Gradient Schemes for incompressible steady Navier-Stokes problem

Robert Eymard, Pierre Feron, Cindy Guichard

MAMERN VI–2015: 6^{th} International Conference on Approximation Methods
and Numerical Modelling in Environment and Natural Resources
Pau (France), June 1-5, 2015

Gradient Schemes for incompressible steady Navier-Stokes problem

Robert Eymard[1], **Pierre Feron**[1] **and Cindy Guichard**[2]

[1] Laboratoire d'Analyse et de Mathématiques Appliquées (LAMA), UMR 8050 CNRS, UPEM, UPEC, 5 boulevard Descartes Champs-sur-Marne 77454 Marne-la-Vallée Cedex 2, France.
robert.eymard, pierre.feron@u-pem.fr

[2] Laboratoire Jacques-Louis Lions (LJLL), UMR 7598 CNRS, Sorbonne Universités, UPMC Univ Paris 6, F-75005 Paris, and INRIA, ANGE project-team, Rocquencourt - B.P. 105 F78153 Le Chesnay cedex, and CEREMA, ANGE project-team, 134 rue de Beauvais F-60280 Margny-Lès-Compiègne, France. cindy.guichard@ljll.math.upmc.fr

Keywords: Gradient Schemes, incompressible steady Navier-Stokes problem, Crouzeix-Raviart Scheme, Taylor-Hood scheme, MAC scheme.

Abstract. *We apply the Gradient Schemes framework to the approximation of the incompressible steady Navier-Stokes problem. We show that some classical schemes (Crouzeix-Raviart, conforming Taylor-Hood and MAC) enter into this framework.*

1 Introduction

The gradient scheme framework has been shown to apply to linear and non-linear elliptic and parabolic problems in [8, 4, 5, 7]. This framework has the benefit of providing common convergence and error estimates results which hold for a wide variety of numerical methods (finite element methods, non-conforming and mixed finite element methods, hybrid and mixed mimetic finite difference methods…). Checking a minimal set of properties for a given numerical method suffices to prove that it belongs to the gradient schemes framework, and therefore that it is convergent on the aforementioned problem. The aim of this paper is to propose one extension of this framework to the incompressible steady Navier-Stokes problem:

$$\left\{\begin{array}{rll} \eta\overline{u} - \nu\Delta\overline{u} + (\overline{u}\cdot\nabla)\overline{u} + \nabla\overline{p} & = f - \mathrm{div}(G) & \text{in } \Omega \\ \mathrm{div}\,\overline{u} & = 0 & \text{in } \Omega \\ \overline{u} & = 0 & \text{on } \partial\Omega \end{array}\right. \tag{1}$$

where $\eta \geq 0$, $\nu > 0$ is the coefficient of kinematic viscosity, $\overline{u}$ represents the velocity field, $\overline{p}$ is the pressure and

$$\begin{array}{l} \Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d \ (1 \leq d \leq 3), \\ f \in \boldsymbol{L}^2(\Omega) \text{ and } G \in \boldsymbol{L}^2(\Omega)^d. \end{array} \tag{2}$$

In the following, if F is a vector space we denote by $\boldsymbol{F}$ the space F^d. Thus, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$ and $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega)^d$, and we define the spaces

$$E(\Omega) = \{\overline{v} \in \boldsymbol{H}_0^1(\Omega),\ \mathrm{div}\overline{v} = 0\},$$

and

$$L_0^2(\Omega) = \{\overline{v} \in L^2(\Omega), \int_\Omega \overline{v}(x)dx = 0\},$$

Definition 1.1 (Weak solution to the incompressible steady Navier-Stokes problem) *Under Hypotheses (2), $(\overline{u}, \overline{p})$ is a weak solution to (1) if*

$$\left\{\begin{array}{l} \overline{u} \in \boldsymbol{H}_0^1(\Omega), \overline{p} \in L_0^2(\Omega), \\ \eta \displaystyle\int_\Omega \overline{u}\cdot\overline{v}\ dx + \nu \int_\Omega \nabla\overline{u} : \nabla\overline{v}\ dx + b(\overline{u},\overline{u},\overline{v}) \\ \qquad - \displaystyle\int_\Omega \overline{p}\,\mathrm{div}\overline{v}\ dx = \int_\Omega (f\cdot\overline{v} + G : \nabla\overline{v})\ dx,\ \forall \overline{v} \in \boldsymbol{H}_0^1(\Omega), \\ \displaystyle\int_\Omega q\,\mathrm{div}\overline{u}\ dx = 0,\ \forall q \in L_0^2(\Omega), \end{array}\right. \tag{3}$$

where "$\cdot$" is the dot product on $\mathbb{R}^d$, if $\tau = (\tau^{(i,j)})_{i,j=1,\ldots,d} \in \mathbb{R}^{d\times d}$ and $\sigma = (\sigma^{(i,j)})_{i,j=1,\ldots,d} \in \mathbb{R}^{d\times d}$, $\tau : \sigma = \displaystyle\sum_{i,j=1}^d \tau^{(i,j)}\sigma^{(i,j)}$ is the doubly contracted product on $\mathbb{R}^{d\times d}$ and $b(u,v,w) = \displaystyle\sum_{i,j=1}^d \int_\Omega u_i(\partial_i v_j)w_j\ dx$.

Lemma 1.2 (Properties of b**)** *Under Hypotheses (2), b is a trilinear continuous form on $\boldsymbol{H}_0^1(\Omega)^3$ and*

$$b(\overline{u},\overline{v},\overline{v}) = 0,\ \forall \overline{u} \in E(\Omega),\ \overline{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4}$$

$$b(\overline{u},\overline{v},\overline{w}) = -b(\overline{u},\overline{w},\overline{v}),\ \forall \overline{u} \in E(\Omega),\ (\overline{v},\overline{w}) \in \boldsymbol{H}_0^1(\Omega). \tag{5}$$

as it is mentioned in [14, Ch.II, Lemma 1.2 and 1.3]

Remark 1.3 *Under Hypothese (2), the existence of a weak solution $(\overline{u},\overline{p})$ to Problem (1) in the sense of Definition 1.1 follows from [14, Ch.II, Theorem 1.2]. Moreover, [14, Ch.II, Theorem 1.2] gives us the uniqueness of the weak solution $(\overline{u},\overline{p})$ dealing with a condition on ν, f and G.*

2 Gradient Discretisation for the incompressible steady Navier-Stokes problem

Definition 2.1 *A gradient discretisation D for the incompressible steady Navier-Stokes problem, with homogeneous Dirichlet's boundary conditions, is defined by $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D)$, where the discrete spaces and operators are assumed to verify the following properties.*

1. *$X_{D,0}$ is a finite-dimensional vector space on $\mathbb{R}$.*
2. *Y_D is a finite-dimensional vector space on $\mathbb{R}$.*
3. *The linear mapping $\Pi_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ is the reconstruction of the approximate velocity field.*
4. *The linear mapping $\chi_D : Y_D \to L^2(\Omega)$ is the reconstruction of the approximate pressure, and must be chosen such that $\|\chi_D \cdot\|_{L^2(\Omega)}$ is a norm on Y_D. We then set $Y_{D,0} = \{q \in Y_D, \int_\Omega \chi_D q \,\mathrm{d}x = 0\}$.*
5. *The linear mapping $\nabla_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\|\cdot\|_D := \|\nabla_D \cdot\|_{\boldsymbol{L}^2(\Omega)^d}$ is a norm on $X_{D,0}$.*
6. *The linear mapping $\mathrm{div}_D : X_{D,0} \to L^2(\Omega)$ is the discrete divergence operator.*

The *coercivity* of a sequence of gradient discretisations ensures that a discrete Sobolev inequality, a control of the discrete divergence and a discrete Ladyzenskaja-Babuka-Brezzi (LBB) condition can be established, all uniform along the sequence of discretisations. Note that the following definition is different from the one which is given in [3], due to the presence of the nonlinear term.

Definition 2.2 (Coercivity) *Let D be a discretisation in the sense of Definition 2.1. Let $q \in \mathbb{N}$ and let C_D and β_D be defined by*

$$C_D = \max_{v \in X_{D,0}, \|v\|_D = 1} \|\Pi_D v\|_{\boldsymbol{L}^q(\Omega)} + \max_{v \in X_{D,0}, \|v\|_D = 1} \|\mathrm{div}_D v\|_{L^2(\Omega)}, \tag{6}$$

where $2 \le q \le \infty$ if $d = 2$ and $2 \le q \le 6$ if $d = 3$.

$$\beta_D = \min\{ \max_{v \in X_{D,0}, \|v\|_D = 1} \int_\Omega \chi_D q \, \mathrm{div}_D v \, \mathrm{d}x \ : \ q \in Y_{D,0} \\ \text{such that } \|\chi_D q\|_{L^2(\Omega)} = 1\}. \tag{7}$$

*A sequence $(D_m)_{m \in \mathbb{N}}$ of gradient discretisations is said to be **coercive** if there exist $C_S \ge 0$ and $\beta > 0$ such that $C_{D_m} \le C_S$ and $\beta_{D_m} \ge \beta$, for all $m \in \mathbb{N}$.*

The following definition is not needed in [3], since, thanks to the linearity of the Stokes problem, only weak convergence results are needed, and strong convergence is resulting from the problem (by convergence of norms).

Definition 2.3 (Compactness) *Let D be a gradient discretisation in the sense of Definition 2.1. A sequence $(D_m)_{m \in \mathbb{N}}$ of gradient discretisations is said to be compact if, for all sequence $(u_m)_{m \in \mathbb{N}} \in X_{D_m,0}$ such that $\|u_m\|_{D_m}$ is bounded, the sequence $(\Pi_{D_m} u_m)_{m \in \mathbb{N}}$ is relatively compact in $\boldsymbol{L}^2(\Omega)$.*

The *consistency* of a sequence of gradient discretisations states that the discrete space and operators "fill in" the continuous space as the discretisation is refined.

Definition 2.4 (Consistency) *Let D be a gradient discretisation in the sense of Definition 2.1, and let $S_D : \boldsymbol{H}_0^1(\Omega) \to [0, +\infty)$, and $\widetilde{S}_D : L_0^2(\Omega) \to [0, +\infty)$ be defined by*

$$\forall \varphi \in \boldsymbol{H}_0^1(\Omega), \quad S_D(\varphi) = \min_{v \in X_{D,0}} \Big(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} \\ + \|\mathrm{div}_D v - \mathrm{div}\varphi\|_{L^2(\Omega)} \Big)$$

and

$$\forall \psi \in L_0^2(\Omega), \quad \widetilde{S}_D(\psi) = \min_{z \in Y_{D,0}} \|\chi_D z - \psi\|_{L^2(\Omega)}.$$

A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be ***consistent*** *if, for all $\varphi \in \boldsymbol{H}_0^1(\Omega)$, $S_{D_m}(\varphi)$ tends to 0 as $m \to \infty$ and, for all $\psi \in L_0^2(\Omega)$, $\widetilde{S}_{D_m}(\psi)$ tends to 0 as $m \to \infty$.*

Definition 2.5 (Limit-conformity) *Let D be a gradient discretisation in the sense of Definition 2.1 and let $\overline{W}_D : Z(\Omega) \mapsto [0, +\infty)$, with*

$$Z(\Omega) = \{(\varphi, \psi) \in \boldsymbol{L}^2(\Omega)^d \times L^2(\Omega), \mathrm{div}\varphi - \nabla\psi \in \boldsymbol{L}^2(\Omega)\},$$

be defined by

$$\forall (\varphi, \psi) \in Z(\Omega)\,,$$
$$\overline{W}_D(\varphi, \psi) = \max_{\substack{v\in X_{D,0}\\ \|v\|_D=1}} \Big(\int_\Omega [\nabla_D v : \varphi + \Pi_D v \cdot (\mathrm{div}\varphi - \nabla\psi) - \psi \mathrm{div}_D v] \,\mathrm{d}x \Big).$$

A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be ***limit-conforming*** *if, for all $(\varphi, \psi) \in Z(\Omega)$, $\overline{W}_{D_m}(\varphi, \psi)$ tends to 0 as $m \to \infty$.*

3 Gradient Scheme

Definition 3.1 (Discretisation of the trilinear form) *Let D be a gradient discretisation in the sense of Definition 2.1, we define $B_D : X_{D,0}^3 \mapsto \boldsymbol{L}^2(\Omega)$ such that*

$$B_D(u, v, w) = \sum_{i,j=1}^{d} \int_\Omega \Pi_D^{(i)} u \nabla_D^{(i,j)} v \Pi_D^{(j)} w \; dx.$$

We define our discrete bilinear form b_D following the same idea as the Finite Elements method:

$$b_D(u, v) = \frac{1}{2} \left(B_D(u, u, v) - B_D(u, v, u) \right).$$

Remark 3.2 (Property of the discrete bilinear form) *With the previous definition of b_D, we can remark that we get the same property as the continuous trilinear form which is that for all $u \in X_{D,0}$, we get that $b_D(u, u) = 0$.*

The gradient scheme for the incompressible steady Navier-Stokes problem is based on a discretisation of the weak formulation (3), in which the continuous spaces and operators are replaced with discrete ones (in (3), we wrote the property "div$\overline{u} = 0$" using test functions to make clearer this parallel between the weak formulation and the gradient scheme). If D is a gradient discretisation in the sense of Definition 2.1 and b_D is defined by Definition 3.1, the scheme is given by:

$$\left\{\begin{array}{l} u \in X_{D,0}\,,\ p \in Y_{D,0}, \\ \displaystyle \eta \int_\Omega \Pi_D u \cdot \Pi_D v + \nu \int_\Omega \nabla_D u : \nabla_D v \; dx + b_D(u,v) \\ \displaystyle \qquad\qquad - \int_\Omega \chi_D p \,\mathrm{div}_D v \,\mathrm{d}x = \int_\Omega \left(f \cdot \Pi_D v + G : \nabla_D v\right) \,\mathrm{d}x,\ \forall v \in X_{D,0}, \\ \displaystyle \int_\Omega \chi_D q \,\mathrm{div}_D u \,\mathrm{d}x = 0,\ \ \forall q \in Y_{D,0}. \end{array}\right. \tag{8}$$

Our main result for the incompressible steady Navier-Stokes problem is the following theorem.

Theorem 3.3 (Convergence of the scheme) *Under Hypotheses (2), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 2.1, which is consistent, limit-conforming, coercive and compact in the sense of Definition 2.4, 2.5, 2.2 and 2.3. Then for any m there exists at least a solution (u_{D_m}, p_{D_m}) to (8) with $D = D_m$ and b_D defined by Definition 3.1. Moreover, as $m \to \infty$, there exists a subsequence of $(D_m)_{m\in\mathbb{N}}$ again denoted $(D_m)_{m\in\mathbb{N}}$ and there exists $(\overline{u}, \overline{p})$, weak solution of the incompressible steady Navier-Stokes problem (1) in the sense of Definition 1.1, such that*

- $\Pi_{D_m} u_{D_m}$ *converges to* $\overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$,
- $\nabla_{D_m} u_{D_m}$ *converges to* $\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega)^d$,
- $\chi_{D_m} p_{D_m}$ *converges to* $\overline{p}$ *in* $L^2(\Omega)$.

4 Examples of gradient discretisations

In this section, we assume that the boundary of $\Omega \subset \mathbb{R}^d$ is polygonal.

4.1 The MAC scheme on rectangular parallelepipedic meshes

The Marker-And-Cell (MAC) scheme [11, 12, 15] can be easily defined on domains whose boundary is the union of subparts parallel to the axes. We assume that it is possible to grid Ω using a finite number of rectangular parallelepipedic gridblocks. We then define the gradient discretization $D = (X_{D,0}, Y_D, \Pi_D, \chi_D, \nabla_D, \mathrm{div}_D)$ (the detailed notations are given in [3] in a 2D case) by:

1. $X_{D,0}$ is the set of families of real values at the center of all internal faces of the mesh, discretizing the velocity in the normal direction to the face,

2. Y_D is the set of all families of real values at the center of the gridblocks, discretizing the pressure,

3. Π_D is the piecewise constant reconstruction of the velocity in the d staggered rectangular parallelepipedic grids, whose centers of the gridblocks are the centers of the faces normal to each of the d basis vectors of $\mathbb{R}^d$,

4. χ_D is the reconstruction of the pressure, piecewise constant in all the gridblocks,

5. $\nabla_D u = (\nabla_D^{(i,j)} u)_{i,j=1,\ldots,d}$ is a piecewise constant approximation of the j-th derivative of the i-th component of the velocity, defined by a standard finite difference formula,

6. $\mathrm{div}_D u = \mathrm{Tr}(\nabla_D u) = \sum_{i=1}^d \nabla_D^{(i,i)} u$.

We then have the following result.

Proposition 4.1 (Gradient Scheme properties of the MAC scheme)
Let $D_m = (X_{D_m,0}, Y_{D_m}, \Pi_{D_m}, \chi_{D_m}, \nabla_{D_m}, \mathrm{div}_{D_m})$ be defined as in the beginning of this section, with h_{D_m} tending to 0 as $m \to \infty$. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming, coercive and compact in the sense of Definitions 2.4, 2.5, 2.2 and 2.3.

Proof
The proof of the consistency and limit-conformity as well as the proof of the lower bound on β_D can be found in [3].

Since the definition of ∇_D is corresponding to the discrete gradient of a finite volume scheme on a mesh satisfying the usual orthogonality property, the bound on C_D is a consequence of the discrete Sobolev inequality obtained in [1] or [6, Lemma 9.5 p. 790] (the control of div_D by ∇_D is then trivial from its definition).

The compactness property is resulting from [6, Lemma 9.3 p. 770]. $\square$

4.2 The Crouzeix-Raviart scheme

We consider a simplicial mesh $\mathcal{T}$. The Crouzeix-Raviart scheme [2] can be seen as a gradient scheme with the gradient discretisation defined by:

1. $X_{D,0}$ is the vector space containing the families of elements of $\mathbb{R}^d$ defined at the center of all internal faces of the mesh,
2. Y_D is the vector space containing the families of real values defined at the center of all simplices,
3. The linear mapping Π_D is the nonconforming piecewise affine reconstruction of each component of the velocity,
4. The linear mapping χ_D is the piecewise constant reconstruction in the simplices,
5. The linear mapping ∇_D is the so-called "broken gradient" of the velocity, defined as the piecewise constant field of the velocity's gradients in each simplex,
6. The linear mapping div_D is the discrete divergence operator, with piecewise constant values in the cells equal to the balance of the normal velocities over the cell's faces.

Proposition 4.2 (Gradient Scheme properties of the Crouzeix-Raviart scheme)

Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of triangulations of Ω satisfying a regularity condition. We define $D_m = (X_{D_m,0}, Y_{D_m}, \Pi_{D_m}, \chi_{D_m}, \nabla_{D_m}, \mathrm{div}_{D_m})$ as above for $\mathcal{T} = \mathcal{T}_m$, and we assume that $h_{\mathcal{T}_m} \to 0$ as $m \to \infty$. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming, coercive and compact in the sense of Definitions 2.4, 2.5, 2.2 and 2.3.

Proof The coercivity is a consequence of the results given in [10]. The consistency and limit-conformity are proved in [3]. The compactness property is proved in [9, Theorem 3.3]. □

4.3 Conforming Taylor–Hood scheme

The Taylor–Hood scheme [13] on a simplicial mesh $\mathcal{T}$ (triangles in 2D or tetrahedra in 3D) can be seen as the gradient scheme corresponding to the gradient discretisation $D = (X_{D,0}, Y_D, \Pi_D, \chi_D, \nabla_D, \mathrm{div}_D)$ defined by:

1. $X_{D,0}$ is the vector space of the degrees of freedom of the $\mathbb{P}^2$ finite element for the d components of the velocity vanishing at the boundary, and Y_D is the vector space of the degrees of freedom of the $\mathbb{P}^1$ finite element for the pressure,

2. Π_D and χ_D are respectively the conforming reconstructions of the velocity and the pressure obtained through the $\mathbb{P}^2$ and $\mathbb{P}^1$ finite element basis functions,

3. ∇_D and div_D are the conforming operators $\nabla_D = \nabla \circ \Pi_D$ and $\mathrm{div}_D = \mathrm{div} \circ \Pi_D$.

Proposition 4.3 (Gradient Scheme properties of the Taylor-Hood scheme) *Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of triangulations of Ω satisfying a regularity condition. We assume that every mesh element has at least d edges in Ω and that $h_{\mathcal{T}_m} \to 0$ as $m \to \infty$. Let $D_m = (X_{D_m,0}, Y_{D_m}, \Pi_{D_m}, \chi_{D_m}, \nabla_{D_m}, \mathrm{div}_{D_m})$ corresponding to the conforming Taylor–Hood scheme for $\mathcal{T}_m$. Then D_m is a gradient discretisation in the sense of Definition 2.1 and the family $(D_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming, coercive and compact in the sense of Definitions 2.4, 2.5, 2.2 and 2.3.*

Proof Since the scheme is conforming, the coercivity and the compactness properties are a consequence of the continuous Sobolev inequalities and Rellich theorem. The consistency and limit-conformity are proved in [3]. □

REFERENCES

[1] Y. Coudière, T. Gallouët, and R. Herbin. Discrete Sobolev Inequalities and L^p Error Estimates for Approximate Finite Volume Solutions of Convection Diffusion Equations. *Mathematical Modelling and Numerical Analysis*, 35:767–778, 1998.

[2] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 7(R-3):33–75, 1973.

[3] J. Droniou, R. Eymard and P. Féron. Gradient schemes for Stokes problems. https://hal.archives-ouvertes.fr/hal-01070703, 2014.

[4] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations. *Math. Models Methods Appl. Sci. (M3AS)*, 2013.

[5] R. Eymard, P. Féron, T. Gallouët, C. Guichard and R. Herbin. Gradient schemes for the Stefan problem. *International Journal On Finite Volumes*, vol. 10 special, 2013.

[6] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. In P. G. Ciarlet and J.-L. Lions, editors, *Techniques of Scientific Computing, Part III*, Handbook of Numerical Analysis, VII, pages 713–1020. North-Holland, Amsterdam, 2000.

[7] R. Eymard and R. Herbin. Gradient Scheme Approximations for Diffusion Problems. *Finite Volumes for Complex Applications VI Problems & Perspectives*, Springer, 2011.

[8] R. Eymard, C. Guichard, and R. Herbin. Small-stencil 3D schemes for diffusive flows in porous media. *ESAIM: Mathematical Modelling and Numerical Analysis*, 46:265–290, 2012.

[9] T. Gallouët, R. Herbin and J.-C. Latché. A convergent finite element-finite volume scheme for the compressible Stokes problem. I. The isothermal case. *Math. Comp.* 78, no. 267, 1333–1352, 2009.

[10] T. Gallouët, R. Herbin, D. Maltese, and A. Novotny. Error estimates for a numerical approximation to the compressible barotropic Navier-Stokes equations. https://hal.archives-ouvertes.fr/hal-01108579, 2015.

[11] F. Harlow and J. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with a free surface. *Physics of Fluids*, 8:2182–2189, 1965.

[12] S. Patankar. Numerical heat transfer and fluid flow. volume XIII of *Series in Computational Methods in Mechanics and Thermal Sciences*. Washington - New York - London: Hemisphere Publishing Corporation; New York. McGraw-Hill Book Company, 1980.

[13] C. Taylor and P. Hood. A numerical solution of the Navier-Stokes equations using the finite element technique. *Internat. J. Comput. & Fluids*, 1(1):73–100, 1973.

[14] R. Temam. Navier-Stokes Equations. *American Mathematical Soc.*, 1984.

[15] P. Wesseling. Principles of computational fluid dynamics. *Springer Series in Computational Mathematics, vol. 29*, Springer-Verlag, Berlin, 2001.