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► **To cite this version:**

Olivier Roussel, Marc Renaud, Michel Taïx. Closed-forms of Kirchhoff elastic rods shape and sensitivity in the planar case. [Research Report] LAAS-CNRS. 2015. hal-01133395v3

**HAL Id: hal-01133395**

**<https://hal.science/hal-01133395v3>**

Submitted on 7 Apr 2015

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# Closed-forms of Kirchhoff elastic rods shape and sensitivity in the planar case

Olivier Roussel<sup>1</sup>, Marc Renaud<sup>1</sup> and Michel Taïx<sup>1\*</sup>

## Abstract

In this report we give closed-forms of Kirchhoff 3-D elastic rods curvature in terms of elliptic functions and, by treating planar rods as a special case, we show we can also obtain closed-forms of planar rods shape, sensitivity and total elastic energy.

## 1 General case of 3-D rods

Consider an inextensible, non-shearable and unit length linearly elastic rod. The shape of the rod traces a curve that we will describe by the mapping  $q : [0, 1] \rightarrow SE(3)$ . The position along the rod is parametrized by  $t \in [0, 1]$  and we will name "base" and "tip" of the rod its extremity at  $t = 0$  and  $t = 1$  respectively. Let the mappings  $u_1(t), u_2(t), u_3(t)$  such that  $u_i : [0, 1] \rightarrow \mathbb{R}$  be axial and bending rod strains respectively, and  $c_1, c_2, c_3$  be the constants that reflect its elasticity properties. As in [3], we say the elastic rod is in static equilibrium in the sense of Kirchhoff if it locally minimizes the elastic energy defined by

$$E_{el} = \frac{1}{2} \int_0^1 \sum_{i=1}^3 c_i u_i^2 dt. \quad (1.1)$$

Without loss of generality, we will also assume that the base of the rod is held fixed at the origin, i.e.  $q(0) = e$  where  $e$  is the identity element of  $SE(3)$ . Under these assumptions, we will denote by  $\mathcal{B}$  the set of positions that the other extremity of the rod  $q(1)$  can reach. As shown in [2], the problem of static equilibrium of such rods can be formulated as an optimal control problem by

$$\begin{aligned} & \underset{q, u}{\text{minimize}} && \frac{1}{2} \int_0^1 \sum_{i=1}^3 c_i u_i^2 dt \\ & \text{subject to} && \dot{q} = q \left( \sum_{i=1}^3 u_i X_i + X_4 \right) \\ & && q(0) = e, \quad q(1) = b \end{aligned} \quad (1.2)$$

for some  $b \in \mathcal{B}$  and where

$$\begin{aligned} X_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & X_2 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & X_3 &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ X_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & X_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & X_6 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

is a basis for  $\mathfrak{se}(3)$ , the Lie algebra of  $SE(3)$ . Note that when solving this optimal control problem, the rod tip position  $b$  is not an input.

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In these conditions, the Maximum Principle states that solutions to this optimal control problem are the projections of extremal curves defined on the cotangent bundle  $T^*SE(3)$  onto  $SE(3)$ . Thanks to the Lie Group structure of  $SE(3)$ , the Hamiltonian can be reduced on the dual of the Lie algebra  $\mathfrak{se}(3)^*$  and the corresponding (time-varying) Hamiltonian vector fields  $\mu : [0, 1] \rightarrow \mathfrak{se}(3)^*$  can be expressed by

$$\begin{cases} \dot{\mu}_1 = \frac{\mu_3\mu_2}{c_3} - \frac{\mu_2\mu_3}{c_2} \\ \dot{\mu}_2 = \mu_6 + \frac{\mu_1\mu_3}{c_1} - \frac{\mu_1\mu_3}{c_3} \\ \dot{\mu}_3 = -\mu_5 + \frac{\mu_1\mu_2}{c_2} - \frac{\mu_1\mu_2}{c_1} \\ \dot{\mu}_4 = \frac{\mu_3\mu_5}{c_3} - \frac{\mu_2\mu_6}{c_2} \\ \dot{\mu}_5 = \frac{\mu_1\mu_6}{c_1} - \frac{\mu_3\mu_4}{c_3} \\ \dot{\mu}_6 = \frac{\mu_2\mu_4}{c_2} - \frac{\mu_1\mu_5}{c_1} \end{cases} \quad (1.3)$$

where vector fields  $\mu$  are related to controls  $u_i$  by  $u_i = c_i^{-1}\mu_i$  for  $i \in \{1, 2, 3\}$ .

Let  $\mathcal{A}$  be the set homeomorphic to  $\mathbb{R}^6$  and  $a \in \mathcal{A}$  such that  $a_i \triangleq \mu_i(0)$ ,  $i \in \{1, \dots, 6\}$ . It has been shown in [2] that coordinates in  $\mathcal{A}$  offer a global parameterization to the set of static equilibrium configuration for the rod. In other words, we can describe configurations of quasi-static 3-D elastic rods using the 6-dimensional configuration space  $\mathcal{A}$ .

Assuming isotropy and normalized elasticity constants such that  $c_i = 1$  for  $i \in \{1, 2, 3\}$ , we have from (1.3)  $\dot{\mu}_1 = 0$ . Then  $\mu_1$  is a constant of motion with  $\mu_1 = a_1$  and

$$\begin{cases} \dot{\mu}_2 = \mu_6 \\ \dot{\mu}_3 = -\mu_5 \\ \dot{\mu}_4 = \mu_3\mu_5 - \mu_2\mu_6 \\ \dot{\mu}_5 = a_1\mu_6 - \mu_3\mu_4 \\ \dot{\mu}_6 = \mu_2\mu_4 - a_1\mu_5 \end{cases} \quad (1.4)$$

The signed curvature  $\kappa$  and the torsion  $\tau$  of the curve can be expressed in terms of  $\mu$  by

$$\kappa^2 = \mu_2^2 + \mu_3^2 \quad \tau = \mu_1 - \frac{\mu_2\mu_5 + \mu_3\mu_6}{\mu_2^2 + \mu_3^2}$$

and, as mentioned in [2], the differential system (1.4) is equivalent to

$$2\ddot{\kappa} + \kappa^3 - 2\kappa(\tau - \lambda_1)^2 = \lambda_2\kappa \quad (1.5a)$$

$$\kappa^2(\tau - \lambda_1) = \lambda_3 \quad (1.5b)$$

where the constants of integration are given by

$$\begin{cases} \lambda_1 \triangleq \frac{a_1}{2} \\ \lambda_2 \triangleq a_2^2 + a_3^2 + 2a_4 - \frac{a_1^2}{2} \\ \lambda_3 \triangleq \frac{a_1}{2}(a_2^2 + a_3^2) - (a_2a_5 + a_3a_6). \end{cases}$$

Substituting (1.5b) into (1.5a) and integrating, we obtain

$$\dot{\kappa}^2 + \frac{1}{4}\kappa^4 + \lambda_3^2\kappa^{-2} - \frac{\lambda_2}{2}\kappa^2 = \lambda_4 \quad (1.6)$$

where the constant of integration  $\lambda_4$  is given by

$$\lambda_4 \triangleq a_5^2 + a_6^2 - \frac{1}{4}(a_2^2 + a_3^2)^2 + \frac{1}{2}(a_2^2 + a_3^2)(a_1^2 - 2a_4) - a_1(a_2a_5 + a_3a_6)$$

By making the change of variable  $v = \kappa^2$ , (1.6) transforms to

$$\dot{v}^2 + v^3 - 2\lambda_2v^2 - 4\lambda_4v + 4\lambda_3^2 = 0 \quad (1.7)$$

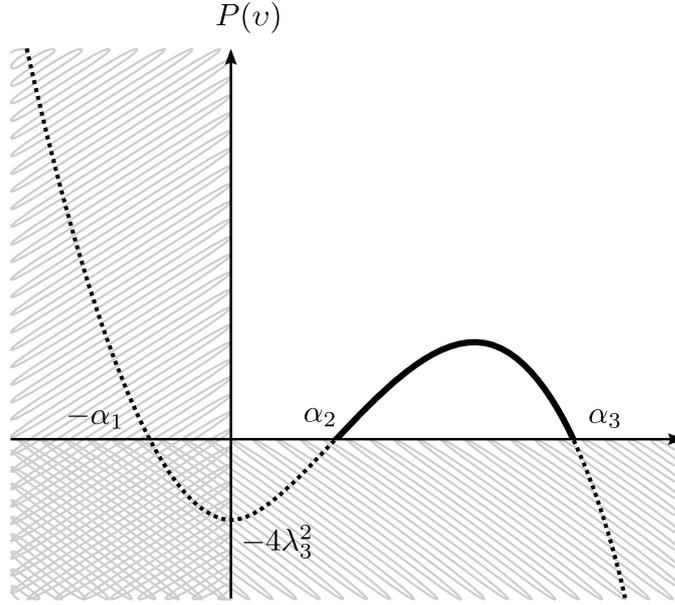


Figure 1: Plot of the cubic polynomial  $P(v)$  with respect to the squared curvature  $v$ . Hatched regions correspond to impossible values illustrating that the only valid range for  $v$  is given by  $\alpha_2$  and  $\alpha_3$ , the zeros of  $P(v)$ .

As already stated in [4], this equation is in the form  $\dot{v}^2 = P(v)$  with  $P$  is the cubic polynomial

$$P(v) \triangleq -v^3 + 2\lambda_2 v^2 + 4\lambda_4 v - 4\lambda_3^2. \quad (1.8)$$

Let  $-\alpha_1, \alpha_2, \alpha_3$  be the zeros of the polynomial  $P(v)$  such that

$$-\alpha_1 \leq 0 \leq \alpha_2 \leq \alpha_3. \quad (1.9)$$

As  $P(\pm\infty) = \mp\infty$  and  $P(0) = -4\lambda_3^2 \leq 0$ ,  $P(v)$  is in the form illustrated in figure 1. Also, we have  $v \geq 0$  and  $P(v) \geq 0$  as they are both squares, so  $v \in [\alpha_2, \alpha_3]$ . The polynomial  $P(v)$  can be rewritten for its zeros by

$$P(v) = -(v + \alpha_1)(v - \alpha_2)(v - \alpha_3).$$

We can express the polynomial zeros  $-\alpha_1, \alpha_2, \alpha_3$  from the constants of integrations  $\lambda_i$  by

$$\begin{aligned} \alpha_1 - \alpha_2 - \alpha_3 &= -2\lambda_2 \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 - \alpha_2\alpha_3 &= 4\lambda_4 \\ \alpha_1\alpha_2\alpha_3 &= 4\lambda_3^2. \end{aligned} \quad (1.10)$$

The squared curvature  $v$  can be expressed in terms of elliptic functions by

$$v(t) = \alpha_3 \left( 1 - n \operatorname{sn}^2(rt + \varphi|m) \right) \quad (1.11)$$

the parameter  $m$ , the characteristic  $n$  and  $r$  can be expressed from the polynomial zeros by

$$m = \frac{\alpha_3 - \alpha_2}{\alpha_3 + \alpha_1} \quad n = \frac{\alpha_3 - \alpha_2}{\alpha_3} \quad r = \frac{1}{2} \sqrt{\alpha_3 + \alpha_1} \quad (1.12)$$

Given

$$\varpi \triangleq \sqrt{\frac{1}{n} \left( 1 - \frac{a_2^2 + a_3^2}{\alpha_3} \right)} \quad (1.13)$$

the phase  $\varphi$  can be retrieved from  $a_2^2 + a_3^2 = \alpha_3(1 - n \operatorname{sn}^2(\varphi|m))$ , and is given by

$$\varphi = \operatorname{sgn}(a_3 a_5 - a_2 a_6) \operatorname{arcsn}(\varpi|m) \quad (1.14)$$

where  $\operatorname{arcsn}$  is the inverse of the Jacobi elliptic function  $\operatorname{sn}$ .

Note that from (1.9), we have  $0 \leq m \leq n \leq 1$ .

As outlined in [3], it has been shown the Hamiltonian vector fields in (1.4) is integrable and we have proved it can be expressed in the following form

$$\begin{cases} \mu_2 = \kappa \sin \psi \\ \mu_3 = \kappa \cos \psi \\ \mu_4 = \frac{1}{2}(\lambda_2 + \frac{a_1^2}{2} - v) \\ \mu_5 = -\dot{\kappa} \cos \psi + \kappa \dot{\psi} \sin \psi \\ \mu_6 = \dot{\kappa} \sin \psi + \kappa \dot{\psi} \cos \psi. \end{cases} \quad (1.15)$$

where

$$\psi(t) = \lambda_1 t - \frac{\lambda_3}{\alpha_3 r} \left( \Pi(n, \operatorname{am}(rt + \varphi|m)) - \Pi(n, \operatorname{am}(\varphi|m)) \right) + \psi(0)$$

with  $\Pi(n, u|m)$  the elliptic integral of the third kind and  $\operatorname{am}(u|m)$  is the Jacobi amplitude.

## 2 Planar case

Although neither the curve  $q(t)$  nor the rod sensitivity  $\frac{\partial q(t)}{\partial a}$  can be explicitly expressed in the general 3-D case, we will show in this section that closed forms can be obtained in the planar case which can be treated as a particular case of the previously presented model.

### 2.1 Curvature and internal wrenches

Considering only planar curves  $q(t)$  in the  $xy$ -plane with  $q = (0, 0, \theta, x, y, 0)^T$ , Hamiltonian vector fields defined in (1.3) simplify to

$$\begin{cases} \dot{\mu}_1 = 0 \\ \dot{\mu}_2 = 0 \\ \dot{\mu}_3 = -\mu_5 \\ \dot{\mu}_4 = \mu_3 \mu_5 \\ \dot{\mu}_5 = -\mu_3 \mu_4 \\ \dot{\mu}_6 = 0 \end{cases} \quad (2.1)$$

Closed-forms of rod internal wrenches  $\mu(t)$  defined in (1.15) reduce to

$$\begin{cases} \mu_1 = 0 \\ \mu_2 = 0 \\ \mu_3 = \kappa \\ \mu_4 = -\frac{1}{2}(\kappa^2 + \lambda_2) \\ \mu_5 = -\dot{\kappa} \\ \mu_6 = 0 \end{cases} \quad (2.2)$$

And constants of integration defined in (1) simplify to

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = a_3^2 + 2a_4 \\ \lambda_3 = 0 \\ \lambda_4 = a_5^2 - a_3^2(\frac{1}{4}a_3^2 + a_4) \end{cases}$$

We retrieve the same results as we would have obtain by applying the same problem formulation on the Lie Group  $SE(2)$  rather than  $SE(3)$ . Therefore, in the rest of this section we will restrict to solutions of (1.2) that are similar to trajectories on  $SE(2)$ , which are generated by the subset of initial conditions  $\{a \in \mathcal{A} : (a_1, a_2, a_6) = (0, 0, 0)\}$ .

In the following equations, when referring to an elliptic function  $\text{pq}$ , we will simplify the notation  $\text{pq}(u|m)$  to  $\text{pq } u$ . Also, let us define

$$\Gamma(t) \triangleq rt + \varphi$$

and the following constants of motion that be needed in the following developments

$$\begin{aligned} \varepsilon &\triangleq \text{sgn}(a_3) \\ \Gamma_0 &\triangleq \Gamma(0) \\ \delta &\triangleq \lambda_2^2 + 4\lambda_4 \end{aligned}$$

The expression of the phase  $\varphi$  given in (1.14) simplifies to

$$\varphi = \text{sgn}(a_3 a_5) \arcsn(\varpi|m)$$

where  $\varpi$  given in (1.13) reduces to

$$\varpi = \sqrt{\frac{1}{n} \left( 1 - \frac{a_3^2}{\alpha_3} \right)}.$$

### 2.1.1 Expression of the curvature

In the planar case, as  $\lambda_3 = 0$ , the polynomial  $P(v)$  simplifies to

$$P(v) = -v^3 + 2\lambda_2 v^2 + 4\lambda_4 v$$

so  $P(v)$  has one trivial zero at  $v = 0$ .

From (1.9), we can distinguish three cases as outlined in [4] and [5]:

- Case I:  $\lambda_4 > 0$

Using (1.10), we have that

$$\lambda_1(\lambda_2 + \lambda_3) > \lambda_2 \lambda_3.$$

This imposes the choice for the zeros to

$$\begin{cases} \alpha_1 = -\lambda_2 + \sqrt{\delta} \\ \alpha_2 = 0 \\ \alpha_3 = \lambda_2 + \sqrt{\delta}. \end{cases}$$

From (1.12) we get  $n = 1$ , so the squared curvature formula in (1.11) simplifies to

$$\begin{aligned} v(t) &= \alpha_3 (1 - \text{sn}^2 \Gamma(t)) \\ &= \alpha_3 \text{cn}^2 \Gamma(t). \end{aligned}$$

Then the signed curvature is given by

$$\kappa(t) = \varepsilon\sqrt{\alpha_3} \operatorname{cn} \Gamma(t). \quad (2.3)$$

The curvature  $\kappa(t)$  oscillates between  $\sqrt{\alpha_3}$  and  $-\sqrt{\alpha_3}$  and the resulting curve  $q(t)$  is called a "wavelike" elastica.

- Case II:  $\lambda_4 < 0$  Using (1.10), we have

$$\lambda_1(\lambda_2 + \lambda_3) < \lambda_2\lambda_3.$$

This imposes the choice for the zeros to

$$\begin{cases} \alpha_1 &= 0 \\ \alpha_2 &= \lambda_2 - \sqrt{\delta} \\ \alpha_3 &= \lambda_2 + \sqrt{\delta}. \end{cases}$$

From (1.12) we get  $n = m$ , so the squared curvature formula in (1.11) simplifies to

$$\begin{aligned} v(t) &= \alpha_3 (1 - m \operatorname{sn}^2 \Gamma(t)) \\ &= \alpha_3 \operatorname{dn}^2 \Gamma(t). \end{aligned}$$

Then the signed curvature is given by

$$\kappa(t) = \varepsilon\sqrt{\alpha_3} \operatorname{dn} \Gamma(t) \quad (2.4)$$

The curvature  $\kappa(t)$  is non-vanishing and the resulting curve  $q(t)$  is called a "orbit-like" elastica.

- Case III:  $\lambda_4 = 0$  This borderline case implies the polynomial  $P(v)$  reduces to

$$P(v) = -v^3 + 2\lambda_2 v^2$$

which has a double zero.

Using first equation of (1.10), only one choice is possible for the zeros  $\alpha_i$ :

$$\begin{cases} \alpha_1 = \alpha_2 = 0 \\ \alpha_3 = |2\lambda_2|. \end{cases}$$

which leads to the signed curvature

$$\kappa(t) = \varepsilon\sqrt{\alpha_3} \operatorname{sech} \Gamma(t) \quad (2.5)$$

This corresponds to the borderline case where the curvature is non-periodic.

### 2.1.2 Reduction to a unique formulation of the curvature

These cases can be reduced to a single formulation of the curvature by allowing the parameter  $m$  to be any positive or null real and applying the Jacobi's real transformation (see [1] §16.11). By relaxing the constraint on the zeros  $\alpha_i$  given in (1.9), and keeping only one fixed choice on the zeros that we will denote by  $\alpha'$  such that

$$\begin{cases} \alpha'_1 = 0 \\ \alpha'_2 = \lambda_2 - \sqrt{\delta} \\ \alpha'_3 = \lambda_2 + \sqrt{\delta}. \end{cases}$$

In this form,  $\alpha'_3$  is positive as  $\sqrt{\delta} > |\lambda_2|$  but  $\alpha'_2$  can now be negative. Note that we still have  $\alpha'_3 \geq \alpha'_2$ .

Using same forms as in (1.12), the elliptic parameter  $m'$  and  $r'$  by

$$m' = \frac{\alpha'_3 - \alpha'_2}{\alpha'_3} \quad r' = \frac{1}{2}\sqrt{\alpha'_3}$$

but as mentioned before, the new elliptic parameter  $m'$  is only constrained to in  $[0, \infty)$ . Then, the signed curvature can be expressed by a unique expression by

$$\kappa(t) = \varepsilon \sqrt{\alpha'_3} \operatorname{dn}(r'(t + \phi)|m') \quad (2.6)$$

When  $m' > 1$ , the Jacobi's real transformation can be applied to reduce to a parameter  $m$  such that  $0 \leq m \leq 1$  and we retrieve the previously described cases.

### 2.1.3 Explicit formulation of rod total elastic energy

Recall from (1.1) the total elastic energy of the rod is given by

$$\begin{aligned} E_{el} &= \frac{1}{2} \int_0^1 u_3(t)^2 dt \\ &= \frac{1}{2} \int_0^1 \kappa(t)^2 dt \end{aligned}$$

Using the unique formulation of the curvature  $\kappa(t)$  given in (2.6) can be integrated to give an explicit formulation in terms of the elliptic integral of the second kind by

$$E_{el} = \frac{\alpha'_3}{2} E(r'(t + \phi)|m')$$

## 2.2 Integration of the curve $q(t)$

From the differential system defined in (1.2), it follows that

$$\dot{\theta} = u_3 = \kappa \quad \dot{x} = \cos \theta \quad \dot{y} = \sin \theta.$$

Using (2.2), the integration of the curvature is given by

$$\cos \theta(t) = \beta_1(0)\beta_1(t) + 4\beta_2(0)\beta_2(t) \quad (2.7a)$$

$$\sin \theta(t) = 2\varepsilon(\beta_1(0)\beta_2(t) - \beta_2(0)\beta_1(t)) \quad (2.7b)$$

$$x(t) = \beta_1(0) \int \beta_1(t) + 4\beta_2(0) \int \beta_2(t) \quad (2.7c)$$

$$y(t) = 2\varepsilon \left( \beta_1(0) \int \beta_2(t) - \beta_2(0) \int \beta_1(t) \right). \quad (2.7d)$$

The functions  $\beta_1(t)$  and  $\beta_2(t)$  can be explicitly given using Jacobi elliptic functions and the elliptic integral of second kind  $E(u|m)$  in the three cases previously described as follows

- Case I:  $\lambda_4 > 0$

Integrating the curvature in (2.3) (see [1] §16.24) leads to

$$\theta(t) = 2\varepsilon (\arccos(\operatorname{dn} \Gamma(t)) - \arccos(\operatorname{dn} \Gamma_0))$$

Let  $A(t) \triangleq \arccos(\operatorname{dn} \Gamma(t))$  and  $A(0) \triangleq \arccos(\operatorname{dn} \Gamma_0)$ , then

$$\begin{aligned}\cos A(t) &= \operatorname{dn} \Gamma(t) \\ \sin A(t) &= \pm \sqrt{1 - \operatorname{dn}^2 \Gamma(t)} \\ &= \sqrt{m} \operatorname{sn} \Gamma(t)\end{aligned}$$

Given that  $\frac{\theta(t)}{2} = \varepsilon (A(t) - A(0))$ , we have

$$\begin{aligned}\cos \frac{\theta(t)}{2} &= \cos A(t) \cos A(0) + \sin A(t) \sin A(0) \\ &= \operatorname{dn} \Gamma(t) \operatorname{dn} \Gamma_0 + m \operatorname{sn} \Gamma(t) \operatorname{sn} \Gamma_0 \\ \sin \frac{\theta(t)}{2} &= \sin A(t) \cos A(0) - \cos A(t) \sin A(0) \\ &= \varepsilon \sqrt{m} (\operatorname{sn} \Gamma(t) \operatorname{dn} \Gamma_0 - \operatorname{sn} \Gamma_0 \operatorname{dn} \Gamma(t))\end{aligned}$$

Using half-angle formulas, we get

$$\begin{aligned}\cos \theta(t) &= \cos^2 \frac{\theta(t)}{2} - \sin^2 \frac{\theta(t)}{2} \\ &= (2 \operatorname{dn}^2 \Gamma_0 - 1) (2 \operatorname{dn}^2 \Gamma(t) - 1) + 4m \operatorname{dn} \Gamma(t) \operatorname{sn} \Gamma(t) \operatorname{dn} \Gamma_0 \operatorname{sn} \Gamma_0 \\ \sin \theta(t) &= 2 \cos \frac{\theta(t)}{2} \sin \frac{\theta(t)}{2} \\ &= 2\varepsilon \sqrt{m} ((2 \operatorname{dn}^2 \Gamma_0 - 1) \operatorname{dn} \Gamma(t) \operatorname{sn} \Gamma(t) - (2 \operatorname{dn}^2 \Gamma(t) - 1) \operatorname{dn} \Gamma_0 \operatorname{sn} \Gamma_0)\end{aligned}$$

which is in the form (2.7) with  $\beta_1$  and  $\beta_2$  given by

$$\beta_1(t) \triangleq 2 \operatorname{dn}^2 \Gamma(t) - 1 \tag{2.8a}$$

$$\beta_2(t) \triangleq \sqrt{m} \operatorname{sn} \Gamma(t) \operatorname{dn} \Gamma(t) \tag{2.8b}$$

and can be integrated to

$$\int \beta_1(t) = 2r^{-1} (E(\operatorname{am} \Gamma(t)) - E(\operatorname{am} \Gamma_0)) - t \tag{2.9a}$$

$$\int \beta_2(t) = -r^{-1} (\operatorname{cn} \Gamma(t) - \operatorname{cn} \Gamma_0) \tag{2.9b}$$

$$\tag{2.9c}$$

- Case II:  $\lambda_4 < 0$

Integrating the curvature in (2.4) (see [1] §16.24) leads to

$$\theta(t) = 2\varepsilon (\arcsin(\operatorname{sn} \Gamma(t)) - \arcsin(\operatorname{sn} \Gamma_0))$$

Let  $A(t) \triangleq \arcsin(\operatorname{sn} \Gamma(t))$  and  $A(0) \triangleq \arcsin(\operatorname{sn} \Gamma_0)$ , then

$$\begin{aligned}\cos A(t) &= \pm \sqrt{1 - \operatorname{sn}^2 \Gamma(t)} \\ &= \operatorname{cn} \Gamma(t) \\ \sin A(t) &= \operatorname{sn} \Gamma(t)\end{aligned}$$

Given that  $\frac{\theta(t)}{2} = \varepsilon (A(t) - A(0))$ , we have

$$\begin{aligned}\cos \frac{\theta(t)}{2} &= \cos A(t) \cos A(0) + \sin A(t) \sin A(0) \\ &= \operatorname{cn} \Gamma(t) \operatorname{cn} \Gamma_0 + \operatorname{sn} \Gamma(t) \operatorname{sn} \Gamma_0 \\ \sin \frac{\theta(t)}{2} &= \sin A(t) \cos A(0) - \cos A(t) \sin A(0) \\ &= \varepsilon (\operatorname{sn} \Gamma(t) \operatorname{cn} \Gamma_0 - \operatorname{cn} \Gamma_0 \operatorname{sn} \Gamma(t))\end{aligned}$$

Using half-angle formulas, we get

$$\begin{aligned}\cos \theta(t) &= \cos^2 \frac{\theta(t)}{2} - \sin^2 \frac{\theta(t)}{2} \\ &= (1 - 2 \operatorname{dn}^2 \Gamma(t)) (1 - 2 \operatorname{dn}^2 \Gamma_0) + 4 \operatorname{cn} \Gamma(t) \operatorname{sn} \Gamma(t) \operatorname{cn} \Gamma_0 \operatorname{sn} \Gamma_0 \\ \sin \theta(t) &= 2 \cos \frac{\theta(t)}{2} \sin \frac{\theta(t)}{2} \\ &= 2\varepsilon ((1 - 2 \operatorname{sn}^2 \Gamma_0) \operatorname{cn} \Gamma(t) \operatorname{sn} \Gamma(t) - (1 - 2 \operatorname{dn}^2 \Gamma(t)) \operatorname{cn} \Gamma_0 \operatorname{sn} \Gamma_0)\end{aligned}$$

which is in the form (2.7) with  $\beta_1$  and  $\beta_2$  given by

$$\beta_1(t) \triangleq 1 - 2 \operatorname{sn}^2 \Gamma(t) \quad (2.10a)$$

$$\beta_2(t) \triangleq \operatorname{sn} \Gamma(t) \operatorname{cn} \Gamma(t) \quad (2.10b)$$

and can be integrated to

$$\int \beta_1(t) = m^{-1} (t(m-2) + 2r^{-1} (\operatorname{E}(\operatorname{am} \Gamma(t)) - \operatorname{E}(\operatorname{am} \Gamma_0))) \quad (2.11a)$$

$$\int \beta_2(t) = -r^{-1} (\operatorname{cn} \Gamma(t) - \operatorname{cn} \Gamma_0) \quad (2.11b)$$

$$(2.11c)$$

- Case III:  $\lambda_4 = 0$

Integrating the curvature in (2.5) (see [1] §16.24) leads to

$$\theta(t) = 2\varepsilon (\arctan(\sinh \Gamma(t)) - \arctan(\sinh \Gamma_0))$$

Let  $A(t) \triangleq \arctan(\sinh \Gamma(t))$  and  $A(0) \triangleq \arctan(\sinh \Gamma_0)$ , then

$$\begin{aligned}\cos A(t) &= \pm (1 + \sinh^2 \Gamma(t))^{-\frac{1}{2}} \\ &= \operatorname{sech} \Gamma(t) \\ \sin A(t) &= \tanh \Gamma(t)\end{aligned}$$

Given that  $\frac{\theta(t)}{2} = \varepsilon (A(t) - A(0))$ , we have

$$\begin{aligned}\cos \frac{\theta(t)}{2} &= \cos A(t) \cos A(0) + \sin A(t) \sin A(0) \\ &= \operatorname{sech} \Gamma(t) \operatorname{sech} \Gamma_0 + \tanh \Gamma(t) \tanh \Gamma_0 \\ \sin \frac{\theta(t)}{2} &= \sin A(t) \cos A(0) - \cos A(t) \sin A(0) \\ &= \varepsilon (\tanh \Gamma(t) \operatorname{sech} \Gamma_0 - \tanh \Gamma_0 \operatorname{sech} \Gamma(t))\end{aligned}$$

Using half-angle formulas, we get

$$\begin{aligned}\cos \theta(t) &= \cos^2 \frac{\theta(t)}{2} - \sin^2 \frac{\theta(t)}{2} \\ &= (2 \operatorname{sech}^2 \Gamma(t) - 1) (2 \operatorname{sech}^2 \Gamma_0 - 1) + 4 \operatorname{sech} \Gamma(t) \tanh \Gamma(t) \operatorname{sech} \Gamma_0 \tanh \Gamma_0 \\ \sin \theta(t) &= 2 \cos \frac{\theta(t)}{2} \sin \frac{\theta(t)}{2} \\ &= 2 \varepsilon \left( (2 \operatorname{sech}^2 \Gamma_0 - 1) \operatorname{sech} \Gamma(t) \tanh \Gamma(t) - (2 \operatorname{sech}^2 \Gamma(t) - 1) \operatorname{sech} \Gamma_0 \tanh \Gamma_0 \right)\end{aligned}$$

which is in the form (2.7) with  $\beta_1$  and  $\beta_2$  given by

$$\beta_1(t) \triangleq 2 \operatorname{sech}^2 \Gamma(t) - 1 \quad (2.12a)$$

$$\beta_2(t) \triangleq \operatorname{sech} \Gamma(t) \tanh \Gamma(t) \quad (2.12b)$$

which integrate to

$$\int \beta_1(t) = 2r^{-1} (\tanh \Gamma(t) - \tanh \Gamma_0) - t \quad (2.13a)$$

$$\int \beta_2(t) = -r^{-1} (\operatorname{sech} \Gamma(t) - \operatorname{sech} \Gamma_0). \quad (2.13b)$$

### 2.3 Explicit formulation of elastic rod sensitivity

In the 3-D case, the elastic rod sensitivity is given by the 6-dimensional Jacobian matrix

$$\mathbf{J}(t, a) = \begin{pmatrix} \frac{\partial q_1}{\partial a_1} & \dots & \frac{\partial q_1}{\partial a_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_6}{\partial a_1} & \dots & \frac{\partial q_6}{\partial a_6} \end{pmatrix} \quad (2.14)$$

In the planar case, this simplifies to

$$\mathbf{J}(t, a) = \begin{pmatrix} *_{2,2} & 0_{2,3} & *_{2,1} \\ 0_{3,2} & \mathbf{J}_{3,3}^P(t, a) & 0_{3,1} \\ *_{1,2} & 0_{1,3} & *_{1,1} \end{pmatrix} \quad (2.15)$$

where  $*$  represents indeterminate values.

As we can only obtain closed-forms of the rod shape and thus of rod sensitivity in this special, we will focus in this section on the 3-dimensional block  $\mathbf{J}_{3,3}^P(t, a)$  of  $\mathbf{J}(t, a)$  for  $i, j \in \{3, 4, 5\}$ . Differentiating the general form of the curve  $q(t)$  in (2.7) leads to

$$\frac{\partial \cos(\theta(t))}{\partial a} = \beta_1(0) \frac{\partial \beta_1(t)}{\partial a} + \frac{\partial \beta_1(0)}{\partial a} \beta_1(t) + 4 \left( \beta_2(0) \frac{\partial \beta_2(t)}{\partial a} + \frac{\partial \beta_2(0)}{\partial a} \beta_2(t) \right) \quad (2.16a)$$

$$\frac{\partial \sin(\theta(t))}{\partial a} = 2 \varepsilon \left( \beta_1(0) \frac{\partial \beta_2(t)}{\partial a} + \beta_2(t) \frac{\partial \beta_1(0)}{\partial a} - \beta_2(0) \frac{\partial \beta_1(t)}{\partial a} - \beta_1(t) \frac{\partial \beta_2(0)}{\partial a} \right) \quad (2.16b)$$

$$\frac{\partial x(t)}{\partial a} = \beta_1(0) \frac{\partial \int \beta_1(t)}{\partial a} + \int \beta_1(t) \frac{\partial \beta_1(0)}{\partial a} + 4 \left( \beta_2(0) \frac{\partial \int \beta_2(t)}{\partial a} + \int \beta_2(t) \frac{\partial \beta_2(0)}{\partial a} \right) \quad (2.16c)$$

$$\frac{\partial y(t)}{\partial a} = 2 \varepsilon \left( \beta_1(0) \frac{\partial \int \beta_2(t)}{\partial a} + \int \beta_2(t) \frac{\partial \beta_1(0)}{\partial a} - \beta_2(0) \frac{\partial \int \beta_1(t)}{\partial a} - \int \beta_1(t) \frac{\partial \beta_2(0)}{\partial a} \right) \quad (2.16d)$$

Regardless the three cases of curve elastica, we can derivate with respect to  $a$  the following forms:

- The elliptic parameters  $m$ ,  $n$  and  $r$

$$\frac{\partial m}{\partial a} = \frac{1}{(\alpha_3 + \alpha_1)^2} \left( \left( \frac{\partial \alpha_3}{\partial a} - \frac{\partial \alpha_2}{\partial a} \right) (\alpha_3 + \alpha_1) - \left( \frac{\partial \alpha_3}{\partial a} + \frac{\partial \alpha_1}{\partial a} \right) (\alpha_3 - \alpha_2) \right) \quad (2.17a)$$

$$\frac{\partial r}{\partial a} = \frac{1}{4\sqrt{\alpha_3 + \alpha_1}} \left( \frac{\partial \alpha_3}{\partial a} + \frac{\partial \alpha_1}{\partial a} \right) \quad (2.17b)$$

$$\frac{\partial n}{\partial a} = \frac{1}{\alpha_3^2} \left( \frac{\partial \alpha_3}{\partial a} \alpha_2 - \frac{\partial \alpha_2}{\partial a} \alpha_3 \right) \quad (2.17c)$$

- The phase  $\varphi$

Given

$$\frac{\partial \varpi}{\partial a} = \frac{1}{2n\varpi} \left( \frac{a_3^2}{\alpha_3^2} \begin{pmatrix} \frac{\partial \alpha_3}{\partial a_3} - 2\frac{\alpha_3}{a_3} \\ \frac{\partial \alpha_3}{\partial a_4} \\ \frac{\partial \alpha_3}{\partial a_5} \end{pmatrix} - \frac{1}{n} \left( 1 - \frac{a_3^2}{\alpha_3} \right) \frac{\partial n}{\partial a} \right) \quad (2.18)$$

and the first order derivatives of the function  $\operatorname{arcsn}(z|m)$

$$\frac{\partial \operatorname{arcsn}(z|m)}{\partial z} = \frac{1}{\sqrt{1-z^2}\sqrt{1-mz^2}}$$

$$\frac{\partial \operatorname{arcsn}(z|m)}{\partial m} = \frac{1}{2(m-1)m} \left( \frac{m\sqrt{1-z^2}z}{\sqrt{1-mz^2}} - E(\operatorname{arcsin} z|m) - (m-1)F(\operatorname{arcsin} z|m) \right)$$

with  $\operatorname{cd}(z|m)$  is the Jacobi elliptic function defined by

$$\operatorname{cd} z = \frac{\operatorname{cn} z}{\operatorname{dn} z},$$

we can express the derivative of the function  $\operatorname{arcsn}(\varpi|m)$  with respect to  $a$  using the chain rule

$$\frac{\partial \operatorname{arcsn} \varpi}{\partial a} = \frac{\partial \operatorname{arcsn} \varpi}{\partial \varpi} \frac{\partial \varpi}{\partial a} + \frac{\partial \operatorname{arcsn} \varpi}{\partial m} \frac{\partial m}{\partial a}$$

$$= \frac{1}{\sqrt{1-mz^2}} \left( \frac{1}{\sqrt{1-z^2}} \frac{\partial \varpi}{\partial a} + \frac{mz\sqrt{1-z^2} - E(\operatorname{arcsin} \varpi) - (m-1)F(\operatorname{arcsin} \varpi)}{2(m-1)m} \frac{\partial m}{\partial a} \right)$$

Then, the general expression of the derivative of the phase  $\varphi$  with respect to  $a$  is

$$\frac{\partial \varphi}{\partial a} = \operatorname{sgn}(a_3 a_5) \frac{\partial \operatorname{arcsn}(\varpi|m)}{\partial a}$$

- The function  $\Gamma(t)$

$$\frac{\partial \Gamma(t)}{\partial a} = t \frac{\partial r}{\partial a} + \frac{\partial \varphi}{\partial a}$$

- The Jacobi elliptic function  $\operatorname{sn}(\Gamma(t)|m)$

Given the first order derivatives of the function  $\operatorname{sn}(z|m)$

$$\frac{\partial \operatorname{sn}(z|m)}{\partial z} = \operatorname{cn}(z|m) \operatorname{dn}(z|m)$$

$$\frac{\partial \operatorname{sn}(z|m)}{\partial m} = \frac{\operatorname{dn}(z|m) \operatorname{cn}(z|m) ((1-m)z - E(\operatorname{am}(z|m)|m)) + m \operatorname{cd}(z|m) \operatorname{sn}(z|m)}{2m(1-m)}$$

we can compute directly

$$\begin{aligned}\frac{\partial \operatorname{sn} \Gamma(t)}{\partial a} &= \frac{\partial \operatorname{sn} \Gamma(t)}{\partial \Gamma(t)} \frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \operatorname{sn} \Gamma(t)}{\partial m} \frac{\partial m}{\partial a} \\ &= \operatorname{cn} \Gamma(t) \operatorname{dn} \Gamma(t) \left( \frac{\partial \Gamma(t)}{\partial a} + \frac{(m-1)\Gamma(t) + \operatorname{E}(\operatorname{am} \Gamma(t)) - m \operatorname{cd} \Gamma(t) \operatorname{sn} \Gamma(t)}{2m(m-1)} \frac{\partial m}{\partial a} \right).\end{aligned}$$

- The Jacobi elliptic function  $\operatorname{cn}(\Gamma(t)|m)$

Given the first order derivatives of the function  $\operatorname{cn}(z|m)$

$$\begin{aligned}\frac{\partial \operatorname{cn}(z|m)}{\partial z} &= -\operatorname{sn}(u|m) \operatorname{dn}(u|m) \\ \frac{\partial \operatorname{cn}(z|m)}{\partial m} &= \frac{\operatorname{dn}(z|m) \operatorname{sn}(z|m) ((m-1)z + \operatorname{E}(\operatorname{am}(z|m)|m) - m \operatorname{cd}(z|m) \operatorname{sn}(z|m))}{2m(1-m)}\end{aligned}$$

we can compute directly

$$\begin{aligned}\frac{\partial \operatorname{cn} \Gamma(t)}{\partial a} &= \frac{\partial \operatorname{cn} \Gamma(t)}{\partial \Gamma(t)} \frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \operatorname{cn} \Gamma(t)}{\partial m} \frac{\partial m}{\partial a} \\ &= -\operatorname{sn} \Gamma(t) \operatorname{dn} \Gamma(t) \left( \frac{\partial \Gamma(t)}{\partial a} + \frac{(m-1)\Gamma(t) + \operatorname{E}(\operatorname{am} \Gamma(t)) - m \operatorname{cd} \Gamma(t) \operatorname{sn} \Gamma(t)}{2m(m-1)} \frac{\partial m}{\partial a} \right).\end{aligned}$$

- The Jacobi elliptic function  $\operatorname{dn}(\Gamma(t)|m)$

Given the first order derivatives of the function  $\operatorname{dn}(z|m)$

$$\begin{aligned}\frac{\partial \operatorname{dn}(z|m)}{\partial z} &= -m \operatorname{cn}(u|m) \operatorname{sn}(u|m) \\ \frac{\partial \operatorname{dn}(z|m)}{\partial m} &= \frac{\operatorname{sn}(z|m) \operatorname{cn}(z|m) ((m-1)z + m \operatorname{E}(\operatorname{am}(z|m)|m) - m \operatorname{dn}(z|m) \operatorname{sc}(z|m))}{2(1-m)}\end{aligned}$$

with  $\operatorname{sc}(z|m)$  is the Jacobi elliptic function defined by

$$\operatorname{sc} z = \frac{\operatorname{sn} z}{\operatorname{cn} z},$$

we can compute directly

$$\begin{aligned}\frac{\partial \operatorname{dn} \Gamma(t)}{\partial a} &= \frac{\partial \operatorname{dn} \Gamma(t)}{\partial \Gamma(t)} \frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \operatorname{dn} \Gamma(t)}{\partial m} \frac{\partial m}{\partial a} \\ &= -m \operatorname{cn} \Gamma(t) \operatorname{sn} \Gamma(t) \left( \frac{\partial \Gamma(t)}{\partial a} \right. \\ &\quad \left. + \frac{(m-1)\Gamma(t) + m \operatorname{E}(\operatorname{am} \Gamma(t)) - m \operatorname{dn} \Gamma(t) \operatorname{sc} \Gamma(t)}{2m(m-1)} \frac{\partial m}{\partial a} \right).\end{aligned}$$

- The elliptic integral of the second kind  $\operatorname{E}(\operatorname{am}(\Gamma(t)|m)|m)$

We first need to express the derivative of the amplitude  $\operatorname{am}(\Gamma(t)|m)$  with respect to  $a$ .

Given first derivatives of the Jacobi amplitude  $\operatorname{am}(z|m)$

$$\begin{aligned}\frac{\partial \operatorname{am}(z|m)}{\partial z} &= \operatorname{dn}(u|m) \\ \frac{\partial \operatorname{am}(z|m)}{\partial m} &= \frac{\operatorname{dn}(z|m) ((m-1)z + \operatorname{E}(\operatorname{am}(z|m)|m)) - m \operatorname{cn}(z|m) \operatorname{sn}(z|m)}{2m(1-m)}\end{aligned}$$

we get the derivative of  $\text{am}(\Gamma(t)|m)$  with respect to  $a$  by applying the chain rule as usual

$$\begin{aligned}\frac{\partial \text{am} \Gamma(t)}{\partial a} &= \frac{\partial \text{am} \Gamma(t)}{\partial \Gamma(t)} \frac{\partial \Gamma(t)}{\partial a} + \frac{\partial \text{am} \Gamma(t)}{\partial m} \frac{\partial m}{\partial a} \\ &= \text{dn} \Gamma(t) \left( \frac{\partial \Gamma(t)}{\partial a} + \frac{(m-1)\Gamma(t) + \text{E}(\text{am} \Gamma(t)) - m \text{cd} \Gamma(t) \text{sn} \Gamma(t)}{2m(m-1)} \frac{\partial m}{\partial a} \right).\end{aligned}$$

Then, given first derivatives of the elliptic integral of the second kind  $\text{E}(z|m)$

$$\begin{aligned}\frac{\partial \text{E}(z|m)}{\partial z} &= \sqrt{1 - m \sin^2 z} \\ \frac{\partial \text{E}(z|m)}{\partial m} &= \frac{\text{E}(z|m) - \text{F}(z|m)}{2m}\end{aligned}$$

Noting the simplification

$$\begin{aligned}\frac{\partial \text{E}(\text{am} \Gamma(t))}{\partial \text{am} \Gamma(t)} &= \sqrt{1 - m \sin^2(\text{am} \Gamma(t))} \\ &= \sqrt{1 - \text{sn}^2 \Gamma(t)} \\ &= \text{dn} \Gamma(t),\end{aligned}$$

and that

$$\text{F}(\text{am} \Gamma(t)) = \Gamma(t),$$

we finally have all the expressions to compute the derivative of  $\text{E}(\text{am} \Gamma(t))$  with respect to  $a$  by applying the chain rule

$$\begin{aligned}\frac{\partial \text{E}(\text{am} \Gamma(t))}{\partial a} &= \frac{\partial \text{E}(\text{am} \Gamma(t))}{\partial \text{am} \Gamma(t)} \frac{\partial \text{am} \Gamma(t)}{\partial a} + \frac{\partial \text{E}(\text{am} \Gamma(t))}{\partial m} \frac{\partial m}{\partial a} \\ &= \text{dn}^2 \Gamma(t) \left( \frac{\partial \Gamma(t)}{\partial a} + \frac{\text{E}(\text{am} \Gamma(t)) - \text{cd} \Gamma(t) \text{sn} \Gamma(t)}{2(m-1)} \frac{\partial m}{\partial a} \right)\end{aligned}$$

Then, most of these forms simplify in the three previously introduced cases and we can give the explicit forms of the derivatives of functions  $\beta_1(t)$  and  $\beta_2(t)$  (and their respective integrals) with respect to  $a$ :

- Case I:  $\lambda_4 > 0$

From (2.8) and (2.9), we get

$$\begin{aligned}\frac{\partial \beta_1(t)}{\partial a} &= 4 \text{dn} \Gamma(t) \frac{\partial \text{dn} \Gamma(t)}{\partial a} \\ \frac{\partial \beta_2(t)}{\partial a} &= \text{dn} \Gamma(t) \frac{\partial \text{sn} \Gamma(t)}{\partial a} + \text{sn} \Gamma(t) \frac{\partial \text{dn} \Gamma(t)}{\partial a} \\ \frac{\partial \int \beta_1(t)}{\partial a} &= \frac{2}{r} \left( \left( \frac{\partial \text{E}(\Gamma(t))}{\partial a} - \frac{\partial \text{E}(\Gamma_0)}{\partial a} \right) - \frac{1}{r} \left( \text{E}(\Gamma(t)) - \text{E}(\Gamma_0) \right) \frac{\partial r}{\partial a} \right) \\ \frac{\partial \int \beta_2(t)}{\partial a} &= \frac{1}{r} \left( \frac{1}{r} (\text{cn} \Gamma(t) - \text{cn} \Gamma_0) \frac{\partial r}{\partial a} - \left( \frac{\partial \text{cn} \Gamma(t)}{\partial a} - \frac{\partial \text{cn} \Gamma_0}{\partial a} \right) \right)\end{aligned}$$

- Case II:  $\lambda_4 < 0$

From (2.10) and (2.11), we get

$$\begin{aligned}\frac{\partial \beta_1(t)}{\partial a} &= -4 \operatorname{sn} \Gamma(t) \frac{\partial \operatorname{sn} \Gamma(t)}{\partial a} \\ \frac{\partial \beta_2(t)}{\partial a} &= \operatorname{cn} \Gamma(t) \frac{\partial \operatorname{sn} \Gamma(t)}{\partial a} + \operatorname{sn} \Gamma(t) \frac{\partial \operatorname{cn} \Gamma(t)}{\partial a} \\ \frac{\partial \int \beta_1(t)}{\partial a} &= \frac{1}{m} \left( \frac{t(1-m)}{m} \frac{\partial m}{\partial a} + \frac{2}{r} (\operatorname{E}(\operatorname{am} \Gamma(t)) - \operatorname{E}(\operatorname{am} \Gamma_0)) \left( \frac{1}{r} \frac{\partial r}{\partial a} - \frac{1}{m} \frac{\partial m}{\partial a} \right) \right. \\ &\quad \left. + \frac{2}{r} \left( \frac{\partial \operatorname{E}(\Gamma(t))}{\partial a} - \frac{\partial \operatorname{E}(\Gamma_0)}{\partial a} \right) \right) \\ \frac{\partial \int \beta_2(t)}{\partial a} &= \frac{1}{rm} \left( (\operatorname{dn} \Gamma(t) - \operatorname{dn} \Gamma_0) \left( \frac{1}{m} \frac{\partial m}{\partial a} + \frac{1}{r} \frac{\partial r}{\partial a} \right) - \left( \frac{\partial \operatorname{dn} \Gamma(t)}{\partial a} - \frac{\partial \operatorname{dn} \Gamma_0}{\partial a} \right) \right)\end{aligned}$$

- Case III:  $\lambda_4 = 0$

From (2.12) and (2.13), we get

$$\begin{aligned}\frac{\partial \beta_1(t)}{\partial a} &= -2 \operatorname{sech}^2 \Gamma(t) \tanh \Gamma(t) \\ \frac{\partial \beta_2(t)}{\partial a} &= \operatorname{sech} \Gamma(t) (1 - 2 \tanh^2 \Gamma(t)) \\ \frac{\partial \int \beta_1(t)}{\partial a} &= \frac{2}{r} \left( \frac{1}{r} \frac{\partial r}{\partial a} (\tanh \Gamma(t) - \tanh \Gamma_0) - (\tanh^2 \Gamma(t) - \tanh^2 \Gamma_0) \right) \\ \frac{\partial \int \beta_2(t)}{\partial a} &= \frac{1}{r} \left( \tanh \Gamma(t) \operatorname{sech} \Gamma(t) - \tanh \Gamma_0 \operatorname{sech} \Gamma_0 - \frac{1}{r} (\operatorname{sech} \Gamma(t) - \operatorname{sech} \Gamma_0) \right)\end{aligned}$$

## Acknowledgments

This work was supported by the French National Research Agency under the project Flecto (ANR- Digital Models).

## References

- [1] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions*. Dover, 1964.
- [2] T. Bretl and Z. McCarthy. Quasi-static manipulation of a kirchhoff elastic rod based on a geometric analysis of equilibrium configurations. *I. J. Robotic Res.*, 33(1):48–68, 2014.
- [3] V. Jurdjevic. *Integrable Hamiltonian systems on complex Lie groups*. Memoirs of the American Mathematical Society. 2005.
- [4] J. Langer and D. A. Singer. The total squared curvature of closed curves. *J. Differential Geom.*, 20(1):1–22, 1984.
- [5] D. A. Singer. Lectures on elastic curves and rods, 2007.