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On diffeomorphism extension

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Abstract

We state and prove a result concerning the fact that a diffeomorphism can be “extended” in such a way that its image is \mathbb{R}^n .

Before stating the theorem, we introduce the following property :

Definition 1 (Conditions (C)) *An open subset E of \mathbb{R}^m is said to verify condition (C) if there exist a C^1 function $\kappa : \mathbb{R}^m \rightarrow \mathbb{R}$, a bounded¹ C^1 vector field χ , and a closed set K_0 contained in E such that:*

1. $E = \{z \in \mathbb{R}^n, \kappa(z) < 0\}$
2. K_0 is globally attractive for χ
3. we have the following transversality assumption:

$$\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \quad \forall z \in \mathbb{R}^m : \kappa(z) = 0.$$

We now state the main theorem of this note :

Theorem 1 (Image extension) *Let $\psi : \mathcal{D} \subset \mathbb{R}^m \rightarrow \psi(\mathcal{D}) \subset \mathbb{R}^m$ be a diffeomorphism. If $\psi(\mathcal{D})$ verifies condition (C) or \mathcal{D} is C^2 -diffeomorphic to \mathbb{R}^m and ψ is C^2 , then for any compact set K in \mathcal{D} there exists a diffeomorphism $\psi_e : \mathcal{D} \rightarrow \mathbb{R}^m$ satisfying :*

$$\psi_e(\mathcal{D}) = \mathbb{R}^m \quad , \quad \psi_e(z) = \psi(z) \quad \forall z \in K.$$

The proof of this theorem, given in Section 2, relies on two preliminary lemmas presented in Section 1.

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¹If not replace χ by $\frac{\chi}{\sqrt{1+|\chi|^2}}$.

1 Technical lemmas

1.1 Construction of a diffeomorphism from an open set to \mathbb{R}^m

In this section, we give sufficient conditions to build a diffeomorphism from \mathbb{R}^m to an open subset E which leaves $E_\varepsilon \subset E$ unchanged. This construction is made explicit in the proof.

The complementary, closure, interior and boundary of a set S are denoted S^c , \overline{S} , $\overset{\circ}{S}$ and ∂S , respectively, with $\partial S = \overline{S} \setminus \overset{\circ}{S}$.

The Hausdorff distance d_H between two sets A and B is defined by :

$$d_H(A, B) = \max \left\{ \sup_{x_A \in A} \inf_{x_B \in B} |x_A - x_B|, \sup_{x_B \in B} \inf_{x_A \in A} |x_A - x_B| \right\}$$

With $Z(z, t)$ we denote the (unique) solution, at time t , to $\dot{z} = \chi(z)$ going through z at time 0.

Lemma 1 *Let E be an open strict subset of \mathbb{R}^m verifying (C), with a C^s vector field χ . Then, for any strictly positive real number ε , there exists a C^s -diffeomorphism $\phi: \mathbb{R}^m \rightarrow E$, such that, with*

$$\Sigma = \bigcup_{t \in [0, \varepsilon]} Z(\partial E, t),$$

we have $\phi(z) = z$ for all $z \in E_\varepsilon = E \cap \Sigma^c$ and $d_H(\partial E_\varepsilon, \partial E) \leq \varepsilon \sup_z |\chi(z)|$.

Proof : We start by establishing some properties.

- E is forward invariant by χ . This is a direct consequence of points 1 and 3 of the condition (C).
- Σ is closed. Take a sequence (z_k) of points in Σ converging to z^* . By definition, there exists a sequence (t_k) , such that :

$$t_k \in [0, \varepsilon] \quad \text{and} \quad Z(z_k, -t_k) \in \partial E \quad \forall k \in \mathbb{N}.$$

Since $[0, \varepsilon]$ is compact, one can extract a subsequence $(t_{\sigma(k)})$ converging to t^* in $[0, \varepsilon]$, and by continuity of the function $(z, t) \mapsto Z(z, -t)$, $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$ tends to $Z(z^*, -t^*)$ which is in ∂E , since ∂E is closed. Finally, because t^* is in $[0, \varepsilon]$, z^* is in Σ by definition.

– Σ is contained in $\text{cl}(E)$. Since, E is forward invariant by χ , and so is $\text{cl}(E)$ (see [4, Theorem 16.3]). This implies

$$\partial E \subset \Sigma = \bigcup_{t \in [0, \varepsilon]} Z(\partial E, t) \subset \text{cl}(E) = E \cup \partial E.$$

At this point, it is useful to note that, because Σ is a closed subset of the open set E , we have $\Sigma \cap E = \Sigma \setminus \partial E$. This implies :

$$E \setminus E_\varepsilon = (E_\varepsilon)^c \cap E = (E^c \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \setminus \partial E, \quad (1)$$

and $E = E_\varepsilon \cup (\Sigma \setminus \partial E)$.

With all these properties at hand, we define now two functions t_z and θ_z . The assumptions of global attractiveness of the closed set K_0 contained in E open, of transversality of χ to ∂E ,

and the property of forward-invariance of E , imply that, for all z in E^c , there exists a unique non negative real number t_z satisfying:

$$\kappa(Z(z, t_z)) = 0 \iff Z(z, t_z) \in \partial E.$$

The same arguments in reverse time allow us to see that, for all z in Σ , t_z exists, is unique and in $[-\varepsilon, 0]$. This way, the function $z \rightarrow t_z$ is defined on $(E_\varepsilon)^c$. Next, for all z in $(E_\varepsilon)^c$, we define :

$$\theta_z = Z(z, t_z).$$

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions $z \mapsto t_z$ and $z \mapsto \theta_z$ are C^s on $(E_\varepsilon)^c$.

Remark 1 κ having constant rank 1 in a neighborhood of ∂E , this set is a closed, regular submanifold of \mathbb{R}^m . The arguments above show that $z \mapsto (\theta_z, t_z)$ is a diffeomorphism between E^c and $\partial E \times [0, +\infty[$. Since ∂E is a deformation retract of E^c and the open unit ball is diffeomorphic to \mathbb{R}^m [?], if E were bounded, E^c could be seen as a h -cobordism between ∂E and the unit sphere \mathbb{S}^{m-1} and t_z as a Morse function with no critical point in E^c . See [5] for instance.

Now we evaluate t_z for z in $\partial\Sigma$. Let z be arbitrary in $\partial\Sigma$ and therefore in Σ which is closed. Assume its corresponding t_z is in $] -\varepsilon, 0[$. The Implicit Function Theorem shows that $z \mapsto t_z$ and $z \mapsto \theta_z$ are defined and continuous on a neighborhood of z . Therefore, there exists a strictly positive real number r satisfying

$$\forall y \in \mathcal{B}_r(z), \exists t_y \in] -\varepsilon, 0[: Z(y, t_y) \in \partial E .$$

This implies that the neighborhood $\mathcal{B}_r(z)$ of z is contained in Σ , in contradiction with the fact that z is on the boundary of Σ .

This shows that, for all z in $\partial\Sigma$, t_z is either 0 or $-\varepsilon$. We write this as $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$, with the notation $(\partial\Sigma)_i = \left\{ z \in \Sigma : t_z = -\varepsilon \right\}$.

Now we want to prove $\partial E_\varepsilon \subset (\partial\Sigma)_i$. To obtain this result, we start by showing :

$$\partial E_\varepsilon \cap \partial E = \emptyset \quad \text{and} \quad \partial E_\varepsilon \subset \partial\Sigma . \tag{2}$$

Suppose the existence of z in $\partial E_\varepsilon \cap \partial E$. z being in ∂E , its corresponding t_z is 0. By the Implicit Function Theorem, there exists a strictly positive real number r such that,

$$\forall y \in \mathcal{B}_r(z), \exists t_y \in] -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[: Z(y, t_y) \in \partial E .$$

But, by definition, any y , for which there exists t_y in $] -\frac{\varepsilon}{2}, 0]$, is in Σ . If instead t_y is strictly positive, then necessarily y is in E^c , because E is forward-invariant by χ and a solution starting in E cannot reach ∂E in positive finite time. We have obtained : $\mathcal{B}_r(z) \subset \Sigma \cup E^c = (E_\varepsilon)^c$. $\mathcal{B}_r(z)$ being a neighborhood of z , this contradicts the fact that z is in the boundary of E_ε .

At this point, we have proved that $\partial E_\varepsilon \cap \partial E = \emptyset$, and, because E_ε is contained in E , this implies $\partial E_\varepsilon \subset E$. With this, (2) will be established by proving that we have $\partial E_\varepsilon \subset \partial\Sigma$. Let z be arbitrary in ∂E_ε and therefore in E which is open. There exists a strictly positive real number r such that we have :

$$\mathcal{B}_r(z) \cap E_\varepsilon = \mathcal{B}_r(z) \cap (E \cap \Sigma^c) \neq \emptyset, \quad \mathcal{B}_r(z) \cap E_\varepsilon^c = \mathcal{B}_r(z) \cap (E^c \cup \Sigma) \neq \emptyset, \quad \mathcal{B}_r(z) \subset E .$$

This implies $\mathcal{B}_r(z) \cap \Sigma^c \neq \emptyset$ and $\mathcal{B}_r(z) \cap \Sigma \neq \emptyset$ and therefore that z is in $\partial\Sigma$.

We have established $\partial E_\varepsilon \cap \partial E = \emptyset$, $\partial E_\varepsilon \subset \partial\Sigma$ and $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$. This does imply :

$$\partial E_\varepsilon \subset (\partial\Sigma)_i = \{z \in E : t_z = -\varepsilon\} . \quad (3)$$

This allows us to extend by continuity the definition of t_z to \mathbb{R}^m by letting $t_z = -\varepsilon$ for all $z \in E_\varepsilon$.

Thanks to all these preparatory steps, we are finally ready to define a function $\phi : \mathbb{R}^m \rightarrow E$ as:

$$\phi(z) = \begin{cases} Z(z, t_z + \nu(t_z)), & \text{if } z \in (E_\varepsilon)^c , \\ z, & \text{if } z \in E_\varepsilon , \end{cases} \quad (4)$$

where ν is an arbitrary C^s and strictly decreasing function defined on \mathbb{R} satisfying:

$$\nu(t) = -t \quad \forall t \leq -\varepsilon \quad , \quad \lim_{t \rightarrow +\infty} \nu(t) = 0 .$$

The image of ϕ is contained in E since we have $E_\varepsilon \subset E$ and :

$$\begin{aligned} t_z + \nu(t_z) &> t_z \quad \forall z \in E_\varepsilon^c , \\ Z(z, t_z) &\in \partial E , \\ Z(z, t) &\in E \quad \forall (z, t) \in \partial E \times \mathbb{R}_{>0} . \end{aligned}$$

The continuity of the functions $(z, t) \in \mathbb{R}^m \times \mathbb{R} \mapsto Z(z, t) \in \mathbb{R}$ and $z \in E_\varepsilon^c \mapsto t_z \in [-\varepsilon, +\infty[$ implies that this function ϕ is continuous at least on $\mathbb{R}^m \setminus \partial E_\varepsilon$. Also, for any z in ∂E_ε , t_z is defined and equal to $-\varepsilon$ (see (3)). So, for any strictly positive real number η , there exists a real number r such that :

$$\begin{aligned} |t_y + \varepsilon| &\leq \eta & \forall y \in \mathcal{B}_r(z) , \\ \nu(t_y) + \varepsilon &\leq \eta & \forall y \in \mathcal{B}_r(z) , \\ \phi(y) &= y & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon , \\ \phi(y) &= Z(y, t_y + \nu(t_y)) & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon^c . \end{aligned}$$

Since we have :

$$\phi(z) = Z(z, t_z + \nu(t_z)) = Z(z, -\varepsilon + \nu(-\varepsilon)) = z ,$$

we conclude that ϕ is also continuous at z .

By differentiating, we obtain :

– at any interior point z of $(E_\varepsilon)^c$

$$\frac{\partial \phi}{\partial z}(z) = \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) ;$$

– at any z in E_ε (which is open) $\frac{\partial \phi}{\partial z}(z) = I$. Also, for any z in ∂E_ε , we have :

$$\begin{aligned} \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) &= \frac{\partial Z}{\partial z}(z, 0) + \chi(Z(z, 0)) \frac{\partial t_z}{\partial z}(z)(1 - 1) , \\ &= I . \end{aligned}$$

This implies that ϕ is C^1 on \mathbb{R}^m .

We now show that ϕ is invertible. Let y be arbitrary in $E \cap E_\varepsilon^c = E \cap \Sigma$. There exists t_y in $[-\varepsilon, 0[$. The function ν being strictly monotonic, $\nu^{-1}(t_y)$ exists and is in $[-\varepsilon, +\infty[$. This allows us to define properly ϕ^{-1} as :

$$\phi^{-1}(y) = \begin{cases} Z(y, t_y - \nu^{-1}(-t_y)), & \text{if } y \in E \setminus E_\varepsilon \\ y, & \text{if } y \in E_\varepsilon \end{cases} \quad (5)$$

This function is an inverse of ϕ as can be seen by reverting the flow induced by χ when needed. Also, with the same arguments as before, we can prove that it is C^1 .

This implies that ϕ is a diffeomorphism from \mathbb{R}^m to E .

Besides, the functions $z \mapsto Z(z, t)$ for $t > 0$, $z \mapsto t_z$ and ν being C^s , ϕ is C^s at any interior point of $(E_\varepsilon)^c$. By continuity of $\nu^{(r)}$ for $r \leq s$, it can be verified that ϕ is also C^s on the boundary ∂E_ε . So, ϕ is a C^s -diffeomorphism from \mathbb{R}^m to E .

Finally, we note that, for any point z_ε in ∂E_ε , there exists a point z in ∂E satisfying :

$$|z_\varepsilon - z| = \left| \int_0^\varepsilon \chi(Z(z, s)) ds \right| \leq \varepsilon \sup_\zeta |\chi(\zeta)| .$$

And conversely, for any z in ∂E , there exist z_ε in ∂E_ε satisfying :

$$|z_\varepsilon - z| = \left| \int_0^\varepsilon \chi(Z(z, s)) ds \right| \leq \varepsilon \sup_\zeta |\chi(\zeta)| .$$

It follows that

$$d_H(\partial E_\varepsilon, \partial E) \leq \varepsilon \sup_\zeta |\chi(\zeta)| \quad (6)$$

and ε may be chosen as small as needed. \square

Two direct consequences of Lemma 1 are :

Corollary 1 *Let $\psi : E \rightarrow \psi(E)$ be a diffeomorphism, with E satisfying (C). For any ε strictly positive, there exist an open subset E_ε of E and a diffeomorphism $\psi_\varepsilon : \mathbb{R}^m \rightarrow \psi(E)$ satisfying :*

$$\begin{aligned} d_H(\partial E_\varepsilon, \partial E) &\leq \varepsilon , \\ \psi_\varepsilon(z) &= \psi(z) \quad \forall z \in E_\varepsilon . \end{aligned}$$

Proof : With ϕ given by Lemma 1, we pick $\psi_\varepsilon = \psi \circ \phi$. \square

Corollary 2 *Let $\psi : E \rightarrow \psi(E)$ be a diffeomorphism, with $\psi(E)$ satisfying (C). For any ε strictly positive, there exist an open subset $\psi(E)_\varepsilon$ of $\psi(E)$ and a diffeomorphism $\psi_\varepsilon : E \rightarrow \mathbb{R}^n$ satisfying :*

$$\begin{aligned} d_H(\partial(\psi(E)_\varepsilon), \partial\psi(E)) &\leq \varepsilon , \\ \psi_\varepsilon(z) &= \psi(z) \quad \forall z \in \psi^{-1}(\psi(E)_\varepsilon) . \end{aligned}$$

Proof : With ϕ given by Lemma 1 from \mathbb{R}^n to $\psi(E)$, we pick $\psi_e = \phi^{-1} \circ \psi$. \square

Remark 2 In Corollary 2, ψ being a diffeomorphism on an open set E , we know that the image of any compact subset K of E is a compact subset of $\psi(E)$ which is also open by Brouwer's invariance theorem. Therefore, with (6), we can find ε such that $\psi(K) \subset (\psi(E))_\varepsilon$ and thus, $\psi_e(z) = \psi(z)$ for all x in K .

1.2 Diffeomorphism extension from a ball

Let $R > 0$. We denote \mathcal{B}_R the open ball in \mathbb{R}^n of radius R and centered at 0.

Lemma 2 Consider a C^2 diffeomorphism $\psi : \mathcal{B}_R \rightarrow \psi(\mathcal{B}_R) \subset \mathbb{R}^n$. For any ε strictly positive, there exists a diffeomorphism $\psi_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\psi_e(x) = \psi(x)$ for all x in $\mathcal{c}\mathcal{L}(\mathcal{B}_{R-\varepsilon})$.

Proof : Without loss of generality we may assume that $\psi(0) = 0$. Consider the mapping $\varphi : \mathcal{B}_R \times [0, 1] \rightarrow \mathbb{R}^n$ defined as

$$\varphi(x, t) = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\psi(xt)}{t}, \quad \varphi(x, 0) = x.$$

Note that for all t the mapping $\varphi_t(x) = \varphi(x, t)$ is a diffeomorphism from \mathcal{B}_R toward $\varphi_t(\mathcal{B}_R)$. Indeed, given x_a and x_b such that $\varphi(x_a, t) = \varphi(x_b, t)$ it yields $\psi(x_a t) = \psi(x_b t)$. Note that the couple $(x_a t, x_b t)$ is in \mathcal{B}_R . The mapping ψ being injective on this set, it yields $x_a = x_b$. Moreover,

$$\frac{\partial \varphi_t}{\partial x}(x) = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\partial \psi}{\partial x}(xt), \quad t > 0, \quad \frac{\partial \varphi_0}{\partial x}(x) = \text{Id}$$

Hence, this mapping is full rank in \mathcal{B}_R . Consequently, for all t in $[0, 1]$, this mapping is a diffeomorphism \mathcal{B}_R toward $\varphi_t(\mathcal{B}_R)$. Consequently, for all t in $[0, 1]$ we can introduce φ_t^{-1} its inverse map.

Note moreover that

$$\overline{\varphi(x, t)} = \frac{\partial \varphi}{\partial t}(x, t) = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(x, t)$$

where ρ is the function given as

$$\rho(x, t) = \frac{1}{t^2} \left[\frac{\partial \psi}{\partial x}(xt)xt - \psi(xt) \right], \quad \rho(x, 0) = \frac{1}{2} x' \left(\frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

The mapping ψ being C^2 , and $\psi(0) = 0$ it yields,

$$\psi(xt) = \frac{\partial \psi}{\partial x}(0)xt + x' \left(\frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x \frac{t^2}{2} + o(t^2)$$

and,

$$\lim_{t \rightarrow 0} \frac{\frac{\partial \psi}{\partial x}(xt)x - \frac{\partial \psi}{\partial x}(0)x}{t} = x' \left(\frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

Hence, the function ρ is well defined and locally Lipschitz. Consequently, $\varphi(x, t)$ is (the unique) solution of the time varying system defined for (z, t) in $\varphi_t(\mathcal{B}_R) \times [0, 1]$ as

$$\dot{z} = \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t)$$

This time varying system can be extended to \mathbb{R}^n as

$$\dot{z} = \begin{cases} 0, & z \notin \varphi_t(\mathcal{B}_R) \\ \chi(\varphi_t^{-1}(z)) \left(\frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t), & z \in \varphi_t(\mathcal{B}_R) \end{cases} \quad (7)$$

where $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a C^1 function such that

$$\chi(x) = \begin{cases} 0, & x \notin \mathcal{B}_R \\ 1, & x \in \mathcal{B}_{R-\varepsilon} \end{cases}$$

Notice that the vector field is zero outside $S = \bigcup_{\substack{z \in \mathcal{B}_R \\ t \in [0,1]}} \varphi_t(z)$, which is a compact set. Indeed, if it is not, one can construct (x_n) and (t_n) , such that for all $n \in \mathbb{N}$,

$$\frac{|\psi(x_n t_n)|}{t_n} \geq n.$$

Since $\text{cl}(\mathcal{B}_R)$ is compact, we extract $x_{\sigma(n)} t_{\sigma(n)}$ tending towards $x^* \in \text{cl}(\mathcal{B}_R)$. Necessarily, $t_{\sigma(n)}$ tends to 0 and $x^* = 0$ since (x_n) is bounded. But this is impossible because, $\frac{\psi(x_n t_n)}{t_n}$ is equivalent to $\frac{\partial \psi}{\partial x}(0) x_n$ around zero.

Therefore, the maximal solutions to this system are defined (at least) in $[0, 1]$ for all initial conditions, and through backward integration, we obtain that $\varphi(\mathbb{R}^n, 1) = \mathbb{R}^n$.

We finally consider

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) Z(x, 1)$$

where $Z(x, 1)$ is the solution of the system (7) evaluated at time 1 and initiated from x at time 0. This mapping being a linear transformation of a (time varying) flow, it is a diffeomorphism, and $\psi_e(\mathbb{R}^n) = \mathbb{R}^n$. Note moreover that for all $x \in \mathcal{B}_{R-\varepsilon}$, we have

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) \varphi(x, 1) = \psi(x).$$

□

2 Proof of Theorem 1

Consider a diffeomorphism $\psi : \mathcal{D} \rightarrow \psi(\mathcal{D}) \subsetneq \mathbb{R}^m$. We want to extend the image of ψ to \mathbb{R}^m , i-e find a diffeomorphism $\psi_e : \mathcal{D} \rightarrow \mathbb{R}^m$ such that:

$$- \psi_e(\mathcal{D}) = \mathbb{R}^m$$

- $\psi_e(z) = \psi(z)$ for all z in $K \subset \mathcal{D}$, where K is any compact subset of \mathcal{D} .

Let us successively study the following two cases :

- **First case:** $\psi(\mathcal{D})$ satisfies (C) The result follows directly from Corollary 2 and Remark 2. In practice, the reader may find an explicit construction of such an extension in the proof of Lemma 1.
- **Second case:** \mathcal{D} is C^2 -diffeomorphic to \mathbb{R}^m and ψ is C^2 :

Let $\phi_1 : \mathcal{D} \rightarrow \mathbb{R}^m$ denote the corresponding diffeomorphism. Let R_1 be a strictly positive real number such that the open ball $\mathcal{B}_{R_1}(0)$ contains $\phi_1(K)$. Let R_2 be a real number strictly larger than R_1 . With Lemma 1 again, and since $\mathcal{B}_{R_2}(0)$ verifies condition (C), there exists of C^2 -diffeomorphism $\phi_2 : \mathcal{B}_{R_2}(0) \rightarrow \mathbb{R}^m$ satisfying

$$\phi_2(z) = z \quad \forall z \in \mathcal{B}_{R_1}(0) .$$

At this point, we have obtained a C^2 -diffeomorphism $\phi = \phi_2^{-1} \circ \phi_1 : \mathcal{D} \rightarrow \mathcal{B}_{R_2}(0)$. Consider $\lambda = \psi \circ \phi^{-1} : \mathcal{B}_{R_2}(0) \rightarrow \psi(\mathcal{D})$. According to Lemma 2, we can extend λ to $\lambda_e : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that $\lambda_e = \psi \circ \phi^{-1}$ on $\mathcal{B}_{R_1}(0)$. Finally, consider $\psi_e = \lambda_e \circ \phi_1 : \mathcal{D} \rightarrow \mathbb{R}^m$. Since, by construction of ϕ_2 , $\phi = \phi_1$ on $\phi_1^{-1}(\mathcal{B}_{R_1}(0))$ which contains K , we have $\psi_e = \psi$ on K .

Remark 3 Note that the second construction is more complex than the first one, for it contains several extensions and in particular that of Lemma 2 which is difficult to implement. For this reason, one may prefer applying the first case whenever it is possible.

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