d_1} & 0 \\ 0 & x^{d_2} \end{pmatrix}$$

we find

$$\nabla|_{\mathbb{P}^1 \setminus \{0\}} : Y' \mapsto dY' + \Omega' \cdot Y' \quad \text{with} \quad \Omega' = M^{-1}\Omega M + M^{-1}dM$$

which must be holomorphic at ∞ . If we now write $\Omega = A(x) \frac{dx}{\prod_{i=1}^n (x-t_i)}$, this holomorphy conditions says that

$$A(x) = - \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} x^{n-1} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

$$\text{with} \quad \begin{cases} b \text{ polynomial of degree } \leq n-2-(d_2-d_1) \\ a, d \text{ polynomials of degree } \leq n-2 \\ c \text{ polynomial of degree } \leq n-2+(d_2-d_1) \end{cases}$$

Fuchsian systems are just logarithmic connections on the trivial bundle $d_1 = d_2 = 0$.

Example 3 (Case $n = 4$). When $E = \mathcal{O} \oplus \mathcal{O}(1)$, then holomorphy at infinity gives the following constraints

$$A_1 + A_2 + A_3 + A_4 = \begin{pmatrix} 0 & 0 \\ \star & -1 \end{pmatrix} \quad \text{and} \quad B(x) \equiv 0.$$

When $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$, then we get constraints

$$A_1 + A_2 + A_3 + A_4 = \begin{pmatrix} 1 & 0 \\ \star & -1 \end{pmatrix}, \quad B(x) \equiv \begin{pmatrix} 0 & 0 \\ \star & 0 \end{pmatrix} \quad (\text{constant matrices})$$

$$\text{and} \quad \sum_i \frac{A_i}{x-t_i} = \begin{pmatrix} \star & \frac{b}{\prod_i (x-t_i)} \\ \star & \star \end{pmatrix}.$$

Finally, when $d_2 - d_1 > 2$, the $[1, 2]$ -coefficient of all A_i 's and $B(x)$ must be zero so that we are in the reducible case. In other words, under our generic assumption on θ , we have $d_2 - d_1 = 0, 1, 2$. For degree $d = 0$ or 1 , we are led to the following possibilities

- $d = 0$ and $(d_1, d_2) = (0, 0)$ or $(-1, 1)$,
- $d = 1$ and $(d_1, d_2) = (0, 1)$.

We now impose spectral data by

$$\text{tr}(A_i) = \theta_1^+ + \theta_i^- \quad \text{and} \quad \det(A_i) = \theta_1^+ \cdot \theta_i^-.$$

The parabolic structure is therefore given by

$$l_i = \mathbb{C} \cdot \begin{pmatrix} b_i \\ \theta_i^+ - a_i \end{pmatrix} \quad \text{or} \quad \mathbb{C} \cdot \begin{pmatrix} \theta_i^+ - d_i \\ c_i \end{pmatrix} \quad \text{where} \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

except when $\theta_i^+ = \theta_i^-$ and A_i is a scalar matrix, in which case l_i may be arbitrarily chosen: it is an extra data in this case.

3.2. Bundle automorphisms. So far, we have described the space of parabolic connections up to Birkhoff normalization of the bundle. Then, we would like to quotient by the action of bundle automorphisms on connections. They can be described as follows.

- When $d_1 = d_2$, then E is just the twist of the trivial bundle by $\mathcal{O}(d_1)$ and automorphisms are the same: $\mathrm{GL}_2(\mathbb{C})$ acts by conjugacy on the connection matrix (13), i.e. on matrices A_i 's (B is scalar and the action is trivial on it).
- When $d_1 < d_2$, then the automorphism group is given in the trivialization chart $E|_{\mathbb{C}} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ by

$$\left\{ M = \begin{pmatrix} \lambda_1 & 0 \\ f(x) & \lambda_2 \end{pmatrix} \text{ with } f(x) \text{ polynomial of degree } \leq d_2 - d_1 \right\},$$

and it is acting on connection matrix (13) by

$$\Omega \mapsto \Omega' = M^{-1}\Omega M + M^{-1}dM.$$

The action on parabolics is obvious.

Example 4. Assume $n = 4$ and $(d_1, d_2) = (-1, 1)$; for simplicity, assume also $(\theta_i^+, \theta_i^-) = (\theta_i, -\theta_i)$. The matrix connection writes $\Omega = A(x) \frac{dx}{\prod_i (x - t_i)}$ where

$$A(x) = \begin{pmatrix} x^3 + a(x) & b \\ c(x) & -x^3 - a(x) \end{pmatrix} \quad \text{with} \quad \begin{cases} b \in \mathbb{C} \\ a \text{ polynomial of degree } \leq 2 \\ c \text{ polynomial of degree } \leq 4 \end{cases}$$

Note that $b \neq 0$ otherwise the connection would be reducible. Conjugating by a diagonal matrix M , we can assume $b = 1$. The action of unipotent isomorphisms is described by

$$M = \begin{pmatrix} 1 & 0 \\ f(x) & 1 \end{pmatrix} : \begin{pmatrix} x^3 + a & 1 \\ c & -x^3 - a \end{pmatrix} \mapsto \begin{pmatrix} x^3 + a + f & 1 \\ c - 2(x^3 + a)f - f^2 + f' & -x^3 - a - f \end{pmatrix}$$

where f is any degree 2 polynomial. Choosing $f = -a$, we derive a unique normal form

$$\Omega = \begin{pmatrix} x^3 & 1 \\ \tilde{c}(x) & -x^3 \end{pmatrix} \frac{dx}{\prod_i (x - t_i)} \quad \text{where} \quad \frac{\tilde{c}(x)}{\prod_i (x - t_i)} = c_0 + \sum_i \frac{\tau_i \theta_i^2}{x - t_i} \quad \text{with} \quad c_0 \in \mathbb{C}$$

and constants τ_i are defined by

$$(14) \quad \frac{1}{\prod_i (x - t_i)} = \sum_i \frac{1}{\tau_i (x - t_i)}, \quad \text{i.e. } \tau_i = \prod_{j \neq i} (t_i - t_j).$$

It follows that the moduli space of connections **on the fixed bundle** $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ is the affine line $\mathbb{A}^1 \ni c_0$ (extra parameter in the above normal form). To get the full moduli space $\mathcal{C}on^\theta(\mathbb{P}^1, \{t_1, t_2, t_3, t_4\})$ of degree 0 bundles with the same eigenvalues (satisfying $(\theta_i^+, \theta_i^-) = (\theta_i, -\theta_i)$ and $\pm\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4 \notin \mathbb{Z}$), we have to patch together this affine line \mathbb{A}^1 with the affine cubic surface S^θ of Fuchsian systems given by Proposition 1 (see remark ending example 3). Following Theorem 2, one should obtain a smooth irreducible quasi-projective surface.

3.3. Isomorphisms between moduli spaces. There are many isomorphisms between moduli spaces of connections $\mathcal{C}on^\theta(X, D)$ modifying spectral data. First, one can **twist** a parabolic connection (E, ∇, l) by a rang 1 logarithmic connection (L, ζ) :

- L is a line bundle of degree k , $L = \mathcal{O}(k)$,
- $\zeta : L \rightarrow L \otimes \Omega^1(D)$ is determined by its eigenvalues λ_i satisfying

$$\lambda_1 + \cdots + \lambda_n + k = 0.$$

The action on connection matrix (13) is given by

$$\Omega \mapsto \Omega + \sum_i \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix} \frac{dx}{x - t_i}$$

and this defines an isomorphism

$$(15) \quad \mathcal{C}on^\theta(X, D) \xrightarrow{\otimes(L, \zeta)} \mathcal{C}on^{\theta'}(X, D) \quad \text{where} \quad \begin{cases} (\theta_i^+)' = \theta_i^+ + \lambda_i \\ (\theta_i^-)' = \theta_i^- + \lambda_i \end{cases}$$

The degree is changed by $d \mapsto d' = d + 2k$. In case of even degree, we can assume up to such an isomorphism that $d = 0$ and moreover $\theta_i^+ + \theta_i^- = 0$ for all i (we get a \mathfrak{sl}_2 -connection). In the odd case, we can assume $d = 1$; then, in case $n = 4$, necessarily $E = \mathcal{O} \oplus \mathcal{O}(1)$ (see remark ending example 3).

The other family of isomorphisms comes from **elementary transformations** of parabolic bundles. Given a parabolic bundle (E, \mathbf{l}) over (X, D) , then we define $(E', \mathbf{l}') := \text{Elm}_{t_i}^-(E, \mathbf{l})$ by

- E' is the vector bundle defined by the subsheaf $E' \subset E$ generated by those sections directed by l_i , i.e. defined by the exact sequence of morphisms of sheaves

$$0 \rightarrow E' \rightarrow E \rightarrow E/l_i \rightarrow 0$$

where l_i is viewed as a sky-scraper sheaf.

- the new parabolic direction $l'_i \subset E'|_{t_i}$ is defined by the kernel of the inclusion morphism $E' \rightarrow E$.

A connection ∇ on E induces a connection ∇' on the subsheaf $E' \subset E$; if (E, ∇, \mathbf{l}) is parabolic, ∇ preserves the direction l_i and ∇' is still parabolic with respect to (E', \mathbf{l}') . Over t_i , the new eigenvalues are $(\theta_i^+, \theta_i^-)' := (\theta_i^- + 1, \theta_i^+)$ (l'_i is now in the eigenspace corresponding to $\theta_i^- + 1$). Over other t_j 's, the morphism $E' \rightarrow E$ is a local bundle isomorphism and the eigenvalues remain unchanged. This operation induces a map

$$(16) \quad \mathcal{C}on^\theta(X, D) \xrightarrow{\text{Elm}_{t_i}^-} \mathcal{C}on^{\theta'}(X, D) \quad \text{where} \quad \begin{cases} (\theta_i^+)' = \theta_i^- + 1 \\ (\theta_i^-)' = \theta_i^+ \\ \text{all other } (\theta_j^\pm)' = \theta_j^\pm \end{cases}$$

In local trivialization of the bundle E around t_i , with basis (e_1, e_2) such that e_1 generates the parabolic direction l_i at t_i , we get a parabolic connection matrix of the form

$$\Omega = \begin{pmatrix} \theta_i^+ & \star \\ 0 & \theta_i^- \end{pmatrix} \frac{dx}{x - t_i} + \text{holomorphic matrix}, \quad \text{and} \quad l_i = \mathbb{C} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The new vector bundle E' is generated near t_i by $(e_1, (x - t_i)e_2)$ so that the connection matrix of ∇' is given by

$$\Omega' = M^{-1}\Omega M + M^{-1}dM \quad \text{where} \quad M = \begin{pmatrix} 1 & 0 \\ 0 & x - t_i \end{pmatrix}$$

which gives

$$\Omega' = \begin{pmatrix} \theta_i^+ & 0 \\ \star & \theta_i^- + 1 \end{pmatrix} \frac{dx}{x - t_i} + \text{holomorphic matrix}, \quad \text{and} \quad l_i = \mathbb{C} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One easily check that applying twice $\text{Elm}_{t_i}^-$ gives the twist by the unique logarithmic connection on $\mathcal{O}(-1)$ having a single pole (with residue +1) at t_i :

$$\text{Elm}_{t_i}^- \circ \text{Elm}_{t_i}^-(E, \nabla, \mathbf{l}) = (\mathcal{O}(-1), \frac{dx}{x - t_i}) \otimes (E, \nabla, \mathbf{l}).$$

Note that $\text{Elm}_{t_i}^-$ decreases the degree of the vector bundle by -1 (whence the sign); one can also define $\text{Elm}_{t_i}^+(E, \mathbf{l}) := (\mathcal{O}(-1), \frac{dx}{x-t_i}) \otimes \text{Elm}_{t_i}^-(E, \mathbf{l})$ increasing the degree of the vector bundle by $+1$, it is actually the inverse map

$$\text{Elm}_{t_i}^+ \circ \text{Elm}_{t_i}^-(E, \nabla, \mathbf{l}) \simeq (E, \nabla, \mathbf{l}).$$

In fact, all these isomorphisms commute together.

One can also define an isomorphism permuting θ_i^+ and θ_i^- , modifying the parabolic structure (we switch to the other eigenspace) and get a non abelian group. We omit this from our discussion. Finally, up to isomorphism, we can always assume degree $d = 0$ and $\theta_i^+ + \theta_i^- = 0$ for all i . However, this may be not the best choice as we shall see.

4. THE PAINLEVÉ CASE $n = 4$

In order to get an explicit irreducible moduli space, it is obviously better to choose $d = 1$ instead of $d = 0$ since then, we can work on the single vector bundle $E = \mathcal{O} \oplus \mathcal{O}(1)$ (see remark ending example 3).

4.1. An explicit construction. We start proceeding like in example 4. The connection matrix writes $\Omega = A(x) \frac{dx}{\prod_i (x-t_i)}$ where

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & -x^3 + d(x) \end{pmatrix} \quad \text{with} \quad \begin{cases} b \text{ polynomial of degree } \leq 1 \\ a, d \text{ polynomials of degree } \leq 2 \\ c \text{ polynomial of degree } \leq 3 \end{cases}$$

Again, $b(x) \neq 0$ otherwise the connection is reducible. Assume for the moment that $b(x)$ is not constant so that we can normalize it to $b(x) = x - q$. The action by unipotent matrices is not enough to kill $a(x)$ however (the freedom $f(x)$ like in example 4 is now of degree 1). The idea is to apply an elementary transformation Elm_q^+ at the parabolic direction given by $\mathcal{O}(1)$. Concretely, we apply the bundle birational isomorphism given by

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x-q} \end{pmatrix}$$

and obtain the new connection matrix

$$M^{-1}\Omega M + M^{-1}dM = \left(\begin{array}{cc} \frac{a(x)}{\prod_i (x-t_i)} & \frac{1}{\prod_i (x-t_i)} \\ \frac{(x-q)c(x)}{\prod_i (x-t_i)} & \frac{-x^3+d(x)}{\prod_i (x-t_i)} - \frac{1}{x-q} \end{array} \right) dx$$

On the new bundle $E' = \mathcal{O} \oplus \mathcal{O}(2)$, we have more automorphisms and we are able to kill the polynomial coefficient $a(x)$ by means of unipotent automorphisms. We now get the normal form

$$(17) \quad \Omega' = \left(\begin{array}{cc} 0 & \frac{1}{\prod_i (x-t_i)} \\ \frac{\tilde{c}(x)}{(x-q)\prod_i (x-t_i)} & \frac{-x^3+d(x)}{\prod_i (x-t_i)} - \frac{1}{x-q} \end{array} \right) dx$$

with $\tilde{c}(x)$ polynomial of degree 5. Taking into account the spectral data, we get

$$\Omega = \sum_i \left(\begin{array}{cc} 0 & \frac{1}{\tau_i} \\ -\tau_i \theta_i^+ \theta_i^- & \theta_i^+ + \theta_i^- \end{array} \right) \frac{dx}{x-t_i} + \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ c_0 & 0 \end{pmatrix} dx$$

for constants $p, c_0 \in \mathbb{C}$. Here, we have also assumed $q \in \mathbb{C} \setminus \{t_1, t_2, t_3, t_4\}$. Indeed, when $q = t_i$, we apply an elementary transformation over the pole which might change the eigenvalues. Finally, note that $x = q$ must be an apparent singular

point, which means, by Fuchs' local theory, that around $x = q$ the kernel of the residual part is also an eigenvector for the constant part of the connection matrix:

$$\Omega \cdot \begin{pmatrix} 1 \\ p \end{pmatrix} = \star \begin{pmatrix} 1 \\ p \end{pmatrix} + o(x - q);$$

this fixes the value of c_0 . Finally, the initial connection is determined by the parameters (p, q) which have the following geometrical meaning:

- q stands for the position of the apparent singular point of the new connection,
- p stands for the kernel of the residue of ∇ over $x = q$.

Note also that, over $x = t_i$, the two eigendirections of ∇ are given by

$$\begin{pmatrix} 1 \\ \tau_i \theta_i^+ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ \tau_i \theta_i^- \end{pmatrix}$$

corresponding respectively to eigenvalues θ_i^+ and θ_i^- .

Let us identify $\mathcal{O}(2) = \Omega^1(D)$, and consider the projectivization $\mathbb{P}(\mathcal{O} \oplus \Omega^1(D))$ of our vector bundle E as the fiber-compactification of $\Omega^1(D)$: the line bundles in E generated by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively correspond to the zero section, the section $\frac{dx}{\prod_i(x-t_i)}$ and the section at infinity. If S denotes the associated ruled surface (total space of the projective bundle), then the total space of $\Omega^1(D)$ identifies with $S \setminus \Sigma$ where Σ is the section at infinity. On fibers $F_i : \{x = t_i\}$, a natural affine coordinate is given by the residue: the section $\frac{dx}{\prod_i(x-t_i)}$ has residue $\frac{1}{\tau_i}$ at $x = t_i$. Therefore, the two eigendirections over $x = t_i$ correspond to the points $s_i^+, s_i^- \in F_i$ satisfying $x = t_i$ and $\text{Res}_{t_i}(s_i^\pm) = \theta_i^\pm$. Over $x = q$, the kernel of the residual matrix also defines a point s :

$$s_i^\pm : \begin{pmatrix} 1 \\ \tau_i \theta_i^\pm \end{pmatrix} \quad \text{and} \quad s : \begin{pmatrix} 1 \\ p \prod_i(q - t_i) \end{pmatrix}.$$

We have just defined a birational map

$$\text{Con}^\theta(X, D) \dashrightarrow S; (E, \nabla, \mathbf{l}) \mapsto s.$$

Now consider the blow-up of the surface $\pi : \hat{S} \rightarrow S$ at all 8 points s_i^\pm : we denote by E_i^\pm the exceptional divisors and $\hat{\Sigma}, \hat{F}_i$ the strict transforms. When $\theta_i^+ = \theta_i^-$, then first blow-up the point $s_i := s_i^+ = s_i^-$, and then blow-up the intersection point between the exceptional divisor E_i and the strict transform of the fiber F_i ; denote by E'_i the last exceptional divisor. Then the birational map above induces an isomorphism:

$$\text{Con}^\theta(X, D) \xrightarrow{\sim} M^\theta := \hat{S} \setminus \hat{\Sigma} \cup \hat{F}_1 \cup \hat{F}_2 \cup \hat{F}_3 \cup \hat{F}_4.$$

For a general point $s \in M^\theta$, formula (17) gives a parabolic connection with an extra apparent singular point; after elementary transformation over $x = q$, we get a parabolic connection in $\text{Con}^\theta(X, D)$. When $q \rightarrow \infty$, there is a limit point in $\text{Con}^\theta(X, D)$ provided that s tends to an affine point of the fiber. When $q \rightarrow t_i$, one can find a limit point in $\text{Con}^\theta(X, D)$ if, and only if, s tends to one of the exceptional divisors, not to \hat{F}_i ; over the point s_i^- , we get all parabolic connections in $\text{Con}^\theta(X, D)$ such that the parabolic l_i lies in the destabilizing subbundle $\mathcal{O}(1)$.

With our notations, the 2-form $dp \wedge dq$ has polar divisor $2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4$ and is non zero at any other point: it defines a non degenerate holomorphic volume form on the moduli space that should better viewed as a symplectic structure.

4.2. Description and properties of moduli spaces. Fix $(X, D) = (\mathbb{P}^1, \{t_1, t_2, t_3, t_4\})$ as before and consider the Hirzebruch surface S given by the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \Omega^1(D))$. This surface may be obtained from the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \Omega^1)$ by applying 4 elementary transformations directed by the section at infinity \mathcal{O} ; from this point of view, the Liouville form on the total space of Ω^1 induces a rational 2-form ω on $\mathbb{P}(\mathcal{O} \oplus \Omega^1(D))$ with polar divisor $2\Sigma + F_1 + F_2 + F_3 + F_4$: as before, Σ is the section defined by \mathcal{O} and F_i is the fiber over t_i . An affine chart is defined by

$$(x, y) : \mathbb{P}(\mathcal{O} \oplus \Omega^1) \setminus \Sigma \cup F_\infty \rightarrow \mathbb{C} \times \mathbb{C}$$

where F_∞ is the fiber over $x = \infty$ and y is normalized so that

- $y = 0$ corresponds to the section defined by $0 \in H^0(X, \mathcal{O} \oplus \Omega^1(D))$,
- $y = 1$ corresponds to the section defined by $\frac{dx}{\prod_i(x-t_i)} \in H^0(X, \mathcal{O} \oplus \Omega^1(D))$

(and $y = \infty$ is defined by \mathcal{O}). For each finite point $y \in F_i$, we can associate the residue $\text{Res}_{t_i}(y \frac{dx}{\prod_i(x-t_i)}) = \frac{y}{\tau_i}$ (see definition (14)). This gives us a natural parametrization of fibers F_i .

Choose 2 points s_i^\pm on each fiber F_i (possibly, $s_i^+ = s_i^-$). Then, blow-up these 8 points and denote by \hat{S} the blow-up surface, and still denote by $\hat{\Sigma}$ and \hat{F}_i the strict transforms. Then consider the open part

$$M^\theta := \hat{S} \setminus \hat{\Sigma} \cup \hat{F}_1 \cup \hat{F}_2 \cup \hat{F}_3 \cup \hat{F}_4.$$

The 2-form ω extends as a holomorphic symplectic form on M with polar divisor

$$(\omega)_\infty = 2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4.$$

Consider on each fiber F_i the point m_i defined by the arithmetic mean of s_i^+ and s_i^- . Precisely, if we denote $\text{Res}_{t_i}(s_i^\pm) = \theta_i^\pm$, then m_i is defined by $\text{Res}_{t_i}(m_i) = \frac{\theta_i^+ + \theta_i^-}{2}$.

Since $S \setminus \Sigma$ is the total space of $\Omega^1(D)$, there is a one-to-one correspondence between global holomorphic sections of $\Omega^1(D)$ and global sections of $S \rightarrow \mathbb{P}^1$ not intersecting Σ . These sections form a 3-dimensional family. Assume first that there exists such a section passing through all 4 points m_i ; this is equivalent to say $\sum_i(\theta_i^+ + \theta_i^-) = 0$. In this case, it can be checked that $2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4$ is the degenerate fiber of an elliptic fibration on \hat{S} . In this case, the open part M may be viewed as a moduli space of Higgs bundles and this fibration is the well-known Hitchin fibration. The fibration is defined by those elements of the linear system $|2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4|$ on S passing through the 8 points s_i^\pm .

Assume now that $\sum_i(\theta_i^+ + \theta_i^-) \neq 0$. Then, after applying a bundle isomorphism of the form $y \mapsto cy$, we may assume $\sum_i(\theta_i^+ + \theta_i^-) = 1$. If $\theta_1^\pm + \theta_2^\pm + \theta_3^\pm + \theta_4^\pm \notin \mathbb{Z}$ whatever the choice of signs, the surface M^θ identifies with the moduli space of connections $\text{Con}^\theta(X, D)$ discussed in the previous section. On the other hand, when $\theta_1^\pm + \theta_2^\pm + \theta_3^\pm + \theta_4^\pm \in \mathbb{Z}$ for some (maybe several) choices of signs, then M^θ can still be viewed as a moduli space of stable connections (we have to delete some reducible connection to obtain a GIT moduli space, see [3]). By the Riemann-Hilbert correspondance (see [3]), there is an analytic mapping

$$RH : M^\theta \longrightarrow \chi \subset \mathbb{C}^3$$

where χ is an affine cubic surface. For generic θ , this is an analytic isomorphism; for special values of θ like above, the affine surface χ is singular and the map RH is a minimal analytic resolution of singularities. It follows that the only occurrence of complete curves in M^θ is for special values of θ : they come from exceptional divisors in the resolution of singularities of χ : they are rational, and there are at most 4 such curves in a given M^θ . In particular, the divisor $2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4$ is no more contained in an elliptic fibration.

The surface M^θ is not affine however. In fact, there are no non-constant regular functions on M^θ (see [1]). This mainly follows from numerical properties

$$(2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4) \cdot \hat{\Sigma} = (2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4) \cdot \hat{F}_i = 0;$$

indeed, a rational function on \hat{S} whose polar divisor is contained in the support of $2\hat{\Sigma} + \hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4$ would define a fibration of complete curves inside the complement M^θ , contradicting the (almost) non existence of complete curves. This generalize Serre example.

4.3. How complete curves arise for special parameters. When $\theta_i^+ = \theta_i^-$ for some i , then the two points s_i^+ and s_i^- coincide in S . In this case, we have to blow-up twice over $s_i^+ = s_i^-$ to get the surface \hat{S} . Precisely, we first blow-up this point to get an exceptional divisor E_i , and then blow-up the intersection point between E_i and the strict transform of F_i ; we get a second exceptional divisor E'_i and still denote by E_i and \hat{F}_i the strict transforms in the resulting surface \hat{S} . We define similarly the open set $M^\theta := \hat{S} \setminus \hat{\Sigma} \cup \hat{F}_1 \cup \hat{F}_2 \cup \hat{F}_3 \cup \hat{F}_4$. Then, E_i is a rational curve in M^θ having -2 self-intersection number. This curve corresponds to the locus of those parabolic connections (E, ∇, \mathbf{l}) for which the residual matrix at t_i is the scalar matrix $\theta_i^+ \cdot I$.

Similarly, when $\theta_i^+ = \theta_i^- + 1$, from Fuchs' local theory, we can check that a generic element (E, ∇, \mathbf{l}) of $\mathcal{C}on^\theta(X, D)$ will have a logarithmic singular point at t_i (the local monodromy has a non trivial Jordan block); however, for some special parabolic connections, the singular point is not logarithmic anymore (it has diagonal monodromy). The locus of those special connections is a complete and smooth rational curve in M^θ , namely the strict transform of the curve $C \subset S$ characterized as follows:

- $q : C \rightarrow \mathbb{P}^1$ has degree 2 and C does not intersect Σ ;
- C intersects F_i at s_i^+ and has two smooth local branches at this point;
- C intersects other fibers F_j at both s_j^+ and s_j^- .

The first condition fixes the linear system on S that contains the curve: $C \in |2\Sigma + F_1 + F_2 + F_3 + F_4|$. The two other conditions characterize C in this linear system. After blowing-up, the resulting curve \hat{C} becomes smooth and does not intersect strict transforms F_j 's anymore.

When $\theta_i^+ - \theta_i^- \in \mathbb{Z}$, the story is the same. Let us just characterize the special curve C in the case $\theta_i^+ = \theta_i^- + 2$:

- $q : C \rightarrow \mathbb{P}^1$ has degree 4 and C does not intersect Σ ;
- C intersects F_i three times at s_i^+ and one time at s_i^- ;
- C intersects other fibers F_j twice at both s_j^+ and s_j^- .

This can be checked by straightforward formal computation.

Another occurrence of complete curves in M^θ comes from reducible connections. When say $\theta_1^+ + \theta_2^+ + \theta_3^+ + \theta_4^+ = 0$, then some reducible connections arise in $\mathcal{C}on^\theta(X, D)$: each of them stabilize a trivial subbundle $\mathcal{O} \subset E$ and induces on it a logarithmic rank 1 connection with eigenvalue θ_i^+ at t_i . The locus of these reducible connections form a section \hat{C} for the projection $q : M^\theta \rightarrow \mathbb{P}^1$. The corresponding curve $C \subset S$ is the section of $q : C \rightarrow \mathbb{P}^1$ that does not intersect Σ and intersecting each F_i at s_i^+ .

4.4. Link with Fuchsian systems. As we shall see here, if we delete one of the exceptional divisors $E_i^\pm \rightarrow s_i^\pm$ from M^θ , then the surface becomes affine.

Consider the moduli space $\mathcal{C}on^\theta(X, D)$ with eigenvalues θ satisfying

$$\begin{cases} \theta_i^+ = \theta_i \\ \theta_i^- = -\theta_i \end{cases} \quad \text{except} \quad \begin{cases} \theta_4^+ = -\theta_4 \\ \theta_4^- = \theta_4 - 1 \end{cases}$$

After applying an elementary transformation

$$\text{Elm}_{t_4}^- : \text{Con}^\theta(X, D) \xrightarrow{\sim} \text{Con}^{\theta'}(X, D)$$

we obtain new eigenvalues

$$\begin{cases} (\theta_i^+)' = \theta_i \\ (\theta_i^-)' = -\theta_i \end{cases} \quad \text{for all } i = 1, 2, 3, 4.$$

This latter moduli space splits into the disjoint union of those connection defined on the trivial bundle, namely Fuchsian systems discussed in section 2, and those defined on $E = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ (see section 4). In fact, a parabolic connection $(E = \mathcal{O} \oplus \mathcal{O}(1), \nabla, \mathbf{l})$ of $\text{Con}^\theta(X, D)$ is sent to a connection on the non trivial bundle by the elementary transformation if, and only if, its parabolic l_4 lie on the subbundle $\mathcal{O}(1) \subset E$. The locus of those special parabolic connections in M^θ is given by the exceptional divisor $E_4^- \rightarrow s_4^-$ (or we better should say the affine part of this divisor once we have deleted the intersection point with F_4). Therefore, E_4^- identifies via $\text{Elm}_{t_4}^-$ to the moduli space \mathbb{A}^1 of those connections on $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ discussed in section 4. Moreover, after deleting E_4^- from M^θ , we get an isomorphism with the moduli space of Fuchsian systems computed in section 2, i.e. a cubic affine surface:

$$M^\theta \setminus E_4^- \xrightarrow{\text{Elm}_{t_4}^-} \text{Fuchs}^{\theta'}(X, D) \xrightarrow{\sim} S^\theta \subset \mathbb{C}^4.$$

We can do in a similar way with other elementary transformations $\text{Elm}_{t_i}^\pm$ (that we have to compose with a convenient twist in order to fit with \mathfrak{sl}_2 -systems discussed in section 2) and we promptly deduce that M^θ becomes a cubic affine surface once we delete any one of the exceptional divisors E_i^\pm .

5. THE GARNIER CASE $n > 4$

The computation of section 4.1 generalizes as follows. Fix $D = t_1 + \dots + t_n$ a reduced divisor on $X := \mathbb{P}^1$, $n \geq 4$. Fix eigenvalues $\theta, \theta_i^\pm \in \mathbb{C}$ for each $i = 1, \dots, n$, with $\sum_{i=1}^n (\theta_i^+ + \theta_i^-) + 1 = 0$. Assume θ generic for simplicity:

$$\theta_i^+ - \theta_i^- \notin \mathbb{Z} \quad \text{and} \quad \theta_1^\pm + \dots + \theta_n^\pm \notin \mathbb{Z}.$$

Consider the moduli space $\text{Con}^\theta(X, D)$ of corresponding connections. Note that we can omit parabolic structure from the discussion: under genericity of θ , the parabolic structure \mathbf{l} is determined by the connection (E, ∇) .

We would like to explicitly describe the generic connection and derive an explicit open subset of $\text{Con}^\theta(X, D)$. Let us first assume $E = \mathcal{O} \oplus \mathcal{O}(1)$. The special subbundle $\mathcal{O}(1) \subset E$ is stabilized by ∇ at exactly $n - 3$ points counted with multiplicity; these are the zeroes of the composition map

$$L = \mathcal{O}(1) \xrightarrow{\text{inclusion}} E \xrightarrow{\nabla} E \otimes \Omega^1(D) \xrightarrow{\text{quotient}} (E/L) \otimes \Omega^1(D).$$

These points are also the extra apparent singular points for the fuchsian scalar equation derived from ∇ and the cyclic vector L . Let us consider the case where these zeroes and the poles t_i 's are two-by-two distinct; denote the by q_1, \dots, q_{n-3} the apparent points. We also assume $q_j \neq \infty$ for simplicity.

Consider now the parabolic structure $\tilde{\mathbf{l}}$ defined over each q_j by the special line bundle L . Still assuming all t_i 's and q_j 's two-by-two distinct, apply a positive elementary transformation at each q_j directed by $\tilde{\mathbf{l}}$ (i.e. by $L = \mathcal{O}(1) \subset E$). Let $(E', \nabla', \tilde{\mathbf{l}}')$ be the resulting parabolic connection; note that $E' = \mathcal{O} \oplus \mathcal{O}(n - 2)$.

Proposition 5. *The connection (E', ∇') is given, up to bundle automorphism, by the connection matrix (we follow notations of section 3.1)*

$$\Omega = \sum_{i=1}^n \begin{pmatrix} 0 & \frac{1}{\tau_i} \\ -\tau_i \theta_i^+ \theta_i^- & \theta_i^+ + \theta_i^- \end{pmatrix} \frac{dx}{x-t_i} + \sum_{j=1}^{n-3} \begin{pmatrix} 0 & 0 \\ p_j & -1 \end{pmatrix} \frac{dx}{x-q_j} + \begin{pmatrix} 0 & 0 \\ c(x) & 0 \end{pmatrix} dx$$

where $c(x)$ is polynomial of degree $n-4$, determined by the fact that all poles q_j 's are non logarithmic (with scalar monodromy). The connection is determined by the new parabolic structure $\tilde{\mathbf{l}}'$ defined by kernel of residual matrices over the q_j 's.

Proof. We proceed like in section 4.1. On the initial bundle $E = \mathcal{O} \oplus \mathcal{O}(1)$, the connection matrix writes $\Omega = A(x) \frac{dx}{\prod_i (x-t_i)}$ where

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & -x^{n-1} + d(x) \end{pmatrix} \quad \text{with} \quad \begin{cases} \deg(b) \leq n-3 \\ \deg(a), \deg(d) \leq n-2 \\ \deg(c) \leq n-1 \end{cases}$$

Then, up to a diagonal automorphism, we have $b(x) = \prod_{j=1}^{n-3} (x-q_j)$. We now apply the $n-3$ elementary transformations and get the new connection matrix

$$\left(\begin{array}{cc} \frac{a(x)}{\prod_i (x-t_i)} & \frac{1}{\prod_i (x-t_i)} \\ \frac{c(x) \prod_j (x-q_j)}{\prod_i (x-t_i)} & -\frac{-x^{n-1} + d(x)}{\prod_i (x-t_i)} - \sum_{j=1}^{n-3} \frac{1}{x-q_j} \end{array} \right) dx$$

On the new bundle $E' = \mathcal{O} \oplus \mathcal{O}(n-2)$, we can use unipotent automorphisms of E' to kill the polynomial coefficient $a(x)$. We now get the unique normal form

$$(18) \quad \Omega' = \left(\begin{array}{cc} 0 & \frac{1}{\prod_i (x-t_i)} \\ \frac{\tilde{c}(x)}{\prod_i (x-t_i) \prod_j (x-q_j)} & \frac{-x^{n-1} + \tilde{d}(x)}{\prod_i (x-t_i)} - \sum_{j=1}^{n-3} \frac{1}{x-q_j} \end{array} \right) dx$$

with $\tilde{c}(x)$ and $\tilde{d}(x)$ polynomials of degree $3n-7$ and $n-2$ respectively. Now, taking into account the spectral data, we get the normal form of the statement. \square

From this statement, one easily derive the following open subset of $\mathcal{C}on^\theta(X, D)$. Consider the surface S defined by the total space of $\mathbb{P}(\mathcal{O} \oplus \Omega^1(D))$ and denote by Σ the section defined by $\Omega^1(D)$: $S \setminus \Sigma$ naturally identifies with the total space of $\Omega^1(D)$. In particular, the affine part of the fiber F_i over t_i has the natural chart $\text{Res}_{t_i} : F_i \setminus \Sigma \xrightarrow{\sim} \mathbb{C}$ given by taking the residue of sections of $\Omega^1(D)$; we define two points $s_i^\pm \in F_i$ by $\text{Res}_{t_i}(s_i^\pm) = \theta_i^\pm$. Let $\pi = \hat{S} \rightarrow S$ be the blow-up of S at all $2n$ points and denote by $\hat{\Sigma}$ and \hat{F}_i the strict transforms. Then consider the open part

$$M^\theta := \hat{S} \setminus (\hat{\Sigma} \cup \hat{F}_1 \cup \dots \cup \hat{F}_n)$$

and the symmetric product

$$(M^\theta)^{(n-3)} := M^\theta \times \dots \times M^\theta / \text{Perm}(1, \dots, n).$$

Then the above normal form defines an embedding

$$(M^\theta)^{(n-3)} \setminus \Delta \hookrightarrow \mathcal{C}on^\theta(X, D)$$

where Δ is the ‘‘codimension one diagonal’’ defined by those $\{q_i, p_i\}$ for which $q_i = q_j$ for some $i \neq j$. This is the point of view developed by Oblezin in [5] where the compactification of this picture is discussed. Another point of view has been recently studied by Saito and the author in [4].

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