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► **To cite this version:**

| Burak Ekici. IMP with exceptions over decorated logic. 2017. hal-01132831v4

**HAL Id: hal-01132831**

**<https://hal.science/hal-01132831v4>**

Preprint submitted on 10 Apr 2017 (v4), last revised 12 Oct 2018 (v9)

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# *IMP with exceptions over decorated logic*

Burak Ekici

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In this paper, we facilitate the reasoning about impure programming languages, by annotating terms with “decorations” that describe what computational (side) effect evaluation of a term may involve. In a point-free categorical language, called the “decorated logic”, we formalize the mutable state and the exception effects first separately, exploiting a nice duality between them, and then combined. The combined decorated logic is used as the target language for the denotational semantics of the IMP+Exc imperative programming language, and allows us to prove equivalences between programs written in IMP+Exc. The combined logic is encoded in Coq, and this encoding is used to certify some program equivalence proofs.

**Keywords:** Computational effects, state, exceptions, program equivalence proofs, decorated logic, Coq.

## 1 Introduction

In programming languages theory, a program is said to have computational effects if, besides a return value, it has observable interactions with the outside world. For instance, using/modifying the program state, raising/recovering exceptions, reading/writing data from/to some file, etc. In order to formally reason about behaviors of a program with computational effects, one has to take into account these interactions. One difficulty in such a reasoning is the mismatch between the syntax of operations with effects and their interpretation. Typically, an operation in an effectful language with arguments in  $X$  that returns a value in  $Y$  is not interpreted as a function from  $X$  to  $Y$ , due to the effects, unless the operation is pure.

The best known *algebraic approach* to formalize computational effects was initiated by Moggi in his seminal paper (27). He showed that the effectful operations of an impure language can be interpreted as arrows of a Kleisli category for an appropriate monad  $(T, \eta, \mu)$  over a base category  $\mathcal{C}$  with finite products. For instance, in Moggi’s *computational metalanguage*, an operation in an impure language with arguments in  $X$  that returns a value in  $Y$  is now interpreted as an arrow from  $\llbracket X \rrbracket$  to  $T\llbracket Y \rrbracket$  in  $\mathcal{C}$  where  $\llbracket X \rrbracket$  is the object of *values* of type  $X$  and  $T\llbracket Y \rrbracket$  is the object of *computations* that return values of type  $Y$ . The use of monads to formalize effects (such as state, exceptions, input/output and non-deterministic choice) was popularized by Wadler in (39), and implemented in the programming languages Haskell and F#. Using monad transformers (19), it is usually possible to “combine” different effects formalized by monads. Moggi’s *computational metalanguage* was extended into the *basic effect calculus* with a notion of *computation type* as in Filinski’s effect PCF (13) and in Levy’s call-by-push-value (CBPV) (21). In their paper (11), Egger et al., defined their effect calculus, named *extended effect calculus* as a canonical calculus incorporating the ideas of Moggi, Filinski and Levy. Following Moggi, they included a type constructor for computations. Following Filinski and Levy, they classified types as value types and computation types.

Being dual to monads, comonads have been used to formalize context-dependent computations. Intuitively, an effect which observes features may arise from a comonad, while an effect which constructs

features may arise from a monad (17). Uustalu and Vene have structured stream computations (38), Orchard et al. array computations (29) and Tzevelekos game semantics (36) via the use of comonads. In (30), Petricek et al. proposed a unified calculus for tracking context dependence in functional languages together with a categorical semantics based on indexed comonads. In his report (28), Orchard proposed a method for choosing between monads and comonads when formalizing computational effects. A computation can be seen as a composition of context-dependence and effectfulness (38). In (5), Brookes and Van Stone showed that such combinations may correspond to distributive laws of a comonad over a monad. This has been applied to clocked causal data-flow computation, combining causal data-flow and exceptions by Uustalu and Vene in (37).

Moggi's approach, using monads in effect modeling, has been extended to Lawvere theories which first appeared in Lawvere's 1963 PhD dissertation (20). Three years later, in (24), Linton showed that every Lawvere theory induces a monad on the category of sets, and more generally on any category which satisfies the local representability condition (23). Therefore, Moggi's seminal paper (27), formalizing computational effects by monads, made it possible for monadic effects to be formalized through Lawvere theories. To this extend, Plotkin and Power, in (31), have shown that effects such as the global and the local state could be formalized by *signatures* of effectful terms and an *equational theory* explaining the interactions between them. Melliès has refined this *equational theory* in (26) showing that some of the equations modeling the mutable global state can be omitted. In (15, 16), Hyland et al. studied the combination of computational effects in terms of Lawvere theories.

Plotkin and Pretnar (31, 32, 33) extended Moggi's classification of terms (*values* and *computations*) with a third level called *handlers* for the computational effects that can be represented by an algebraic theory (*algebraic effects*). Initially, they introduce an *handler* for the exception handling, and then account for its generalization to the other handlers to cope with other algebraic effects such as stream redirection, explicit non-determinism, CCS, parameter passing, timeout and rollback (33, §3). For each algebraic effect, *handling constructs* are used to apply handlers to effectful computations where effectful computations can be interpreted as algebraic operations while handling constructs as homomorphisms from free algebras. This use of handling constructs is inspired from Benton and Kennedy's paper (3) where a construct specifically for exceptions is introduced. Notice also that formalization of the exception effect can also be made from a co-algebraic point of view as in (18). Also, exception handling is used in (35) to get a Hoare logic for exceptions .

Apart from all these, there is an older formal way of modeling computational effects called the *effect systems*. In their 1988 paper (25), Lucassen and Gifford presented an approach to programming languages for parallel computers. The key idea was to use an *effect system* to discover expression scheduling constraints. There, every expression comes with three components: *types* to represent the kinds of the return values, *effects* to summarize the observable interactions of expressions and *regions* to highlight the areas of the memory where expressions may have effects. To this extend, one can simply reason that if two expressions do not have overlapping effects, then they can obviously be scheduled in parallel. The reasoning is done by some inference rules for *types* and *effects* based on the second order typed  $\lambda$ -calculus.

In (6), Duval et al. proposes yet another paradigm to formalize computational effects by mixing effect systems and algebraic theories, named *the decorated logic*. The key point of this paradigm is that every term comes with a decoration which exposes its features with respect to a single computational effect or to several ones keeping their interpretations close to syntax in reasoning with effects. In addition, an *equational theory* highlights the interactions among terms with two sorts of equations: *weak* equations relate terms with respect only to their results and *strong* equations relate terms with respect both to their

results and effects. By and large, decorated logic provides an equational reasoning in between programs written in imperative languages after being a target language for a denotational semantics of the studied language.

The main point of this paper is to design the decorated logic for the global state and the exception effects, and then combine them to serve as a target language for denotational semantics of imperative programming languages mixing mentioned effects only. In this regards, this paper is organized as follows: in Section 2, we introduce an imperative programming language, that mixes the state and the exception effects, by defining its small-step operational semantics. The language we study there is called IMP+Exc which extends the IMP (or while) language with a mechanism to handle *exceptions*. Then, we formally design the decorated logic for the state and the exception effects in Sections 3 and 4, respectively. In Section 5, we combine these logics. Finally, Section 6 details the use of the combined decorated logic as the target language for the IMP+Exc denotational semantics. This allows us to do an equational reasoning between termination-guaranteed IMP+Exc programs. I.e., proving two different looking programs are actually doing the same job with respect to the state and exception effects. Also, we certify such proofs with the Coq Proof Assistant. Figure 1 summarizes the approach of the paper.

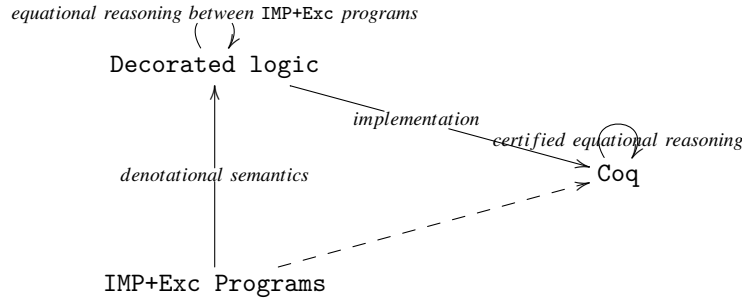


Figure 1: The approach

## 2 IMP with exceptions

IMP is a standard Turing complete imperative programming language natively providing global variables of integer (Z), Boolean (B) and unit (U) data types, standard integer and Boolean arithmetic enriched with a set of commands that is made of do-nothing, assignment, sequence, conditionals and looping operations. Below, we detail its syntax where  $n$  represents a constant integer term while  $x$  is an integer global variable. Note also that abbreviations  $a_{exp}$  and  $b_{exp}$  respectively denote arithmetic and Boolean expressions as well as  $cmd$  stands for the commands.

$$\begin{aligned}
 a_{exp}: a_1 a_2 & ::= n \mid x \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \times a_2 \\
 b_{exp}: b_1 b_2 & ::= true \mid false \mid a_1 \stackrel{?}{=} a_2 \mid a_1 \stackrel{?}{\neq} a_2 \mid a_1 \stackrel{?}{>} a_2 \mid a_1 \stackrel{?}{<} a_2 \mid b_1 \wedge b_2 \mid b_1 \vee b_2 \mid \neg b_1 \\
 cmd: c_1 c_2 & ::= SKIP \mid x \stackrel{\Delta}{=} e \mid c_1; c_2 \mid if\ b\ then\ c_1\ else\ c_2 \mid while\ b\ do\ c_1
 \end{aligned}$$

Figure 2: Standard IMP syntax

Neither arithmetic nor Boolean expressions are allowed to modify the state: they are either pure or read-only. We present, in Figure 3, the small-step semantics for evaluation of arithmetic expressions in IMP where we use a small-step transition function  $\rightarrow_a: \text{aexp} \times \mathcal{S} \rightarrow \text{aexp}$ . This function computes a new arithmetic expression out of an input arithmetic expression and the current program state denoted  $\mathcal{S}$  which includes contents of variables at a given time.

$$\begin{array}{c} (\text{const}) \frac{}{(n, s) \rightarrow_a n} \quad (\text{var}) \frac{}{(x, s) \rightarrow_a x(s)} \quad (\text{op-sym}) \frac{(a_1, s) \rightarrow_a n_1 \quad (a_2, s) \rightarrow_a n_2}{(a_1 \text{ op } a_2, s) \rightarrow_a n_1 \text{ op}_{\mathbb{Z}} n_2} \end{array}$$

Figure 3: Small-step natural semantics for arithmetic expressions

The symbol  $\text{op}$  represents the operation symbols ( $+$ ,  $-$  or  $\times$ ) given by the standard syntax in Figure 2 while  $\text{op}_{\mathbb{Z}}: \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}$  denotes the corresponding binary operations in  $\mathbb{Z}$ . Similarly, in Figure 4, we present the small-step semantics for evaluation of Boolean expressions in IMP where we use a small-step transition function  $\rightarrow_b: \text{bexp} \times \mathcal{S} \rightarrow \text{bexp}$ . This function simply computes a new Boolean expression out of an input Boolean expression and the current program state.

$$\begin{array}{c} (\text{true}) \frac{}{(\text{true}, s) \rightarrow_b \text{true}} \quad (\text{false}) \frac{}{(\text{false}, s) \rightarrow_b \text{false}} \\ (\text{op1}) \frac{(b_1, s) \rightarrow_b n_1 \quad (b_2, s) \rightarrow_b n_2}{(b_1 \text{ opb } b_2, s) \rightarrow_b n_1 \text{ opb}_{\mathbb{B}} n_2} \quad (\text{op2}) \frac{(b_1, s) \rightarrow_b n_1}{(\neg b_1, s) \rightarrow_b \text{neg } n_1} \end{array}$$

Figure 4: Small-step natural semantics for arithmetic expressions

The symbols  $\text{true}$  and  $\text{false}$  are Boolean operation symbols given by the standard syntax in Figure 2 while  $\text{true}$  and  $\text{false}$  are Boolean constructors. Similarly,  $\text{opb}$  represents the binary operation symbols (all except ' $\neg$ '), while  $\text{opb}_{\mathbb{B}}: \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}$  denotes the corresponding Boolean operations, and  $\text{neg}: \mathbb{B} \rightarrow \mathbb{B}$  is the Boolean negation. The small-step operational semantics for evaluation of commands are given in Figure 5 where we use a small-step transition function  $\rightsquigarrow: \text{cmd} \times \mathcal{S} \rightarrow \text{cmd} \times \mathcal{S}$  which is interpreted as at the state  $s$ , one step execution of the command  $c$  changes the state into  $s'$  and the command  $c'$  is in execution.

$$\begin{array}{c} (\text{sequence}) \frac{s, c_1 \rightsquigarrow s', c'_1}{s, (c_1; c_2) \rightsquigarrow s', (c'_1; c_2)} \quad (\text{skip}) \frac{}{s, (\text{SKIP}; c) \rightsquigarrow s, c} \\ (\text{assign}) \frac{(a, s) \rightarrow_a n}{s, (x := a) \rightsquigarrow s[x \leftarrow n], \text{SKIP}} \\ (\text{cond}_1) \frac{(b, s) \rightarrow_b \text{true}}{s, (\text{if } b \text{ then } c_1 \text{ else } c_2) \rightsquigarrow s, c_1} \quad (\text{cond}_2) \frac{(b, s) \rightarrow_b \text{false}}{s, (\text{if } b \text{ then } c_1 \text{ else } c_2) \rightsquigarrow s, c_2} \\ (\text{while}_1) \frac{(b, s) \rightarrow_b \text{true}}{s, (\text{while } b \text{ do } c) \rightsquigarrow s, (c; \text{while } b \text{ do } c)} \quad (\text{while}_2) \frac{(b, s) \rightarrow_b \text{false}}{s, (\text{while } b \text{ do } c) \rightsquigarrow s, \text{SKIP}} \end{array}$$

Figure 5: Small-step operational semantics for commands

Note that all the small-step transition functions we used so far are meant to reduce any given configuration just a single step. I.e.,  $(a, s) \rightarrow_a n$  means that one step reduction of the arithmetic expression  $a$  at a state  $s$  results in another arithmetic expression  $n$ . Besides all these, we need to elucidate that a command  $c$  terminates at a state  $s$  if  $s, c \rightsquigarrow^* s'$ , SKIP for some state  $s'$ , where  $\rightsquigarrow^*$  is the transitive closure of the transition function  $\rightsquigarrow$ . Mind also that there is no run-time error since any command apart from the SKIP is allowed to execute at any state  $s$ , and SKIP alone is used to indicate the final step of some command set.

## 2.1 A mechanism to handle exceptions

Extending the IMP language with a mechanism that allows raising exceptions and recover from them, we just enrich the command set with THROW and TRY/CATCH blocks. In addition to the ones in Figure 2, we also consider following commands in Figure 6 where  $e$  is an exception name coming from a finite set

$$\text{cmd: } c_1 \ c_2 \quad ::= \quad \dots \mid \text{THROW } e \mid \text{TRY } c_1 \ \text{CATCH } e \Rightarrow c_2$$

Figure 6: Syntax for exceptional commands

EName which exists by assumption. There is also a type  $EV_e$  of exceptional values (parameters) for each exception name  $e$ . The small-step operational semantics for THROW and TRY/CATCH commands are shown in Figure 7.

$$\begin{array}{c} \text{(throw)} \frac{\forall e : \text{EName}}{s, (\text{THROW } e; c) \rightsquigarrow s, \text{THROW } e} \quad \text{(tc}_0\text{)} \frac{\forall e : \text{EName}, c_1 \in \text{cmd} \setminus \{\text{THROW } e\}}{s, \text{TRY } c_1 \ \text{CATCH } e \Rightarrow c_2 \rightsquigarrow s, c_1} \\ \text{(tc}_1\text{)} \frac{\forall e : \text{EName}}{s, \text{TRY } (\text{THROW } e) \ \text{CATCH } e \Rightarrow c \rightsquigarrow s, c} \quad \text{(tc}_2\text{)} \frac{\forall e_1 \ e_2 : \text{EName} \quad e_1 \neq e_2}{s, \text{TRY } (\text{THROW } e_1) \ \text{CATCH } e_2 \Rightarrow c \rightsquigarrow s, \text{THROW } e_1} \end{array}$$

Figure 7: Small-step operational semantics for additional commands

Exceptional commands are pure in terms of the state effect: they neither use nor modify the program state. However, they introduce another sort of computational effect: the exception. In prior, we stated that the command SKIP alone indicates the termination of a program. Now, we extend this by saying THROW  $e$  is also an end but an abnormal end. Intuitively, if an exceptional value of name  $e$  is raised in the TRY block and recovered immediately in the CATCH, the program then resumes with the continuation provided. Unlikely, an exceptional value of name  $e_1$  gets propagated if in the CATCH block, another exceptional value of name  $e_2$  is attempted to be recovered.

Recall that in Section 6, we independently define a denotational semantics for the IMP+Exc language using the decorated logic for the state and the exception effects as the target language. We formalize this target (in a generic way) logic in Section 5 which basically combines the logics presented in Sections 3 and 4.

## 3 The Decorated Logic for the state effect ( $\mathcal{L}_{st}$ )

The decorated logic, as a generic framework, is an extension to Moggi's monadic equational logic (27) with the use of decorations on terms and equalities. It allows to do an *equational reasoning* between programs written in imperative programming languages.

Specifically, the decorated logic for the state effect ( $\mathcal{L}_{st}$ ) is obtained from equational logics by classifying terms and equalities (7). Terms are classified as *pure* terms, *accessors* and *modifiers*. The decoration of a term characterizes the way it may interact with the state, and used as a superscript. The decoration (0) is reserved for *pure* terms, while (1) is for *read-only (accessor)* and (2) is for *read-write (modifier)* terms. Equalities are classified as *strong* and *weak*, denoted respectively by the symbols  $\equiv$  and  $\sim$ . An equality between two terms relates the results they return together with the manipulations they make on the state. Two terms are called strongly equal if they return the same result with the same state manipulation; they are called weakly equal if they return the same result with different state manipulations.

<b>Grammar of the decorated logic for the state:</b>		$(i \in \text{Loc})$
Types:	$\mathbf{t}, \mathbf{s} ::=$	$X \mid Y \mid \dots \mid \mathbf{t} \times \mathbf{s} \mid \mathbb{1} \mid \mathbf{V}_i$
Decorations for terms:	$(d) ::=$	$(0) \mid (1) \mid (2)$
Terms:	$\mathbf{f}, \mathbf{g} ::=$	$\mathbf{a}^{(d)} \mid \mathbf{b}^{(d)} \mid \dots \mid \mathbf{g} \circ \mathbf{f}^{(d)} \mid \langle \mathbf{f}, \mathbf{g} \rangle_1^{(d)} \mid$ $\text{lookup}_i^{(1)} \mid \text{update}_i^{(2)} \mid (\text{tpure} \cdot)^{(0)}$
Equations:	$\text{eq} ::=$	$\mathbf{f}^{(d)} \equiv \mathbf{g}^{(d)} \mid \mathbf{f}^{(d)} \sim \mathbf{g}^{(d)}$

**Figure 8:**  $\mathcal{L}_{st}$ : syntax

Figure 8 shows the grammar of the  $\mathcal{L}_{st}$  where  $\mathbb{1}$  is the singleton type while  $\mathbf{V}_i$  is the type of values that can be stored in any location  $i$ . We assume that there is a finite set of locations called  $\text{Loc}$ . Given types  $X$  and  $Y$ , we have  $X \times Y$  representing type products.

Terms are closed under composition ( $\circ$ ) and pairing ( $\langle \_, \_ \rangle_1$ ). I.e., for all terms  $\mathbf{f} : X \rightarrow Y$  and  $\mathbf{g} : Y \rightarrow Z$ , we have  $\mathbf{g} \circ \mathbf{f} : X \rightarrow Z$ . Similarly, for all  $\mathbf{f} : X \rightarrow Y$  and  $\mathbf{g} : X \rightarrow Z$ , there is  $\langle \mathbf{f}, \mathbf{g} \rangle_1 : X \rightarrow Y \times Z$ . Notice that the pair subscript 1 denotes the left pairs. One can define in a symmetric way the right pairs for terms  $\mathbf{f} : X \rightarrow Y$  and  $\mathbf{g} : X \rightarrow Z$  as  $\langle \mathbf{f}, \mathbf{g} \rangle_r := \text{permut} \circ \langle \mathbf{g}, \mathbf{f} \rangle_1$  where  $\text{permut} := \langle \pi_2, \pi_1 \rangle_1$ . In the same way, one can respectively obtain left and right products of terms  $\mathbf{f} : X_1 \rightarrow Y_1$  and  $\mathbf{g} : X_2 \rightarrow Y_2$  as  $\mathbf{f} \times_1 \mathbf{g} := \langle \mathbf{f} \circ \pi_1, \mathbf{g} \circ \pi_2 \rangle_1$  and  $\mathbf{f} \times_r \mathbf{g} := \langle \mathbf{f} \circ \pi_1, \mathbf{g} \circ \pi_2 \rangle_r$ . The term products are used to impose some order of term evaluation since the evaluation result depends on the order that the mutable state is used. See sequential products in (7, §2.3). The decoration of a pair (product) or a composition depends on the decoration of its components, always taking the larger. I.e.,  $\forall \mathbf{f}^{(1)} : X \rightarrow Y$  and  $\mathbf{g}^{(2)} : Y \rightarrow Z$ ,  $\mathbf{g} \circ \mathbf{f} : X \rightarrow Z$  takes the decoration (2).

The interface terms are  $\text{lookup}_i : \mathbb{1} \times \mathbf{S} \rightarrow \mathbf{V}_i$  and  $\text{update}_i : \mathbf{V}_i \times \mathbf{S} \rightarrow \mathbb{1} \times \mathbf{S}$  where  $\mathbf{S}$  denotes the distinguished object of states which never appears in the decorated setting. The use of decorations provides a new schema where term signatures are constructed without any occurrence of the state object. For instance,  $\text{lookup}_i^{(1)} : \mathbb{1} \rightarrow \mathbf{V}_i$  is an accessor while  $\text{update}_i^{(2)} : \mathbf{V}_i \rightarrow \mathbb{1}$  is a modifier. This way, we keep signatures close to their syntax and compose compatible terms as usual. The term  $\text{lookup}$  reads the value stored in a given location while  $\text{update}$  modifies it. We can call them *the unique sources of impurity*, since only the terms including  $\text{lookup}$  or  $\text{update}$  are impure; meaning those do not include them are pure with respect to the state effect.

The  $\text{tpure}$  is a special constructor used to introduce decorated pure terms into the logic  $\mathcal{L}_{st}$ . It inputs a non-decorated pure term (from a purely functional environment) and drops it in with the decoration (0). The identity term  $\text{id}$ , the canonical pair projections  $\pi_1$  and  $\pi_2$ , the empty pair term  $\langle \ \rangle$  (which throws out

the given value), and constant terms are defined using the `tpure` constructor, for all types  $X$  and  $Y$ , as:

$$\begin{array}{lll}
\text{id}_x^{(0)} & : & X \rightarrow X \quad := \quad \text{tpure id} \\
\pi_1^{(0)} & : & X \times Y \rightarrow X \quad := \quad \text{tpure fst} \\
\pi_2^{(0)} & : & X \times Y \rightarrow Y \quad := \quad \text{tpure snd} \\
\langle \rangle_x^{(0)} & : & X \rightarrow \mathbb{1} \quad := \quad \text{tpure } (\lambda x : X. \text{void} : \mathbb{1}) \\
\text{constant}_x^{(0)} & : & \mathbb{1} \rightarrow X \quad := \quad \text{tpure } (\lambda \_ . x : X)
\end{array}$$

The intended model of the above grammar is built with respect to the set of states  $S$  where a pure term  $p^{(0)} : X \rightarrow Y$  is interpreted as a function  $p : X \rightarrow Y$ , an accessor  $a^{(1)} : X \rightarrow Y$  as a function  $a : X \times S \rightarrow Y$ , and a modifier  $m^{(2)} : X \rightarrow Y$  as a function  $m : X \times S \rightarrow Y \times S$ . The complete and detailed category theoretical model is given in (12, §5.1).

#### Rules of the decorated logic for the state:

$$\begin{array}{l}
(0\text{-to-1}) \frac{f^{(0)}}{f^{(1)}} \quad (1\text{-to-2}) \frac{f^{(1)}}{f^{(2)}} \quad (\text{pwrepl}) \frac{f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y \quad g^{(0)} : Y \rightarrow Z}{g \circ f_1 \sim g \circ f_2} \\
(\text{wsubs}) \frac{g^{(2)} : X \rightarrow Y \quad f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z}{f_1 \circ g \sim f_2 \circ g} \quad (\text{wrefl}) \frac{f^{(2)}}{f \sim f} \quad (\text{wsym}) \frac{f^{(2)} \sim g^{(2)}}{g \sim f} \\
(\text{wtrans}) \frac{f^{(2)} \sim g^{(2)} \quad g^{(2)} \sim h^{(2)}}{f \sim h} \quad (\text{replsubs}) \frac{f_1^{(2)} \equiv f_2^{(2)} : X \rightarrow Y \quad g_1^{(2)} \equiv g_2^{(2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv g_2 \circ f_2} \\
(\text{refl}) \frac{f^{(2)}}{f \equiv f} \quad (\text{sym}) \frac{f^{(2)} \equiv g^{(2)}}{g \equiv f} \quad (\text{trans}) \frac{f^{(2)} \equiv g^{(2)} \quad g^{(2)} \equiv h^{(2)}}{f \equiv h} \\
(\text{w\_unit}) \frac{f^{(2)} : X \rightarrow \mathbb{1}}{f \sim \langle \rangle_x} \quad (\text{stow}) \frac{f^{(2)} \equiv g^{(2)}}{f \sim g} \quad (\text{wtos}) \frac{f_1^{(1)} \sim f_2^{(1)}}{f_1 \equiv f_2} \\
(\text{ax}_1) \frac{}{\text{lookup}_i^{(1)} \circ \text{update}_i^{(2)} \sim \text{id}_{V_i}^{(0)}} \quad (\text{ax}_2) \frac{\forall i, j \in \text{Loc}, i \neq j}{\text{lookup}_i^{(1)} \circ \text{update}_j^{(2)} \sim \text{lookup}_i^{(1)} \circ \langle \rangle_{V_i}^{(0)}} \\
(\text{effect}) \frac{f_1^{(2)}, f_2^{(2)} : X \rightarrow Y \quad f_1^{(2)} \sim f_2^{(2)} \quad \langle \rangle_Y^{(0)} \circ f_1^{(2)} \equiv \langle \rangle_Y^{(0)} \circ f_2^{(2)}}{f_1 \equiv f_2} \\
(\text{local\_global}) \frac{f_1^{(2)}, f_2^{(2)} : X \rightarrow \mathbb{1} \quad \forall i \in \text{Loc}, \text{lookup}_i^{(1)} \circ f_1^{(2)} \sim \text{lookup}_i^{(1)} \circ f_2^{(2)}}{f_1 \equiv f_2} \\
(\text{w\_lpair\_eq}) \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : X \rightarrow Z}{\pi_1 \circ \langle f_1, f_2 \rangle_1 \sim f_1} \quad (\text{s\_lpair\_eq}) \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : X \rightarrow Z}{\pi_2 \circ \langle f_1, f_2 \rangle_1 \equiv f_2} \\
(\text{tcomp}) \frac{f : Y \rightarrow Z \quad g : X \rightarrow Y}{(\text{tpure } f)^{(0)} \circ (\text{tpure } g)^{(0)} \equiv (\text{tpure } (f \circ g))^{(0)}}
\end{array}$$

Figure 9:  $\mathcal{L}_g$ : rules



The syntax given in Figure 8 is enriched with a set of rules which are presented in Figure 9. Hierarchically, a *pure* term can be seen as an *accessor* which never reads the state (0-to-1), and similarly an *accessor* can be seen as a *modifier* which never modifies the state (1-to-2). Weak equalities do not form a congruence: the term replacement cannot be done unless the replaced term is pure. I.e., given an equation  $f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y$  and a term  $g : Y \rightarrow Z$ , it is possible to get the equation  $g \circ f_1 \sim g \circ f_2$  only when the term  $g$  is pure. At this stage, we have no information about the modifications that  $f_1$  and  $f_2$  make on the memory state. Therefore, the post executed impure term  $g$  would destroy this result equality, for instance by reading the location  $i$  on which  $f_1$  and  $f_2$  has performed different modifications (pwrepl). However, the term substitution can be done regardless of the term decoration. I.e., given the equation  $f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z$  and a term  $g : X \rightarrow Y$ , it is possible to get the equation  $f_1 \circ g \sim f_2 \circ g$  independent from the decoration of the term  $g$ . We already know that  $f_1$  and  $f_2$  return the same result, executing any term  $g$  in advance would not end them returning different results (wsubs). Strong equalities form a congruence by allowing both term substitutions and replacements regardless of the term decorations (replsubs).

Both strong and weak equalities are defined to be *equivalence relations* assuming that they are *reflexive*, *transitive* and *symmetric*.

Any term  $f : X \rightarrow \mathbb{1}$  with no result returned “void” (the unique inhabitant of  $\mathbb{1}$  type) has an obvious result equality with the canonical empty pair  $\langle \rangle_X$  (w\_unit). It is obviously free to convert strong equations into weak ones (stow). However, one has to make sure that the equated terms are not modifiers in order to convert weak equations into strong ones with no further evidence (wtos).

The fundamental equations are given with the rules (ax<sub>1</sub>) and (ax<sub>2</sub>). The former states that by updating the location  $i$  with a value  $v$  and then observing the same location, one gets the value  $v$ . This outputs the same value with the identity term  $\text{id}_{v_i}$ , if it takes  $v$  as an argument. However, notice that these two ways of getting the value  $v$  have different state manipulations which makes them *weakly equal*. The latter, (ax<sub>2</sub>), is to assume that updating the location  $j$  with a value  $v$  and then reading the content of a different location  $i$  would return the same value with first throwing out the value  $v$  then observing the content of location  $i$ . They definitely have different manipulations on the state so that they are *weakly equal*.

Two modifiers  $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$  modify the state in the same way if and only if  $\langle \rangle_Y \circ f_1 \equiv \langle \rangle_Y \circ f_2 : X \rightarrow \mathbb{1}$ , where  $\langle \rangle_Y : Y \rightarrow \mathbb{1}$  throws out the returned value. So that  $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$  are *strongly equal* if and only if  $f_1 \sim f_2$  and  $\langle \rangle_Y \circ f_1 \equiv \langle \rangle_Y \circ f_2$  (effect).

Locally, the strong equality between two modifiers  $f_1^{(2)}, f_2^{(2)} : X \rightarrow \mathbb{1}$  can also be expressed as a pair of weak equations:  $f_1 \sim f_2$  and  $\forall i : \text{Loc}, \text{lookup}_i \circ \langle \rangle_Y \circ f_1 \sim \text{lookup}_i \circ \langle \rangle_Y \circ f_2$ . The latter intuitively means that  $f_1$  and  $f_2$  leaves the memory with the same values stored in all (finitely many) locations after being executed. Given that both return “void” there is no explicitly need to check if  $f_1 \sim f_2$ . It suffices to see whether  $\forall i : \text{Loc}, \text{lookup}_i \circ \langle \rangle_Y \circ f_1 \sim \text{lookup}_i \circ \langle \rangle_Y \circ f_2$  to end up with  $f_1 \equiv f_2$  (local\_global).

With (w\_lpair\_eq) and (w\_rpair\_eq) term pairs are characterized: the (left) pair structure  $\langle f_1, f_2 \rangle_1$  cannot be used when  $f_1$  and  $f_2$ , both are modifiers, since it may lead to a conflict on the returned result. However, it can be used only when  $f_1$  is an accessor. We state by (w\_lpair\_eq) that  $\langle f_1, f_2 \rangle_1^{(2)}$  has only result equality with  $f_1^{(1)}$  and by (w\_rpair\_eq) that it has both result and effect equality with  $f_2^{(2)}$ .

The rule tcomp states that the tpure constructor preserves the composition of pure terms up to the strong equality. Meaning that one can first compose pure terms outside the decorated environment (in Coq’s logic for instance) and use the tpure constructor to translate them into the  $\mathcal{L}_{st}$ , or translate the terms into the  $\mathcal{L}_{st}$  first, and then compose them there. Notice that the red colored composition symbol ( $\circ$ ), in the

rule conclusion, stands for any composition operation for non-decorated pure terms.

Note that these rules are designed to be sound with respect to a categorical model, and detailed in (12, §5.2, §5.3, §5.4, §5.5). However, their syntactic completeness is not immediate. In (8), we define a new syntactic completeness property, embedding a consistency check, called the relative Hilbert-Post completeness. In (12, §5.4) prove that this set of rules is complete with due respect.

### 3.1 Decorated properties of the memory state

Plotkin and Power have introduced, in (31, §3), an equational representation of the mutable state whose decorated versions are given as follows:

- (1)<sub>d</sub> Annihilation lookup-update. *Reading the content of a location  $i$  and then updating it with the obtained value is just like doing nothing.*  $\forall i \in \text{Loc}, \text{update}_i^{(2)} \circ \text{lookup}_i^{(1)} \equiv \text{id}_i^{(0)} : \mathbb{1} \rightarrow \mathbb{1}$ .
- (2)<sub>d</sub> Interaction lookup-lookup. *Reading twice the same location  $i$  is the same as reading it once.*  $\forall i \in \text{Loc}, \text{lookup}_i^{(1)} \circ \langle \rangle_{V_i}^{(0)} \circ \text{lookup}_i^{(1)} \equiv \text{lookup}_i^{(1)} : \mathbb{1} \rightarrow V_i$ .
- (3)<sub>d</sub> Interaction update-update. *Storing value the values  $x$  and  $y$  in a row to the same location  $i$  is just like storing  $y$  in it.*  $\forall i \in \text{Loc}, \text{update}_i^{(2)} \circ \pi_2^{(0)} \circ (\text{update}_i^{(2)} \times_r \text{id}_i^{(0)}) \equiv \text{update}_i^{(2)} \circ \pi_2^{(0)} : V_i \times V_i \rightarrow \mathbb{1}$ .
- (4)<sub>d</sub> Interaction update-lookup. *Storing the value  $x$  in a location  $i$  and then reading the content of  $i$ , gives the value  $x$ .*  $\forall i \in \text{Loc}, \text{lookup}_i^{(1)} \circ \text{update}_i^{(2)} \sim \text{id}_{V_i}^{(0)} : V_i \rightarrow V_i$ .
- (5)<sub>d</sub> Commutation lookup-lookup. *The order of reading two different locations  $i$  and  $j$  does not matter.*  $\forall i \neq j \in \text{Loc}, (\text{id}_{V_i}^{(0)} \times_r \text{lookup}_j^{(1)}) \circ \pi_1^{-1(0)} \circ \text{lookup}_i^{(1)} \equiv \text{permut}_{j,i}^{(0)} \circ (\text{id}_{V_j}^{(0)} \times_r \text{lookup}_i^{(1)}) \circ \pi_1^{-1(0)} \circ \text{lookup}_j^{(1)} : \mathbb{1} \rightarrow V_i \times V_j$  where  $\pi_1^{-1(0)} := \langle \text{id}, \langle \rangle \rangle_i^{(0)}$ .
- (6)<sub>d</sub> Commutation update-update. *The order of storing in two different locations  $i$  and  $j$  does not matter.*  $\forall i \neq j \in \text{Loc}, \text{update}_j^{(2)} \circ \pi_2^{(0)} \circ (\text{update}_i^{(2)} \times_r \text{id}_{V_j}^{(0)}) \equiv \text{update}_i^{(2)} \circ \pi_1^{(0)} \circ (\text{id}_{V_i}^{(0)} \times_1 \text{update}_j^{(2)}) : V_i \times V_j \rightarrow \mathbb{1}$ .
- (7)<sub>d</sub> Commutation update-lookup. *The order of storing in a location  $i$  and reading another location  $j$  does not matter.*  $\forall i \neq j \in \text{Loc}, \text{lookup}_j^{(1)} \circ \text{update}_i^{(2)} \equiv \pi_2^{(0)} \circ (\text{update}_i^{(2)} \times_r \text{id}_{V_j}^{(0)}) \circ (\text{id}_{V_i}^{(0)} \times_1 \text{lookup}_j^{(1)}) \circ \pi_1^{-1(0)} : V_i \rightarrow V_j$ .
- (8)<sub>d</sub> Commutation lookup-constant. *Just after storing a constant  $c$  in a location  $i$ , observing the content of  $i$  is the same as regenerating the constant  $c$ .*  $\forall i \in \text{Loc}, \forall c \in V_i; \text{lookup}_i^{(1)} \circ \text{update}_i^{(2)} \circ \text{constant } c^{(0)} \equiv \text{constant } c^{(0)} \circ \text{update}_i^{(2)} \circ \text{constant } c^{(0)} : \mathbb{1} \rightarrow V_i$ .

These are the archetype properties that we have proved within the scope of the  $\mathcal{L}_{st}$ . To see these proofs, check out Ekici's PhD thesis (12, §5.3). Besides, we have implemented the  $\mathcal{L}_{st}$  in Coq to certify mentioned proofs. Section 3.2 details this implementation.

### 3.2 $\mathcal{L}_{st}$ in Coq

In this section, we aim to highlight some crucial points of the  $\mathcal{L}_{st}$  implementation in Coq. It mainly consists of four steps: (1) implementing the terms, (2) assigning the decorations over terms, (3) stating the rules, and (4) proving properties of the memory state referred in Section 3.1.

We represent the set of memory locations by a Coq parameter `Loc : Type`. Since memory locations may contain different types of values, we also assume an arrow type `Val : Loc → Type` that is the type of values contained in each location.

```
Parameters (Loc: Type) (Val: Loc → Type).
```

We define the terms of  $\mathcal{L}_{st}$  using an inductive predicate called `term`. It establishes a new Coq Type out of two input Types. The type `term Y X` is dependent. It depends on the Type instances `X` and `Y`, and represents the arrow type `X → Y` in the decorated framework.

```
Inductive term: Type → Type → Type ≙
| tpure: ∀ {X Y: Type}, (X → Y) → term Y X
| comp:  ∀ {X Y Z: Type}, term X Y → term Y Z → term X Z
| pair:  ∀ {X Y Z: Type}, term X Z → term Y Z → term (X*Y) Z
| lookup: ∀ i:Loc, term (Val i) unit
| update: ∀ i:Loc, term unit (Val i).
Infix "o" ≙ comp (at level 70).
```

The constructor `tpure` takes a Coq side (pure) function and translates it into the decorated environment. `comp` constructor deals with the composition of two compatible terms. I.e., given a pair of terms `f : term X Y` and `g : term Y Z`, then the composition `f o g` would be an instance of the type `term X Z`. For the sake of conciseness, infix ‘`o`’ is used to denote the term composition. Similarly, the (left) `pair` constructor is to constitute pairs of compatible terms. I.e., given `f : term Y X` and `g : term Z X`, we have `pair ⟨f, g⟩1 : term (Y × Z) X`. Instead of the symbol  $\langle \_, \_ \rangle_l$ , we use the keyword `pair` in the implementation. The terms `lookup` and `update` come as no surprise; just that the singleton type  $\mathbb{1}$  and the type of values  $V_i$  are respectively called `unit` and `Val i` in the code. The terms such as the identity, the pair projections, the empty pair and the constant function can be derived from the native Coq functions, with the use of `tpure` constructor as:

```
Definition id {X: Type}      : term X X      ≙ tpure id.
Definition pi1 {X Y: Type}  : term X (X*Y) ≙ tpure fst.
Definition pi2 {X Y: Type}  : term Y (X*Y) ≙ tpure snd.
Definition forget {X}       : term unit X   ≙ tpure (fun _ => tt).
Definition constant {X: Type} (v: X): term X unit ≙ tpure (fun _ => v).
```

Remark also that the pair projections are named `pi1` and `pi2` while the unique mapping  $\langle \_ \rangle_X$  from any type `X` to  $\mathbb{1}$  is named `forget` in the implementation.

The decorations are enumerated: `pure` (0), `ro` (1) and `rw` (2) and inductively assigned to terms via the new predicate called `is`. It builds a proposition out of a `term` and a decoration. I.e.,  $\forall i : \text{Loc}, \text{is } \text{ro } (\text{lookup } i)$  is a Prop instance, ensuring that `lookup i` is an accessor.

```

Inductive kind  $\triangleq$  pure | ro | rw.
Inductive is: kind  $\rightarrow$   $\forall$  X Y, term X Y  $\rightarrow$  Prop  $\triangleq$ 
| is_tpure:  $\forall$  X Y (f: X  $\rightarrow$  Y), is pure (@tpure X Y f)
| is_comp:  $\forall$  k X Y Z (f: term X Y) (g: term Y Z), is k f  $\rightarrow$  is k g  $\rightarrow$  is k (f o g)
| is_pair:  $\forall$  k X Y Z (f: term X Z) (g: term Y Z), is ro f  $\rightarrow$  is k f  $\rightarrow$  is k g  $\rightarrow$ 
is k (pair f g)

| is_lookup:  $\forall$  i, is ro (lookup i)
| is_update:  $\forall$  i, is rw (update i)
| is_pure_ro:  $\forall$  X Y (f: term X Y), is pure f  $\rightarrow$  is ro f
| is_ro_rw:  $\forall$  X Y (f: term X Y), is ro f  $\rightarrow$  is rw f.

```

Any term that is built by the `tpure` constructor is pure. The decoration of any composed or paired off term depends on its components and always takes the upper decoration ( $\text{pure} < \text{ro} < \text{rw}$ ). E.g., given a modifier term and a read-only term, their composition will be a modifier, as well. The decoration of a pair construction depends on its second component, since the first one should at most be a read-only term. Hence, we cannot form pairs of two modifier terms. The pair construction always takes the upper decoration. For instance, given a pure term and a read-only term, their pair will be a read-only term, too. We declare the term `lookup` as an accessor while `update` being a modifier. The last two constructors define the decoration hierarchies.

It is trivial to derive that any `tpure` built term is pure. I.e., the purity of the first pair projection can be proven as follows:

```

Lemma is_pi1 X Y: is pure (@pi1 X Y).
Proof. apply is_tpure. Qed.

```

We now state the rules up to weak and strong equalities by defining them in a mutually inductive way: mutuality here is used to enable the constructors including both weak and strong equalities. We use the notation  $\text{==}$  and  $\sim$  to denote strong and weak equalities, respectively.

```

Definition idem X Y (x y: term X Y)  $\triangleq$  x = y.
Inductive strong:  $\forall$  X Y, relation (term X Y)  $\triangleq$ 
| refl X Y: Reflexive (@strong X Y)
| sym:  $\forall$  X Y, Symmetric (@strong X Y)
| trans:  $\forall$  X Y, Transitive (@strong X Y)
| replsubs:  $\forall$  X Y Z, Proper (@strong X Y  $\Rightarrow$  @strong Y Z  $\Rightarrow$  @strong X Z) comp
| wtos:  $\forall$  X Y (f g: term X Y), is ro f  $\rightarrow$  is ro g  $\rightarrow$  f  $\sim$  g  $\rightarrow$  f == g
| s_lpair_eq:  $\forall$  X Y' Y (f1: term Y X) (f2: term Y' X), is ro f1  $\rightarrow$  pi2 o pair f1 f2 == f2
| effect:  $\forall$  X Y (f g: term Y X), forget o f == forget o g  $\rightarrow$  f  $\sim$  g  $\rightarrow$  f == g
| local_global:  $\forall$  X (f g: term unit X), ( $\forall$  i: Loc, lookup i o f  $\sim$  lookup i o g)  $\rightarrow$  f == g
| tcomp:  $\forall$  X Y Z (f: Z  $\rightarrow$  Y) (g: Y  $\rightarrow$  X), tpure (compose g f) == tpure g o tpure f

with weak:  $\forall$  X Y, relation (term X Y)  $\triangleq$ 
| wsym:  $\forall$  X Y, Symmetric (@weak X Y)
| wtrans:  $\forall$  X Y, Transitive (@weak X Y)
| pwepl:  $\forall$  A B C (g: term C B), (is pure g)  $\rightarrow$  Proper (@weak B A  $\Rightarrow$  @weak C A) (comp g)
| wsubs:  $\forall$  A B C, Proper (@weak C B  $\Rightarrow$  @idem B A  $\Rightarrow$  @weak C A) comp
| stow:  $\forall$  X Y (f g: term X Y), f == g  $\rightarrow$  f - g
| w_lpair_eq:  $\forall$  X Y' Y (f1: term Y X) (f2: term Y' X), is ro f1  $\rightarrow$  pi1 o pair f1 f2  $\sim$  f1
| w_unit:  $\forall$  X (f g: term unit X), f - g
| ax1:  $\forall$  i, lookup i o update i  $\sim$  id
| ax2:  $\forall$  i j, i  $\neq$  j  $\rightarrow$  lookup j o update i  $\sim$  lookup j o forget
where "x == y"  $\triangleq$  (strong x y) and "x ~ y"  $\triangleq$  (weak x y).

```

The rule `tcomp` states that the `tpure` constructor preserves the composition of pure terms up to the strong equality: one can first compose pure terms on Coq side (using higher order function `compose`) and then

apply `tpure` constructor to translate them into decorated settings or can translate the terms first and then compose them in decorated settings.

This framework allows us to express and prove, in Coq, the decorated versions of the properties mentioned in Section 3.1. E.g., the commutation `lookup-uptade` looks like:

```
(** Commutation update update **)
Theorem CUU: ∀ i j: Loc, i≠j → update j o (pi2 o (rprod (update i) (@id (Val j)))) ==
update i o (pi1 o (lprod (@id (Val i)) (update j))).
```

where

```
Definition permut {X Y}: term (X*Y) (Y*X) ≜ pair pi2 pi1.
Definition rpair {X Y Z} (f: term Y X) (g: term Z X): term (Y*Z) X ≜ permut o pair g f.
Definition lprod {X Y X' Y'} (f: term X X') (g: term Y Y'): term (X*Y) (X'*Y') ≜ pair (f o pi1) (g o pi2).
Definition rprod {X Y X' Y'} (f: term X X') (g: term Y Y') ≜
permut o pair (g o pi2) (f o pi1).
```

The full Coq proofs of such properties can be found [here](#), and the entire implementation [there](#).

## 4 The Decorated Logic for the exception effect ( $\mathcal{L}_{exc}$ )

Similar to the  $\mathcal{L}_{st}$ , the decorated logic for the exception effect ( $\mathcal{L}_{exc}$ ) is obtained from equational logics by classifying terms and equations (9). Terms are classified as *pure* terms, *propagators* and *catchers*. The decoration of a term characterizes the way it may interact with exceptions, and used as a superscript. The decoration (0) is reserved for terms which are *pure*, while (1) is for *propagators* (or *throwers*) and (2) is for *catchers*. Equalities are classified as *strong* and *weak*, denoted respectively by the symbols  $\equiv$  and  $\sim$ . A fundamental feature of the exceptions mechanism is the distinction between *ordinary* (*non-exceptional*) values and *exceptions* (or *exceptional values*). An equality between two terms relates their behaviors on ordinary and exceptional values. Two terms are called strongly equal if they behave the same both on ordinary and exceptional values; they are called weakly equal if they behave the same on ordinary values but different on exceptional ones.

It has been shown in (10) that the core part of this proof system is dual to one for the state. Based on this nice duality, we build the logic  $\mathcal{L}_{exc}$ , and detail it in the following.

<b>Grammar of the decorated logic for the exception:</b>		( $e \in EName$ )
Types:	$t, s ::=$	$X \mid Y \mid \dots \mid t+s \mid \odot \mid EV_e$
Decoration for terms:	$(d) ::=$	$(0) \mid (1) \mid (2)$
Terms:	$f, g ::=$	$a^{(d)} \mid b^{(d)} \mid \dots \mid g \circ f^{(d)} \mid [f \mid g]_1^{(d)}$ $\text{tag}_e^{(1)} \mid \text{untag}_e^{(2)} \mid (\downarrow f)^{(1)} \mid (\text{tpure } \cdot)^{(0)}$
Equations:	$\text{eq} ::=$	$f^{(d)} \equiv g^{(d)} \mid f^{(d)} \sim g^{(d)}$

**Figure 10:**  $\mathcal{L}_{exc}$ : syntax

Figure 10 shows the grammar of  $\mathcal{L}_{exc}$  where  $\mathbb{O}$  is the empty (uninhabited) type while  $EV_e$  is the type of parameters for each exception name  $e$ . We assume that there is a finite set of exception names called  $EName$ . Given types  $X$  and  $Y$ , we have  $X+Y$  denoting co-product (disjoint union or sum) types.

Terms are closed under composition ( $\circ$ ) and co-pairing ( $[\_ | \_ ]_1$ ). I.e., for all terms  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , we have  $g \circ f: X \rightarrow Z$ . Similarly, for all  $f: X \rightarrow Y$  and  $g: Z \rightarrow Y$ , there is  $[f | g]_1: X+Z \rightarrow Y$ . Notice that the co-pair subscript 1 denotes the left co-pairs. One can define in a symmetric way the right co-pairs for terms  $f: X \rightarrow Y$  and  $g: Z \rightarrow Y$  as  $[f | g]_r := [g, f]_1 \circ \text{permut}$  where  $\text{permut} := [\text{in}_2 | \text{in}_1]_1$ . Similarly, one can respectively obtain left and right co-products (sums) of terms  $f: X_1 \rightarrow Y_1$  and  $g: X_2 \rightarrow Y_2$  as  $f+_1g := [\text{in}_1 \circ f | \text{in}_2 \circ g]_1$  and  $f+_rg := [\text{in}_1 \circ f | \text{in}_2 \circ g]_r$ . The term co-pairs, thus co-products, are used to model the *case distinction* among terms. The decoration of a co-pair (co-product) or a composition depends on the decoration of its components, always taking the larger. I.e.,  $\forall f^{(1)}: X \rightarrow Y$  and  $g^{(2)}: Y \rightarrow Z$ ,  $g \circ f: X \rightarrow Z$  takes the decoration (2).

The interface terms are  $\text{tag}_e: EV_e \rightarrow \mathbb{O}+E$  and  $\text{untag}_e: \mathbb{O}+E \rightarrow EV_e+E$  where  $E$  denotes the distinguished object of exceptions which never appears in the decorated setting. The use of decorations provides a new schema where term signatures are constructed without any occurrence of it. For instance,  $\text{tag}_e^{(1)}: EV_e \rightarrow \mathbb{O}$  is a thrower while  $\text{untag}_e^{(2)}: \mathbb{O} \rightarrow EV_e$  is a catcher. This way, we keep signatures close to their syntax and compose compatible terms as usual. The term  $\text{tag}_e$  encapsulates an ordinary value with an exception of name  $e$  while the term  $\text{untag}_e$  tries to recover the value from the exceptional case.

The ‘ $\downarrow$ ’ symbol denotes the downcast term that takes as input a term and prevents it from catching exceptions. It is used when to define the `try/catch` block in this settings. See Definition 4.2.

The `tpure` is a special constructor used to introduce decorated pure terms into the logic  $\mathcal{L}_{exc}$ . It inputs a non-decorated pure term and drops it in with the decoration (0). The identity term `id`, the canonical co-pair inclusions `in1` and `in2`, and the empty co-pair term  $[\ ]_X$  (used to convert the type of input exceptional value into the given type:  $X$  in this case) are defined using the `tpure` constructor, for all types  $X$  and  $Y$ , as follows:

$$\begin{aligned} \text{id}_X^{(0)} &: X \rightarrow X &:= & \text{tpure id} \\ \text{in}_1^{(0)} &: X \rightarrow X+Y &:= & \text{tpure inl} \\ \text{in}_2^{(0)} &: Y \rightarrow X+Y &:= & \text{tpure inr} \\ [\ ]_X^{(0)} &: \mathbb{O} \rightarrow X &:= & \text{tpure } (\lambda \_ : \mathbb{O}. x : X) \end{aligned}$$

**Definition 4.1.** For each type  $Y$  and exception name  $e$ , the propagator  $\text{throw}_{Y,e}^{(1)}$  is defined as:

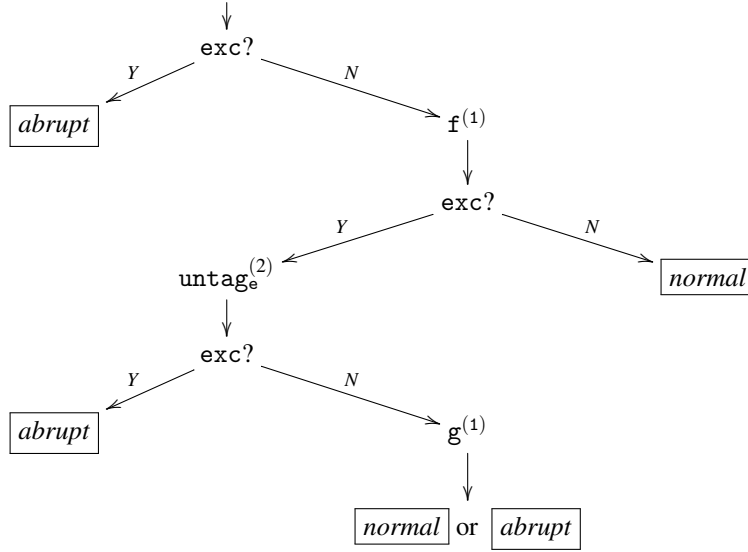
$$\text{throw}_{Y,e}^{(1)} := [\ ]_Y^{(0)} \circ \text{tag}_e^{(1)}: EV_e \rightarrow Y$$

Intuitively, raising an exception with name  $e$  is first tagging the given ordinary value as an exception of name  $e$  and then coerces the output type into  $Y$  for the continuation issues.

**Definition 4.2.** For each propagators  $f^{(1)}: X \rightarrow Y$ ,  $g^{(1)}: EV_e \rightarrow Y$  and each exception name  $e$ , the propagator  $\text{try}(f)\text{catch}(e \Rightarrow g)^{(1)}$  is defined in three steps, as follows:

$$\begin{aligned} \text{Catch}(e \Rightarrow g)^{(2)} &:= [\text{id}_Y^{(0)} | g^{(1)} \circ \text{untag}_e^{(2)}]_1: & Y+\mathbb{O} \rightarrow Y \\ \text{Try}(f)\text{Catch}(e \Rightarrow g)^{(2)} &:= \text{Catch}(e \Rightarrow g)^{(2)} \circ \text{in}_{Y,\mathbb{O}}^{(0)} \circ f^{(1)}: & X \rightarrow Y \\ \text{try}(f)\text{catch}(e \Rightarrow g)^{(1)} &:= \downarrow(\text{Try}(f)\text{Catch}(e \Rightarrow g)^{(2)}): & X \rightarrow Y \end{aligned}$$

To handle an exception, the intermediate expressions  $\text{Catch}(e \Rightarrow g)$  and  $\text{Try}(f)\text{Catch}(e \Rightarrow g)$  are private catchers and the expression  $\text{try}(f)\text{catch}(e \Rightarrow g)$  is a public propagator: the downcast operator prevents it from catching exceptions with name  $e$  which might have been raised before the  $\text{try}(f)\text{catch}(e \Rightarrow g)$  expression. The definition of  $\text{try}(f)\text{catch}(e \Rightarrow g)$  corresponds to the Java mechanism for exceptions (14, Ch. 14) and (18) with the following control flow, where  $\text{exc?}$  means “is this value an exception?”, an *abrupt* termination returns an uncaught exception and a *normal* termination returns an ordinary value.



The intended model of the above grammar is built with respect to the set of exceptions  $E$  where a pure term  $p^{(0)} : X \rightarrow Y$  is interpreted as a function  $p : X \rightarrow Y$ , a propagator  $pp^{(1)} : X \rightarrow Y + E$ , and a catcher  $c^{(2)} : X \rightarrow Y$  as a function  $c : X + E \rightarrow Y + E$ . The complete and detailed category theoretical model is given in (12, §6.1).

The syntax given in Figure 10 is enriched with a set of rules which are presented in Figure 11. Hierarchically, a *pure* term can be seen as a *propagator* which never throws exceptions (0-to-1), and similarly a *propagator* can be seen as a *catcher* which never recovers from exceptions (1-to-2). Weak equalities do not form a congruence: the term substitution cannot be done unless the substituted term is pure. I.e., given the equation  $f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z$  and a term  $g : X \rightarrow Y$ , it is possible to get the equation  $f_1 \circ g \sim f_2 \circ g$  only when the term  $g$  is pure. At this stage, we have no information about the behaviors of  $f_1$  and  $f_2$  on exceptional values. Therefore, the pre-executed term  $g$  would destroy this result equality unless being pure, for instance, by throwing an exception of name  $e$  for which  $f_1$  and  $f_2$  can perform different behaviors: one can propagate, while the other can recover from it (pwrepl). However, the term replacement can be done regardless of the term decoration. I.e., given the equation  $f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y$  and a term  $g : Y \rightarrow Z$ , it is possible to get the equation  $g \circ f_1 \sim g \circ f_2$  independent from the decoration of the term  $g$ . We already now that  $f_1$  and  $f_2$  behave the same on ordinary values, executing any term  $g$  afterwards would not end the composition behave different on ordinary values (wrepl). Strong equalities form a congruence by allowing both term substitutions and replacements regardless of the term decorations (replsubs).

**Rules of the decorated logic for the exception:**

$$\begin{array}{c}
(0\text{-to-}1) \frac{f^{(0)}}{f^{(1)}} \quad (1\text{-to-}2) \frac{f^{(1)}}{f^{(2)}} \quad (\text{pwsubs}) \frac{g^{(0)} : X \rightarrow Y \quad f_1^{(2)} \sim f_2^{(2)} : Y \rightarrow Z}{f_1 \circ g \sim f_2 \circ g} \\
(\text{wrepl}) \frac{f_1^{(2)} \sim f_2^{(2)} : X \rightarrow Y \quad g^{(2)} : Y \rightarrow Z}{g \circ f_1 \sim g \circ f_2} \quad (\text{wrefl}) \frac{f^{(2)}}{f \sim f} \quad (\text{wsym}) \frac{f^{(2)} \sim g^{(2)}}{g \sim f} \\
(\text{wtrans}) \frac{f^{(2)} \sim g^{(2)} \quad g^{(2)} \sim h^{(2)}}{f \sim h} \quad (\text{replsubs}) \frac{f_1^{(2)} \equiv f_2^{(2)} : X \rightarrow Y \quad g_1^{(2)} \equiv g_2^{(2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv g_2 \circ f_2} \\
(\text{refl}) \frac{f^{(2)}}{f \equiv f} \quad (\text{sym}) \frac{f^{(2)} \equiv g^{(2)}}{g \equiv f} \quad (\text{trans}) \frac{f^{(2)} \equiv g^{(2)} \quad g^{(2)} \equiv h^{(2)}}{f \equiv h} \quad (\text{w\_empty}) \frac{f^{(2)} : \mathbb{O} \rightarrow X}{f \sim []_X} \\
(\text{estow}) \frac{f^{(2)} \equiv g^{(2)}}{f \sim g} \quad (\text{ewtos}) \frac{f_1^{(1)} \sim f_2^{(1)}}{f_1 \equiv f_2} \quad (\text{w\_downcast}) \frac{f^{(2)} : Y \rightarrow X}{(\downarrow f)^{(1)} \sim f} \\
(\text{eax}_1) \frac{}{\text{untag}_e^{(2)} \circ \text{tag}_e^{(1)} \sim \text{id}_{\text{EV}_e}^{(0)}} \quad (\text{eax}_2) \frac{\forall e_1, e_2 \in \text{EName}, e_1 \neq e_2}{\text{untag}_{e_1}^{(2)} \circ \text{tag}_{e_2}^{(1)} \sim []_{\text{EV}_e}^{(0)} \circ \text{tag}_{e_2}^{(1)}} \\
(\text{effect}) \frac{f_1^{(2)}, f_2^{(2)} : Y \rightarrow X \quad f_1^{(2)} \sim f_2^{(2)} \quad f_1^{(2)} \circ []_Y^{(0)} \equiv f_2^{(2)} \circ []_Y^{(0)}}{f_1 \equiv f_2} \\
(\text{elocal\_global}) \frac{f_1^{(2)}, f_2^{(2)} : \mathbb{O} \rightarrow X \quad \forall e \in \text{EName}, f_1^{(2)} \circ \text{tag}_e^{(1)} \sim f_2^{(2)} \circ \text{tag}_e^{(1)}}{f_1 \equiv f_2} \\
(\text{w\_lco pair\_eq}) \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_1 \sim f_1} \quad (\text{s\_lco pair\_eq}) \frac{f_1^{(1)} : X \rightarrow Y \quad f_2^{(2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_2 \equiv f_2} \\
(\text{tcomp}) \frac{f : Y \rightarrow Z \quad g : X \rightarrow Y}{(\text{tpure } f)^{(0)} \circ (\text{tpure } g)^{(0)} \equiv (\text{tpure } (f \circ g))^{(0)}}
\end{array}$$

**Figure 11:**  $\mathcal{L}_{\text{exc}}$  rules

Both strong and weak equalities are defined to be *equivalence relations* assuming that they are *reflexive*, *transitive* and *symmetric*.

Any term  $f : \mathbb{O} \rightarrow X$  with no input parameter has an equivalence on ordinary values with the empty co-pair  $[]_X$  ( $\text{w\_empty}$ ). It is obviously free to convert strong equations into weak ones ( $\text{estow}$ ). However, one has to make sure that the equated terms are not catchers in order to convert weak equations into strong ones with no further evidence ( $\text{ewtos}$ ).

The rule ( $\text{w\_downcast}$ ) states that the term  $(\downarrow f)$  behaves as  $f$ , if the argument is ordinary. It prevents  $f$  from catching the exceptional arguments otherwise.

The fundamental equations are given with the rules ( $\text{eax}_1$ ) and ( $\text{eax}_2$ ). The former states that encapsulating an ordinary value with an exception of name  $e$  followed by an immediate recovery would be equivalent to “doing nothing” in terms of ordinary values. Clearly, this is only a weak equation since its sides behave different on exceptional values: left hand side may recover but right hand side definitely propagates. The latter, ( $\text{eax}_2$ ), is to assume that encapsulating an ordinary value  $v$  with an exception of name  $e_2$  and then trying to recover it from a different exception of name  $e_1$  would just lead  $e_2$  to be propagated. Similarly, if



the ordinary value  $v$  is encapsulated with  $e_2$  with no recovery attempt afterwards would again lead  $e_2$  to be propagated. These two operations behave the same on ordinary values but different on exceptional ones. For instance, left hand side recovers the input value encapsulated with the exception name  $e_1$  while right hand side propagates it.

Two catchers  $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$  behave the same on exceptional values if and only if  $f_1 \circ [ ]_X \equiv f_2 \circ [ ]_X$ , where  $[ ]_X : \mathbb{O} \rightarrow X$  throws out the exceptional values. So that  $f_1^{(2)}, f_2^{(2)} : X \rightarrow Y$  are *strongly equal* if and only if  $f_1 \sim f_2$  and  $f_1 \circ [ ]_X \equiv f_2 \circ [ ]_X$  (eeffect).

Strong equality between two catchers  $f_1^{(2)}, f_2^{(2)} : \mathbb{O} \rightarrow X$  can also be expressed as a pair of weak equations:  $f_1 \sim f_2$  and  $\forall e \in \text{EName}, f_1 \circ \text{tag}_e \sim f_2 \circ \text{tag}_e$ . The latter intuitively means that  $f_1$  and  $f_2$  behaves the same on all (finitely many) exceptional values when executed. Given that both behave the same on ordinary arguments (due to (w\_empty)), there is no explicitly need to check if  $f_1 \sim f_2$ . It suffices to see whether  $\forall e \in \text{EName}, f_1 \circ \text{tag}_e \sim f_2 \circ \text{tag}_e$  to end up with  $f_1 \equiv f_2$  (elocal\_global).

With (w\_lcopair\_eq) and (w\_rcopair\_eq) term co-pairs (sums) are characterized: the (left) co-pair structure  $[f_1 \mid f_2]_1$  cannot be used when  $f_1$  and  $f_2$ , both are catchers, since it may lead to a conflict on exceptional values. When  $f_1$  is a propagator, with (w-copair-eq), we assume that ordinary values of type  $X$  are treated by  $[f_1 \mid f_2]_1^{(2)}$  as they would be by  $f_1^{(1)}$  and with (s-copair-eq) that ordinary values of type  $Z$  and exceptional values are treated by  $[f_1 \mid f_2]_1^{(2)}$  as they would be by  $f_2^{(2)}$ .

The rule tcomp states that the tpure constructor preserves the composition of pure terms up to the strong equality. Meaning that one can first compose pure terms outside the decorated environment (in Coq's logic for instance) and use the tpure constructor to translate them into the  $\mathcal{L}_{st}$ , or translate the terms into the  $\mathcal{L}_{st}$  first, and then compose them there.

Similar to the rules of the logic  $\mathcal{L}_{st}$ , the rules of the logic  $\mathcal{L}_{exc}$  also designed to be sound with respect to a categorical model, and detailed in (12, §6.2, §6.3, §6.4, §6.5). In addition, in (8) we prove that this set of rules is complete with respect to the notion of relative Hilbert-Post completeness.

#### 4.1 Decorated properties of the exception effect

Similar to the one for the state effect presented in Section 3.1, we propose an equational representation of the exception effect with the following decorated equations:

- (1)<sub>d</sub> Annihilation tag-untag. *Untagging an exception of name  $e$  and then raising it again is just like doing nothing.*  $\forall e \in \text{EName}, \text{tag}_e^{(1)} \circ \text{untag}_e^{(2)} \equiv \text{id}_{\mathbb{O}}^{(0)} : \mathbb{O} \rightarrow \mathbb{O}$ .
- (2)<sub>d</sub> Commutation untag-untag. *Untagging two distinct exception names can be done in any order.*  
 $\forall e \neq r \in \text{EName}, (\text{untag}_e +_r \text{id}_{EV_r})^{(2)} \circ \text{in}_2^{(0)} \circ \text{untag}_r^{(2)} \equiv$   
 $(\text{id}_{EV_e} +_1 \text{untag}_r)^{(2)} \circ \text{in}_1^{(0)} \circ \text{untag}_e^{(2)} : \mathbb{O} \rightarrow EV_e + EV_r$ .
- (3)<sub>d</sub> Propagator-propagates. *A propagator term always propagates the exception.*  
 $\forall e \in \text{EName}, a^{(1)} : X \rightarrow Y, a^{(1)} \circ [ ]_X^{(0)} \circ \text{tag}_e^{(1)} \equiv [ ]_Y^{(0)} \circ \text{tag}_e^{(1)} : EV_e \rightarrow Y$ .
- (4)<sub>d</sub> Recovery. *The parameter used for throwing an exception may be recovered.*  
 $(\forall f^{(1)}, g^{(1)} : X \rightarrow \mathbb{O}, [ ]_Y^{(0)} \circ f^{(1)} \equiv [ ]_Y^{(0)} \circ g^{(1)} \implies f^{(1)} \equiv g^{(1)}) \implies$   
 $(\forall e \in \text{EName}, u_1^{(0)}, u_2^{(0)} : X \rightarrow EV_e, ([ ]_Y^{(0)} \circ \text{tag}_e^{(1)} \circ u_1^{(0)} \equiv [ ]_Y^{(0)} \circ \text{tag}_e^{(1)} \circ u_2^{(0)}) \implies u_1^{(0)} \equiv u_2^{(0)})$ .

- (5)<sub>d</sub> *Try*. The strong equation is compatible with `try/catch`.  
 $\forall e \in \text{EName}, a_1^{(1)}, a_2^{(1)} : X \rightarrow Y, b^{(1)} : \text{EV}_e \rightarrow Y, a_1^{(1)} \equiv a_2^{(1)} \implies$   
 $(\downarrow ([\text{id}_Y \mid b \circ \text{untag}_e]_1 \circ \text{in}_1 \circ [\ ]_Y \circ \text{tag}_e \circ a_1)^{(1)} \equiv \downarrow ([\text{id}_Y \mid b \circ \text{untag}_e]_1 \circ \text{in}_1 \circ [\ ]_Y \circ \text{tag}_e \circ a_2)^{(1)}).$
- (6)<sub>d</sub> *Try*<sub>0</sub>. Pure code inside `try` never triggers the code inside `catch`.  
 $\forall e \in \text{EName}, u^{(0)} : X \rightarrow Y, b^{(1)} : \text{EV}_e \rightarrow Y,$   
 $\downarrow ([\text{id}_Y \mid b \circ \text{untag}_e]_1 \circ \text{in}_1 \circ u)^{(1)} \equiv \text{id}_Y^{(0)} \circ u^{(0)} : X \rightarrow Y.$
- (7)<sub>d</sub> *Try*<sub>1</sub>. The code inside `catch` is executed as soon as an exception is thrown inside `try`.  
 $\forall e \in \text{EName}, u^{(0)} : X \rightarrow \text{EV}_e, b^{(1)} : \text{EV}_e \rightarrow Y,$   
 $\downarrow ([\text{id}_Y \mid b \circ \text{untag}_e]_1 \circ \text{in}_1 \circ [\ ]_Y \circ \text{tag}_e \circ u)^{(1)} \equiv b^{(1)} \circ u^{(0)} : X \rightarrow Y.$
- (8)<sub>d</sub> *Try*<sub>2</sub>. An exception gets propagated, if the exception name is not pattern matched in `catch`.  
 $\forall (e \neq f) \in \text{EName}, u^{(0)} : X \rightarrow \text{EV}_f, b^{(1)} : \text{EV}_e \rightarrow Y,$   
 $\downarrow ([\text{id}_Y \mid b \circ \text{untag}_e]_1 \circ \text{in}_1 \circ [\ ]_Y \circ \text{tag}_f \circ u)^{(1)} \equiv [\ ]_Y^{(0)} \circ \text{tag}_f^{(1)} \circ u^{(0)} : X \rightarrow Y.$

These are the archetype properties that we have proved within the scope of the  $\mathcal{L}_{exc}$ . To see these proofs, check out (12, §6.7). Besides, we have implemented the  $\mathcal{L}_{exc}$  in Coq to certify mentioned proofs. Section 4.2 briefly discusses this implementation.

## 4.2 $\mathcal{L}_{exc}$ in Coq

Coq implementation of  $\mathcal{L}_{exc}$  follows the same approach with the one for  $\mathcal{L}_{st}$ . We represent the set of exception names by a Coq parameter `EName : Type`. An arrow type `EVa1 : EName → Type` is assumed as the type of values (parameters) for each exception name. We then inductively define terms and, assign decorations over them. There, we respectively use keywords `epure`, `ppg` and `ctc` instead of (0), (1) and (2). The rules up to weak and strong equalities are stated in a mutually inductive way to write constructors including both types of equalities, similar to the approach presented in Section 3.2. We choose not to replay the entire Coq encoding here, but we formalize Definitions 4.1 and 4.2 in Coq terms as follows:

```

Definition throw (X: Type) (e: EName) ≙ (@empty X) o tag e.
Definition try_catch (X Y: Type) (e: EName) (f: term Y X) (g: term Y (Val e)) ≙
  downcast (copair (@id Y) (g o untag e) o in1 o f).

```

The encodings of other terms are contained in [this file](#).

We can conclude that such a framework allows us to express and prove, in Coq, the decorated versions of the properties mentioned in Section 4.1. E.g., the propagator-propagates looks like:

```

(** Propagator propagates **)
Lemma PPT: ∀ X Y (e: EName) (a: term Y X), is_ppg a → a o ((@empty X) o tag e) == (@empty Y) o tag e.

```

The full Coq proofs of such properties can be found [here](#), and the entire implementation [there](#).

## 5 Combining $\mathcal{L}_{st}$ and $\mathcal{L}_{exc}$

In order to formally cope with different computational effects, one needs to compose the related formal models. For instance, using monad transformers (19), it is usually possible to combine effects formalized by monads, as encoded in Haskell. Handler compositions allow combining effects modelled by algebraic handlers, as implemented in Eff (1, 2, 34) and Idris (4). To combine effects formalized in decorated settings, we just need to compose the related logics. In this section, we formally study the combination of the state and the exception effects using the logics  $\mathcal{L}_{st}$  and  $\mathcal{L}_{exc}$ . We call the newly born logic *the decorated logic for the state and the exception*, and denote it  $\mathcal{L}_{st+exc}$ . To start with, we give below the syntax of  $\mathcal{L}_{st+exc}$ :

**Grammar of the decorated logic for the state and the exception:** ( $i \in \text{Loc}$ ) ( $e \in \text{EName}$ )

Types:	$\mathbf{t}, \mathbf{s}$	$::=$	$X \mid Y \mid \dots \mid \mathbf{t} \times \mathbf{s} \mid \mathbf{t} + \mathbf{s} \mid \mathbb{1} \mid \mathbb{0} \mid \mathbf{V}_i \mid \mathbf{E}\mathbf{V}_e$
Decoration for terms:	$(d_1, d_2)$	$::=$	$(0, 0) \mid (0, 1) \mid (0, 2) \mid (1, 0) \mid (1, 1) \mid$ $(1, 2) \mid (2, 0) \mid (2, 1) \mid (2, 2)$
Terms:	$\mathbf{f}, \mathbf{g}$	$::=$	$\mathbf{a}^{(d_1, d_2)} \mid \mathbf{b}^{(d_1, d_2)} \mid \dots \mid \mathbf{g} \circ \mathbf{f}^{(d_1, d_2)} \mid$ $\langle \mathbf{f}, \mathbf{g} \rangle_1^{(d_1, d_2)} \mid [\mathbf{f} \mid \mathbf{g}]_1^{(d_1, d_2)} \mid$ $\text{lookup}_i^{(1, 0)} \mid \text{update}_i^{(2, 0)} \mid$ $\text{tag}_e^{(0, 1)} \mid \text{untag}_e^{(0, 2)} \mid$ $(\downarrow \mathbf{f})^{(0, 1)} \mid (\text{tpure } \bullet)^{(0, 0)}$
Equations:	$\text{eq}$	$::=$	$\mathbf{f}^{(d_1, d_2)} \equiv \equiv \mathbf{g}^{(d_1, d_2)} \mid \mathbf{f}^{(d_1, d_2)} \equiv \sim \mathbf{g}^{(d_1, d_2)} \mid$ $\mathbf{f}^{(d_1, d_2)} \sim \equiv \mathbf{g}^{(d_1, d_2)} \mid \mathbf{f}^{(d_1, d_2)} \sim \sim \mathbf{g}^{(d_1, d_2)}$

**Figure 12:**  $\mathcal{L}_{st+exc}$ : syntax

The decorations are paired off to cover all possible combinations: the decoration symbol on the left is given in terms of the state effect while the one on the right is of the exception. I.e.,  $\mathbf{f}^{(1, 2)}$  says that  $\mathbf{f}$  may *access* to the state alongside *catching* exceptions. The decoration of a (co)-pair ((co)-product) or a composition pair depends on the decorations of its components, always taking the larger. I.e.,  $\forall \mathbf{f}^{(1, 2)} : X \rightarrow Y$  and  $\mathbf{g}^{(2, 1)} : Y \rightarrow Z$ ,  $\mathbf{g} \circ \mathbf{f} : X \rightarrow Z$  takes the decoration  $(2, 2)$ .

Types and terms are manually unionized in such a way that the interface terms for the state effect are pure with respect to the exception and vice versa:  $\text{lookup}_i^{(1, 0)}$ ,  $\text{update}_i^{(2, 0)}$ ,  $\text{tag}_e^{(0, 1)}$  and  $\text{untag}_e^{(0, 2)}$ . As in Sections 3 and 4, we use the special  $\text{tpure}$  constructor to define pure terms such as the identity  $\text{id}$ , the canonical pair projections  $\pi_1$  and  $\pi_2$ , the empty pair  $\langle \rangle$  (throws out the given value), the canonical co-pair

inclusions  $\text{in}_1$  and  $\text{in}_2$ , the empty co-pair  $[]$  and constant terms, for all types  $X$  and  $Y$ , as follows:

$$\begin{array}{llll}
 \text{id}_X^{(0,0)} & : & X \rightarrow X & := & \text{tpure id} \\
 \pi_1^{(0,0)} & : & X \times Y \rightarrow X & := & \text{tpure fst} \\
 \pi_2^{(0,0)} & : & X \times Y \rightarrow Y & := & \text{tpure snd} \\
 \langle \rangle_X^{(0,0)} & : & X \rightarrow \mathbb{1} & := & \text{tpure } (\lambda x : X. \text{void} : \mathbb{1}) \\
 \text{in}_1^{(0,0)} & : & X \rightarrow X+Y & := & \text{tpure inl} \\
 \text{in}_2^{(0,0)} & : & Y \rightarrow X+Y & := & \text{tpure inr} \\
 []_X^{(0,0)} & : & \mathbb{0} \rightarrow X & := & \text{tpure } (\lambda \_ : \mathbb{0}. x : X) \\
 \text{constant}_X^{(0,0)} & : & \mathbb{1} \rightarrow X & := & \text{tpure } (\lambda \_ . x : X)
 \end{array}$$

The rule combinations need a bit of reformulation as we summarize below:

- The decoration symbol (0) freely converts into (1) and (2), while the symbol (1) just into (2) when the other symbol is fixed. I.e.,  $f^{(0,2)}$  freely converts into  $f^{(1,2)}$ . See all cases below:

$$\begin{array}{l}
 - \frac{f^{(0,d)}}{f^{(1,d)}}, \frac{f^{(1,d)}}{f^{(2,d)}} \text{ for } d \in \{0, 1, 2\} \\
 - \frac{f^{(d,0)}}{f^{(d,1)}}, \frac{f^{(d,1)}}{f^{(d,2)}} \text{ for } d \in \{0, 1, 2\}
 \end{array}$$

- We have all possible combinations of equality sorts:  $\equiv\equiv$ ,  $\equiv\sim$ ,  $\sim\equiv$  and  $\sim\sim$ . The first equality symbol relates terms with respect to the state effect. I.e.,  $f \equiv\sim g$  means that  $f$  and  $g$  are strongly equal with respect to the state, while being weakly equal with respect to the exception. Below we present the conversion rules between these four sorts. The burden here is that a strong equality symbol can always be freely converted into a weak one independent of according to which effect it relates terms. But, to convert a weak equality symbol into a strong one, we need to make sure that the related terms are decorated either with (0) or (1) with respect to the effect they are weakly related.

$$\begin{array}{l}
 - (\equiv\equiv\text{-to-}\equiv\sim) \frac{f^{(2,2)} \equiv\equiv g^{(2,2)}}{f \equiv\sim g}, \quad (\equiv\equiv\text{-to-}\sim\equiv) \frac{f^{(2,2)} \equiv\equiv g^{(2,2)}}{f \sim\equiv g} \\
 - (\equiv\sim\text{-to-}\sim\sim) \frac{f^{(2,2)} \equiv\sim g^{(2,2)}}{f \sim\sim g}, \quad (\sim\equiv\text{-to-}\sim\sim) \frac{f^{(2,2)} \sim\equiv g^{(2,2)}}{f \sim\sim g} \\
 - (\sim\equiv\text{-to-}\equiv\equiv) \frac{f^{(1,2)} \sim\equiv g^{(1,2)}}{f \equiv\equiv g}, \quad (\equiv\sim\text{-to-}\equiv\equiv) \frac{f^{(2,1)} \equiv\sim g^{(2,1)}}{f \equiv\equiv g} \\
 - (\sim\sim\text{-to-}\equiv\equiv) \frac{f^{(1,2)} \sim\sim g^{(1,2)}}{f \equiv\equiv g}, \quad (\sim\sim\text{-to-}\sim\equiv) \frac{f^{(2,1)} \sim\sim g^{(2,1)}}{f \sim\equiv g}
 \end{array}$$

- The rules of the logic  $\mathcal{L}_{st+exc}$  are presented in Figure 13 as a union of the ones given in Figures 9 and 11 in terms of new equality sorts and refined term decorations. There, we replay the whole rule bodies, and implicitly assume that all equality sorts are equivalence relations respecting the properties *reflexivity*, *symmetry*, and *transitivity*.

**Rules of the decorated logic for the state and the exception:**

$$\begin{array}{c}
\text{(pwrapl)} \frac{f_1^{(2,2)} \sim \equiv f_2^{(2,2)} : X \rightarrow Y \quad g^{(0,2)} : Y \rightarrow Z}{g \circ f_1 \sim \equiv g \circ f_2} \quad \text{(wsubs)} \frac{g^{(2,2)} : X \rightarrow Y \quad f_1^{(2,2)} \sim \equiv f_2^{(2,2)} : Y \rightarrow Z}{f_1 \circ g \sim \equiv f_2 \circ g} \\
\text{(pwsubs)} \frac{g^{(2,0)} : X \rightarrow Y \quad f_1^{(2,2)} \sim f_2^{(2,2)} : Y \rightarrow Z}{f_1 \circ g \sim f_2 \circ g} \quad \text{(wrepl)} \frac{f_1^{(2,2)} \sim f_2^{(2,2)} : X \rightarrow Y \quad g^{(2,2)} : Y \rightarrow Z}{g \circ f_1 \sim g \circ f_2} \\
\text{(replsubs)} \frac{f_1^{(2,2)} \equiv \equiv f_2^{(2,2)} : X \rightarrow Y \quad g_1^{(2,2)} \equiv \equiv g_2^{(2,2)} : Y \rightarrow Z}{g_1 \circ f_1 \equiv \equiv g_2 \circ f_2} \\
\text{(w\_unit)} \frac{f^{(2,2)} : X \rightarrow \mathbb{1}}{f \sim \equiv \langle \rangle_X} \quad \text{(w\_empty)} \frac{f^{(2,2)} : \mathbb{0} \rightarrow X}{f \equiv \sim []_X} \quad \text{(w\_downcast)} \frac{f^{(2,2)} : Y \rightarrow X}{(\downarrow f)^{(2,1)} \equiv \sim f} \\
\text{(ax}_1\text{)} \frac{}{\text{lookup}_i^{(1,0)} \circ \text{update}_i^{(2,0)} \sim \equiv \text{id}_{V_i}^{(0,0)}} \quad \text{(ax}_2\text{)} \frac{}{\text{lookup}_i^{(1,0)} \circ \text{update}_j^{(2,0)} \sim \equiv \text{lookup}_i^{(1,0)} \circ \langle \rangle_{V_i}^{(0,0)}} \\
\text{(eax}_1\text{)} \frac{}{\text{untag}_e^{(0,2)} \circ \text{tag}_e^{(0,1)} \sim \equiv \text{id}_{EV_e}^{(0,0)}} \quad \text{(eax}_2\text{)} \frac{\forall e_1, e_2 \in \text{EName}, e_1 \neq e_2}{\text{untag}_{e_1}^{(0,2)} \circ \text{tag}_{e_2}^{(0,1)} \equiv \sim []_{EV_e}^{(0,0)} \circ \text{tag}_{e_2}^{(0,1)}} \\
\text{(effect)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : X \rightarrow Y \quad f_1^{(2,2)} \sim \equiv f_2^{(2,2)} \quad \langle \rangle_Y^{(0,0)} \circ f_1^{(2,2)} \equiv \equiv \langle \rangle_Y^{(0,0)} \circ f_2^{(2,2)}}{f_1 \equiv \equiv f_2} \\
\text{(eeffect)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : Y \rightarrow X \quad f_1^{(2,2)} \equiv \sim f_2^{(2,2)} \quad f_1^{(2,2)} \circ []_Y^{(0,0)} \equiv \equiv f_2^{(2,2)} \circ []_Y^{(0,0)}}{f_1 \equiv \equiv f_2} \\
\text{(local\_global)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : X \rightarrow \mathbb{1} \quad \forall i \in \text{Loc}, \text{lookup}_i^{(1,0)} \circ f_1^{(2,2)} \sim \equiv \text{lookup}_i^{(1,0)} \circ f_2^{(2,2)}}{f_1 \equiv \equiv f_2} \\
\text{(elocal\_global)} \frac{f_1^{(2,2)}, f_2^{(2,2)} : \mathbb{0} \rightarrow X \quad \forall e \in \text{EName}, f_1^{(2,2)} \circ \text{tag}_e^{(0,1)} \equiv \sim \circ f_2^{(2,2)} \circ \text{tag}_e^{(0,1)}}{f_1 \equiv \equiv f_2} \\
\text{(w\_lpair\_eq)} \frac{f_1^{(1,2)} : X \rightarrow Y \quad f_2^{(2,2)} : X \rightarrow Z}{\pi_1 \circ \langle f_1, f_2 \rangle_1 \sim \equiv f_1} \quad \text{(s\_lpair\_eq)} \frac{f_1^{(1,2)} : X \rightarrow Y \quad f_2^{(2,2)} : X \rightarrow Z}{\pi_2 \circ \langle f_1, f_2 \rangle_1 \equiv \equiv f_2} \\
\text{(w\_lcopy\_eq)} \frac{f_1^{(2,1)} : X \rightarrow Y \quad f_2^{(2,2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_1 \equiv \sim f_1} \quad \text{(s\_lcopy\_eq)} \frac{f_1^{(2,1)} : X \rightarrow Y \quad f_2^{(2,2)} : Z \rightarrow Y}{[f_1 \mid f_2] \circ \text{in}_2 \equiv \equiv f_2} \\
\text{(tcomp)} \frac{f : Y \rightarrow Z \quad g : X \rightarrow Y}{(\text{tpure } f)^{(0,0)} \circ (\text{tpure } g)^{(0,0)} \equiv \equiv (\text{tpure } (f \circ g))^{(0,0)}}
\end{array}$$

Figure 13:  $\mathcal{L}_{st+exc}$ : rules

### 5.1 Decorated properties of the state and exception effects

The properties given in Sections 3.1 and 4.1 are now stated with the refined term decorations, and related with the  $\equiv$  equation. I.e., propagator-propagates and annihilation untag-untag look like:

$$\begin{aligned} \forall e \in \text{EName}, a^{(0,1)} : X \rightarrow Y, a^{(0,1)} \circ [ ]_X^{(0,0)} \circ \text{tag}_e^{(0,1)} &\equiv [ ]_Y^{(0,0)} \circ \text{tag}_e^{(0,1)} : \text{EV}_e \rightarrow Y. \\ \forall i \neq j \in \text{Loc}, \text{update}_j^{(2,0)} \circ \pi_2^{(0,0)} \circ (\text{update}_i^{(2,0)} \times_r \text{id}_{V_j}^{(0,0)}) &\equiv \\ \text{update}_i^{(2,0)} \circ \pi_1^{(0,0)} \circ (\text{id}_{V_i}^{(0,0)} \times_l \text{update}_j^{(2,0)}) : V_i \times V_j &\rightarrow \mathbb{1}. \end{aligned}$$

These are the archetype properties that we can prove within the scope of the  $\mathcal{L}_{st+exc}$ . However, we prefer not to prove them for this generic framework; instead, we first specialize them in a way to serve as a target language for a denotational semantics of IMP+Exc, and then prove them for the specialized version. Also, we encode the specialized version in Coq and certify related proofs. Section 6 gives the related details.

## 6 IMP+Exc over decorated logic

Now, it comes to define a denotational semantics for the IMP+Exc language, with the combined decorated logic for the state and the exception ( $\mathcal{L}_{st+exc}$ ) as the target language. Recall that by doing this, we aim to prove some (strong) equalities between terminating programs written in IMP+Exc with respect to the state and the exception effects.

In IMP+Exc, the values that can be stored in any location (variable)  $i$  are just integers. So that any occurrence of  $(V_i)$  in term signatures of  $\mathcal{L}_{st+exc}$  is replaced by  $\mathbb{Z}$ . I.e.,  $\text{lookup}^{(1,0)} : \mathbb{1} \rightarrow \mathbb{Z}$  while  $\mathbb{Z}$ . I.e.,  $\text{update}^{(2,0)} : \mathbb{Z} \rightarrow \mathbb{1}$ . Here, we start with defining a denotational semantics of IMP+Exc expressions over combined decorated settings using a translator function  $\text{dExp}$ . This function takes an expression as input and outputs a decorated term of type  $\text{term } \mathbb{Z} \mathbb{1}$  or  $\text{term } \mathbb{B} \mathbb{1}$  depending on the input expression type:

$$\begin{aligned} \text{dExp } n &\Rightarrow (\text{constant } n)^{(0,0)} \\ \text{dExp } x &\Rightarrow (\text{lookup } x)^{(1,0)} \\ \text{dExp } (a_1 + a_2) &\Rightarrow (\text{tpure add})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\ \text{dExp } (a_1 \times a_2) &\Rightarrow (\text{tpure mlt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\ \text{dExp } (a_1 - a_2) &\Rightarrow (\text{tpure subt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\ \text{dExp } (\text{true}) &\Rightarrow (\text{constant true})^{(0,0)} \\ \text{dExp } (\text{false}) &\Rightarrow (\text{constant false})^{(0,0)} \\ \text{dExp } (a_1 \stackrel{?}{=} a_2) &\Rightarrow (\text{tpure chkeq})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\ \text{dExp } (a_1 \stackrel{?}{\neq} a_2) &\Rightarrow (\text{tpure chkneq})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\ \text{dExp } (a_1 \stackrel{?}{>} a_2) &\Rightarrow (\text{tpure chkggt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \\ \text{dExp } (a_1 \stackrel{?}{<} a_2) &\Rightarrow (\text{tpure chklt})^{(0,0)} \circ \langle \text{dExp } a_1, \text{dExp } a_2 \rangle_1^{(1,0)} \end{aligned}$$

$$\begin{aligned}
\text{dExp } (b_1 \wedge b_2) &\Rightarrow (\text{tpure andB})^{(0,0)} \circ \langle \text{dExp } b_1, \text{dExp } b_2 \rangle_1^{(1,0)} \\
\text{dExp } (b_1 \vee b_2) &\Rightarrow (\text{tpure orB})^{(0,0)} \circ \langle \text{dExp } b_1, \text{dExp } b_2 \rangle_1^{(1,0)} \\
\text{dExp } (\neg b) &\Rightarrow (\text{tpure notB})^{(0,0)} \circ \text{dExp } b^{(1,0)}
\end{aligned}$$

where *true* and *false* are Boolean constructors. The constructor `tpure` is applied to given unary functions. For instance `add: (ℤ × ℤ) → ℤ` takes an instance of an integer tuple and returns their sum. To see the definition of the other functions in a Coq implementation, please check out [this file](#).

We have some additional rules to make use of some pure algebraic operations in the combined decorated setting presented in Figure 14 where the pure term `lpi: ℤ → ℤ`, within the rule (`imp-li`), is used to connect successive loop iterations as long as the loop conditional evaluates into decorated logic's *true* (*constant true*). Also, the pure term `pbl: ℤ → ℤ + ℤ` forms a bridge between the usual Boolean data type and its correspondence in the decorated settings which is the type `ℤ + ℤ`.

$$\begin{aligned}
\text{lpi } (b : \text{term } \mathbb{Z} (\mathbb{Z} + \mathbb{Z})) (f : \text{term } \mathbb{Z} \mathbb{Z}) &:= \text{tpure } (\lambda x : \mathbb{Z}. x). \\
\text{pbl} &:= \text{tpure } (\text{bool\_to\_two}) \\
\text{where } \text{bool\_to\_two } (b : \text{bool}) &:= (\text{if } b \text{ then } (\text{inl void}) \text{ else } (\text{inr void})). \\
\text{such that } \text{void} : \mathbb{Z} &\text{ is the unique constructor of the type } \mathbb{Z}, \text{ and} \\
\text{inl, inr} : \mathbb{Z} \rightarrow (\mathbb{Z} + \mathbb{Z}) &\text{ are the canonical inclusions}
\end{aligned}$$

Note also that in (`imp2`) and (`imp4`) by replacing *false* into *true* we get (`imp3`) and (`imp5`) that are not explicitly stated in Figure 14.

$$\begin{aligned}
(\text{imp}_1) &\frac{\forall p, q : \mathbb{Z}, (f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z})}{\text{tpure } f \circ \langle \text{constant } p, \text{constant } q \rangle_1 \equiv \equiv (\text{constant } f(p, q))} \\
(\text{imp}_2) &\frac{\forall p, q : \mathbb{Z}, (f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{B}) \quad f(p, q) = \text{false}}{\text{tpure } f \circ \langle \text{constant } p, \text{constant } q \rangle_1 \equiv \equiv \text{constant } \text{false}} \\
(\text{imp}_4) &\frac{\forall p, q : \mathbb{B}, (f : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}) \quad f(p, q) = \text{false}}{\text{tpure } f \circ \langle \text{constant } p, \text{constant } q \rangle_1 \equiv \equiv \text{constant } \text{false}} \\
(\text{imp-li}) &\frac{\forall (b : \text{term } \mathbb{Z} (\mathbb{Z} + \mathbb{Z})) (f : \text{term } \mathbb{Z} \mathbb{Z})}{\text{lpi } b \quad f \equiv \equiv [(\text{lpi } b \quad f) \circ f \text{ id}]_1 \circ b} \\
(\text{imp}_6) &\frac{(\forall x, f \quad x = g \quad x)}{\text{tpure } f \equiv \equiv \text{tpure } g}
\end{aligned}$$

**Figure 14:** Additional rules on puer terms: IMP+Exc specific

**Lemma 6.1.**  $\text{pbl}^{(0,0)} \circ (\text{constant } \text{false})^{(0,0)} \equiv \equiv \text{in}_2$ .

**Proof:** unfolding all term definitions, we have  $\text{tpure } (\lambda : \text{bool}. \text{if } b \text{ then } (\text{inl } \mathbb{Z}) \text{ else } (\text{inr } \mathbb{Z})) \circ \text{tpure } (\lambda _ : \mathbb{Z}. \text{true}) \equiv \equiv \text{tpure } \text{inl}$ . Now, it is trivial to obtain  $\text{tpure } \text{inl} \equiv \equiv \text{tpure } \text{inl}$  by first rewriting `tcomp` from left to right, and then applying `imp6`.  $\square$

**Lemma 6.2.**  $\text{pbl}^{(0,0)} \circ (\text{constant } \text{true})^{(0,0)} \equiv \text{in}_1$ .

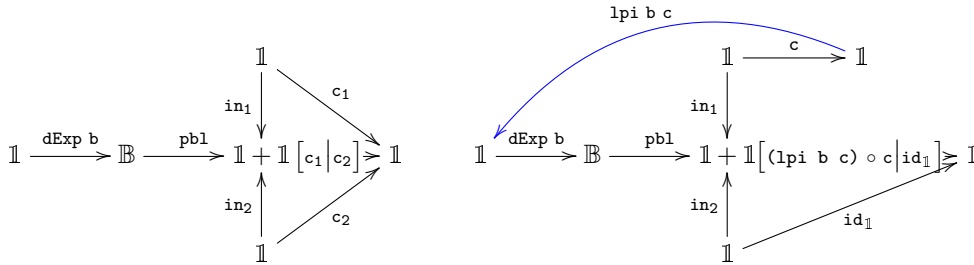
**Proof:** It follows the same steps with the proof of Lemma 6.1  $\square$

The fact that IMP+Exc commands are of type  $\mathbb{1} \rightarrow \mathbb{1}$ , and since  $\text{throw}_e^{(0,1)} := [\ ]_Y \circ \text{tag}_e^{(0,1)} : \text{EV}_e \rightarrow Y$  is a command, we replace  $\text{EV}_e$  and  $Y$  with  $\mathbb{1}$ . This means that we stick to a single exceptional value (parameter), for each exception name  $e \in \text{EName}$ .

Below, we recursively define the IMP+Exc commands within  $\mathcal{L}_{st+exc}$  using a translator function  $\text{dCmd}$  which establishes a decorated term of type  $\text{term } \mathbb{1} \mathbb{1}$  out of an input command.

$$\begin{aligned}
\text{dCmd (SKIP)} &\Rightarrow (\text{id } \mathbb{1})^{(0,0)} \\
\text{dCmd (x } \triangleq \text{ a)} &\Rightarrow (\text{update}_{e_x})^{(2,0)} \circ (\text{dExp a})^{(1,0)} \\
\text{dCmd (c}_1; \text{c}_2) &\Rightarrow (\text{dCmd c}_2)^{(2,0)} \circ (\text{dCmd c}_1)^{(2,0)} \\
\text{dCmd (if b then c}_1 \text{ else c}_2) &\Rightarrow \left[ \text{dCmd c}_1 \mid \text{dCmd c}_2 \right]_1^{(2,0)} \circ \text{pbl}^{(0,0)} \circ (\text{dExp b})^{(1,0)} \\
\text{dCmd (while e do c)} &\Rightarrow \left[ (\text{lpi (pbl} \circ (\text{dExp b})) (\text{dCmd c})) \circ (\text{dCmd c}) \mid \text{id} \right]_1^{(2,0)} \\
&\quad \circ \text{pbl}^{(0,0)} \circ (\text{dExp b})^{(1,0)} \\
\text{dCmd (THROW e)} &\Rightarrow \text{throw } e^{(0,1)} \\
\text{dCmd (TRY c}_1 \text{ CATCH e } \Rightarrow \text{c}_2) &\Rightarrow \text{try(f)catch(e } \Rightarrow \text{g)}^{(0,1)}
\end{aligned}$$

In Figure 15, the diagram on the left schematizes the command `if b then c1 else c2`: if the Boolean expression `dExp b` evaluates into `(constant true)` then by Lemma 6.2, we have the command `c1` in execution, `c2` otherwise by Lemma 6.1. As for the loops, it is well known that as long as the looping condition evaluates into `(constant true)`, loop body gets executed. This is depicted in Figure 15 (the diagram on the right), as the arrow `lpi b c` is each time replaced by the whole diagram itself. This is made possible by the rule `(imp-li)`. If the looping condition evaluates into `(constant false)`, using Lemma 6.1, we then have the term `id1` in execution forcing the loop to terminate. Notice that the case distinction in the diagrams are provided by the term inclusions.



**Figure 15:** `(cond b c1 c2)` and `(while b do c)` in  $\mathcal{L}_{st+exc}$



Figure 16 respectively visualizes the formal behaviors of THROW and TRY/CATCH commands where the basis is the core decorated terms for the exception effect. They are formulated as in Definitions 4.1 and 4.2 with a single difference in their signatures: domains and co-domains are now set to  $\mathbb{1}$ .

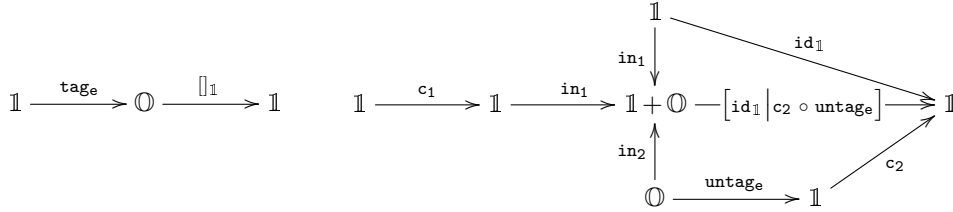


Figure 16: (THROW  $e$ ) and (TRY  $c_1$  CATCH  $e \Rightarrow c_2$ ) in  $\mathcal{L}_{st+exc}$

We now implement the IMP+Exc denotational semantics using the  $\mathcal{L}_{st+exc}$  as the target language in a Coq encoding. Expressions are inductively forming a new Coq Type by taking another Coq Type as input, and called Exp. Indeed, the type Exp is a dependent type. This means that the type Exp A depends on the term A : Type. I.e., when  $A := \mathbb{B}$ , we build the type for Boolean expressions while the case  $A := \mathbb{Z}$  is to construct the type for arithmetic expressions. As for the type constructors, we use the syntactic operators given within the contexts of aexp and bexp in Figure 2. The only difference is in the constructor names: instead of the corresponding notations, they are given in plain text. However, it is easy to match them one another since they are given in the same order. I.e., ttrue is used instead of the true expression symbol. Note also that true stands for the Boolean constructor.

```

Inductive Exp : Type → Type  $\triangleq$ 
| const :  $\forall A, A \rightarrow \text{Exp } A$ 
| var : Loc  $\rightarrow \text{Exp } Z$ 
| plus : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } Z$ 
| subtr : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } Z$ 
| mult : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } Z$ 
| ttrue : Exp bool
| tfalse : Exp bool
| eq : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } \text{bool}$ 
| neq : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } \text{bool}$ 
| gt : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } \text{bool}$ 
| lt : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } \text{bool}$ 
| ge : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } \text{bool}$ 
| le : Exp Z  $\rightarrow \text{Exp } Z \rightarrow \text{Exp } \text{bool}$ 
| and : Exp bool  $\rightarrow \text{Exp } \text{bool} \rightarrow \text{Exp } \text{bool}$ 
| or : Exp bool  $\rightarrow \text{Exp } \text{bool} \rightarrow \text{Exp } \text{bool}$ 
| neg : Exp bool  $\rightarrow \text{Exp } \text{bool}$ .

```

Let us encode the dExp function in Coq using a fixpoint which is also called dExp:

```

Fixpoint dExp A (e : Exp A) : term A unit  $\triangleq$ 
match e with
| const Z n  $\Rightarrow$  constant n
| var x  $\Rightarrow$  lookup x
| plus a1 a2  $\Rightarrow$  tpure add o pair (dExp Z a1) (dExp Z a2)
| subtr a1 a2  $\Rightarrow$  tpure subt o pair (dExp Z a1) (dExp Z a2)
| mult a1 a2  $\Rightarrow$  tpure mlt o pair (dExp Z a1) (dExp Z a2)
| ttrue  $\Rightarrow$  constant true

```

```

| ffalse      ⇒ constant false
| eq a1 a2    ⇒ tpure chkeq o pair (dExp Z a1) (dExp Z a2)
| neq a1 a2   ⇒ tpure chkneq o pair (dExp Z a1) (dExp Z a2)
| gt a1 a2    ⇒ tpure chkgt o pair (dExp Z a1) (dExp Z a2)
| lt a1 a2    ⇒ tpure chklt o pair (dExp Z a1) (dExp Z a2)
| ge a1 a2    ⇒ tpure chkge o pair (dExp Z a1) (dExp Z a2)
| le a1 a2    ⇒ tpure chkle o pair (dExp Z a1) (dExp Z a2)
| and b1 b2   ⇒ tpure andB o pair (dExp bool b1) (dExp bool b2)
| or b1 b2    ⇒ tpure orB o pair (dExp bool b1) (dExp bool b2)
| neg b       ⇒ tpure notB o (dExp bool b)
end.

```

A similar idea of implementation follows for the commands. We inductively define a Coq type `Cmd` of IMP+Exc commands whose constructors are the members of IMP+Exc command set as presented in Figures 2 and 6. Notice that some commands are encoded with different names. I.e., the assignment command ‘ $\triangleq$ ’ is called `assign`, the sequencing command ‘`;`’ is called `sequence` while “if then else” block is named `cond` in the implementation. However, it is easy to match them one another since they are presented in the same order.

```

Inductive Cmd : Type ≜
| SKIP      : Cmd
| sequence  : Cmd → Cmd → Cmd
| assign    : Loc → Exp Z → Cmd
| cond      : Exp bool → Cmd → Cmd → Cmd
| while     : Exp bool → Cmd → Cmd
| THROW     : EName → Cmd
| TRY_CATCH : EName → Cmd → Cmd → Cmd.

```

We now encode the `dCmd` function in Coq using a fixpoint which is also called `dCmd`:

```

Fixpoint dCmd (c: Cmd): (term unit unit) ≜
match c with
| skip      ⇒ (@id unit)
| sequence c0 c1 ⇒ (dCmd c1) o (dCmd c0)
| assign j e0 ⇒ (update j) o (dExp Z e0)
| cond b c2 c3 ⇒ copair (dCmd c2) (dCmd c3) o (pb1 o (dExp bool b))
| while b c4 ⇒ (copair (lpi (pb1 o (dExp bool b)) (dCmd c4) o (dCmd c4)) (@id unit)) o (pb1 o (dExp bool b))
| THROW e ⇒ (throw unit e)
| TRY_CATCH e c1 c2 ⇒ (try_catch e (dCmd c1) (dCmd c2))
end.

```

Now, we retain sufficient material to state and prove equivalences between programs written in IMP+Exc, and certify such proofs in Coq.

## 6.1 Program equivalence proofs

In this section, we finally prove equivalences of bunch of programs written in IMP+Exc, using the denotational semantics characterized within the scope of the logic  $\mathcal{L}_{st+exc}$ . Note that for the sake of simplicity, we will use  $u_x$ ,  $l_x$ ,  $(t \text{ op})$  and  $(c \text{ p})$  instead of  $(\text{update } x)^{(2,0)}$ ,  $(\text{lookup } x)^{(1,0)}$ ,  $(\text{tpure op})^{(0,0)}$  and  $(\text{constant p})^{(0,0)}$ , respectively.

*Remark 6.3.* Recall that the use of term products is to impose some order of term evaluation on the mutable state. IMP+Exc specific properties of the mutable state are slightly different than their generic versions

(mentioned in Section 3.1) due to the fact that the language does not allow parallel term evaluations, meaning that every term is evaluated in the sequence they are given. Therefore, we no more need to use term products in property statements. The properties we use through out the following proofs are re-stated in Figure 17. The full certified Coq proofs of these properties can be found [here](#).

1. `interaction update-update`  $\forall x \in \text{Loc } p, q : \mathbb{Z}, u_x \circ (c p) \circ u_x \circ (c q) \equiv u_x \circ (c p)$
2. `commutation update-update`  $\forall x \neq y \in \text{Loc } p, q : \mathbb{Z}, u_x \circ (c p) \circ u_y \circ (c q) \equiv u_y \circ (c q) \circ u_x \circ (c p)$
3. `commutation-lookup-constant-update`  $\forall x \in \text{Loc}, p, q \in \mathbb{Z}, \langle l_x, (c q) \rangle_1 \circ u_x \circ (c p) \equiv \langle (c p), (c q) \rangle_1 \circ u_x \circ (c p)$

**Figure 17:** Primitive properties of the state: IMP+Exc specific

*Remark 6.4.* Below, we state three lemme using the IMP+Exc notation introduced in Figures 2 and 6. However, we introduce a new set of notations for the Coq encoding to increase the readability score even a little: browse this set of notations [here](#) where, i.e., the `assign` command is denoted by ‘`::=`’ while the `sequence` command by ‘`;;`’. These notations do not appear through out the paper, but might be of help in reading the lemma statements in the Coq encoding. Notice also that they are not so pretty, due to the fact that Coq internally reserves prettier notations for other issues.

**Lemma 6.5.** *For all exceptionally pure commands  $f, g \in \text{cmd} \setminus \{\text{THROW}, \text{TRY}/\text{CATCH}\}$  and  $b \in \{\text{true}, \text{false}\}$ , if program pieces `prog1` and `prog2` are given as in the following listings, then  $\text{dCmd}(\text{prog1}) \equiv \text{dCmd}(\text{prog2})$ .*

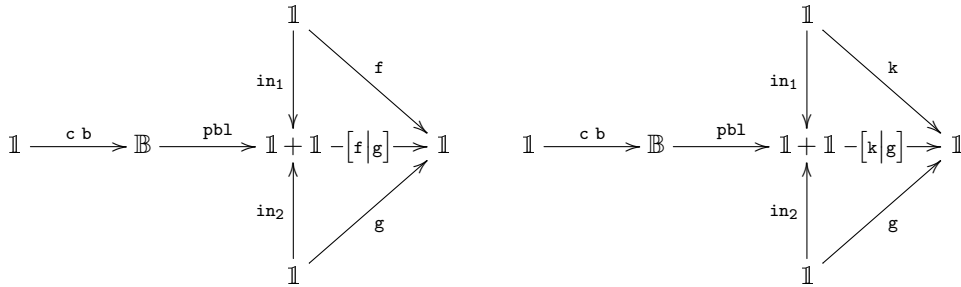
**Listing 1:** `prog1`

```
/* prog1 */
if b then f else g;
```

**Listing 2:** `prog2`

```
/* prog2 */
if b then (if b then f else g)
else g;
```

**Proof:** We sketch the diagrams of both programs below:



where  $k = (\text{if } b \text{ then } f \text{ else } g)$ . The statement we would like to prove is

$$[f|g]_1 \circ \text{pb1} \circ c b \equiv [k|g]_1 \circ \text{pb1} \circ c b. \quad (1)$$

Using the decorated rules of the logic  $\mathcal{L}_{st+exc}$ , in the below given order, our aim is to simplify both sides of the statement into the same shape. The proof proceeds by a case analysis on  $b$ .

If  $b = \text{false}$ , by unfolding the definitions of `pbl` and (*c false*), we have

$$[f|g]_1 \circ t (\text{bool\_to\_two}) \circ t (\lambda x : \text{unit}. \text{false}) \equiv [k|g]_1 \circ t (\text{bool\_to\_two}) \circ t (\lambda x : \text{unit}. \text{false}). \quad (2)$$

We rewrite (`tcomp`), and get

$$[f|g]_1 \circ t (\lambda x : \text{unit}. \text{bool\_to\_two } \text{false}) \equiv [k|g]_1 \circ t (\lambda x : \text{unit}. \text{bool\_two } \text{false}). \quad (3)$$

Now, we cut

$$t (\lambda x : \text{unit}. \text{bool\_to\_two } \text{false}) \equiv \text{in}_2 \quad (4)$$

and rewrite it back in the goal. So that we obtain

$$[f|g]_1 \circ \text{in}_2 \equiv [k|g]_1 \circ \text{in}_2. \quad (5)$$

Then, we use (`s_lcopair_eq`), and finally have  $g \equiv g$  which is trivial since  $\equiv$  is reflexive. It remains to show that the cut statement in Equation 4 is true. By simplifying `t (λx : unit. bool_to_two false)` and unfolding `in2`, we have

$$t (\lambda x : \text{unit}. \text{inr } x) \equiv t (\text{inr}). \quad (6)$$

Now, we apply (`imp6`) and get

$$\forall x : \text{unit}, \text{inr } x = \text{inr } x \quad (7)$$

which is trivial since the Leibniz equality '=' is reflexive.

If  $b = \text{true}$ , by following above procedure with `true` (instead of `false`) we first handle

$$[f|g]_1 \circ \text{in}_1 \equiv [k|g]_1 \circ \text{in}_1 \quad (8)$$

and then freely convert  $\equiv$  into  $\equiv \sim$ . There, rewriting the rule (`w_lcopair_eq`) yields  $f \equiv \sim k$ . We unfold `k` with  $b = \text{true}$  and get

$$f \equiv \sim [f|g]_1 \circ \text{in}_1. \quad (9)$$

Now by rewriting (`w_lcopair_eq`), we have  $f \equiv \sim f$ , which is again trivial, since  $\equiv \sim$  is reflexive.  $\square$

**Lemma 6.6.** *For all  $x : \text{Loc}$ , if program pieces `prog3` and `prog4` are given as in the following listings, then  $\text{dCmd } (\text{prog3}) \equiv \text{dCmd } (\text{prog4})$ .*

**Listing 3:** `prog3`

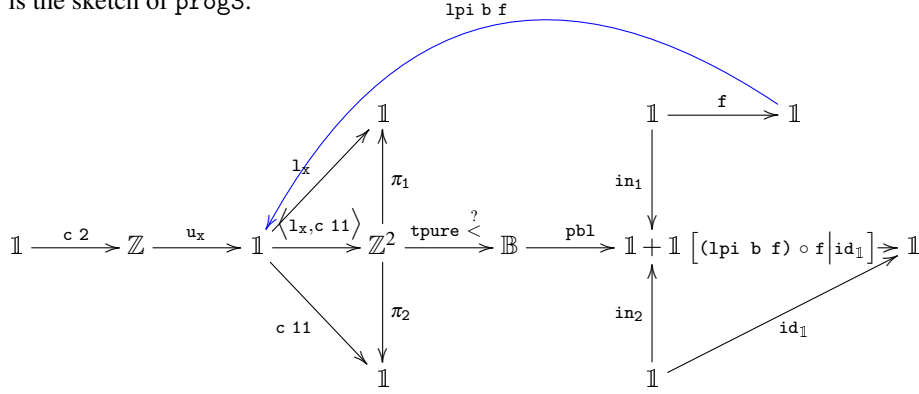
```
/* prog3 */
x  $\triangleq$  2;
while (x < 11)
  do (x  $\triangleq$  x + 4);
```

**Listing 4:** `prog4`

```
/* prog4 */
x  $\triangleq$  14;
```

**Proof:** In the proof structure we intend to reduce `prog3`, first dealing with the pre-loop assignments and the looping pre-condition. Since it evaluates into *true*, in the second step we identify things related to the first loop iteration. The third step primarily studies the second and then the third loop iterations after which the looping pre-condition switches to *false*. Finally, we explain the program termination and show that `prog3` does exactly the same state manipulation with `prog4`. Note also that we do not need to check the results they returned, since all IMP+Exc commands, thus programs, return `void : U`.

Below is the sketch of prog3:



where  $f = (x \triangleq x + 4)$  and  $b = (x < 11)$ . Using the decorated rules of the logic  $\mathcal{L}_{st+exc}$ , given in the below order, we simplify this diagram into

$$1 \xrightarrow{c\ 14} Z \xrightarrow{u_x} 1$$

which is actually prog4 when sketched.

1. Initially, we have

$$[(lpi\ b\ f) \circ f | id_1] \circ pbl \circ (t < ?) \circ \langle l_x, (c\ 11) \rangle \circ u_x \circ (c\ 2) \equiv u_x \circ (c\ 14). \quad (10)$$

Let us simplify it as far as possible. By using commutation – lookup – constant – update (see Figure 17), we obtain

$$[(lpi\ b\ f) \circ f | id_1] \circ pbl \circ (t < ?) \circ \langle (c\ 2), (c\ 11) \rangle \circ u_x \circ (c\ 2) \equiv u_x \circ (c\ 14). \quad (11)$$

Since the looping pre-condition  $(t < ?) \circ \langle (c\ 2), (c\ 11) \rangle$  evaluates into  $(c\ true)$ , due to  $(imp_3)$ , we have

$$[(lpi\ b\ f) \circ f | id_1] \circ pbl \circ (c\ true) \circ u_x \circ (c\ 2) \equiv u_x \circ (c\ 14). \quad (12)$$

By rewriting the Lemma 6.2, we get

$$[(lpi\ b\ f) \circ f | id_1] \circ in_1 \circ u_x \circ (c\ 2) \equiv u_x \circ (c\ 14). \quad (13)$$

Here, we first convert  $\equiv$  into  $\equiv \sim$  then rewrite  $(w\_lco\ pair\_eq)$ , and end up with

$$(lpi\ b\ f) \circ f \circ u_x \circ (c\ 2) \equiv \sim u_x \circ (c\ 14) \quad (14)$$

in which the second appearance of  $f$  unfolds into

$$(lpi\ b\ f) \circ u_x \circ (t +) \circ \langle l_x, c\ 4 \rangle \circ u_x \circ (c\ 2) \equiv \sim u_x \circ (c\ 14). \quad (15)$$

Since, there is no exceptional case, we are freely back to  $\equiv$ . By rewriting `commutation –lookup –constant –update`, we obtain

$$(\text{lp}i \text{ b } f) \circ u_x \circ (t +) \circ \langle c \ 2, c \ 4 \rangle \circ u_x \circ (c \ 2) \equiv u_x \circ (c \ 14). \quad (16)$$

The rule `(imp1)` gives

$$(\text{lp}i \text{ b } f) \circ u_x \circ (c \ 6) \circ u_x \circ (c \ 2) \equiv u_x \circ (c \ 14). \quad (17)$$

Now, by rewriting the lemma `interaction-update-update` (see Figure 17), we get

$$(\text{lp}i \text{ b } f) \circ u_x \circ (c \ 6) \equiv u_x \circ (c \ 14). \quad (18)$$

2. For the second loop iteration, rewriting `(imp-li)` gives

$$[(\text{lp}i \text{ b } f) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \overset{?}{<}) \circ \langle l_x, (c \ 11) \rangle \circ u_x \circ (c \ 6) \equiv u_x \circ (c \ 14). \quad (19)$$

where looping pre-condition evaluates into  $(c \ \text{true})$ . Therefore, we iterate the above procedure, given in the step 1, once again and derive

$$(\text{lp}i \text{ b } f) \circ u_x \circ (c \ 10) \equiv u_x \circ (c \ 14). \quad (20)$$

3. In the third iteration, rewriting the `(imp-li)` gives

$$[(\text{lp}i \text{ b } f) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \overset{?}{<}) \circ \langle l_x, (c \ 11) \rangle \circ u_x \circ (c \ 10) \equiv u_x \circ (c \ 14). \quad (21)$$

As in step 2, the looping pre-condition evaluates into  $(c \ \text{true})$  forcing us to reiterate the above procedure, given in the step 1, which results in

$$(\text{lp}i \text{ b } f) \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (22)$$

4. In the fourth step, rewriting the `(imp-li)` gives

$$[(\text{lp}i \text{ b } f) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \overset{?}{<}) \circ \langle l_x, (c \ 11) \rangle \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (23)$$

By rewriting `commutation –lookup –constant –update`, we obtain

$$[(\text{lp}i \text{ b } f) \circ f | \text{id}_1] \circ \text{pbl} \circ (t \overset{?}{<}) \circ \langle (c \ 14), (c \ 11) \rangle \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (24)$$

Finally here, the looping pre-condition  $(t \overset{?}{<}) \circ \langle (c \ 14), (c \ 11) \rangle$  evaluates into  $(c \ \text{false})$  which reduces into

$$[(\text{lp}i \text{ b } f) \circ f | \text{id}_1] \circ \text{pbl} \circ (c \ \text{false}) \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (25)$$

We rewrite the Lemma 6.1, and get

$$[(\text{lp}i \text{ b } f) \circ f | \text{id}_1] \circ \text{in}_2 \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14). \quad (26)$$

Now, we rewrite `(s_lcopair_eq)`, and handle

$$\text{id}_1 \circ u_x \circ (c \ 14) \equiv u_x \circ (c \ 14) \quad (27)$$

which is trivial, since the identity term disappears when to compose and  $\equiv$  is reflexive.  $\square$

**Lemma 6.7.** For each  $x y : \text{Loc}$ ,  $e : \text{EName}$ , if program pieces  $\text{prog5}$  and  $\text{prog6}$  are given as in the following listings, then  $\text{dCmd}(\text{prog5}) \equiv \text{dCmd}(\text{prog6})$ .

Listing 5: prog5

```

/* prog3 */
x  $\triangleq$  1;
y  $\triangleq$  20;
TRY(
  while (true)
  do
    ( if (x  $\leq$  0) then (THROW e)
      else x  $\triangleq$  x - 1
    )
) CATCH e  $\Rightarrow$  (y  $\triangleq$  7);

```

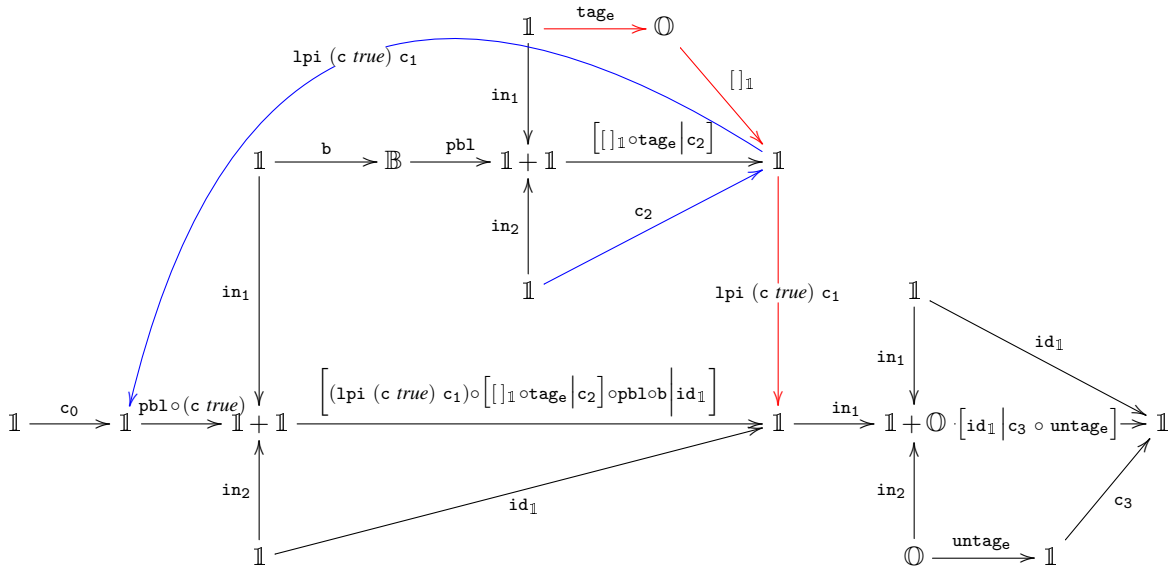
Listing 6: prog6

```

/* prog4 */
x  $\triangleq$  0;
y  $\triangleq$  7;

```

**Proof:** In the proof structure, we first tackle with the downcast operator. The second task is to deal with the first loop iteration which has the state but no exception effect. In the third, we study the second iteration of the loop where an exception is thrown, and the abrupt loop termination afterwards. Finally, in the fourth step, we explain the exception recovery and the normal program termination. Below is the sketch of prog5:



where  $b = (x \leq 0)$ ,  $c_0 = (x \triangleq 1; y \triangleq 20)$ ,  $c_1 = (\text{if } (x \leq 0) \text{ then } (\text{THROW } e) \text{ else } (x \triangleq x - 1))$ ,  $c_2 = (x \triangleq x - 1)$ ,  $c_3 = (y \triangleq 7)$ , blue arrows indicate the normal loop iterations while red ones are to identify the program behavior after the exception of name  $e$  is raised. Using the rules of the logic  $\mathcal{L}_{st+exc}$ , we reduce the above diagram into

$$\mathbb{1} \xrightarrow{c_0} \mathbb{Z} \xrightarrow{u_x} \mathbb{1} \xrightarrow{c_7} \mathbb{Z} \xrightarrow{u_y} \mathbb{1}$$

which is actually the  $\text{prog6}$  when sketched.

1. Initially, we have

$$\downarrow \left( [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \circ \text{pbl} \circ (c \text{ true}) \right) \\ \circ u_y \circ (c 20) \circ u_x \circ (c 1) \equiv u_y \circ (c 7) \circ u_x \circ (c 0). \quad (28)$$

We first convert  $\equiv$  into  $\equiv \sim$ , then rewrite the (w\_downcast) rule and get

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \\ \circ \text{pbl} \circ (c \text{ true}) \circ u_y \circ (c 20) \circ u_x \circ (c 1) \equiv \sim u_y \circ (c 7) \circ u_x \circ (c 0). \quad (29)$$

Rewriting commutation-update-update, on both sides, gives

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \\ \circ \text{pbl} \circ (c \text{ true}) \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (30)$$

Rewriting Lemma 6.2 yields

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ \left[ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ \text{b} | \text{id}_1 \right] \\ \circ \text{in}_1 \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (31)$$

2. Now; we rewrite the rule (w\_lcopair\_eq), and handle

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \\ \circ \text{pbl} \circ \text{b} \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0). \quad (32)$$

By unfolding b, we have

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \\ \circ \text{pbl} \circ (t \stackrel{?}{\leq}) \circ \langle l_x (c 0) \rangle \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (33)$$

Rewriting the lemma commutation-lookup-constant-update, we obtain

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ [[ ]_1 \circ \text{tag}_e | c_2] \\ \circ \text{pbl} \circ (t \stackrel{?}{\leq}) \circ \langle (c 1), (c 0) \rangle \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (34)$$

We rewrite the rule (imp<sub>2</sub>), and get

$$[\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ \circ [[ ]_1 \circ \text{tag}_e | c_2] \circ \text{pbl} \circ (c \text{ false}) \circ u_x \circ (c 1) \circ u_y \circ (c 20) \equiv \sim u_x \circ (c 0) \circ u_y \circ (c 7). \quad (35)$$



Rewriting the Lemma 6.1 yields

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ & \quad \circ [ [ ]_1 \circ \text{tag}_e | c_2 ] \circ \text{in}_2 \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (36)$$

We now rewrite (s\_lcopair\_eq) which gives

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ & \quad \circ c_2 \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (37)$$

Here, by unfolding  $c_2$ , we have

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ u_x \circ (t \ -) \circ \langle 1_x, (c \ 1) \rangle \\ & \quad \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (38)$$

Rewriting the lemma `commutation – lookup – constant – update` gives

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ u_x \circ (t \ -) \circ \langle (c \ 1), (c \ 1) \rangle \\ & \quad \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (39)$$

We rewrite (imp<sub>1</sub>), and get

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \\ & \quad \circ u_x \circ (c \ 0) \circ u_x \circ (c \ 1) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (40)$$

We again rewrite the lemma `commutation-update-update`, and obtain

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \text{ true}) c_1) \circ u_x \circ (c \ 0) \\ & \quad \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (41)$$

3. We re-iterate the loop via (imp-li), and have

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ [ (\text{lpi } (c \text{ true}) c_1) \circ c_1 | \text{id} ] \\ & \quad \circ \text{pbl} \circ (c \ \text{true}) \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \end{aligned} \quad (42)$$

We rewrite Lemma 6.2, (w\_lcopair\_eq), then unfold  $c_1$ , and get:

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \ \text{true}) c_1) \circ [\text{throw e } \mathbb{1} | c_2] \\ & \quad \circ \text{pbl} \circ (t \ \stackrel{?}{\leq}) \circ \langle 1_x, (c \ 0) \rangle \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \end{aligned} \quad (43)$$

By rewriting `commutation – lookup – constant – update`, (imp<sub>3</sub>) and Lemma 6.2, we have

$$\begin{aligned} & [\text{id}_1 | c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ (\text{lpi } (c \ \text{true}) c_1) \circ [\text{throw e } \mathbb{1} | c_2] \circ \text{in}_1 \\ & \quad \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \end{aligned} \quad (44)$$

By (w\_lcopair\_eq), the exception is raised:

$$\begin{aligned} [\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ ((\text{lpi } (c \text{ true}) \ c_1) \circ \text{throw } e \ \mathbb{1}) \\ \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \end{aligned} \quad (45)$$

Due to the raised exception, the infinite loop gets abruptly terminated at this step. We first unfold THROW then rewrite propagator-propagates (see Section 5.1), and get

$$[\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_1 \circ [ ]_1 \circ \text{tag}_e \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 20). \quad (46)$$

4. Here, we first cut  $\text{in}_1 \circ [ ]_1 \equiv \equiv \text{in}_2$ , and rewrite it back in the equation. Thus, we have

$$[\text{id}_1 \mid c_3 \circ \text{untag}_e] \circ \text{in}_2 \circ \text{tag}_e \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (47)$$

By rewriting (s\_lcopair\_eq), we obtain

$$c_3 \circ \text{untag}_e \circ \text{tag}_e \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (48)$$

Since  $u_x \circ (c \ 0) \circ u_y \circ (c \ 20)$  is pure with respect to the exception, we rewrite (eax<sub>1</sub>), and get

$$c_3 \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (49)$$

Unfolding the definition of the command  $c_3 = (u_y \circ (c \ 7))$ , we have

$$u_y \circ (c \ 7) \circ u_x \circ (c \ 0) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (50)$$

We now rewrite commutation-update-update on the left, and handle

$$u_x \circ (c \ 0) \circ u_y \circ (c \ 7) \circ u_y \circ (c \ 20) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (51)$$

Finally, it suffices to rewrite interaction-update-update,

$$u_x \circ (c \ 0) \circ u_y \circ (c \ 7) \equiv \sim u_x \circ (c \ 0) \circ u_y \circ (c \ 7). \quad (52)$$

which is trivial since the equality symbol  $\equiv \sim$  is reflexive. It still remains to prove the previous cut  $\text{in}_1 \circ [ ]_1 \equiv \equiv \text{in}_2$ : since everything is pure with respect to the exception, we have

$$\text{in}_1 \circ [ ]_1 \equiv \sim \text{in}_2. \quad (53)$$

Now, rewriting the rule (w\_empty) gives  $[ ]_{1+1} \equiv \sim [ ]_{1+1}$ .  $\square$

The full Coq proofs of above lemmas can be found [here](#), and the entire implementation [there](#).

## 7 Concluding remarks

We have presented frameworks for formalizing the treatment of the state and the exception effects, first separately, and then combined, using the decorated logic. Decorations describe what computational effect evaluation of a term may involve, and form a bridge between the syntax and its interpretation in reasoning about terms by making computational effects explicit in the decorated syntax. We have designed a denotational semantics for the IMP+Exc language using the combined decorated logic  $\mathcal{L}_{st+exc}$  as the target language. This way, we managed prove strong equalities between IMP+Exc programs. We have also encoded the combined logic in the Coq proof assistant to certify related proofs.

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