

IMP with exceptions over decorated logic

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In this paper, we facilitate the reasoning about impure programming languages, by annotating terms with “decorations” that describe what computational (side) effect evaluation of a term may involve. In a point-free categorical language, called the “decorated logic”, we formalize the mutable state and the exception effects first separately, exploiting a nice duality between them, and then combined. The combined decorated logic is used as the target language for the denotational semantics of the IMP+Exc imperative programming language, and allows us to prove equivalences between programs written in IMP+Exc. The combined logic is encoded in Coq, and this encoding is used to certify some program equivalence proofs.

Keywords: Computational effects, state, exceptions, program equivalence proofs, decorated logic, Coq.

1 Introduction

In programming languages theory, a program is said to have computational effects if, besides a return value, it has observable interactions with the outside world. For instance, using/modifying the program state, raising/recovering exceptions, reading/writing data from/to some file, etc. In order to formally reason about behaviors of a program with computational effects, one has to take into account these interactions. One difficulty in such a reasoning is the mismatch between the syntax of operations with effects and their interpretation. Typically, an operation in an effectful language with arguments in X that returns a value in Y is not interpreted as a function from X to Y , due to the effects, unless the operation is pure.

The best known *algebraic approach* to formalize computational effects was initiated by Moggi in his seminal paper (25). He showed that the effectful operations of an impure language can be interpreted as arrows of a Kleisli category for an appropriate monad (T, η, μ) over a base category \mathcal{C} with finite products. For instance, in Moggi’s *computational metalanguage*, an operation in an impure language with arguments in X that returns a value in Y is now interpreted as an arrow from $\llbracket X \rrbracket$ to $T\llbracket Y \rrbracket$ in \mathcal{C} where $\llbracket X \rrbracket$ is the object of *values* of type X and $T\llbracket Y \rrbracket$ is the object of *computations* that return values of type Y . The use of monads to formalize effects (such as state, exceptions, input/output and non-deterministic choice) was popularized by Wadler in (37), and implemented in the programming languages Haskell and F#. Using monad transformers (18), it is usually possible to “combine” different effects formalized by monads. Moggi’s *computational metalanguage* was extended into the *basic effect calculus* with a notion of *computation type* as in Filinski’s effect PCF (12) and in Levy’s call-by-push-value (CBPV) (20). In their paper (10), Egger et al., defined their effect calculus, named *extended effect calculus* as a canonical calculus incorporating the ideas of Moggi, Filinski and Levy. Following Moggi, they included a type constructor for computations. Following Filinski and Levy, they classified types as value types and computation types.

Being dual to monads, comonads have been used to formalize context-dependent computations. Intuitively, an effect which observes features may arise from a comonad, while an effect which constructs

features may arise from a monad (16). Uustalu and Vene have structured stream computations (36), Orchard et al. array computations (27) and Tzevelekos game semantics (34) via the use of comonads. In (28), Petricek et al. proposed a unified calculus for tracking context dependence in functional languages together with a categorical semantics based on indexed comonads. In his report (26), Orchard proposed a method for choosing between monads and comonads when formalizing computational effects. A computation can be seen as a composition of context-dependence and effectfulness (36). In (5), Brookes and Van Stone showed that such combinations may correspond to distributive laws of a comonad over a monad. This has been applied to clocked causal data-flow computation, combining causal data-flow and exceptions by Uustalu and Vene in (35).

Moggi's approach, using monads in effect modeling, has been extended to Lawvere theories which first appeared in Lawvere's 1963 PhD dissertation (19). Three years later, in (22), Linton showed that every Lawvere theory induces a monad on the category of sets, and more generally on any category which satisfies the local representability condition (21). Therefore, Moggi's seminal paper (25), formalizing computational effects by monads, made it possible for monadic effects to be formalized through Lawvere theories. To this extend, Plotkin and Power, in (29), have shown that effects such as the global and the local state could be formalized by *signatures* of effectful terms and an *equational theory* explaining the interactions between them. Melliès has refined this *equational theory* in (24) showing that some of the equations modeling the mutable global state can be omitted. In (14, 15), Hyland et al. studied the combination of computational effects in terms of Lawvere theories.

Plotkin and Pretnar (29, 30, 31) extended Moggi's classification of terms (*values* and *computations*) with a third level called *handlers* for the computational effects that can be represented by an algebraic theory (*algebraic effects*). Initially, they introduce an *handler* for the exception handling, and then account for its generalization to the other handlers to cope with other algebraic effects such as stream redirection, explicit non-determinism, CCS, parameter passing, timeout and rollback (31, §3). For each algebraic effect, *handling constructs* are used to apply handlers to effectful computations where effectful computations can be interpreted as algebraic operations while handling constructs as homomorphisms from free algebras. This use of handling constructs is inspired from Benton and Kennedy's paper (3) where a construct specifically for exceptions is introduced. Notice also that formalization of the exception effect can also be made from a co-algebraic point of view as in (17). Also, exception handling is used in (33) to get a Hoare logic for exceptions .

Apart from all these, there is an older formal way of modeling computational effects called the *effect systems*. In their 1988 paper (23), Lucassen and Gifford presented an approach to programming languages for parallel computers. The key idea was to use an *effect system* to discover expression scheduling constraints. There, every expression comes with three components: *types* to represent the kinds of the return values, *effects* to summarize the observable interactions of expressions and *regions* to highlight the areas of the memory where expressions may have effects. To this extend, one can simply reason that if two expressions do not have overlapping effects, then they can obviously be scheduled in parallel. The reasoning is done by some inference rules for *types* and *effects* based on the second order typed λ -calculus.

In (6), Duval et al. proposes yet another paradigm to formalize computational effects by mixing effect systems and algebraic theories, named *the decorated logic*. The key point of this paradigm is that every term comes with a decoration which exposes its features with respect to a single computational effect or to several ones keeping their interpretations close to syntax in reasoning with effects. In addition, an *equational theory* highlights the interactions among terms with two sorts of equations: *weak* equations relate terms with respect only to their results and *strong* equations relate terms with respect both to their

results and effects. By and large, decorated logic provides an equational reasoning in between programs written in imperative languages after being a target language for a denotational semantics of the studied language.

The main point of this paper is to design the decorated logic for the global state and the exception effects, and then combine them to serve as a target language for denotational semantics of imperative programming languages mixing mentioned effects only. In this regards, this paper is organized as follows: in Section 2, we introduce an imperative programming language that mixes the state and the exception effects by defining its small-step operational semantics. The language we study there is called IMP+Exc which extends the IMP (or `while`) language with a mechanism to *raise* and *handle exceptions*. In Section 3, we introduce the decorated logic as a generic framework extending Moggi’s monadic equational logic. Then, we formally specialize the decorated logic for the state and the exception effects in Sections 4 and 5, respectively. In Section 6, we combine these logics. Finally, Section 7 details the use of the combined decorated logic as the target language for the IMP+Exc denotational semantics. This provides a rigorous formalism for an equational reasoning between termination-guaranteed IMP+Exc programs. I.e., proving two different looking programs are in fact doing the same job with respect to the state and exception effects. Also, we certify such proofs with the Coq Proof Assistant. Figure 1 summarizes the approach of the paper.

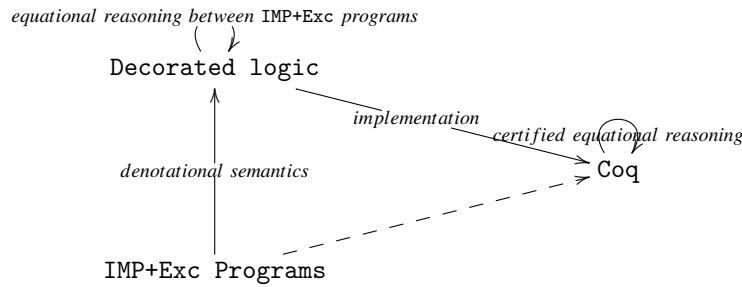


Figure 1: The approach

2 IMP with exceptions

IMP is a standard Turing complete imperative programming language natively providing global variables of integer (Z), Boolean (B) and unit (U) data types, standard integer and Boolean arithmetic enriched with a set of commands that is made of do-nothing, assignment, sequence, conditionals and looping operations. Below, we detail its syntax where n represents a constant integer term while x is an integer global variable. Note also that abbreviations $aexp$ and $bexp$ respectively denote arithmetic and Boolean expressions as well as cmd stands for the commands.

$$\begin{aligned}
 aexp: a_1 a_2 & ::= n \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \times a_2 \\
 bexp: b_1 b_2 & ::= true \mid false \mid a_1 \stackrel{?}{=} a_2 \mid a_1 \stackrel{?}{\neq} a_2 \mid a_1 \stackrel{?}{>} a_2 \mid a_1 \stackrel{?}{<} a_2 \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \mid \neg b_1 \\
 cmd: c_1 c_2 & ::= SKIP \mid x \stackrel{\Delta}{=} e \mid c_1; c_2 \mid \text{if } b \text{ then } c_1 \text{ else } c_2 \mid \text{while } b \text{ do } c_1
 \end{aligned}$$

Figure 2: Standard IMP syntax

Neither arithmetic nor Boolean expressions are allowed to modify the state: they are either pure or read-only. We present, in Figure 3, the small-step semantics for evaluation of arithmetic expressions in IMP where we use a small-step transition function $\rightarrow_a: \text{aexp} \times S \rightarrow \text{aexp}$. This function computes a new arithmetic expression out of an input arithmetic expression and the current program state denoted S which includes contents of variables at a given time.

$$(\text{const}) \frac{}{(n, s) \rightarrow_a n} \quad (\text{var}) \frac{}{(x, s) \rightarrow_a x(s)} \quad (\text{op-sym}) \frac{(a_1, s) \rightarrow_a n_1 \quad (a_2, s) \rightarrow_a n_2}{(a_1 \text{ op } a_2, s) \rightarrow_a n_1 \text{ op}_{\mathbb{Z}} n_2}$$

Figure 3: Small-step natural semantics for arithmetic expressions

The symbol op represents the operation symbols ($+$, $-$ or \times) given by the standard syntax in Figure 2 while $\text{op}_{\mathbb{Z}}: \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}$ denotes the corresponding binary operations in \mathbb{Z} . Similarly, in Figure 4, we present the small-step semantics for evaluation of Boolean expressions in IMP where we use a small-step transition function $\rightarrow_b: \text{bexp} \times S \rightarrow \text{bexp}$. This function simply computes a new Boolean expression out of an input Boolean expression and the current program state.

$$\begin{array}{c} (\text{true}) \frac{}{(\text{true}, s) \rightarrow_b \text{true}} \quad (\text{false}) \frac{}{(\text{false}, s) \rightarrow_b \text{false}} \\ (\text{op1}) \frac{(b_1, s) \rightarrow_b n_1 \quad (b_2, s) \rightarrow_b n_2}{(b_1 \text{ opb } b_2, s) \rightarrow_b n_1 \text{ opb}_{\mathbb{B}} n_2} \quad (\text{op2}) \frac{(b_1, s) \rightarrow_b n_1}{(\neg b_1, s) \rightarrow_b \text{neg } n_1} \end{array}$$

Figure 4: Small-step natural semantics for arithmetic expressions

The symbols true and false are Boolean operation symbols given by the standard syntax in Figure 2 while true and false are Boolean constructors. Similarly, opb represents the binary operation symbols (all except ‘ \neg ’), while $\text{opb}_{\mathbb{B}}: \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}$ denotes the corresponding Boolean operations, and $\text{neg}: \mathbb{B} \rightarrow \mathbb{B}$ is the Boolean negation. The small-step operational semantics for evaluation of commands are given in Figure 5 where we use a small-step transition function $\rightsquigarrow: \text{cmd} \times S \rightarrow \text{cmd} \times S$ which is interpreted as at the state s , one step execution of the command c changes the state into s' and the command c' is in execution.

$$\begin{array}{c} (\text{sequence}) \frac{s, c_1 \rightsquigarrow s', c'_1}{s, (c_1; c_2) \rightsquigarrow s', (c'_1; c_2)} \quad (\text{skip}) \frac{}{s, (\text{SKIP}; c) \rightsquigarrow s, c} \\ (\text{assign}) \frac{(a, s) \rightarrow_a n}{s, (x := a) \rightsquigarrow s[x \leftarrow n], \text{SKIP}} \\ (\text{cond}_1) \frac{(b, s) \rightarrow_b \text{true}}{s, (\text{if } b \text{ then } c_1 \text{ else } c_2) \rightsquigarrow s, c_1} \quad (\text{cond}_2) \frac{(b, s) \rightarrow_b \text{false}}{s, (\text{if } b \text{ then } c_1 \text{ else } c_2) \rightsquigarrow s, c_2} \\ (\text{while}_1) \frac{(b, s) \rightarrow_b \text{true}}{s, (\text{while } b \text{ do } c) \rightsquigarrow s, (c; \text{while } b \text{ do } c)} \quad (\text{while}_2) \frac{(b, s) \rightarrow_b \text{false}}{s, (\text{while } b \text{ do } c) \rightsquigarrow s, \text{SKIP}} \end{array}$$

Figure 5: Small-step operational semantics for commands

Note that all the small-step transition functions we used so far are meant to reduce any given configuration just a single step. I.e., $(a, s) \rightarrow_a n$ means that one step reduction of the arithmetic expression a at a state s results in another arithmetic expression n . Besides all these, we need to elucidate that a command

c terminates at a state s if $s, c \rightsquigarrow^* s'$, SKIP for some state s' , where \rightsquigarrow^* is the transitive closure of the transition function \rightsquigarrow . Mind also that there is no run-time error since any command apart from the SKIP is allowed to execute at any state s , and SKIP alone is used to indicate the final step of some command set.

2.1 A mechanism to handle exceptions

Extending the IMP language with a mechanism that allows raising exceptions and recover from them, we just enrich the command set with THROW and TRY/CATCH blocks. In addition to the ones in Figure 2, we also consider following commands in Figure 6.

$$\text{cmd: } c_1 c_2 ::= \dots \mid \text{THROW } e \mid \text{TRY } c_1 \text{ CATCH } e \Rightarrow c_2$$

Figure 6: Syntax for exceptional commands

where e is an exception name coming from a finite set EName which exists by assumption. There is also a type EV_e of exceptional values (parameters) for each exception name e . The small-step operational semantics for THROW and TRY/CATCH commands are shown in Figure 7.

$$\begin{array}{c} \text{(throw)} \frac{\forall e : \text{EName}}{s, (\text{THROW } e; c) \rightsquigarrow s, \text{THROW } e} \quad \text{(tc}_0\text{)} \frac{\forall e : \text{EName}, c_1 \in \text{cmd} \setminus \{\text{THROW } e\}}{s, \text{TRY } c_1 \text{ CATCH } e \Rightarrow c_2 \rightsquigarrow s, c_1} \\ \text{(tc}_1\text{)} \frac{\forall e : \text{EName}}{s, \text{TRY } (\text{THROW } e) \text{ CATCH } e \Rightarrow c \rightsquigarrow s, c} \quad \text{(tc}_2\text{)} \frac{\forall e_1 e_2 : \text{EName} \quad e_1 \neq e_2}{s, \text{TRY } (\text{THROW } e_1) \text{ CATCH } e_2 \Rightarrow c \rightsquigarrow s, \text{THROW } e_1} \end{array}$$

Figure 7: Small-step operational semantics for additional commands

Exceptional commands are pure in terms of the state effect: they neither use nor modify the program state. However, they introduce another sort of computational effect: the exception. In prior, we stated that the command SKIP alone indicates the termination of a program. Now, we extend this by saying THROW e is also an end but an abnormal end. Intuitively, if an exceptional value of name e is raised in the TRY block and recovered immediately in the CATCH, the program then resumes with the continuation provided. Unlikely, an exceptional value of name e_1 gets propagated if in the CATCH block, another exceptional value of name e_2 is attempted to be recovered.

Recall that in Section 7, we independently define a denotational semantics for the IMP+Exc language using the decorated logic for the state and the exception effects as the target language. We formalize this target (in a generic way) logic in Section 6 which basically combines the logics presented in Sections 4 and 5.

3 Decorated Logic (\mathcal{L}_{dec})

The decorated logic, as a generic framework, is an extension to Moggi's monadic equational logic (25) with the use of decorations on terms and equalities. It provides a rigorous formalism to do an *equational reasoning* between programs written in imperative programming languages after being defined as a target language for their denotational semantics.

3.1 Monadic Equational Logic (\mathcal{L}_{meq})

The *monadic equational logic* (\mathcal{L}_{meq}) is the minimal logic that can be interpreted in a category with objects as types, arrows as terms and equalities as equations. The keyword “*monadic*” has little to do with monads. It rather means that the operations of the logic are *unary* (or mono-adic). Figure 8 presents the syntax of the logic \mathcal{L}_{meq} .

Grammar for the monadic equational logic:

Types: $t ::= X \mid Y \mid \dots$
 Terms: $f, g ::= \text{id}_t \mid a \mid b \mid \dots \mid g \circ f$
 Equations: $\text{eq} ::= f \cong g$

Figure 8: \mathcal{L}_{meq} : syntax

Every term has a source and a target type, e.g., $f: X \rightarrow Y$. Every equation is formed by terms with the same source and target types, e.g., $e: f \cong g$ where $f, g: X \rightarrow Y$.

congruence rules

(refl) $\frac{f}{f \cong f}$ (sym) $\frac{f \cong g}{g \cong f}$ (trans) $\frac{f \cong g \quad g \cong h}{f \cong h}$ (replsubs) $\frac{f_1 \cong f_2: X \rightarrow Y \quad g_1 \cong g_2: Y \rightarrow Z}{g_1 \circ f_1 \cong g_2 \circ f_2}$

categorical rules

(id) $\frac{X}{\text{id}_X: X \rightarrow X}$ (comp) $\frac{f: X \rightarrow Y \quad g: Y \rightarrow Z}{(g \circ f): X \rightarrow Z}$ (ids) $\frac{f: X \rightarrow Y}{f \circ \text{id}_X \cong f}$ (idt) $\frac{f: X \rightarrow Y}{\text{id}_Y \circ f \cong f}$
 (assoc) $\frac{f: X \rightarrow Y \quad g: Y \rightarrow Z \quad h: Z \rightarrow U}{h \circ (g \circ f) \cong (h \circ g) \circ f}$

Figure 9: \mathcal{L}_{meq} : rules

The *congruence rules* say that the relation ‘ \cong ’ is a congruence meaning that it is an equivalence relation (reflexive, symmetric and transitive) which obeys *replacements* and *substitutions* of compatible terms with respect to the composition. The basic categorical rules indicate that there is an identity morphism $\text{id}_X: X \rightarrow X$ for each type X , composition is an associative operation, and composing any term f with id is f , up to \cong , no matter the composition order.

3.2 The decorated logic

The decorated logic, as a generic framework, extends the monadic equational logic with a 3-tier effect system for terms and a 2-tier system for equations made of “up-to-effects” (weak) and “strong” equalities. Figure 10 presents its syntax.

Grammar for the decorated logic:

Types: $t ::= X \mid Y \mid \dots$
 Decoration for terms: $(d) ::= (0) \mid (1) \mid (2)$
 Terms: $f, g ::= a^{(d)} \mid b^{(d)} \mid \dots \mid g \circ f^{(d)} \mid (\text{tpure } \bullet)^{(0)}$
 Equations: $\text{eq} ::= f \equiv g \mid f \sim g$

Figure 10: \mathcal{L}_{dec} : syntax

Syntactically, each term has a source and a target type as well as a decoration which describe what computational side effects evaluation of that term may involve, and used as a superscript (0), (1) or (2): a *pure* term is decorated with (0), an effect *constructor* with (1) and an effect *modifier* term comes with the decoration (2). Each equation is formed by two terms with the same source and target as well as a decoration which is denoted either by “ \sim ” (*weak*) or by “ \equiv ” (*strong*). A weak equality between two terms relates them according only to their results, while a strong equality relates terms according both to their result and the side effect evaluation they involve with respect to the effect in question.

The tpure is a special constructor used to introduce decorated pure terms into the logic \mathcal{L}_{dec} . It inputs a non-decorated pure term from a pure type system (i.e., Coq's logic) and drops it in with the decoration (0). For instance, the identity term id is defined using the tpure constructor, for all types X as follows:

$$\text{id}_X^{(0)} : X \rightarrow X := \text{tpure } (\lambda x : X. x : X).$$

In Figure 11, we present the inference rules associated to the syntax given in Figure 9.

hierarchy rules

$$(0\text{-to-}1) \frac{f^{(0)}}{f^{(1)}} \quad (1\text{-to-}2) \frac{f^{(1)}}{f^{(2)}} \quad (\text{stow}) \frac{f^{(2)} \equiv g^{(2)}}{f \sim g} \quad (\text{wtos}) \frac{f^{(1)} \sim g^{(1)}}{f \equiv g}$$

congruence rules

$$(\text{refl}) \frac{f^{(2)}}{f \equiv f} \quad (\text{sym}) \frac{f^{(2)} \equiv g^{(2)}}{g \equiv f} \quad (\text{trans}) \frac{f^{(2)} \equiv g^{(2)} \quad g^{(2)} \equiv h^{(2)}}{f \equiv h}$$

$$(\text{wrefl}) \frac{f^{(2)}}{f \sim f} \quad (\text{wsym}) \frac{f^{(2)} \sim g^{(2)}}{g \sim f} \quad (\text{wtrans}) \frac{f^{(2)} \sim g^{(2)} \quad g^{(2)} \sim h^{(2)}}{f \sim h}$$

$$(\text{replsubs}) \frac{f_1^{(2)} \equiv f_2^{(2)} : X \rightarrow Y \quad g_1^{(2)} \equiv g_2^{(2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv g_2 \circ f_2}$$

categorical rules

$$(\text{comp}) \frac{f^{(d_1)} : X \rightarrow Y \quad g^{(d_2)} : Y \rightarrow Z}{(g \circ f)^{(\max(d_1, d_2))} : X \rightarrow Z} \quad (\text{assoc}) \frac{f^{(2)} : X \rightarrow Y \quad g^{(2)} : Y \rightarrow Z \quad h^{(2)} : Z \rightarrow U}{h \circ (g \circ f) \equiv (h \circ g) \circ f}$$

$$(\text{ids}) \frac{f^{(2)} : X \rightarrow Y}{f \circ \text{id}_X \equiv f} \quad (\text{idt}) \frac{f^{(2)} : X \rightarrow Y}{\text{id}_Y \circ f \equiv f}$$

$$(\text{tcomp}) \frac{f : Y \rightarrow Z \quad g : X \rightarrow Y}{(\text{tpure } f)^{(0)} \circ (\text{tpure } g)^{(0)} \equiv (\text{tpure } (f \circ g))^{(0)}}$$

Figure 11: \mathcal{L}_{dec} : rules

Hierarchically, a *pure* term can be seen as a *constructor* (0-to-1), and similarly a *constructor* term can be seen as a *modifier* on demand (1-to-2). It is obviously free to convert strong equations into weak ones (stow). However, one has to make sure that the equated terms are not decorated with (2) in order to convert weak equations into strong ones with no further evidence (wtos).

Both strong and weak equalities are defined to be *equivalence relations* with the assumption that they are *reflexive*, *transitive* and *symmetric*. Strong equations form a congruence relation but weak equations do not: we will see this in detail when we specialize the decorated logic for the global state and the exception effects in Sections 4 and 5, respectively.

The categorical rules present properties of the term composition: the decoration of a composition depends on the decoration of its components, always taking the larger. I.e., $\forall f^{(1)} : X \rightarrow Y$ and $g^{(2)} : Y \rightarrow Z$, $g \circ f : X \rightarrow Z$ takes the decoration (2) (comp). Composition is an associative operation (assoc). The identity term disappears when to compose on the right (ids), and on the left (idt). The rule (tcomp) states that the tpure constructor preserves the composition of pure terms terms up to the strong equality. Meaning that one can first compose pure terms outside the decorated environment (in any pure type system) and use the tpure constructor to translate them into the \mathcal{L}_{dec} , or translate the terms into the \mathcal{L}_{dec} first, and then compose them there. Notice that the red colored composition symbol (\circ), in the rule conclusion, stands for the composition operation for non-decorated pure terms.

4 The Decorated Logic for the state effect (\mathcal{L}_{st})

The use and modification of the memory state is the fundamental feature of imperative languages, and considered as a sort of computational side effect. In this section, we present a proof system for the use of the global state which involves access and modify operations, called the decorated logic for the state effect (\mathcal{L}_{st}). This logic is obtained by extending the generic framework presented in Section 3.2. In this case, the decoration (0) is reserved for *pure* terms, while (1) is for *read-only* (*accessor*) and (2) is for *read-write* (*modifier*) terms. Two terms are called strongly equal if they return the same result with the same state manipulation; they are called weakly equal if they return the same result with different state manipulations.

Grammar of the decorated logic for the state:		$(i \in \text{Loc})$
Types:	$\mathfrak{t}, \mathfrak{s} ::=$	$X \mid Y \mid \dots \mid \mathfrak{t} \times \mathfrak{s} \mid \mathbb{1} \mid V_i$
Decorations for terms:	$(d) ::=$	$(0) \mid (1) \mid (2)$
Terms:	$\mathfrak{f}, \mathfrak{g} ::=$	$\mathfrak{a}^{(d)} \mid \mathfrak{b}^{(d)} \mid \dots \mid \mathfrak{g} \circ \mathfrak{f}^{(d)} \mid \langle \mathfrak{f}, \mathfrak{g} \rangle_1^{(d)} \mid$ $\text{lookup}_i^{(1)} \mid \text{update}_i^{(2)} \mid (\text{tpure } \bullet)^{(0)}$
Equations:	$\text{eq} ::=$	$\mathfrak{f}^{(d)} \equiv \mathfrak{g}^{(d)} \mid \mathfrak{f}^{(d)} \sim \mathfrak{g}^{(d)}$

Figure 12: \mathcal{L}_{st} : syntax

Figure 12 shows the grammar of the \mathcal{L}_{st} where $\mathbb{1}$ is the singleton type while V_i is the type of values that can be stored in any location i . We assume that there is a finite set of locations called Loc . Given types X and Y , we have $X \times Y$ representing type products.

Terms are closed under composition (\circ) and pairing ($\langle _, _ \rangle_1$). I.e., for all terms $\mathfrak{f} : X \rightarrow Y$ and $\mathfrak{g} : Y \rightarrow Z$, we have $\mathfrak{g} \circ \mathfrak{f} : X \rightarrow Z$. Similarly, for all $\mathfrak{f} : X \rightarrow Y$ and $\mathfrak{g} : X \rightarrow Z$, there is $\langle \mathfrak{f}, \mathfrak{g} \rangle_1 : X \rightarrow Y \times Z$. Notice that the pair subscript ‘1’ denotes the left pairs. One can define in a symmetric way the right pairs for terms $\mathfrak{f} : X \rightarrow Y$ and $\mathfrak{g} : X \rightarrow Z$ as $\langle \mathfrak{f}, \mathfrak{g} \rangle_r := \text{permut} \circ \langle \mathfrak{g}, \mathfrak{f} \rangle_1$ where $\text{permut} := \langle \pi_2, \pi_1 \rangle_1$. In the same way, one can respectively obtain left and right products of terms $\mathfrak{f} : X_1 \rightarrow Y_1$ and $\mathfrak{g} : X_2 \rightarrow Y_2$ as $\mathfrak{f} \times_1 \mathfrak{g} := \langle \mathfrak{f} \circ \pi_1, \mathfrak{g} \circ \pi_2 \rangle_1$ and $\mathfrak{f} \times_r \mathfrak{g} := \langle \mathfrak{f} \circ \pi_1, \mathfrak{g} \circ \pi_2 \rangle_r$. The term pairs/products are used to impose some order of term evaluation since the evaluation result depends on the order that the mutable state is accessed/modified. I.e., the product of two terms can be intuitively interpreted as they run on the global state in parallel, while sequential products, put forward in (7, §2.3), enforce terms to use the state in sequence. The decoration of a pair/product depends on the decoration of its components, always taking the larger. I.e., $\forall \mathfrak{f}^{(1)} : X \rightarrow Y$ and $\mathfrak{g}^{(2)} : X \rightarrow Z$, the term $\langle \mathfrak{f}, \mathfrak{g} \rangle_1 : X \rightarrow Y \times Z$ takes the decoration (2). Remark that the pairs of modifiers are allowed to be constructed in the logic \mathcal{L}_{st} . However, they cannot be used in the provided equational reasoning, since they may lead to a conflict on the returned result. See the rules (w_lpair_eq) and (s_lpair_eq) in Figure 13.

The interface terms are $\text{lookup}_i : \mathbb{1} \times S \rightarrow V_i$ and $\text{update}_i : V_i \times S \rightarrow \mathbb{1} \times S$ where S denotes the distinguished object of states which never appears in the decorated setting. The use of decorations provides a new schema where term signatures are constructed without any occurrence of the state object. For instance, $\text{lookup}_i^{(1)} : \mathbb{1} \rightarrow V_i$ is an accessor while $\text{update}_i^{(2)} : V_i \rightarrow \mathbb{1}$ is a modifier. This way, we keep signatures close to their syntax and compose compatible terms as usual. The term lookup reads the value stored in a given location while update modifies it. We can call them *the unique sources of impurity*, since only the terms including lookup or update are impure; meaning those do not include them are pure with respect to the state effect.

The identity term id , the canonical pair projections π_1 and π_2 , the empty pair $\langle \rangle$ and constants are translated from a pure type system with type products using the tpure constructor, for all types X and Y , as follows:

$$\begin{aligned} \text{id}_x^{(0)} & : X \rightarrow X & := & \text{tpure } (\lambda x : X. x : X) \\ \pi_1^{(0)} & : X \times Y \rightarrow X & := & \text{tpure } \text{fst} \\ \pi_2^{(0)} & : X \times Y \rightarrow Y & := & \text{tpure } \text{snd} \\ \langle \rangle_x^{(0)} & : X \rightarrow \mathbb{1} & := & \text{tpure } (\lambda x : X. \text{void} : \mathbb{1}) \\ \text{constant}_x^{(0)} & : \mathbb{1} \rightarrow X & := & \text{tpure } (\lambda _ . x : X) \end{aligned}$$

where fst and snd are constructors of product types.

The intended model of the above grammar is built with respect to the set of states S where a pure term $p^{(0)} : X \rightarrow Y$ is interpreted as a function $p : X \rightarrow Y$, an accessor $a^{(1)} : X \rightarrow Y$ as a function $a : X \times S \rightarrow Y$, and a modifier $m^{(2)} : X \rightarrow Y$ as a function $m : X \times S \rightarrow Y \times S$. The complete and detailed category theoretical model is given in (11, §5.1).

Rules of the decorated logic for the state:

$$\begin{aligned} (\text{pwrepl}) \frac{f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y \quad g^{(0)} : Y \rightarrow Z}{g \circ f_1 \sim g \circ f_2} & \quad (\text{wtrans}) \frac{f^{(2)} \sim g^{(2)} \quad g^{(2)} \sim h^{(2)}}{f \sim h} \\ (\text{replsubs}) \frac{f_1^{(2)} \equiv f_2^{(2)} : X \rightarrow Y \quad g_1^{(2)} \equiv g_2^{(2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv g_2 \circ f_2} & \quad (\text{w_unit}) \frac{f^{(2)} : X \rightarrow \mathbb{1}}{f \sim \langle \rangle_x} \\ (\text{ax}_1) \frac{}{\text{lookup}_i^{(1)} \circ \text{update}_i^{(2)} \sim \text{id}_{v_i}^{(0)}} & \quad (\text{ax}_2) \frac{\forall i, j \in \text{Loc}, i \neq j}{\text{lookup}_i^{(1)} \circ \text{update}_j^{(2)} \sim \text{lookup}_i^{(1)} \circ \langle \rangle_{V_i}^{(0)}} \\ (\text{effect}) \frac{f_1^{(2)}, f_2^{(2)} : X \rightarrow Y \quad f_1^{(2)} \sim f_2^{(2)} \quad \langle \rangle_Y^{(0)} \circ f_1^{(2)} \equiv \langle \rangle_Y^{(0)} \circ f_2^{(2)}}{f_1 \equiv f_2} & \\ (\text{local_global}) \frac{f_1^{(2)}, f_2^{(2)} : X \rightarrow \mathbb{1} \quad \forall i \in \text{Loc}, \text{lookup}_i^{(1)} \circ f_1^{(2)} \sim \text{lookup}_i^{(1)} \circ f_2^{(2)}}{f_1 \equiv f_2} & \\ (\text{w_lpair_eq}) \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : X \rightarrow Z}{\pi_1 \circ \langle f_1, f_2 \rangle_1 \sim f_1} & \quad (\text{s_lpair_eq}) \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : X \rightarrow Z}{\pi_2 \circ \langle f_1, f_2 \rangle_1 \equiv f_2} \end{aligned}$$

Figure 13: \mathcal{L}_{st} : rules

The syntax given in Figure 12 is enriched with a set of rules which are presented in Figure 13 in addition to the ones in Figure 11. Weak equalities do not form a congruence: the term replacement cannot be done unless the replaced term is pure. I.e., given an equation $f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y$ and a term $g : Y \rightarrow Z$, it is possible to get the equation $g \circ f_1 \sim g \circ f_2$ only when the term g is pure. At this stage, we have no information about the modifications that f_1 and f_2 make on the memory state. Therefore, the post executed impure term g would destroy this result equality, for instance by reading the location i on which f_1 and f_2 has performed different modifications (pwrepl). However, the term substitution can be done regardless of the term decoration. I.e., given the equation $f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z$ and a term $g : X \rightarrow Y$, it is possible to

get the equation $f_1 \circ g \sim f_2 \circ g$ independent from the decoration of the term g . We already know that f_1 and f_2 return the same result, executing any term g in advance would not end them returning different results (wsubs). Strong equalities form a congruence by allowing both term substitutions and replacements regardless of the term decorations (replsubs).

Any term $f : X \rightarrow \mathbb{1}$ with no result returned “void” (the unique inhabitant of $\mathbb{1}$ type) has an obvious result equality with the canonical empty pair $\langle \rangle_X$ (w_unit).

The fundamental equations are given with the rules (ax₁) and (ax₂). The former states that by updating the location i with a value v and then observing the same location, one gets the value v . This outputs the same value with the identity term id_{V_i} , if it takes v as an argument. However, notice that these two ways of getting the value v have different state manipulations which makes them *weakly equal*. The latter, (ax₂), is to assume that updating the location j with a value v and then reading the content of a different location i would return the same value with first throwing out the value v then observing the content of location i . They definitely have different manipulations on the state so that they are *weakly equal*.

Two modifiers $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$ modify the state in the same way if and only if $\langle \rangle_Y \circ f_1 \equiv \langle \rangle_Y \circ f_2 : X \rightarrow \mathbb{1}$, where $\langle \rangle_Y : Y \rightarrow \mathbb{1}$ throws out the returned value. So that $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$ are *strongly equal* if and only if $f_1 \sim f_2$ and $\langle \rangle_Y \circ f_1 \equiv \langle \rangle_Y \circ f_2$ (effect).

Locally, the strong equality between two modifiers $f_1^{(2)}, f_2^{(2)} : X \rightarrow \mathbb{1}$ can also be expressed as a pair of weak equations: $f_1 \sim f_2$ and $\forall i : \text{Loc}, \text{lookup}_i \circ \langle \rangle_Y \circ f_1 \sim \text{lookup}_i \circ \langle \rangle_Y \circ f_2$. The latter intuitively means that f_1 and f_2 leaves the memory with the same values stored in all (finitely many) locations after being executed. Given that both return “void” there is no explicitly need to check if $f_1 \sim f_2$. It suffices to see whether $\forall i : \text{Loc}, \text{lookup}_i \circ \langle \rangle_Y \circ f_1 \sim \text{lookup}_i \circ \langle \rangle_Y \circ f_2$ to end up with $f_1 \equiv f_2$ (local_global).

With (w_lpair_eq) and (w_rpair_eq) term pairs are characterized: the (left) pair structure $\langle f_1, f_2 \rangle_1$ cannot be used when f_1 and f_2 , both are modifiers, since it may lead to a conflict on the returned result. However, it can be used only when f_1 is an accessor. We state by (w_lpair_eq) that $\langle f_1, f_2 \rangle_1^{(2)}$ has only result equality with $f_1^{(1)}$ and by (w_rpair_eq) that it has both result and effect equality with $f_2^{(2)}$.

Note that these rules are designed to be sound with respect to a categorical model, and detailed in (11, §5.2, §5.3, §5.4, §5.5). However, their syntactic completeness is not immediate. In (8), we define a new syntactic completeness property, subsuming a consistency check, called the relative Hilbert-Post completeness. In (11, §5.4) prove that this set of rules is complete with due respect.

4.1 Decorated properties of the memory state

Plotkin and Power have introduced, in (29, §3), an equational representation of the mutable state whose decorated versions are given as follows:

- (1)_d Annihilation lookup-update. *Reading the content of a location i and then updating it with the obtained value is just like doing nothing.* $\forall i \in \text{Loc}, \text{update}_i^{(2)} \circ \text{lookup}_i^{(1)} \equiv \text{id}_i^{(0)} : \mathbb{1} \rightarrow \mathbb{1}$.
- (2)_d Interaction lookup-lookup. *Reading twice the same location i is the same as reading it once.* $\forall i \in \text{Loc}, \text{lookup}_i^{(1)} \circ \langle \rangle_{V_i}^{(0)} \circ \text{lookup}_i^{(1)} \equiv \text{lookup}_i^{(1)} : \mathbb{1} \rightarrow V_i$.
- (3)_d Interaction update-update. *Storing value the values x and y in a row to the same location i is just like storing y in it.* $\forall i \in \text{Loc}, \text{update}_i^{(2)} \circ \pi_2^{(0)} \circ (\text{update}_i^{(2)} \times_r \text{id}_i^{(0)}) \equiv \text{update}_i^{(2)} \circ \pi_2^{(0)} : V_i \times V_i \rightarrow \mathbb{1}$.

- (4)_d Interaction update-lookup. *Storing the value x in a location i and then reading the content of i, gives the value x.* $\forall i \in \text{Loc}, \text{lookup}_i^{(1)} \circ \text{update}_i^{(2)} \sim \text{id}_{V_i}^{(0)} : V_i \rightarrow V_i$.
- (5)_d Commutation lookup-lookup. *The order of reading two different locations i and j does not matter.* $\forall i \neq j \in \text{Loc}, (\text{id}_{V_i}^{(0)} \times_r \text{lookup}_j^{(1)}) \circ \pi_1^{-1(0)} \circ \text{lookup}_i^{(1)} \equiv \text{permut}_{j,i}^{(0)} \circ (\text{id}_{V_j}^{(0)} \times_r \text{lookup}_i^{(1)}) \circ \pi_1^{-1(0)} \circ \text{lookup}_j^{(1)} : \mathbb{1} \rightarrow V_i \times V_j$ where $\pi_1^{-1(0)} := \langle \text{id}, \langle \rangle \rangle_1^{(0)}$.
- (6)_d Commutation update-update. *The order of storing in two different locations i and j does not matter.* $\forall i \neq j \in \text{Loc}, \text{update}_j^{(2)} \circ \pi_2^{(0)} \circ (\text{update}_i^{(2)} \times_r \text{id}_{V_j}^{(0)}) \equiv \text{update}_i^{(2)} \circ \pi_1^{(0)} \circ (\text{id}_{V_i}^{(0)} \times_1 \text{update}_j^{(2)}) : V_i \times V_j \rightarrow \mathbb{1}$.
- (7)_d Commutation update-lookup. *The order of storing in a location i and reading another location j does not matter.* $\forall i \neq j \in \text{Loc}, \text{lookup}_j^{(1)} \circ \text{update}_i^{(2)} \equiv \pi_2^{(0)} \circ (\text{update}_i^{(2)} \times_r \text{id}_{V_j}^{(0)}) \circ (\text{id}_{V_i}^{(0)} \times_1 \text{lookup}_j^{(1)}) \circ \pi_1^{-1(0)} : V_i \rightarrow V_j$.
- (8)_d Commutation lookup-constant. *Just after storing a constant c in a location i, observing the content of i is the same as regenerating the constant c.* $\forall i \in \text{Loc}, \forall c \in V_i; \text{lookup}_i^{(1)} \circ \text{update}_i^{(2)} \circ \text{constant } c^{(0)} \equiv \text{constant } c^{(0)} \circ \text{update}_i^{(2)} \circ \text{constant } c^{(0)} : \mathbb{1} \rightarrow V_i$.

These are the archetype properties that we have proved within the scope of the \mathcal{L}_{st} . To see these proofs, check out Ekici's PhD thesis (11, §5.3). Besides, we have implemented the \mathcal{L}_{st} in Coq to certify mentioned proofs. Section 4.2 details this implementation.

4.2 \mathcal{L}_{st} in Coq

In this section, we aim to highlight some crucial points of the \mathcal{L}_{st} implementation in Coq. It mainly consists of four steps: (1) implementing the terms, (2) assigning the decorations over terms, (3) stating the rules, and (4) proving properties of the memory state referred in Section 4.1.

We represent the set of memory locations by a Coq parameter $\text{Loc} : \text{Type}$. Since memory locations may contain different types of values, we also assume an arrow type $\text{Val} : \text{Loc} \rightarrow \text{Type}$ that is the type of values contained in each location.

```
Parameters (Loc: Type) (Val: Loc → Type).
```

We define the terms of \mathcal{L}_{st} using an inductive predicate called `term`. It establishes a new Coq Type out of two input Types. The type `term Y X` is dependent. It depends on the Type instances X and Y, and represents the arrow type $X \rightarrow Y$ in the decorated framework.

```
Inductive term: Type → Type → Type ≙
| tpure: ∀ {X Y: Type}, (X → Y) → term Y X
| comp:  ∀ {X Y Z: Type}, term X Y → term Y Z → term X Z
| pair:  ∀ {X Y Z: Type}, term X Z → term Y Z → term (X*Y) Z
| lookup: ∀ i:Loc, term (Val i) unit
| update: ∀ i:Loc, term unit (Val i).
Infix "o" ≙ comp (at level 70).
```

The constructor `tpure` takes a Coq side (pure) function and translates it into the decorated environment. `comp` constructor deals with the composition of two compatible terms. I.e., given a pair of terms $f : \text{term } X \ Y$ and $g : \text{term } Y \ Z$, then the composition $f \circ g$ would be an instance of the type $\text{term } X \ Z$. For the sake of conciseness, infix ‘`o`’ is used to denote the term composition. Similarly, the (left) pair constructor is to constitute pairs of compatible terms. I.e., given $f : \text{term } Y \ X$ and $g : \text{term } Z \ X$, we have pair $\langle f, g \rangle_1 : \text{term } (Y \times Z) \ X$. Instead of the symbol $\langle _, _ \rangle_1$, we use the keyword `pair` in the implementation. The terms `lookup` and `update` come as no surprise; just that the singleton type $\mathbb{1}$ and the type of values V_i are respectively called `unit` and `Val i` in the code. The terms such as the identity, the pair projections, the empty pair and the constant function can be derived from the native Coq functions, with the use of `tpure` constructor as follows:

```

Definition id {X: Type}      : term X X       $\triangleq$  tpure id.
Definition pi1 {X Y: Type}  : term X (X*Y)  $\triangleq$  tpure fst.
Definition pi2 {X Y: Type}  : term Y (X*Y)  $\triangleq$  tpure snd.
Definition forget {X}       : term unit X    $\triangleq$  tpure (fun _ => tt).
Definition constant {X: Type} (v: X): term X unit  $\triangleq$  tpure (fun _ => v).

```

Remark that `id` is overloaded: defined one is the identity of the decorated logic while the other one is the identity of Coq’s logic. The pair projections are named `pi1` and `pi2` while the unique mapping $\langle _ \rangle_X$ from any type X to $\mathbb{1}$ is named `forget` in the implementation.

The decorations are enumerated: `pure` (0), `ro` (1) and `rw` (2) and inductively assigned to terms via the new predicate called `is`. It builds a proposition out of a term and a decoration. I.e., $\forall i : \text{Loc}, \text{is } ro$ (`lookup i`) is a `Prop` instance, ensuring that `lookup i` is an accessor.

```

Inductive kind  $\triangleq$  pure | ro | rw.
Inductive is: kind  $\rightarrow$   $\forall X \ Y, \text{term } X \ Y \rightarrow \text{Prop} \triangleq$ 
| is_tpure:  $\forall X \ Y (f: X \rightarrow Y), \text{is pure } (@\text{tpure } X \ Y \ f)$ 
| is_comp:  $\forall k \ X \ Y \ Z (f: \text{term } X \ Y) (g: \text{term } Y \ Z), \text{is } k \ f \rightarrow \text{is } k \ g \rightarrow \text{is } k \ (f \circ g)$ 
| is_pair:  $\forall k \ X \ Y \ Z (f: \text{term } X \ Z) (g: \text{term } Y \ Z), \text{is } ro \ f \rightarrow \text{is } k \ f \rightarrow \text{is } k \ g \rightarrow \text{is } k \ (\text{pair } f \ g)$ 
| is_lookup:  $\forall i, \text{is } ro \ (\text{lookup } i)$ 
| is_update:  $\forall i, \text{is } rw \ (\text{update } i)$ 
| is_pure_ro:  $\forall X \ Y (f: \text{term } X \ Y), \text{is pure } f \rightarrow \text{is } ro \ f$ 
| is_ro_rw:  $\forall X \ Y (f: \text{term } X \ Y), \text{is } ro \ f \rightarrow \text{is } rw \ f$ .

```

Any term that is built by the `tpure` constructor is pure. The decoration of any composed or paired off term depends on its components and always takes the upper decoration (`pure < ro < rw`). E.g., given a modifier term and a read-only term, their composition will be a modifier, as well. The decoration of a pair construction depends on its second component, since the first one should at most be a read-only term. Hence, we cannot form pairs of two modifier terms. The pair construction always takes the upper decoration. For instance, given a pure term and a read-only term, their pair will be a read-only term, too. We declare the term `lookup` as an accessor while `update` being a modifier. The last two constructors define the decoration hierarchies.

It is trivial to derive that any `tpure` built term is pure. I.e., the purity of the first pair projection can be proven as follows:

```

Lemma is_pi1 X Y: is pure (@pi1 X Y).
Proof. apply is_tpure. Qed.

```

We now state the rules up to weak and strong equalities by defining them in a mutually inductive way: mutuality here is used to enable the constructors including both weak and strong equalities. We use the notation $==$ and \sim to denote strong and weak equalities, respectively.

```

Definition idem X Y (x y: term X Y)  $\triangleq$  x = y.
Inductive strong:  $\forall$  X Y, relation (term X Y)  $\triangleq$ 
| refl X Y: Reflexive (@strong X Y)
| sym:  $\forall$  X Y, Symmetric (@strong X Y)
| trans:  $\forall$  X Y, Transitive (@strong X Y)
| replsubs:  $\forall$  X Y Z, Proper (@strong X Y  $\Rightarrow$  @strong Y Z  $\Rightarrow$  @strong X Z) comp
| ids:  $\forall$  X Y (f: term X Y), f o id == f
| idt:  $\forall$  X Y (f: term X Y), id o f == f
| assoc:  $\forall$  X Y Z T (f: term X Y) (g: term Y Z) (h: term Z T), f o (g o h) == (f o g) o h
| wtos:  $\forall$  X Y (f g: term X Y), is ro f  $\rightarrow$  is ro g  $\rightarrow$  f ~ g  $\rightarrow$  f == g
| s_lpair_eq:  $\forall$  X Y' Y (f1: term Y X) (f2: term Y' X), is ro f1  $\rightarrow$  pi2 o pair f1 f2 == f2
| effect:  $\forall$  X Y (f g: term Y X), forget o f == forget o g  $\rightarrow$  f ~ g  $\rightarrow$  f == g
| local_global:  $\forall$  X (f g: term unit X), ( $\forall$  i: Loc, lookup i o f ~ lookup i o g)  $\rightarrow$  f == g
| tcomp:  $\forall$  X Y Z (f: Z  $\rightarrow$  Y) (g: Y  $\rightarrow$  X), tpure (compose g f) == tpure g o tpure f
with weak:  $\forall$  X Y, relation (term X Y)  $\triangleq$ 
| wsym:  $\forall$  X Y, Symmetric (@weak X Y)
| wtrans:  $\forall$  X Y, Transitive (@weak X Y)
| wrepl :  $\forall$  A B C, Proper (@idem C B  $\Rightarrow$  @weak B A  $\Rightarrow$  @weak C A) comp
| pwrepl:  $\forall$  A B C (g: term C B), (is pure g)  $\rightarrow$  Proper (@weak B A  $\Rightarrow$  @weak C A) (comp g)
| wsubs:  $\forall$  A B C, Proper (@weak C B  $\Rightarrow$  @idem B A  $\Rightarrow$  @weak C A) comp
| stow:  $\forall$  X Y (f g: term X Y), f == g  $\rightarrow$  f ~ g
| w_lpair_eq:  $\forall$  X Y' Y (f1: term Y X) (f2: term Y' X), is ro f1  $\rightarrow$  pi1 o pair f1 f2 ~ f1
| w_unit:  $\forall$  X (f g: term unit X), f ~ g
| ax1:  $\forall$  i, lookup i o update i ~ id
| ax2:  $\forall$  i j, i  $\neq$  j  $\rightarrow$  lookup j o update i ~ lookup j o forget
where "x == y"  $\triangleq$  (strong x y) and "x ~ y"  $\triangleq$  (weak x y).

```

The rule `tcomp` states that the `tpure` constructor preserves the composition of pure terms up to the strong equality: one can first compose pure terms on Coq side (using higher order function `compose`) and then apply `tpure` constructor to translate them into decorated settings or can translate the terms first and then compose them in decorated settings.

This framework allows us to express and prove, in Coq, the decorated versions of the properties mentioned in Section 4.1. E.g., the commutation `lookup`-`uptade` looks like:

```

(** Commutation update update **)
Theorem CUU:  $\forall$  i j: Loc, i  $\neq$  j  $\rightarrow$  update j o (pi2 o (rprod (update i) (@id (Val j)))) ==
update i o (pi1 o (lprod (@id (Val i)) (update j))).

```

where

```

Definition permut {X Y}: term (X*Y) (Y*X)  $\triangleq$  pair pi2 pi1.
Definition rpair {X Y Z} (f: term Y X) (g: term Z X): term (Y*Z) X  $\triangleq$  permut o pair g f.
Definition lprod {X Y X' Y'} (f: term X X') (g: term Y Y'): term (X*Y) (X'*Y')  $\triangleq$  pair (f o pi1) (g o pi2).
Definition rprod {X Y X' Y'} (f: term X X') (g: term Y Y')  $\triangleq$ 
permut o pair (g o pi2) (f o pi1).

```

The full Coq proofs of such properties can be found [here](#), and the entire implementation [there](#).

5 The Decorated Logic for the exception effect (\mathcal{L}_{exc})

Exception handling is provided by most modern programming languages to deal with anomalous or exceptional events which require special processing. In linear algebra, for instance, dynamic evaluation can

be used to apply programs which have been written for the matrices with coefficients modulo some prime number to the matrices with coefficients modulo some composite number. A way to implement dynamic evaluation in modern computing languages is to use the exceptions mechanism which is considered as a computational side effect. In this section, we present a proof system for exceptions, which involves raising and handling operations, called the decorated logic for the exception effect (\mathcal{L}_{exc}). This logic is obtained by extending the generic framework presented in Section 3.2. In this context, the decoration (0) is reserved for *pure* terms, while (1) is for *propagators* and (2) is for *catchers*. A fundamental feature of the exceptions mechanism is the distinction between *ordinary (non-exceptional)* values and *exceptions (or exceptional values)*. Two terms are called strongly equal if they behave the same on ordinary and exceptional values; they are called weakly equal if they behave the same on ordinary values but different on exceptional ones.

It has been shown in (9) that the core part of this proof system is dual to one for the state. Based on this nice duality, we build the logic \mathcal{L}_{exc} , and detail it in the following.

Grammar of the decorated logic for the exception:		($e \in \text{EName}$)
Types:	$t, s ::=$	$X \mid Y \mid \dots \mid t+s \mid \mathbb{O} \mid \text{EV}_e$
Decoration for terms:	$(d) ::=$	$(0) \mid (1) \mid (2)$
Terms:	$f, g ::=$	$a^{(d)} \mid b^{(d)} \mid \dots \mid g \circ f^{(d)} \mid [f \mid g]_1^{(d)}$ $\text{tag}_e^{(1)} \mid \text{untag}_e^{(2)} \mid (\downarrow f)^{(1)} \mid (\text{tpure} \cdot)^{(0)}$
Equations:	$\text{eq} ::=$	$f^{(d)} \equiv g^{(d)} \mid f^{(d)} \sim g^{(d)}$

Figure 14: \mathcal{L}_{exc} : syntax

Figure 14 shows the grammar of \mathcal{L}_{exc} where \mathbb{O} is the empty (uninhabited) type while EV_e is the type of parameters for each exception name e . We assume that there is a finite set of exception names called EName . Given types X and Y , we have $X+Y$ denoting co-product (disjoint union or sum) types.

Terms are closed under composition (\circ) and co-pairing ($[_ \mid _]_1$). I.e., for all terms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we have $g \circ f : X \rightarrow Z$. Similarly, for all $f : X \rightarrow Y$ and $g : Z \rightarrow Y$, there is $[f \mid g]_1 : X+Z \rightarrow Y$. Notice that the co-pair subscript ‘1’ denotes the left co-pairs. One can define in a symmetric way the right co-pairs for terms $f : X \rightarrow Y$ and $g : Z \rightarrow Y$ as $[f \mid g]_r := [g, f]_1 \circ \text{permut}$ where $\text{permut} := [\text{in}_2 \mid \text{in}_1]_1$. Similarly, one can respectively obtain left and right co-products (sums) of terms $f : X_1 \rightarrow Y_1$ and $g : X_2 \rightarrow Y_2$ as $f +_l g := [\text{in}_1 \circ f \mid \text{in}_2 \circ g]_l$ and $f +_r g := [\text{in}_1 \circ f \mid \text{in}_2 \circ g]_r$. The term co-pairs, thus co-products, are used to model the *case distinction* among terms. The decoration of a co-pair (co-product) depends on the decoration of its components, always taking the larger. I.e., $\forall f^{(1)} : X \rightarrow Z$ and $g^{(2)} : Y \rightarrow Z$, $[f \mid g]_1 : X+Y \rightarrow Z$ takes the decoration (2). Remark that the co-pairs of catchers are allowed to be constructed in the logic \mathcal{L}_{exc} . However, they cannot be used in the provided equational reasoning, since they may lead to an ambiguous case distinction over terms returning exceptional arguments. See the rules ($w_l\text{copair_eq}$) and ($s_l\text{copair_eq}$) in Figure 15.

The interface terms are $\text{tag}_e : \text{EV}_e \rightarrow \mathbb{O}+E$ and $\text{untag}_e : \mathbb{O}+E \rightarrow \text{EV}_e+E$ where E denotes the distinguished object of exceptions which never appears in the decorated setting. The use of decorations provides a new schema where term signatures are constructed without any occurrence of it. For instance, $\text{tag}_e^{(1)} : \text{EV}_e \rightarrow \mathbb{O}$ is a thrower while $\text{untag}_e^{(2)} : \mathbb{O} \rightarrow \text{EV}_e$ is a catcher. This way, we keep signatures close to their syntax and compose compatible terms as usual. The term tag_e encapsulates an ordinary value with an exception of name e while the term untag_e tries to recover the value from the exceptional case.

The ‘ \downarrow ’ symbol denotes the downcast term that takes as input a term and prevents it from catching

exceptions. It is used when to define the `try/catch` block in this settings. See Definition 5.2.

The identity term `id`, the canonical co-pair inclusions `in1` and `in2`, and the empty co-pair `[]X` (used to convert the type of input exceptional value into the given type: `X` in this case) are translated from a pure type system with sum types using the `tpure` constructor, for all types `X` and `Y`, as follows:

$$\begin{aligned} \text{id}_X^{(0)} &: X \rightarrow X &:= & \text{tpure } (\lambda x : X. x : X) \\ \text{in}_1^{(0)} &: X \rightarrow X+Y &:= & \text{tpure inl} \\ \text{in}_2^{(0)} &: Y \rightarrow X+Y &:= & \text{tpure inr} \\ []_X^{(0)} &: \mathbb{O} \rightarrow X &:= & \text{tpure } (\lambda _ : \mathbb{O}. x : X) \end{aligned}$$

where `inl` and `inr` are constructors of sum types.

Definition 5.1. For each type `Y` and exception name `e`, the propagator `throwY,e(1)` is defined as:

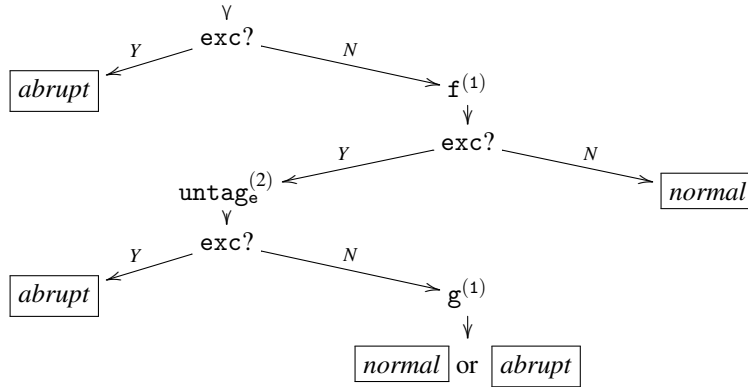
$$\text{throw}_{Y,e}^{(1)} := []_Y^{(0)} \circ \text{tag}_e^{(1)} : EV_e \rightarrow Y$$

Intuitively, raising an exception with name `e` is first tagging the given ordinary value as an exception of name `e` and then coerces the output type into `Y` for the continuation issues.

Definition 5.2. For each propagators `f(1) : X → Y`, `g(1) : EVe → Y` and each exception name `e`, the propagator `try(f)catch(e ⇒ g)(1)` is defined in three steps, as follows:

$$\begin{aligned} \text{Catch}(e \Rightarrow g)^{(2)} &:= [\text{id}_Y^{(0)} \mid g^{(1)} \circ \text{untag}_e^{(2)}]_1 : Y + \mathbb{O} \rightarrow Y \\ \text{Try}(f)\text{Catch}(e \Rightarrow g)^{(2)} &:= \text{Catch}(e \Rightarrow g)^{(2)} \circ \text{in}_{Y,\mathbb{O}}^{(0)} \circ f^{(1)} : X \rightarrow Y \\ \text{try}(f)\text{catch}(e \Rightarrow g)^{(1)} &:= \downarrow(\text{Try}(f)\text{Catch}(e \Rightarrow g)^{(2)}) : X \rightarrow Y \end{aligned}$$

To handle an exception, the intermediate expressions `Catch(e ⇒ g)` and `Try(f)Catch(e ⇒ g)` are private catchers and the expression `try(f)catch(e ⇒ g)` is a public propagator: the downcast operator prevents it from catching exceptions with name `e` which might have been raised before the `try(f)catch(e ⇒ g)` expression. The definition of `try(f)catch(e ⇒ g)` corresponds to the Java mechanism for exceptions (13, §14) and (17) with the following control flow, where `exc?` means “is this value an exception?”, an *abrupt* termination returns an uncaught exception and a *normal* termination returns an ordinary value.



Remark 5.3. The decorated terms $\text{throw}^{(1)}$ and $\text{throw/catch}^{(1)}$ stated in Definitions 5.1 and 5.2 will be used, in Section 7 (see the translator function dCmd), as the target denotational semantics of the IMP+Exc commands THROW and TRY/CATCH.

The intended model of the grammar of the logic \mathcal{L}_{exc} is built with respect to the set of exceptions E where a pure term $p^{(0)} : X \rightarrow Y$ is interpreted as a function $p : X \rightarrow Y$, a propagator $\text{pp}^{(1)} : X \rightarrow Y$ as a function $\text{pp} : X \rightarrow Y+E$, and a catcher $c^{(2)} : X \rightarrow Y$ as a function $c : X+E \rightarrow Y+E$. The complete and detailed category theoretical model is given in (11, §6.1).

Rules of the decorated logic for the exception:

$$\begin{array}{c}
\text{(pwsubs)} \frac{g^{(0)} : X \rightarrow Y \quad f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z}{f_1 \circ g \sim f_2 \circ g} \quad \text{(wrepl)} \frac{f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y \quad g^{(2)} : Y \rightarrow Z}{g \circ f_1 \sim g \circ f_2} \\
\text{(replsubs)} \frac{f_1^{(2)} \equiv f_2^{(2)} : X \rightarrow Y \quad g_1^{(2)} \equiv g_2^{(2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv g_2 \circ f_2} \\
\text{(w_empty)} \frac{f^{(2)} : \mathbb{0} \rightarrow X}{f \sim []_X} \quad \text{(w_downcast)} \frac{f^{(2)} : Y \rightarrow X}{(\downarrow f)^{(1)} \sim f} \\
\text{(eax}_1) \frac{}{\text{untag}_e^{(2)} \circ \text{tag}_e^{(1)} \sim \text{id}_{EV_e}^{(0)}} \quad \text{(eax}_2) \frac{\forall e_1, e_2 \in E\text{Name}, e_1 \neq e_2}{\text{untag}_{e_1}^{(2)} \circ \text{tag}_{e_2}^{(1)} \sim []_{EV_e}^{(0)} \circ \text{tag}_{e_2}^{(1)}} \\
\text{(effect)} \frac{f_1^{(2)}, f_2^{(2)} : Y \rightarrow X \quad f_1^{(2)} \sim f_2^{(2)} \quad f_1^{(2)} \circ []_Y^{(0)} \equiv f_2^{(2)} \circ []_Y^{(0)}}{f_1 \equiv f_2} \\
\text{(elocal_global)} \frac{f_1^{(2)}, f_2^{(2)} : \mathbb{0} \rightarrow X \quad \forall e \in E\text{Name}, f_1^{(2)} \circ \text{tag}_e^{(1)} \sim f_2^{(2)} \circ \text{tag}_e^{(1)}}{f_1 \equiv f_2} \\
\text{(w_lcopy_eq)} \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_1 \sim f_1} \quad \text{(s_lcopy_eq)} \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_2 \equiv f_2}
\end{array}$$

Figure 15: \mathcal{L}_{exc} rules

The syntax given in Figure 14 is enriched with a set of rules which are presented in Figure 15 in addition to the ones in Figure 11. Weak equalities do not form a congruence: the term substitution cannot be done unless the substituted term is pure. I.e., given the equation $f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z$ and a term $g : X \rightarrow Y$, it is possible to get the equation $f_1 \circ g \sim f_2 \circ g$ only when the term g is pure. At this stage, we have no information about the behaviors of f_1 and f_2 on exceptional values. Therefore, the pre-executed term g would destroy this result equality unless being pure, for instance, by throwing an exception of name e for which f_1 and f_2 can perform different behaviors: one can propagate, while the other can recover from it (pwrepl). However, the term replacement can be done regardless of the term decoration. I.e., given the equation $f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y$ and a term $g : Y \rightarrow Z$, it is possible to get the equation $g \circ f_1 \sim g \circ f_2$ independent from the decoration of the term g . We already know that f_1 and f_2 behave the same on ordinary values, executing any term g afterwards would not end the composition behave different on ordinary values (wrepl). Strong equalities form a congruence by allowing both term substitutions and replacements regardless of the term decorations (replsubs).

Any term $f : \mathbb{O} \rightarrow X$ with no input parameter has an equivalence on ordinary values with the empty co-pair $[\]_X$ (w_empty).

The rule ($w_downcast$) states that the term $(\downarrow f)$ behaves as f , if the argument is ordinary. It prevents f from catching the exceptional arguments otherwise.

The fundamental equations are given with the rules (eax_1) and (eax_2). The former states that encapsulating an ordinary value with an exception of name e followed by an immediate recovery would be equivalent to “doing nothing” in terms of ordinary values. Clearly, this is only a weak equation since its sides behave different on exceptional values: left hand side may recover but right hand side definitely propagates. The latter, (eax_2), is to assume that encapsulating an ordinary value v with an exception of name e_2 and then trying to recover it from a different exception of name e_1 would just lead e_2 to be propagated. Similarly, if the ordinary value v is encapsulated with e_2 with no recovery attempt afterwards would again lead e_2 to be propagated. These two operations behave the same on ordinary values but different on exceptional ones. For instance, left hand side recovers the input value encapsulated with the exception name e_1 while right hand side propagates it.

Two catchers $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$ behave the same on exceptional values if and only if $f_1 \circ [\]_X \equiv f_2 \circ [\]_X$, where $[\]_X : \mathbb{O} \rightarrow X$ throws out the exceptional values. So that $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$ are *strongly equal* if and only if $f_1 \sim f_2$ and $f_1 \circ [\]_X \equiv f_2 \circ [\]_X$ (eeffect).

Strong equality between two catchers $f_1^{(2)}, f_2^{(2)} : \mathbb{O} \rightarrow X$ can also be expressed as a pair of weak equations: $f_1 \sim f_2$ and $\forall e : EName, f_1 \circ tag_e \sim f_2 \circ tag_e$. The latter intuitively means that f_1 and f_2 behaves the same on all (finitely many) exceptional values when executed. Given that both behave the same on ordinary arguments (due to (w_empty)), there is no explicitly need to check if $f_1 \sim f_2$. It suffices to see whether $\forall e : EName, f_1 \circ tag_e \sim f_2 \circ tag_e$ to end up with $f_1 \equiv f_2$ (elocal_global).

With ($w_lcopair_eq$) and ($w_rcopair_eq$) term co-pairs (sums) are characterized: the (left) co-pair structure $[f_1 \mid f_2]_1$ cannot be used when f_1 and f_2 , both are catchers, since it may lead to a conflict on exceptional values. When f_1 is a propagator, with (w_copair_eq), we assume that ordinary values of type X are treated by $[f_1 \mid f_2]_1^{(2)}$ as they would be by $f_1^{(1)}$ and with (s_copair_eq) that ordinary values of type Z and exceptional values are treated by $[f_1 \mid f_2]_1^{(2)}$ as they would be by $f_2^{(2)}$.

Similar to the rules of the logic \mathcal{L}_{st} , the rules of the logic \mathcal{L}_{exc} also designed to be sound with respect to a categorical model, and detailed in (11, §6.2, §6.3, §6.4, §6.5). In addition, in (8) we prove that this set of rules is complete with respect to the notion of relative Hilbert-Post completeness.

5.1 Decorated properties of the exception effect

Similar to the one for the state effect presented in Section 4.1, we propose an equational representation of the exception effect with the following decorated equations:

- (1)_d Annihilation tag-untag. *Untagging an exception of name e and then raising it again is just like doing nothing.* $\forall e \in EName, tag_e^{(1)} \circ untag_e^{(2)} \equiv id_{\mathbb{O}}^{(0)} : \mathbb{O} \rightarrow \mathbb{O}$.
- (2)_d Commutation unttag-unttag. *Untagging two distinct exception names can be done in any order.* $\forall e \neq r \in EName, (untag_e^{(0)} +_r id_{EV_r})^{(2)} \circ in_2^{(0)} \circ unttag_r^{(2)} \equiv (id_{EV_e} +_1 unttag_r)^{(2)} \circ in_1^{(0)} \circ unttag_e^{(2)} : \mathbb{O} \rightarrow EV_e + EV_r$.
- (3)_d Propagator-propagates. *A propagator term always propagates the exception.* $\forall e \in EName, a^{(1)} : X \rightarrow Y, a^{(1)} \circ [\]_X^{(0)} \circ tag_e^{(1)} \equiv [\]_Y^{(0)} \circ tag_e^{(1)} : EV_e \rightarrow Y$.

(4)_d *Recovery. The parameter used for throwing an exception may be recovered.*

$$\begin{aligned} & \left(\forall f^{(1)}, g^{(1)} : X \rightarrow \mathbb{O}, \left[\right]_Y^{(0)} \circ f^{(1)} \equiv \left[\right]_Y^{(0)} \circ g^{(1)} \implies f^{(1)} \equiv g^{(1)} \right) \implies \\ & \left(\forall e \in \text{EName}, u_1^{(0)}, u_2^{(0)} : X \rightarrow \text{EV}_e, \left(\left[\right]_Y^{(0)} \circ \text{tag}_e^{(1)} \circ u_1^{(0)} \equiv \left[\right]_Y^{(0)} \circ \text{tag}_e^{(1)} \circ u_2^{(0)} \right) \implies u_1^{(0)} \equiv u_2^{(0)} \right). \end{aligned}$$

(5)_d *Try. The strong equation is compatible with try/catch.*

$$\begin{aligned} & \forall e \in \text{EName}, a_1^{(1)}, a_2^{(1)} : X \rightarrow Y, b^{(1)} : \text{EV}_e \rightarrow Y, a_1^{(1)} \equiv a_2^{(1)} \implies \\ & \left(\downarrow \left(\left[\text{id}_Y \mid b \circ \text{untag}_e \right]_1 \circ \text{in}_1 \circ \left[\right]_Y \circ \text{tag}_e \circ a_1 \right)^{(1)} \equiv \downarrow \left(\left[\text{id}_Y \mid b \circ \text{untag}_e \right]_1 \circ \text{in}_1 \circ \left[\right]_Y \circ \text{tag}_e \circ a_2 \right)^{(1)} \right). \end{aligned}$$

(6)_d *Try₀. Pure code inside try never triggers the code inside catch.*

$$\begin{aligned} & \forall e \in \text{EName}, u^{(0)} : X \rightarrow Y, b^{(1)} : \text{EV}_e \rightarrow Y, \\ & \downarrow \left(\left[\text{id}_Y \mid b \circ \text{untag}_e \right]_1 \circ \text{in}_1 \circ u \right)^{(1)} \equiv \text{id}_Y^{(0)} \circ u^{(0)} : X \rightarrow Y. \end{aligned}$$

(7)_d *Try₁. The code inside catch is executed as soon as an exception is thrown inside try.*

$$\begin{aligned} & \forall e \in \text{EName}, u^{(0)} : X \rightarrow \text{EV}_e, b^{(1)} : \text{EV}_e \rightarrow Y, \\ & \downarrow \left(\left[\text{id}_Y \mid b \circ \text{untag}_e \right]_1 \circ \text{in}_1 \circ \left[\right]_Y \circ \text{tag}_e \circ u \right)^{(1)} \equiv b^{(1)} \circ u^{(0)} : X \rightarrow Y. \end{aligned}$$

(8)_d *Try₂. An exception gets propagated, if the exception name is not pattern matched in catch.*

$$\begin{aligned} & \forall (e \neq f) \in \text{EName}, u^{(0)} : X \rightarrow \text{EV}_f, b^{(1)} : \text{EV}_e \rightarrow Y, \\ & \downarrow \left(\left[\text{id}_Y \mid b \circ \text{untag}_e \right]_1 \circ \text{in}_1 \circ \left[\right]_Y \circ \text{tag}_f \circ u \right)^{(1)} \equiv \left[\right]_Y^{(0)} \circ \text{tag}_f^{(1)} \circ u^{(0)} : X \rightarrow Y. \end{aligned}$$

These are the archetype properties that we have proved within the scope of the \mathcal{L}_{exc} . To see these proofs, check out (11, §6.7). Besides, we have implemented the \mathcal{L}_{exc} in Coq to certify mentioned proofs. Section 5.2 briefly discusses this implementation.

5.2 \mathcal{L}_{exc} in Coq

Coq implementation of \mathcal{L}_{exc} follows the same approach with the one for \mathcal{L}_{st} . We represent the set of exception names by a Coq parameter $\text{EName} : \text{Type}$. An arrow type $\text{EVal} : \text{EName} \rightarrow \text{Type}$ is assumed as the type of values (parameters) for each exception name. We then inductively define terms and, assign decorations over them. There, we respectively use keywords `epure`, `ppg` and `ctc` instead of (0), (1) and (2). The rules up to weak and strong equalities are stated in a mutually inductive way to write constructors including both types of equalities, similar to the approach presented in Section 4.2. We choose not to replay the entire Coq encoding here, but we formalize Definitions 5.1 and 5.2 in Coq terms as follows:

```
Definition throw (X: Type) (e: EName)  $\triangleq$  (@empty X)  $\circ$  tag e.
Definition try_catch (X Y: Type) (e: EName) (f: term Y X) (g: term Y (Val e))  $\triangleq$ 
  downcast (copair (@id Y) (g  $\circ$  untag e)  $\circ$  in1  $\circ$  f).
```

The encodings of other terms are contained in [this file](#).

We can conclude that such a framework allows us to express and prove, in Coq, the decorated versions of the properties mentioned in Section 5.1. E.g., the propagator-propagates looks like:

```
(** Propagator propagates **)
Lemma PPT:  $\forall X Y (e: \text{EName}) (a: \text{term } Y X), \text{is\_ppg } a \rightarrow a \circ ((\text{empty } X) \circ \text{tag } e) == (\text{empty } Y) \circ \text{tag } e.$ 
```

The full Coq proofs of such properties can be found [here](#), and the entire implementation [there](#).

6 Combining \mathcal{L}_{st} and \mathcal{L}_{exc}

In order to formally cope with different computational effects, one needs to compose the related formal models. For instance, using monad transformers (18), it is usually possible to combine effects formalized by monads, as encoded in Haskell. Handler compositions allow combining effects modelled by algebraic handlers, as implemented in Eff (1, 2, 32) and in Idris (4). To combine effects formalized in decorated settings, we just need to compose the related logics. In this section, we formally study the combination of the state and the exception effects using the logics \mathcal{L}_{st} and \mathcal{L}_{exc} . We call the newly born logic *the decorated logic for the state and the exception*, and denote it \mathcal{L}_{st+exc} . To start with, we give below the syntax of \mathcal{L}_{st+exc} :

Grammar of the decorated logic for the state and the exception: $(i \in \text{Loc}) \quad (e \in \text{EName})$

Types:	t, s	$::=$	$X \mid Y \mid \dots \mid t \times s \mid t + s \mid \mathbb{1} \mid \mathbb{0} \mid V_i \mid EV_e$
Decoration for terms:	(d_1, d_2)	$::=$	$(0, 0) \mid (0, 1) \mid (0, 2) \mid (1, 0) \mid (1, 1) \mid$ $(1, 2) \mid (2, 0) \mid (2, 1) \mid (2, 2)$
Terms:	f, g	$::=$	$a^{(d_1, d_2)} \mid b^{(d_1, d_2)} \mid \dots \mid g \circ f^{(d_1, d_2)} \mid$ $\langle f, g \rangle_1^{(d_1, d_2)} \mid [f \mid g]_1^{(d_1, d_2)} \mid$ $\text{lookup}_i^{(1, 0)} \mid \text{update}_i^{(2, 0)} \mid$ $\text{tag}_e^{(0, 1)} \mid \text{untag}_e^{(0, 2)} \mid$ $(\downarrow f)^{(0, 1)} \mid (\text{tpure } \bullet)^{(0, 0)}$
Equations:	eq	$::=$	$f^{(d_1, d_2)} \equiv \equiv g^{(d_1, d_2)} \mid f^{(d_1, d_2)} \equiv \sim g^{(d_1, d_2)} \mid$ $f^{(d_1, d_2)} \sim \equiv g^{(d_1, d_2)} \mid f^{(d_1, d_2)} \sim \sim g^{(d_1, d_2)}$

Figure 16: \mathcal{L}_{st+exc} : syntax

The decorations are paired off to cover all possible combinations: the decoration symbol on the left is given in terms of the state effect while the one on the right is of the exception. I.e., $f^{(1, 2)}$ says that f may *access* to the state alongside *catching* exceptions. The decoration of a (co)-pair ((co)-product) or a composition depends on the decorations of its components, always taking the larger. I.e., $\forall f^{(1, 2)} : X \rightarrow Y$ and $g^{(2, 1)} : Y \rightarrow Z$, $g \circ f : X \rightarrow Z$ takes the decoration $(2, 2)$.

Types and terms are manually unionized in such a way that the interface terms for the state effect are pure with respect to the exception and vice versa: $\text{lookup}_i^{(1, 0)}$, $\text{update}_i^{(2, 0)}$, $\text{tag}_e^{(0, 1)}$ and $\text{untag}_e^{(0, 2)}$. As in Sections 4 and 5, we use the special tpure constructor to define pure terms such as the identity id , the canonical pair projections π_1 and π_2 , the empty pair $\langle \rangle$, the canonical co-pair inclusions in_1 and in_2 , the

empty co-pair $[\]$ and constants are translated from a pure type system with product and sum types using the tpure constructor, for all types X and Y , as:

$$\begin{array}{llll}
\text{id}_X^{(0,0)} & : & X \rightarrow X & := & \text{tpure } (\lambda x : X. x : X) \\
\pi_1^{(0,0)} & : & X \times Y \rightarrow X & := & \text{tpure fst} \\
\pi_2^{(0,0)} & : & X \times Y \rightarrow Y & := & \text{tpure snd} \\
\langle \rangle_X^{(0,0)} & : & X \rightarrow \mathbb{1} & := & \text{tpure } (\lambda x : X. \text{void} : \mathbb{1}) \\
\text{in}_1^{(0,0)} & : & X \rightarrow X+Y & := & \text{tpure inl} \\
\text{in}_2^{(0,0)} & : & Y \rightarrow X+Y & := & \text{tpure inr} \\
[\]_X^{(0,0)} & : & \mathbb{0} \rightarrow X & := & \text{tpure } (\lambda _ : \mathbb{0}. x : X) \\
\text{constant}_X^{(0,0)} & : & \mathbb{1} \rightarrow X & := & \text{tpure } (\lambda _ . x : X)
\end{array}$$

where fst and snd are constructors of product types while inl and inr are of sum types.

The rule combinations need a bit of reformulation as we summarize below:

- The decoration symbol (0) freely converts into (1) and (2), while the symbol (1) just into (2) when the other symbol is fixed. I.e., $f^{(0,2)}$ freely converts into $f^{(1,2)}$. See all cases below:

$$- \frac{f^{(0,d)}}{f^{(1,d)}}, \frac{f^{(1,d)}}{f^{(2,d)}}, \frac{f^{(d,0)}}{f^{(d,1)}}, \frac{f^{(d,1)}}{f^{(d,2)}} \text{ for } d \in \{0, 1, 2\}$$

- We have all possible combinations of equality sorts: $\equiv\equiv$, $\equiv\sim$, $\sim\equiv$ and $\sim\sim$. The first equality symbol relates terms with respect to the state effect. I.e., $f \equiv\equiv g$ means that f and g are strongly equal with respect to the state, while being weakly equal with respect to the exception. Below we present the conversion rules between these four sorts. The burden here is that a strong equality symbol can always be freely converted into a weak one independent of according to which effect it relates terms. But, to convert a weak equality symbol into a strong one, we need to make sure that the related terms are decorated either with (0) or (1) with respect to the effect they are weakly related.

$$\begin{array}{l}
- (\equiv\equiv\text{-to-}\equiv\sim) \frac{f^{(2,2)} \equiv\equiv g^{(2,2)}}{f \equiv\sim g}, \quad (\equiv\equiv\text{-to-}\sim\equiv) \frac{f^{(2,2)} \equiv\equiv g^{(2,2)}}{f \sim\equiv g} \\
- (\equiv\sim\text{-to-}\sim\sim) \frac{f^{(2,2)} \equiv\sim g^{(2,2)}}{f \sim\sim g}, \quad (\sim\equiv\text{-to-}\sim\sim) \frac{f^{(2,2)} \sim\equiv g^{(2,2)}}{f \sim\sim g} \\
- (\sim\equiv\text{-to-}\equiv\equiv) \frac{f^{(1,2)} \sim\equiv g^{(1,2)}}{f \equiv\equiv g}, \quad (\equiv\sim\text{-to-}\equiv\equiv) \frac{f^{(2,1)} \equiv\sim g^{(2,1)}}{f \equiv\equiv g} \\
- (\sim\sim\text{-to-}\equiv\sim) \frac{f^{(1,2)} \sim\sim g^{(1,2)}}{f \equiv\sim g}, \quad (\sim\sim\text{-to-}\sim\equiv) \frac{f^{(2,1)} \sim\sim g^{(2,1)}}{f \sim\equiv g}
\end{array}$$

- The rules of the logic \mathcal{L}_{st+exc} are presented in Figure 17 as a union of the ones given in Figures 13 and 15 in terms of new equality sorts and refined term decorations. There, we replay the whole rule bodies, and implicitly assume that all equality sorts are equivalence relations respecting the properties *reflexivity*, *symmetry*, and *transitivity*.

Rules of the decorated logic for the state and the exception:

$$\begin{array}{c}
\text{(assoc)} \frac{f^{(2,2)} : X \rightarrow Y \quad g^{(2,2)} : Y \rightarrow Z \quad h^{(2,2)} : Z \rightarrow T}{h \circ (g \circ f) \equiv (h \circ g) \circ f} \quad \text{(ids)} \frac{f^{(2,2)} : X \rightarrow Y}{f \circ \text{id}_X \equiv f} \quad \text{(idt)} \frac{f^{(2,2)} : X \rightarrow Y}{\text{id}_Y \circ f \equiv f} \\
\text{(pwrepl)} \frac{f_1^{(2,2)} \sim \equiv f_2^{(2,2)} : X \rightarrow Y \quad g^{(0,2)} : Y \rightarrow Z}{g \circ f_1 \sim \equiv g \circ f_2} \quad \text{(wsubs)} \frac{g^{(2,2)} : X \rightarrow Y \quad f_1^{(2,2)} \sim \equiv f_2^{(2,2)} : Y \rightarrow Z}{f_1 \circ g \sim \equiv f_2 \circ g} \\
\text{(pwsubs)} \frac{g^{(2,0)} : X \rightarrow Y \quad f_1^{(2,2)} \equiv \sim f_2^{(2,2)} : Y \rightarrow Z}{f_1 \circ g \equiv \sim f_2 \circ g} \quad \text{(wrepl)} \frac{f_1^{(2,2)} \equiv \sim f_2^{(2,2)} : X \rightarrow Y \quad g^{(2,2)} : Y \rightarrow Z}{g \circ f_1 \equiv \sim g \circ f_2} \\
\text{(replsubs)} \frac{f_1^{(2,2)} \equiv \equiv f_2^{(2,2)} : X \rightarrow Y \quad g_1^{(2,2)} \equiv \equiv g_2^{(2,2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv \equiv g_2 \circ f_2} \\
\text{(w_unit)} \frac{f^{(2,2)} : X \rightarrow \mathbb{1}}{f \sim \equiv \langle \rangle_X} \quad \text{(w_empty)} \frac{f^{(2,2)} : \mathbb{0} \rightarrow X}{f \equiv \sim []_X} \quad \text{(w_downcast)} \frac{f^{(2,2)} : Y \rightarrow X}{(\downarrow f)^{(2,1)} \equiv \sim f} \\
\text{(ax}_1) \frac{}{\text{lookup}_i^{(1,0)} \circ \text{update}_i^{(2,0)} \sim \equiv \text{id}_{V_i}^{(0,0)}} \quad \text{(ax}_2) \frac{\forall i, j \in \text{Loc}, i \neq j}{\text{lookup}_i^{(1,0)} \circ \text{update}_j^{(2,0)} \sim \equiv \text{lookup}_i^{(1,0)} \circ \langle \rangle_{V_i}^{(0,0)}} \\
\text{(eax}_1) \frac{}{\text{untag}_e^{(0,2)} \circ \text{tag}_e^{(0,1)} \equiv \sim \text{id}_{E_{V_e}}^{(0,0)}} \quad \text{(eax}_2) \frac{\forall e_1, e_2 \in \text{EName}, e_1 \neq e_2}{\text{untag}_{e_1}^{(0,2)} \circ \text{tag}_{e_2}^{(0,1)} \equiv \sim []_{E_{V_{e_1}}}^{(0,0)} \circ \text{tag}_{e_2}^{(0,1)}} \\
\text{(effect)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : X \rightarrow Y \quad f_1^{(2,2)} \sim \equiv f_2^{(2,2)} \quad \langle \rangle_Y^{(0,0)} \circ f_1^{(2,2)} \equiv \equiv \langle \rangle_Y^{(0,0)} \circ f_2^{(2,2)}}{f_1 \equiv \equiv f_2} \\
\text{(effect)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : Y \rightarrow X \quad f_1^{(2,2)} \equiv \sim f_2^{(2,2)} \quad f_1^{(2,2)} \circ []_Y^{(0,0)} \equiv \equiv f_2^{(2,2)} \circ []_Y^{(0,0)}}{f_1 \equiv \equiv f_2} \\
\text{(local_global)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : X \rightarrow \mathbb{1} \quad \forall i \in \text{Loc}, \text{lookup}_i^{(1,0)} \circ f_1^{(2,2)} \sim \equiv \text{lookup}_i^{(1,0)} \circ f_2^{(2,2)}}{f_1 \equiv \equiv f_2} \\
\text{(elocal_global)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : \mathbb{0} \rightarrow X \quad \forall e \in \text{EName}, f_1^{(2,2)} \circ \text{tag}_e^{(0,1)} \equiv \sim \circ f_2^{(2,2)} \circ \text{tag}_e^{(0,1)}}{f_1 \equiv \equiv f_2} \\
\text{(w_lpair_eq)} \frac{f_1^{(1,2)} : X \rightarrow Y \quad f_2^{(2,2)} : X \rightarrow Z}{\pi_1 \circ \langle f_1, f_2 \rangle_1 \sim \equiv f_1} \quad \text{(s_lpair_eq)} \frac{f_1^{(1,2)} : X \rightarrow Y \quad f_2^{(2,2)} : X \rightarrow Z}{\pi_2 \circ \langle f_1, f_2 \rangle_1 \equiv \equiv f_2} \\
\text{(w_lcopy_eq)} \frac{f_1^{(2,1)} : X \rightarrow Y \quad f_2^{(2,2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_1 \sim \equiv f_1} \quad \text{(s_lcopy_eq)} \frac{f_1^{(2,1)} : X \rightarrow Y \quad f_2^{(2,2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_2 \equiv \equiv f_2} \\
\text{(tcomp)} \frac{f : Y \rightarrow Z \quad g : X \rightarrow Y}{(\text{tpure } f)^{(0,0)} \circ (\text{tpure } g)^{(0,0)} \equiv \equiv (\text{tpure } (f \circ g))^{(0,0)}}
\end{array}$$

Figure 17: \mathcal{L}_{st+exc} : rules**6.1 Decorated properties of the state and exception effects**

The properties given in Sections 4.1 and 5.1 are now stated with the refined term decorations, and related with the \equiv equation. I.e., propagator-propagates and annihilation `untag-untag` look like:

$$\begin{aligned}
\forall e \in \text{EName}, a^{(0,1)} : X \rightarrow Y, a^{(0,1)} \circ [\]_X^{(0,0)} \circ \text{tag}_e^{(0,1)} &\equiv [\]_Y^{(0,0)} \circ \text{tag}_e^{(0,1)} : \text{EV}_e \rightarrow Y. \\
\forall i \neq j \in \text{Loc}, \text{update}_j^{(2,0)} \circ \pi_2^{(0,0)} \circ (\text{update}_i^{(2,0)} \times_r \text{id}_{V_j}^{(0,0)}) &\equiv \\
&\text{update}_i^{(2,0)} \circ \pi_1^{(0,0)} \circ (\text{id}_{V_i}^{(0,0)} \times_l \text{update}_j^{(2,0)}) : V_i \times V_j \rightarrow \mathbb{1}.
\end{aligned}$$

These are the archetype properties that we can prove within the scope of the \mathcal{L}_{st+exc} . However, we prefer not to prove them for this generic framework; instead, we first specialize them in a way to serve as a target language for a denotational semantics of IMP+Exc, and then prove them for the specialized version. Also, we encode the specialized version in Coq and certify related proofs. Section 7 gives the related details.

7 IMP+Exc over the combined decorated logic \mathcal{L}_{st+exc}

Now, it comes to define a denotational semantics for the IMP+Exc language, with the combined decorated logic for the state and the exception (\mathcal{L}_{st+exc}) as the target language. Recall that by doing this, we aim to prove some (strong) equalities between terminating programs written in IMP+Exc with respect to the state and the exception effects.

In IMP+Exc, the values that can be stored in any location (variable) i are just integers. So that any occurrence of (V_i) in term signatures of \mathcal{L}_{st+exc} is replaced by \mathbb{Z} . I.e., $\text{lookup}^{(1,0)} : \mathbb{1} \rightarrow \mathbb{Z}$ while \mathbb{Z} . I.e., $\text{update}^{(2,0)} : \mathbb{Z} \rightarrow \mathbb{1}$. Here, we start with defining a denotational semantics of IMP+Exc expressions over combined decorated settings using a translator function dExp . This function takes an expression as input and outputs a decorated term of type $\text{term } \mathbb{Z} \ \mathbb{1}$ or $\text{term } \mathbb{B} \ \mathbb{1}$ depending on the input expression type:

$$\begin{aligned}
\text{dExp } n &\Rightarrow (\text{constant } n)^{(0,0)} \\
\text{dExp } x &\Rightarrow (\text{lookup } x)^{(1,0)} \\
\text{dExp } (a_1 + a_2) &\Rightarrow (\text{tpure add})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (a_1 \times a_2) &\Rightarrow (\text{tpure mlt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (a_1 - a_2) &\Rightarrow (\text{tpure subt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (\text{true}) &\Rightarrow (\text{constant true})^{(0,0)} \\
\text{dExp } (\text{false}) &\Rightarrow (\text{constant false})^{(0,0)} \\
\text{dExp } (a_1 \stackrel{?}{=} a_2) &\Rightarrow (\text{tpure chkeq})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (a_1 \stackrel{?}{\neq} a_2) &\Rightarrow (\text{tpure chkneq})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (a_1 \stackrel{?}{>} a_2) &\Rightarrow (\text{tpure chkggt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (a_1 \stackrel{?}{<} a_2) &\Rightarrow (\text{tpure chklt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\
\text{dExp } (b_1 \wedge b_2) &\Rightarrow (\text{tpure andB})^{(0,0)} \circ \langle \text{dExp } b_1, \text{dExp } b_2 \rangle_1^{(1,0)}
\end{aligned}$$

$$\begin{aligned} \text{dExp } (b_1 \vee b_2) &\Rightarrow (\text{tpure orB})^{(0,0)} \circ \langle \text{dExp } b_1, \text{dExp } b_2 \rangle_1^{(1,0)} \\ \text{dExp } (\neg b) &\Rightarrow (\text{tpure notB})^{(0,0)} \circ \text{dExp } b^{(1,0)} \end{aligned}$$

where *true* and *false* are Boolean constructors. The constructor *tpure* is applied to given unary functions. For instance $\text{add} : (\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathbb{Z}$ takes an instance of an integer tuple and returns their sum. To see the definition of the other functions in a Coq implementation, please check out [this file](#).

We have some additional rules to make use of some pure algebraic operations in the combined decorated setting presented in Figure 18 where the pure term $\text{lpi} : \mathbb{1} \rightarrow \mathbb{1}$, within the rule (*imp-li*), is used to connect successive loop iterations as long as the loop conditional evaluates into decorated logic's *true* (*constant true*). Also, the pure term $\text{pbl} : \mathbb{B} \rightarrow \mathbb{1} + \mathbb{1}$ forms a bridge between the usual Boolean data type and its correspondence in the decorated settings which is the type $\mathbb{1} + \mathbb{1}$.

$$\begin{aligned} \text{lpi } (b : \text{term } \mathbb{1} (\mathbb{1} + \mathbb{1})) (f : \text{term } \mathbb{1} \mathbb{1}) &:= \text{tpure } (\lambda x : \mathbb{1}. x). \\ \text{pbl} &:= \text{tpure } (\text{bool_to_two}) \\ \text{where } \text{bool_to_two } (b : \text{bool}) &:= (\text{if } b \text{ then } (\text{inl void}) \text{ else } (\text{inr void})). \\ \text{such that } \text{void} : \mathbb{1} &\text{ is the unique constructor of the type } \mathbb{1}, \text{ and} \\ \text{inl, inr} : \mathbb{1} &\rightarrow (\mathbb{1} + \mathbb{1}) \text{ are the canonical inclusions} \end{aligned}$$

Note also that in (*imp₂*) and (*imp₄*) by replacing *false* into *true* we get (*imp₃*) and (*imp₅*) that are not explicitly stated in Figure 18.

$$\begin{aligned} (\text{imp}_1) &\frac{\forall p, q : \mathbb{Z}, (f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z})}{\text{tpure } f \circ \langle \text{constant } p, \text{constant } q \rangle_1 \equiv \equiv (\text{constant } f(p, q))} \\ (\text{imp}_2) &\frac{\forall p, q : \mathbb{Z}, (f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{B}) \ f(p, q) = \text{false}}{\text{tpure } f \circ \langle \text{constant } p, \text{constant } q \rangle_1 \equiv \equiv \text{constant } \text{false}} \\ (\text{imp}_4) &\frac{\forall p, q : \mathbb{B}, (f : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}) \ f(p, q) = \text{false}}{\text{tpure } f \circ \langle \text{constant } p, \text{constant } q \rangle_1 \equiv \equiv \text{constant } \text{false}} \\ (\text{imp-li}) &\frac{\forall (b : \text{term } \mathbb{1} (\mathbb{1} + \mathbb{1})) (f : \text{term } \mathbb{1} \mathbb{1})}{\text{lpi } b \ f \equiv \equiv [(\text{lpi } b \ f) \circ f | \text{id}]_1 \circ b} \\ (\text{imp}_6) &\frac{(\forall x, f \ x = g \ x)}{\text{tpure } f \equiv \equiv \text{tpure } g} \end{aligned}$$

Figure 18: Additional rules on pure terms: IMP+Exc specific

Lemma 7.1. $\text{pbl}^{(0,0)} \circ (\text{constant } \text{false})^{(0,0)} \equiv \equiv \text{in}_2$.

Proof: unfolding all term definitions, we have $\text{tpure } (\lambda : \text{bool}. \text{if } b \text{ then } (\text{inl } \mathbb{1}) \text{ else } (\text{inr } \mathbb{1})) \circ \text{tpure } (\lambda _ : \mathbb{1}. \text{true}) \equiv \equiv \text{tpure } \text{inl}$. Now, it is trivial to obtain $\text{tpure } \text{inl} \equiv \equiv \text{tpure } \text{inl}$ by first rewriting *tcomp* from left to right, and then applying *imp₆*. \square

Lemma 7.2. $\text{pbl}^{(0,0)} \circ (\text{constant } \text{true})^{(0,0)} \equiv \equiv \text{in}_1$.

Proof: It follows the same steps with the proof of Lemma 7.1 \square

Remark 7.3. See [this file](#) for the Lemmata 7.1 and 7.2 statements with their proofs in a Coq implementation.

The fact that IMP+Exc commands are of type $\mathbb{1} \rightarrow \mathbb{1}$, and since $\text{throw}_e^{(0,1)} := [\]_Y^{(0,0)} \circ \text{tag}_e^{(0,1)} : \text{EV}_e \rightarrow Y$ is a command, we replace EV_e and Y with $\mathbb{1}$. This means that we stick to a single exceptional value (parameter), for each exception name $e \in \text{EName}$.

Below, we recursively define the IMP+Exc commands within \mathcal{L}_{st+exc} using a translator function dCmd which establishes a decorated term of type $\text{term } \mathbb{1} \ \mathbb{1}$ out of an input command. The translation of any IMP+Exc command cannot be a public catcher, even the one for TRY/CATCH; recall Definition 5.2. That is why we have at most decoration (1) with respect to the exception.

$$\begin{aligned}
\text{dCmd (SKIP)} &\Rightarrow (\text{id } \mathbb{1})^{(0,0)} \\
\text{dCmd (x } \triangleq \text{ a)} &\Rightarrow (\text{update}_x)^{(2,0)} \circ (\text{dExp a})^{(1,0)} \\
\text{dCmd (c}_1; \text{c}_2) &\Rightarrow (\text{dCmd c}_2)^{(2,1)} \circ (\text{dCmd c}_1)^{(2,1)} \\
\text{dCmd (if b then c}_1 \text{ else c}_2) &\Rightarrow \left[\text{dCmd c}_1 \mid \text{dCmd c}_2 \right]_1^{(2,1)} \circ \text{pbl}^{(0,0)} \circ (\text{dExp b})^{(1,0)} \\
\text{dCmd (while b do c)} &\Rightarrow \left[(\text{lpi (pbl } \circ (\text{dExp b})) (\text{dCmd c})) \circ (\text{dCmd c}) \mid \text{id} \right]_1^{(2,1)} \\
&\quad \circ \text{pbl}^{(0,0)} \circ (\text{dExp b})^{(1,0)} \\
\text{dCmd (THROW e)} &\Rightarrow \text{throw e}^{(0,1)} \\
\text{dCmd (TRY c}_1 \text{ CATCH e } \Rightarrow \text{c}_2) &\Rightarrow \text{try (dCmd c}_1) \text{ catch (e } \Rightarrow (\text{dCmd c}_2))^{(2,1)}
\end{aligned}$$

In Figure 19, the diagram on the left schematizes the command `if b then c1 else c2`: if the Boolean expression `dExp b` evaluates into (constant *true*) then by Lemma 7.2, we have the command `c1` in execution, `c2` otherwise by Lemma 7.1. As for the loops, it is well known that as long as the looping condition evaluates into (constant *true*), loop body gets executed. This is depicted in Figure 19 (the diagram on the right), as the arrow `lpi b c` is each time replaced by the whole diagram itself. This is made possible by the rule (`imp-li`). If the looping condition evaluates into (constant *false*), using Lemma 7.1, we then have the term `id1` in execution forcing the loop to terminate. Notice that the case distinction in the diagrams are provided by the term inclusions.

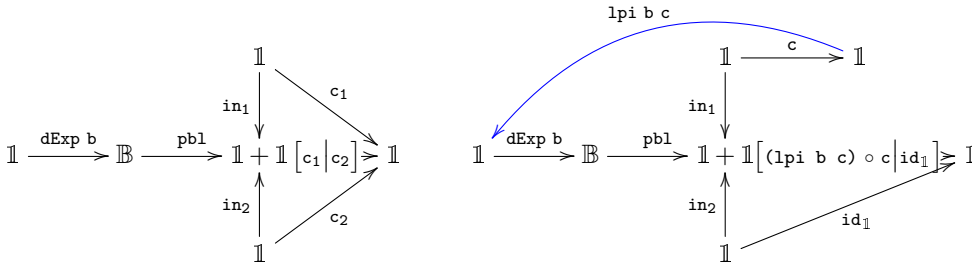


Figure 19: (`cond b c1 c2`) and (`while b do c`) in \mathcal{L}_{st+exc}

Figure 20 respectively visualizes the formal behaviors of THROW and TRY/CATCH commands where the basis is the core decorated terms for the exception effect. They are formulated as in Definitions 5.1 and 5.2 with a single difference in their signatures: domains and co-domains are now set to $\mathbb{1}$.

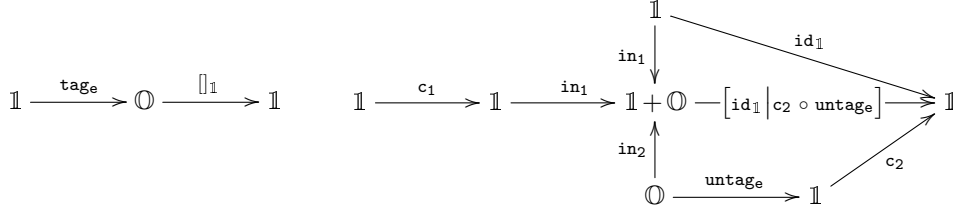


Figure 20: (THROW e) and (TRY c_1 CATCH $e \Rightarrow c_2$) in \mathcal{L}_{st+exc}

We now implement the IMP+Exc denotational semantics using the \mathcal{L}_{st+exc} as the target language in a Coq encoding. Expressions are inductively forming a new Coq Type by taking another Coq Type as input, and called Exp. Indeed, the type Exp is a dependent type. This means that the type Exp A depends on the term A : Type. I.e., when $A := \mathbb{B}$, we build the type for Boolean expressions while the case $A := \mathbb{Z}$ is to construct the type for arithmetic expressions. As for the type constructors, we use the syntactic operators given within the contexts of aexp and bexp in Figure 2. The only difference is in the constructor names: instead of the corresponding notations, they are given in plain text. However, it is easy to match them one another since they are given in the same order. I.e., ttrue is used instead of the true expression symbol. Note also that true stands for the Boolean constructor.

```

Inductive Exp : Type → Type  $\triangleq$ 
| const :  $\forall A, A \rightarrow \text{Exp } A$ 
| var    : Loc → Exp Z
| plus   : Exp Z → Exp Z → Exp Z
| subtr  : Exp Z → Exp Z → Exp Z
| mult   : Exp Z → Exp Z → Exp Z
| ttrue  : Exp bool
| tfalse : Exp bool
| eq     : Exp Z → Exp Z → Exp bool
| neq    : Exp Z → Exp Z → Exp bool
| gt     : Exp Z → Exp Z → Exp bool
| lt     : Exp Z → Exp Z → Exp bool
| ge     : Exp Z → Exp Z → Exp bool
| le     : Exp Z → Exp Z → Exp bool
| and    : Exp bool → Exp bool → Exp bool
| or     : Exp bool → Exp bool → Exp bool
| neg    : Exp bool → Exp bool.

```

Let us interpret the dExp function in Coq using the following fix-point:

```

Fix-point dExp A (e : Exp A) : term A unit  $\triangleq$ 
match e with
| const Z n     $\Rightarrow$  constant n
| var x         $\Rightarrow$  lookup x
| plus a1 a2    $\Rightarrow$  tpure add o pair (dExp Z a1) (dExp Z a2)
| subtr a1 a2   $\Rightarrow$  tpure subt o pair (dExp Z a1) (dExp Z a2)
| mult a1 a2    $\Rightarrow$  tpure mlt o pair (dExp Z a1) (dExp Z a2)
| ttrue         $\Rightarrow$  constant true

```

```

| ffalse      ⇒ constant false
| eq a1 a2    ⇒ tpure chkeq o pair (dExp Z a1) (dExp Z a2)
| neq a1 a2   ⇒ tpure chkneq o pair (dExp Z a1) (dExp Z a2)
| gt a1 a2    ⇒ tpure chkgt o pair (dExp Z a1) (dExp Z a2)
| lt a1 a2    ⇒ tpure chklt o pair (dExp Z a1) (dExp Z a2)
| ge a1 a2    ⇒ tpure chkge o pair (dExp Z a1) (dExp Z a2)
| le a1 a2    ⇒ tpure chkle o pair (dExp Z a1) (dExp Z a2)
| and b1 b2   ⇒ tpure andB o pair (dExp bool b1) (dExp bool b2)
| or b1 b2    ⇒ tpure orB o pair (dExp bool b1) (dExp bool b2)
| neg b       ⇒ tpure notB o (dExp bool b)
end.

```

A similar idea of implementation follows for the commands. We inductively define a Coq type `Cmd` of IMP+Exc commands whose constructors are the members of IMP+Exc command set as presented in Figures 2 and 6. Notice that some commands are encoded with different names. I.e., the assignment command ‘ \triangleq ’ is called `assign`, the sequencing command ‘`;`’ is called `sequence` while “if then else” block is named `cond` in the implementation. However, it is easy to match them one another since they are presented in the same order.

```

Inductive Cmd : Type ≜
| SKIP      : Cmd
| sequence  : Cmd → Cmd → Cmd
| assign    : Loc → Exp Z → Cmd
| cond      : Exp bool → Cmd → Cmd → Cmd
| while     : Exp bool → Cmd → Cmd
| THROW     : EName → Cmd
| TRY_CATCH : EName → Cmd → Cmd → Cmd.

```

We now interpret the `dCmd` function in Coq using the below fix-point:

```

Fixpoint dCmd (c: Cmd): (term unit unit) ≜
match c with
| skip      ⇒ (@id unit)
| sequence c0 c1 ⇒ (dCmd c1) o (dCmd c0)
| assign j e0 ⇒ (update j) o (dExp Z e0)
| cond b c2 c3 ⇒ copair (dCmd c2) (dCmd c3) o (pb1 o (dExp bool b))
| while b c4 ⇒ (copair (lpi (pb1 o (dExp bool b)) (dCmd c4) o (dCmd c4)) (@id unit)) o (pb1 o (dExp bool b))
| THROW e ⇒ (throw unit e)
| TRY_CATCH e c1 c2 ⇒ (try_catch e (dCmd c1) (dCmd c2))
end.

```

Now, we retain sufficient material to state and prove equivalences between programs written in IMP+Exc, and certify such proofs in Coq.

7.1 Program equivalence proofs

In this section, we finally prove equivalences of bunch of programs written in IMP+Exc, using the denotational semantics characterized within the scope of the logic \mathcal{L}_{st+exc} . Note that for the sake of simplicity, we will use u_x , l_x , $(t \text{ op})$ and $(c \text{ p})$ instead of $(\text{update } x)^{(2,0)}$, $(\text{lookup } x)^{(1,0)}$, $(\text{tpure op})^{(0,0)}$ and $(\text{constant p})^{(0,0)}$, respectively.

Remark 7.4. Recall that the use of term products is to impose some order of term evaluation on the mutable state. IMP+Exc specific properties of the mutable state are slightly different than their generic versions

(mentioned in Section 4.1) due to the fact that the language does not allow parallel term evaluations, meaning that every term is evaluated in the sequence they are given. Therefore, we no more need to use term products in property statements. The properties we use through out the following proofs are re-stated in Figure 21. The full certified Coq proofs of these properties can be found [here](#).

1. **interaction update-update** $\forall x \in \text{Loc } p, q : \mathbb{Z}, u_x \circ (c p) \circ u_x \circ (c q) \equiv u_x \circ (c p)$
2. **commutation update-update** $\forall x \neq y \in \text{Loc } p, q : \mathbb{Z}, u_x \circ (c p) \circ u_y \circ (c q) \equiv u_y \circ (c q) \circ u_x \circ (c p)$
3. **commutation-lookup-constant-update** $\forall x \in \text{Loc}, p, q \in \mathbb{Z}, \langle l_x, (c q) \rangle_1 \circ u_x \circ (c p) \equiv \langle (c p), (c q) \rangle_1 \circ u_x \circ (c p)$

Figure 21: Primitive properties of the state: IMP+Exc specific

Remark 7.5. Below, we state three lemmata using the IMP+Exc notation introduced in Figures 2 and 6. However, we introduce a new set of notations for the Coq encoding to increase the readability score even a little: browse this set of notations [here](#) where, i.e., the assign command is denoted by ‘::=’ while the sequence command by ‘;’. These notations do not appear through out the paper, but might be of help in reading the lemma statements in the Coq encoding. Notice also that they are not so pretty, due to the fact that Coq internally reserves prettier notations for other issues.

Lemma 7.6. *For all exceptionally pure commands $f, g \in \text{cmd} \setminus \{\text{THROW}, \text{TRY/CATCH}\}$ and $b \in \{\text{true}, \text{false}\}$, if program pieces prog1 and prog2 are given as in the following listings, then $\text{dCmd}(\text{prog1}) \equiv \text{dCmd}(\text{prog2})$.*

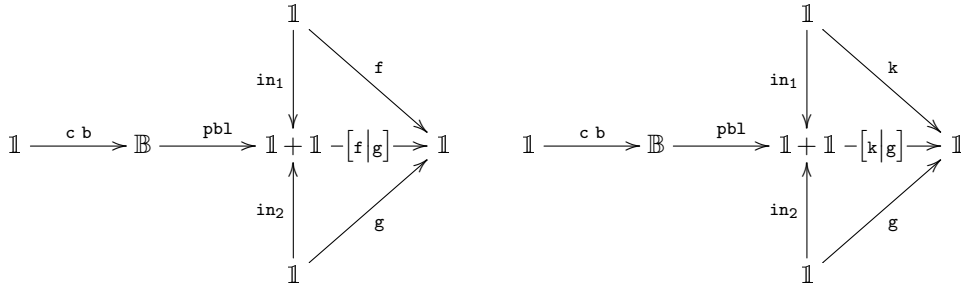
Listing 1: prog1

```
/* prog1 */
if b then f else g;
```

Listing 2: prog2

```
/* prog2 */
if b then (if b then f else g)
else g;
```

Proof: We sketch the diagrams of both programs below:



where $k = (\text{if } b \text{ then } f \text{ else } g)$. The statement we would like to prove is

$$[f|g]_1 \circ \text{pb1} \circ c b \equiv [k|g]_1 \circ \text{pb1} \circ c b. \quad (1)$$

Using the decorated rules of the logic \mathcal{L}_{st+exc} , in the below given order, our aim is to simplify both sides of the statement into the same shape with respect to the equality sort \equiv . The proof proceeds by a case analysis on b .

If $b = \text{false}$, by unfolding the definitions of `pbl` and (*c false*), we have

$$[f|g]_1 \circ t (\text{bool_to_two}) \circ t (\lambda x : \text{unit}.\text{false}) \equiv \equiv [k|g]_1 \circ t (\text{bool_to_two}) \circ t (\lambda x : \text{unit}.\text{false}). \quad (2)$$

We rewrite (`tcomp`), and get

$$[f|g]_1 \circ t (\lambda x : \text{unit}.\text{bool_to_two } \text{false}) \equiv \equiv [k|g]_1 \circ t (\lambda x : \text{unit}.\text{bool_two } \text{false}). \quad (3)$$

Now, we cut

$$t (\lambda x : \text{unit}.\text{bool_to_two } \text{false}) \equiv \equiv \text{in}_2 \quad (4)$$

and rewrite it back in the goal. So that we obtain

$$[f|g]_1 \circ \text{in}_2 \equiv \equiv [k|g]_1 \circ \text{in}_2. \quad (5)$$

Then, we use (`s_lcopair_eq`), and finally have $g \equiv \equiv g$ which is trivial since $\equiv \equiv$ is reflexive. It remains to show that the cut statement in Equation 4 is true. By simplifying $t (\lambda x : \text{unit}.\text{bool_to_two } \text{false})$ and unfolding `in2`, we have

$$t (\lambda x : \text{unit}.\text{inr } x) \equiv \equiv t (\text{inr}). \quad (6)$$

Now, we apply (`imp6`) and get

$$\forall x : \text{unit}, \text{inr } x = \text{inr } x \quad (7)$$

which is trivial since the Leibniz equality '=' is reflexive.

If $b = \text{true}$, by following above procedure with `true` (instead of `false`) we first handle

$$[f|g]_1 \circ \text{in}_1 \equiv \equiv [k|g]_1 \circ \text{in}_1 \quad (8)$$

and then freely convert $\equiv \equiv$ into $\equiv \sim$. There, rewriting the rule (`w_lcopair_eq`) yields $f \equiv \sim k$. We unfold `k` with $b = \text{true}$ and get

$$f \equiv \sim [f|g]_1 \circ \text{in}_1. \quad (9)$$

Now by rewriting (`w_lcopair_eq`), we have $f \equiv \sim f$, which is again trivial, since $\equiv \sim$ is reflexive. \square

Lemma 7.7. *For all $x : \text{Loc}$, if program pieces `prog3` and `prog4` are given as in the following listings, then $\text{dCmd } (\text{prog3}) \equiv \equiv \text{dCmd } (\text{prog4})$.*

Listing 3: `prog3`

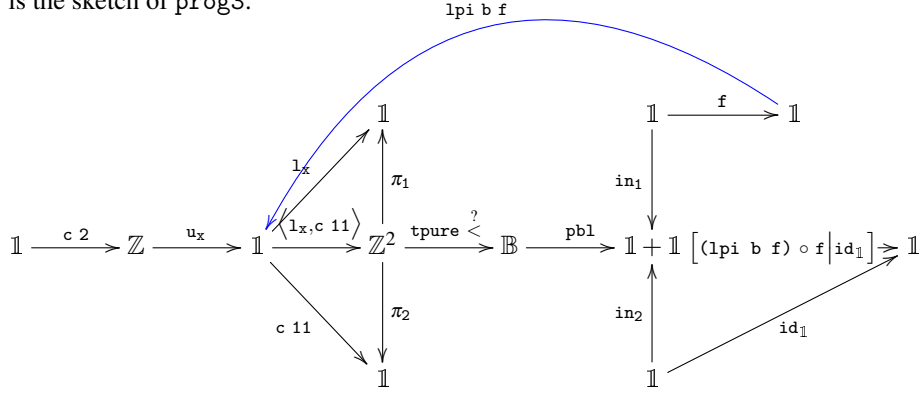
```
/* prog3 */
x  $\triangleq$  2;
while (x < 11)
  do (x  $\triangleq$  x + 4);
```

Listing 4: `prog4`

```
/* prog4 */
x  $\triangleq$  14;
```

Proof: In the proof structure we intend to reduce `prog3`, first dealing with the pre-loop assignments and the looping pre-condition. Since it evaluates into *true*, in the second step we identify things related to the first loop iteration. The third step primarily studies the second and then the third loop iterations after which the looping pre-condition switches to *false*. Finally, we explain the program termination and show that `prog3` does exactly the same state manipulation with `prog4`. Note also that we do not need to check the results they returned, since all `IMP+Exc` commands, thus programs, return `void : U`.

Below is the sketch of prog3:



where $f = (x \triangleq x + 4)$ and $b = (x < 11)$. Using the decorated rules of the logic \mathcal{L}_{st+exc} , given in the below order, we simplify this diagram into the one given below with respect to the equality sort $\equiv\equiv$:

$$1 \xrightarrow{c\ 14} Z \xrightarrow{u_x} 1$$

which is actually prog4 when sketched.

1. Initially, we have

$$[(lpi\ b\ f) \circ f | id_1] \circ pbl \circ (t < ?) \circ \langle l_x, (c\ 11) \rangle \circ u_x \circ (c\ 2) \equiv\equiv u_x \circ (c\ 14). \quad (10)$$

Let us simplify it as far as possible. By rewriting commutation – lookup – constant – update (see Figure 21), we obtain

$$[(lpi\ b\ f) \circ f | id_1] \circ pbl \circ (t < ?) \circ \langle (c\ 2), (c\ 11) \rangle \circ u_x \circ (c\ 2) \equiv\equiv u_x \circ (c\ 14). \quad (11)$$

Since the looping pre-condition $(t < ?) \circ \langle (c\ 2), (c\ 11) \rangle$ evaluates into $(c\ true)$, due to (imp_3) , we have

$$[(lpi\ b\ f) \circ f | id_1] \circ pbl \circ (c\ true) \circ u_x \circ (c\ 2) \equiv\equiv u_x \circ (c\ 14). \quad (12)$$

By rewriting the Lemma 7.2, we get

$$[(lpi\ b\ f) \circ f | id_1] \circ in_1 \circ u_x \circ (c\ 2) \equiv\equiv u_x \circ (c\ 14). \quad (13)$$

Here, we first convert $\equiv\equiv$ into $\equiv\sim$ then rewrite $(w_lco\ pair_eq)$, and end up with

$$(lpi\ b\ f) \circ f \circ u_x \circ (c\ 2) \equiv\sim u_x \circ (c\ 14) \quad (14)$$

in which the second appearance of f unfolds into

$$(lpi\ b\ f) \circ u_x \circ (t +) \circ \langle l_x, c\ 4 \rangle \circ u_x \circ (c\ 2) \equiv\sim u_x \circ (c\ 14). \quad (15)$$

Since, there is no exceptional case, we are freely back to \equiv . By rewriting `commutation -lookup -constant -update`, we obtain

$$(\text{lpi } b \text{ f}) \circ u_x \circ (t +) \circ \langle c \ 2, c \ 4 \rangle \circ u_x \circ (c \ 2) \equiv u_x \circ (c \ 14). \quad (16)$$

The rule `(imp1)` gives

$$(\text{lpi } b \text{ f}) \circ u_x \circ (c \ 6) \circ u_x \circ (c \ 2) \equiv u_x \circ (c \ 14). \quad (17)$$

Now, we rewrite the lemma `interaction-update-update` (see Figure 21) and get

$$(\text{lpi } b \text{ f}) \circ u_x \circ (c \ 6) \equiv u_x \circ (c \ 14). \quad (18)$$

2. For the second loop iteration, rewriting `(imp-li)` gives

$$[(\text{lpi } b \text{ f}) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \stackrel{?}{<}) \circ \langle l_x, (c \ 11) \rangle \circ u_x \circ (c \ 6) \equiv u_x \circ (c \ 14). \quad (19)$$

where looping pre-condition evaluates into $(c \ \text{true})$. Therefore, we iterate the above procedure, given in the step 1, once again and derive

$$(\text{lpi } b \text{ f}) \circ u_x \circ (c \ 10) \equiv u_x \circ (c \ 14). \quad (20)$$

3. In the third iteration, rewriting the `(imp-li)` gives

$$[(\text{lpi } b \text{ f}) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \stackrel{?}{<}) \circ \langle l_x, (c \ 11) \rangle \circ u_x \circ (c \ 10) \equiv u_x \circ (c \ 14). \quad (21)$$

As in step 2, the looping pre-condition evaluates into $(c \ \text{true})$ forcing us to reiterate the above procedure, given in the step 1, which results in

$$(\text{lpi } b \text{ f}) \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (22)$$

4. In the fourth step, rewriting the `(imp-li)` gives

$$[(\text{lpi } b \text{ f}) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \stackrel{?}{<}) \circ \langle l_x, (c \ 11) \rangle \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (23)$$

By rewriting `commutation -lookup -constant -update`, we obtain

$$[(\text{lpi } b \text{ f}) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \stackrel{?}{<}) \circ \langle (c \ 14), (c \ 11) \rangle \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (24)$$

Finally here, the looping pre-condition $(t \stackrel{?}{<}) \circ \langle (c \ 14), (c \ 11) \rangle$ evaluates into $(c \ \text{false})$ yielding

$$[(\text{lpi } b \text{ f}) \circ f | \text{id}_1] \circ \text{pbl} \circ (c \ \text{false}) \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (25)$$

We rewrite the Lemma 7.1, and get

$$[(\text{lpi } b \text{ f}) \circ f | \text{id}_1] \circ \text{in}_2 \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (26)$$

Now, we rewrite `(s_lcopair_eq)`, and handle

$$\text{id}_1 \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14) \quad (27)$$

which is trivial, since the identity term disappears when to compose and \equiv is reflexive. \square

Lemma 7.8. For each $x\ y : \text{Loc}$, $e : \text{EName}$, if program pieces prog5 and prog6 are given as in the following listings, then $\text{dCmd}(\text{prog5}) \equiv \equiv \text{dCmd}(\text{prog6})$.

Listing 5: prog5

```

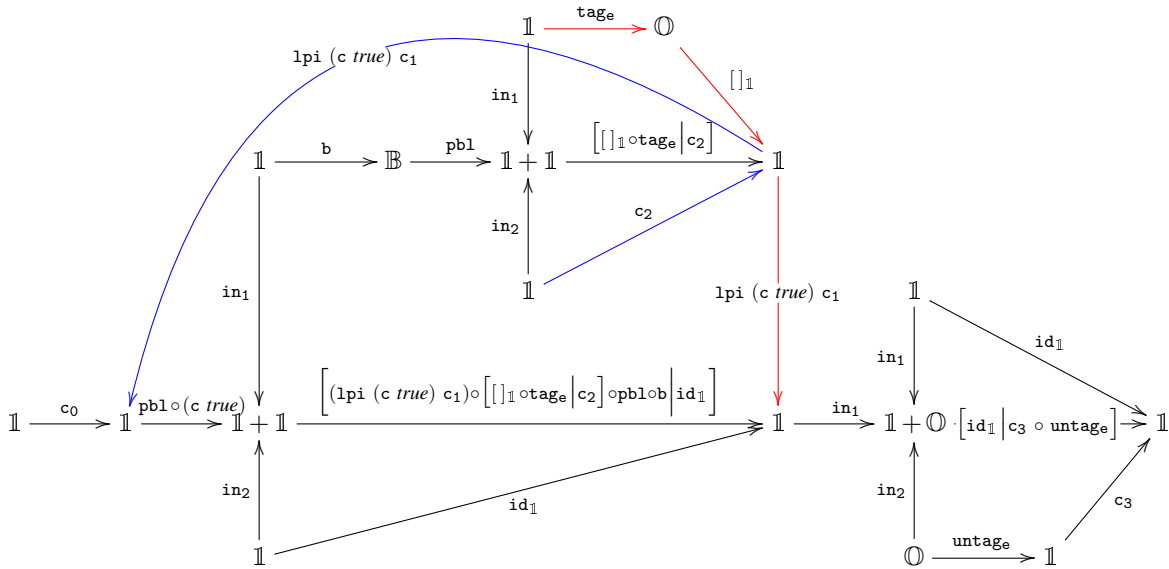
/* prog3 */
x  $\triangleq$  1;
y  $\triangleq$  20;
TRY(
  while (true)
  do
    ( if (x  $\leq$  0) then (THROW e)
      else x  $\triangleq$  x - 1
    )
) CATCH e  $\Rightarrow$  (y  $\triangleq$  7);
    
```

Listing 6: prog6

```

/* prog4 */
x  $\triangleq$  0;
y  $\triangleq$  7;
    
```

Proof: In the proof structure, we first tackle with the downcast operator. The second task is to deal with the first loop iteration which has the state but no exception effect. In the third, we study the second iteration of the loop where an exception is thrown, and the abrupt loop termination afterwards. Finally, in the fourth step, we explain the exception recovery and the normal program termination. Below is the sketch of prog5 :



where $b = (x \leq 0)$, $c_0 = (x \triangleq 1; y \triangleq 20)$, $c_1 = (\text{if } (x \leq 0) \text{ then } (\text{THROW } e) \text{ else } (x \triangleq x - 1))$, $c_2 = (x \triangleq x - 1)$, $c_3 = (y \triangleq 7)$. Notice that blue arrows depict the normal loop iterations while red ones are to identify the program behavior after the exception of name e is raised. Using the rules of the logic \mathcal{L}_{st+exc} , we can reduce the above diagram into the one given below with respect to the equality sort $\equiv \equiv$:

$$\mathbb{1} \xrightarrow{c_0} \mathbb{Z} \xrightarrow{u_x} \mathbb{1} \xrightarrow{c_7} \mathbb{Z} \xrightarrow{u_y} \mathbb{1}$$

which is actually the prog6 when sketched.

1. Initially, we have

$$\downarrow \left([\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[(\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \circ \text{pbl} \circ (c \text{ true}) \right) \\ \circ u_y \circ (c 20) \circ u_x \circ (c 1) \equiv \equiv u_y \circ (c 7) \circ u_x \circ (c 0). \quad (28)$$

We first convert $\equiv \equiv$ into $\equiv \sim$, then rewrite the (w_downcast) rule and get

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[(\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \\ \circ \text{pbl} \circ (c \text{ true}) \circ u_y \circ (c 20) \circ u_x \circ (c 1) \equiv \sim u_y \circ (c 7) \circ u_x \circ (c 0). \quad (29)$$

Rewriting `commutation-update-update`, on both sides, gives

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[(\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \\ \circ \text{pbl} \circ (c \text{ true}) \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (30)$$

Rewriting Lemma 7.2 yields

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[(\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \\ \circ \text{in}_1 \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (31)$$

2. Now; we rewrite the rule (w_lcopair_eq), and handle

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \\ \circ \text{pbl} \circ \text{b} \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0). \quad (32)$$

By unfolding `b`, we have

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \\ \circ \text{pbl} \circ (t \stackrel{?}{\leq}) \circ \langle l_x (c 0) \rangle \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (33)$$

Rewriting the lemma `commutation-lookup-constant-update`, we obtain

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ [[]_1 \circ \text{tag}_e | c_2] \\ \circ \text{pbl} \circ (t \stackrel{?}{\leq}) \circ \langle (c 1), (c 0) \rangle \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (34)$$

We rewrite the rule (`imp2`), and get

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ \circ [[]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ (c \text{ false}) \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (35)$$

Rewriting the Lemma 7.1 yields

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ & \quad \circ [[\]_1 \circ \text{tag}_e \mid c_2] \circ \text{in}_2 \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (36)$$

We now rewrite (s_lcopair_eq) which gives

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ & \quad \circ c_2 \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (37)$$

Here, by unfolding c_2 , we have

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ u_x \circ (t \ -) \circ \langle 1_x, (c \ 1) \rangle \\ & \quad \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (38)$$

Rewriting the lemma commutation – lookup – constant – update gives

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ u_x \circ (t \ -) \circ \langle (c \ 1), (c \ 1) \rangle \\ & \quad \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (39)$$

We rewrite (imp₁), and get

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ & \quad \circ u_x \circ (c \ 0) \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (40)$$

We again rewrite the lemma commutation–update–update, and obtain

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ u_x \circ (c \ 0) \\ & \quad \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (41)$$

3. We re-iterate the loop via (imp-li), and have

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ [(\text{lpi } (c \text{ true}) c_1) \circ c_1 \mid \text{id}] \\ & \quad \circ \text{pbl} \circ (c \ \text{true}) \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (42)$$

We rewrite Lemma 7.2, (w_lcopair_eq), then unfold c_1 , and get:

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \ \text{true}) c_1) \circ [\text{throw } e \ \mathbb{1} \mid c_2] \\ & \quad \circ \text{pbl} \circ (t \ \overset{?}{\leq}) \circ \langle 1_x, (c \ 0) \rangle \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \end{aligned} \quad (43)$$

By rewriting commutation – lookup – constant – update, (imp₃) and Lemma 7.2, we have

$$\begin{aligned} & [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \ \text{true}) c_1) \circ [\text{throw } e \ \mathbb{1} \mid c_2] \circ \text{in}_1 \\ & \quad \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \end{aligned} \quad (44)$$

By (w_lcopair_eq), the exception is raised:

$$\begin{aligned} [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ ((\text{lpi } (c \text{ true}) c_1) \circ \text{throw } e \ \mathbb{1}) \\ \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \end{aligned} \quad (45)$$

Due to the raised exception, the infinite loop gets abruptly terminated at this step. We first unfold THROW then rewrite propagator-propagates (see Section 6.1), and get

$$[\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ []_1 \circ \text{tag}_e \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \quad (46)$$

4. Here, we first cut $\text{in}_1 \circ []_1 \equiv \equiv \text{in}_2$, and rewrite it back in the equation. Thus, we have

$$[\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_2 \circ \text{tag}_e \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (47)$$

By rewriting (s_lcopair_eq), we obtain

$$c_3 \circ \text{untag}_e \circ \text{tag}_e \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (48)$$

Since $u_x \circ (c \ 0) \circ u_y \circ (c \ 20)$ is pure with respect to the exception, we rewrite (eax₁), and get

$$c_3 \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (49)$$

Unfolding the definition of the command $c_3 = (u_y \circ (c \ 7))$, we have

$$u_y \circ (c \ 7) \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (50)$$

We now rewrite commutation-update-update on the left, and handle

$$u_x \circ (c \ 0) \circ u_y \circ (c \ 7) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (51)$$

Finally, it suffices to rewrite interaction-update-update,

$$u_x \circ (c \ 0) \circ u_y \circ (c \ 7) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (52)$$

which is trivial since the equality symbol $\equiv \sim$ is reflexive. It still remains to prove the previous cut $\text{in}_1 \circ []_1 \equiv \equiv \text{in}_2$: since everything is pure with respect to the exception, we have

$$\text{in}_1 \circ []_1 \equiv \sim \text{in}_2. \quad (53)$$

Now, rewriting the rule (w_empty) gives $[]_{1+1} \equiv \sim []_{1+1}$. \square

The full Coq proofs of above lemmata can be found [here](#), and the entire implementation [there](#).

8 Concluding remarks

We have presented frameworks for formalizing the treatment of the state and the exception effects, first separately, and then combined, using the decorated logic. Decorations describe what computational effect evaluation of a term may involve, and form a bridge between the syntax and its interpretation in reasoning about terms by making computational effects explicit in the decorated syntax. We have designed a denotational semantics for the IMP+Exc language using the combined decorated logic \mathcal{L}_{st+exc} as the target language. This way, we managed prove strong equalities between IMP+Exc programs. We have also encoded the combined logic in the Coq proof assistant to certify related proofs.

References

- [1] Bauer, A., Pretnar, M.: An effect system for algebraic effects and handlers. *Logical Methods in Computer Science* 10(4) (2014), [http://dx.doi.org/10.2168/LMCS-10\(4:9\)2014](http://dx.doi.org/10.2168/LMCS-10(4:9)2014)
- [2] Bauer, A., Pretnar, M.: Programming with algebraic effects and handlers. *J. Log. Algebr. Meth. Program.* 84(1), 108–123 (2015), <http://dx.doi.org/10.1016/j.jlamp.2014.02.001>
- [3] Benton, N., Kennedy, A.: Exceptional syntax. *J. Funct. Program.* 11(4), 395–410 (Jul 2001), <http://dx.doi.org/10.1017/S0956796801004099>
- [4] Brady, E.: Programming and reasoning with algebraic effects and dependent types. In: *Proceedings of the 18th ACM SIGPLAN International Conference on Functional Programming*. pp. 133–144. ICFP '13, ACM, New York, NY, USA (2013), <http://doi.acm.org/10.1145/2500365.2500581>
- [5] Brookes, S., Van Stone, K.: Monads and Comonads in Intensional Semantics. Tech. Rep. CMU-CS-93-140, Pittsburgh, PA, USA (1993), <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.45.5695>
- [6] Domínguez, C., Duval, D.: Diagrammatic logic applied to a parameterisation process. *Mathematical Structures in Computer Science* 20(4), 639–654 (2010), <http://dx.doi.org/10.1017/S0960129510000150>
- [7] Dumas, J.G., Duval, D., Ekici, B., Pous, D.: Formal verification in Coq of program properties involving the global state effect. In: Tasson, C. (ed.) *25e Journées Francophones des Langages Applicatifs*, Fréjus (Jan 2014), <http://hal.archives-ouvertes.fr/hal-00869230>
- [8] Dumas, J., Duval, D., Ekici, B., Pous, D., Reynaud, J.: Relative hilbert-post completeness for exceptions. In: *Mathematical Aspects of Computer and Information Sciences - 6th International Conference, MACIS 2015, Berlin, Germany, November 11-13, 2015, Revised Selected Papers*. pp. 596–610 (2015), http://dx.doi.org/10.1007/978-3-319-32859-1_51
- [9] Dumas, J., Duval, D., Fousse, L., Reynaud, J.: A duality between exceptions and states. *Mathematical Structures in Computer Science* 22(4), 719–722 (2012), <http://dx.doi.org/10.1017/S0960129511000752>
- [10] Egger, J., Møgelberg, R.E., Simpson, A.: The enriched effect calculus: syntax and semantics. *J. Log. Comput.* 24(3), 615–654 (2014), <http://dx.doi.org/10.1093/logcom/exs025>
- [11] Ekici, B.: Certification de programmes avec des effets calculatoires. Ph.D. thesis (2015), <https://tel.archives-ouvertes.fr/tel-01250842/document>
- [12] Filinski, A.: Controlling Effects. Ph.D. thesis (1996)
- [13] Gosling, J., Joy, B., Steele, G., Bracha, G.: *Java(TM) Language Specification, The (3rd Edition)* (Java (Addison-Wesley)). Addison-Wesley Professional (2005)
- [14] Hyland, M., Levy, P.B., Plotkin, G.D., Power, J.: Combining algebraic effects with continuations. *Theor. Comput. Sci.* 375(1-3), 20–40 (2007), <http://dx.doi.org/10.1016/j.tcs.2006.12.026>

- [15] Hyland, M., Plotkin, G.D., Power, J.: Combining effects: Sum and tensor. *Theor. Comput. Sci.* 357(1-3), 70–99 (2006), <http://dx.doi.org/10.1016/j.tcs.2006.03.013>
- [16] Jacobs, B., Rutten, J.: An introduction to (co)algebras and (co)induction. In: In: D. Sangiorgi and J. Rutten (eds), *Advanced topics in bisimulation and coinduction*. pp. 38–99 (2011)
- [17] Jacobs, B.: A formalisation of java’s exception mechanism. In: *ESOP’01*. pp. 284–301 (2001)
- [18] Jaskelioff, M.: Modular monad transformers. In: *Programming Languages and Systems, 18th European Symposium on Programming, ESOP 2009, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2009, York, UK, March 22-29, 2009. Proceedings*. pp. 64–79 (2009), http://dx.doi.org/10.1007/978-3-642-00590-9_6
- [19] Lawvere, F.W.: Functorial Semantic of Algebraic Theories (Available with commentary as TAC Reprint 5.). Ph.D. thesis (1963), <http://matija.pretnar.info/pdf/the-logic-and-handling-of-algebraic-effects.pdf>
- [20] Levy, P.B.: Call-by-push-value: A subsuming paradigm. In: *Typed Lambda Calculi and Applications, 4th International Conference, TLCA’99, L’Aquila, Italy, April 7-9, 1999, Proceedings*. pp. 228–242 (1999), http://dx.doi.org/10.1007/3-540-48959-2_17
- [21] Linton, F.: Relative functorial semantics: adjointness results. In: *Lecture notes in mathematics*. vol. 99 (1969)
- [22] Linton, F.E.J.: Some aspects of equational theories. In: *Proc. Conf. on Categorical Algebra*. pp. 84–95. Springer-Verlag, La Jolla (1966)
- [23] Lucassen, J.M., Gifford, D.K.: Polymorphic effect systems. In: *Proceedings of the 15th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. pp. 47–57. POPL ’88, ACM, New York, NY, USA (1988), <http://doi.acm.org/10.1145/73560.73564>
- [24] Mellies, P.: Segal condition meets computational effects. In: *Proceedings of the 25th Annual IEEE Symposium on Logic in Computer Science, LICS 2010, 11-14 July 2010, Edinburgh, United Kingdom*. pp. 150–159 (2010), <http://dx.doi.org/10.1109/LICS.2010.46>
- [25] Moggi, E.: Notions of computation and monads. *Inf. Comput.* 93(1), 55–92 (Jul 1991), [http://dx.doi.org/10.1016/0890-5401\(91\)90052-4](http://dx.doi.org/10.1016/0890-5401(91)90052-4)
- [26] Orchard, D.: Should i use a monad or a comonad? Tech. rep. (2012)
- [27] Orchard, D.A., Bolingbroke, M., Mycroft, A.: Ypnos: Declarative, parallel structured grid programming. In: *Proceedings of the 5th ACM SIGPLAN Workshop on Declarative Aspects of Multicore Programming*. pp. 15–24. DAMP ’10, ACM, New York, NY, USA (2010), <http://doi.acm.org/10.1145/1708046.1708053>
- [28] Petricek, T., Orchard, D., Mycroft, A.: Coeffects: unified static analysis of context-dependence. In: *Proceedings of International Conference on Automata, Languages, and Programming - Volume Part II. ICALP 2013*

- [29] Plotkin, G.D., Power, J.: Notions of computation determine monads. In: Foundations of Software Science and Computation Structures, 5th International Conference, FOSSACS 2002. Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2002 Grenoble, France, April 8-12, 2002, Proceedings. pp. 342–356 (2002), http://dx.doi.org/10.1007/3-540-45931-6_24
- [30] Plotkin, G.D., Pretnar, M.: Handlers of algebraic effects. In: Programming Languages and Systems, 18th European Symposium on Programming, ESOP 2009, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2009, York, UK, March 22-29, 2009. Proceedings. pp. 80–94 (2009), http://dx.doi.org/10.1007/978-3-642-00590-9_7
- [31] Plotkin, G.D., Pretnar, M.: Handling algebraic effects. Logical Methods in Computer Science 9(4) (2013), [http://dx.doi.org/10.2168/LMCS-9\(4:23\)2013](http://dx.doi.org/10.2168/LMCS-9(4:23)2013)
- [32] Pretnar, M.: Inferring algebraic effects. Logical Methods in Computer Science 10(3) (2014), [http://dx.doi.org/10.2168/LMCS-10\(3:21\)2014](http://dx.doi.org/10.2168/LMCS-10(3:21)2014)
- [33] Schröder, L., Mossakowski, T.: Generic exception handling and the Java monad. In: Rattray, C., Maharaj, S., Shankland, C. (eds.) Algebraic Methodology and Software Technology. Lecture Notes in Computer Science, vol. 3116, pp. 443–459. Springer (2004), <http://www.springerlink.com/openurl.asp?genre=article&issn=0302-9743&volume=3116&spage=443>
- [34] Tzevelekos, N.: Nominal game semantics. Ph.D. thesis (2008), cS-RR-09-18
- [35] Uustalu, T., Vene, V.: The essence of dataflow programming. In: Central European Functional Programming School, First Summer School, CEFPS 2005, Budapest, Hungary, July 4-15, 2005, Revised Selected Lectures. pp. 135–167 (2005), http://dx.doi.org/10.1007/11894100_5
- [36] Uustalu, T., Vene, V.: Comonadic notions of computation. Electron. Notes Theor. Comput. Sci. 203(5), 263–284 (Jun 2008), <http://dx.doi.org/10.1016/j.entcs.2008.05.029>
- [37] Wadler, P.: The essence of functional programming. In: Proceedings of the 19th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages. pp. 1–14. POPL '92, ACM, New York, NY, USA (1992), <http://doi.acm.org/10.1145/143165.143169>