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On diffeomorphism extension

Pauline Bernard, Laurent Praly*and Vincent Andrieu†

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Abstract

We state and prove a result concerning the fact that a diffeomorphism can be "extended" in such a way that its image is $\mathbb{R}^n$.

Before stating the theorem, we introduce the following property :

Definition 1 (Conditions $(\mathcal{C})$) *An open subset E of $\mathbb{R}^m$ is said to verify condition $(\mathcal{C})$ if there exist a C^1 function $\kappa : \mathbb{R}^m \to \mathbb{R}$, a bounded[1] C^1 vector field χ, and a closed set K_0 contained in E such that:*

1. $E = \{z \in \mathbb{R}^n, \kappa(z) < 0\}$
2. *K_0 is globally attractive for χ*
3. *we have the following transversality assumption:*

$$\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \qquad \forall z \in \mathbb{R}^m : \ \kappa(z) = 0.$$

We now state the main theorem of this note :

Theorem 1 (Image extension) *Let ψ: $\mathcal{D} \subset \mathbb{R}^m \to \psi(\mathcal{D}) \subset \mathbb{R}^m$ be a diffeomorphism. If $\psi(\mathcal{D})$ verifies condition $(\mathcal{C})$ or $\mathcal{D}$ is C^2-diffeomorphic to $\mathbb{R}^m$ and ψ is C^2, then for any compact set K in $\mathcal{D}$ there exists a diffeomorphism $\psi_e : \mathcal{D} \to \mathbb{R}^m$ satisfying :*

$$\psi_e(\mathcal{D}) = \mathbb{R}^m \quad , \qquad \psi_e(z) = \psi(z) \quad \forall z \in K.$$

The proof of this theorem, given in Section 2, relies on two preliminary lemmas presented in Section 1.

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[1]If not replace χ by $\frac{\chi}{\sqrt{1+|\chi|^2}}$.

1 Technical lemmas

1.1 Construction of a diffeomorphism from an open set to $\mathbb{R}^m$

In this section, we give sufficient conditions to build a diffeomorphism from $\mathbb{R}^m$ to an open subset E which leaves $E_\varepsilon \subset E$ unchanged. This construction is made explicit in the proof.

The complementary, closure, interior and boundary of a set S are denoted S^c, $\overline{S}$, $\overset{\circ}{S}$ and ∂S, respectively, with $\partial S = \overline{S} \backslash \overset{\circ}{S}$.

The Hausdorff distance d_H between two sets A and B is defined by :

$$d_H(A,B) \;=\; \max\left\{ \sup_{x_A \in A} \inf_{x_B \in B} |x_A - x_B| \,,\, \sup_{x \in A} \inf_{x_B \in B} |x_A - x_B| \right\}$$

With $Z(z,t)$ we denote the (unique) solution, at time t, to $\dot{z} = \chi(z)$ going trough z at time 0.

Lemma 1 *Let E be an open strict subset of $\mathbb{R}^m$ verifying $(\mathcal{C})$, with a C^s vector field χ. Then, for any strictly positive real number ε, there exists a C^s-diffeomorphism ϕ: $\mathbb{R}^m \to E$, such that, with*

$$\Sigma = \bigcup_{t \in [0,\varepsilon]} Z(\partial E, t) \;,$$

we have $\phi(z) = z$ for all $z \in E_\varepsilon = E \cap \Sigma^{\mathsf{c}}$ and $d_H(\partial E_\varepsilon, \partial E) \;\leq\; \varepsilon \sup_z |\chi(z)|$.

Proof : We start by establishing some properties.
– *E is forward invariant by χ.* This is a direct consequence of points 1 and 3 of the condition $(\mathcal{C})$.
– *Σ is closed.* Take a sequence (z_k) of points in Σ converging to z^*. By definition, there exists a sequence (t_k), such that :

$$t_k \in [0,\varepsilon] \qquad \text{and} \qquad Z(z_k, -t_k) \in \partial E \qquad \forall k \in \mathbb{N} \;.$$

Since $[0,\varepsilon]$ is compact, one can extract a subsequence $(t_{\sigma(k)})$ converging to t^* in $[0,\varepsilon]$, and by continuity of the function $(z,t) \mapsto Z(z,-t)$, $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$ tends to $Z(z^*, -t^*)$ which is in ∂E, since ∂E is closed. Finally, because t^* is in $[0,\varepsilon]$, z^* is in Σ by definition.
– *Σ is contained in $\mathtt{cl}(E)$.* Since, E is forward invariant by χ, and so is $\mathtt{cl}(E)$ (see [4, Theorem 16.3]). This implies

$$\partial E \;\subset\; \Sigma = \bigcup_{t \in [0,\varepsilon]} Z(\partial E, t) \;\subset\; \mathtt{cl}(E) \;=\; E \cup \partial E \;.$$

At this point, it is useful to note that, because Σ is a closed subset of the open set E, we have $\Sigma \cap E \;=\; \Sigma \backslash \partial E$. This implies :

$$E \backslash E_\varepsilon = (E_\varepsilon)^{\mathsf{c}} \cap E = (E^{\mathsf{c}} \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \backslash \partial E, \tag{1}$$

and $E = E_\varepsilon \underset{\neq}{\cup} (\Sigma \backslash \partial E)$.

With all these properties at hand, we define now two functions t_z and θ_z. The assumptions of global attractiveness of the closed set K_0 contained in E open, of transversality of χ to ∂E,

and the property of forward-invariance of E, imply that, for all z in E^{c}, there exists a unique non negative real number t_z satisfying:

$$\kappa\left(Z(z,t_z)\right)=0 \quad \Longleftrightarrow \quad Z(z,t_z)\in\partial E.$$

The same arguments in reverse time allow us to see that, for all z in Σ, t_z exists, is unique and in $[-\varepsilon,0]$. This way, the function $z\to t_z$ is defined on $(E_\varepsilon)^{\mathsf{c}}$. Next, for all z in $(E_\varepsilon)^{\mathsf{c}}$, we define :

$$\theta_z=Z(z,t_z).$$

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions $z\mapsto t_z$ and $z\mapsto\theta_z$ are C^s on $(E_\varepsilon)^{\mathsf{c}}$.

Remark 1 κ having constant rank 1 in a neighborhood of ∂E, this set is a closed, regular submanifold of $\mathbb{R}^m$. The arguments above show that $z\mapsto(\theta_z,t_z)$ is a diffeomorphism between E^{c} and $\partial E\times[0,+\infty[$. Since ∂E is a deformation retract of E^{c} and the open unit ball is diffeomorphic to $\mathbb{R}^m$ [**?**], if E were bounded, E^{c} could be seen as a h-cobordism between ∂E and the unit sphere $\mathbb{S}^{m-1}$ and t_z as a Morse function with no critical point in E^{c}. See [5] for instance.

Now we evaluate t_z for z in $\partial\Sigma$. Let z be arbitrary in $\partial\Sigma$ and therefore in Σ which is closed. Assume its corresponding t_z is in $]-\varepsilon,0[$. The Implicit Function Theorem shows that $z\mapsto t_z$ and $z\mapsto\theta_z$ are defined and continuous on a neighborhood of z. Therefore, there exists a strictly positive real number r satisfying

$$\forall y\in\mathcal{B}_r(z)\ ,\ \exists t_y\in]-\varepsilon,0[:\ Z(y,t_y)\in\partial E\ .$$

This implies that the neighborhood $\mathcal{B}_r(z)$ of z is contained in Σ, in contradiction with the fact that z is on the boundary of Σ.

This shows that, for all z in $\partial\Sigma$, t_z is either 0 or $-\varepsilon$. We write this as $\partial\Sigma=\partial E\cup(\partial\Sigma)_i$, with the notation $(\partial\Sigma)_i=\left\{z\in\Sigma\ :\ t_z=-\varepsilon\right\}$.

Now we want to prove $\partial E_\varepsilon\subset(\partial\Sigma)_i$. To obtain this result, we start by showing :

$$\partial E_\varepsilon\cap\partial E=\emptyset \quad\text{and}\quad \partial E_\varepsilon\subset\partial\Sigma\ . \tag{2}$$

Suppose the existence of z in $\partial E_\varepsilon\cap\partial E$. z being in ∂E, its corresponding t_z is 0. By the Implicit Function Theorem, there exists a strictly positive real number r such that,

$$\forall y\in\mathcal{B}_r(z)\ ,\ \exists t_y\in\left]-\tfrac{\varepsilon}{2},\tfrac{\varepsilon}{2}\right[\ :Z(y,t_y)\in\partial E\ .$$

But, by definition, any y, for which there exists t_y in $]\frac{-\varepsilon}{2},0]$, is in Σ. If instead t_y is strictly positive, then necessarily y is in E^{c}, because E is forward-invariant by χ and a solution starting in E cannot reach ∂E in positive finite time. We have obtained : $\mathcal{B}_r(z)\subset\Sigma\cup E^{\mathsf{c}}=(E_\varepsilon)^{\mathsf{c}}$. $\mathcal{B}_r(z)$ being a neighborhood of z, this contradicts the fact that z is in the boundary of E_ε.

At this point, we have proved that $\partial E_\varepsilon\cap\partial E=\emptyset$, and, because E_ε is contained in E, this implies $\partial E_\varepsilon\subset E$. With this, (2) will be established by proving that we have $\partial E_\varepsilon\subset\partial\Sigma$. Let z be arbitrary in ∂E_ε and therefore in E which is open. There exists a strictly positive real number r such that we have :

$$\mathcal{B}_r(z)\cap E_\varepsilon\ =\ \mathcal{B}_r(z)\ \cap\ (E\cap\Sigma^{\mathsf{c}})\neq\emptyset\ ,\ \mathcal{B}_r(z)\cap E_\varepsilon^{\mathsf{c}}\ =\ \mathcal{B}_r(z)\ \cap\ (E^{\mathsf{c}}\cup\Sigma)\neq\emptyset\ ,\ \mathcal{B}_r(z)\subset E\ .$$

This implies $\mathcal{B}_r(z) \cap \Sigma^c \neq \emptyset$ and $\mathcal{B}_r(z) \cap \Sigma \neq \emptyset$ and therefore that z is in $\partial\Sigma$.

We have established $\partial E_\varepsilon \cap \partial E = \emptyset$, $\partial E_\varepsilon \subset \partial\Sigma$ and $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$. This does imply :

$$\partial E_\varepsilon \subset (\partial\Sigma)_i = \{z \in E : t_z = -\varepsilon\} . \tag{3}$$

This allows us to extend by continuity the definition of t_z to $\mathbb{R}^m$ by letting $t_z = -\varepsilon$ for all $z \in E_\varepsilon$.

Thanks to all these preparatory steps, we are finally ready to define a function $\phi : \mathbb{R}^m \to E$ as:

$$\phi(z) = \begin{cases} Z\left(z, t_z + \nu(t_z)\right), & \text{if } z \in (E_\varepsilon)^c , \\ z, & \text{if } z \in E_\varepsilon , \end{cases} \tag{4}$$

where ν is an arbitrary C^s and strictly decreasing function defined on $\mathbb{R}$ satisfying:

$$\nu(t) = -t \quad \forall t \leq -\varepsilon \quad , \qquad \lim_{t\to+\infty} \nu(t) = 0.$$

The image of ϕ is contained in E since we have $E_\varepsilon \subset E$ and :

$$\begin{gathered} t_z + \nu(t_z) > t_z \qquad \forall z \in E_\varepsilon^{\mathsf{c}} , \\ Z(z, t_z) \in \partial E , \\ Z(z,t) \in E \qquad \forall (z,t) \in \partial E \times \mathbb{R}_{>0} . \end{gathered}$$

The continuity of the functions $(z,t) \in \mathbb{R}^m \times \mathbb{R} \mapsto Z(z,t) \in \mathbb{R}$ and $z \in E_\varepsilon^{\mathsf{c}} \mapsto t_z \in [-\varepsilon, +\infty[$ implies that this function ϕ is continuous at least on $\mathbb{R}^m \backslash \partial E_\varepsilon$. Also, for any z in ∂E_ε, t_z is defined and equal to $-\varepsilon$ (see (3)). So, for any strictly positive real number η, there exists a real number r such that :

$$\begin{array}{rcll} |t_y + \varepsilon| & \leq & \eta & \forall y \in \mathcal{B}_r(z) , \\ \nu(t_y) + \varepsilon & \leq & \eta & \forall y \in \mathcal{B}_r(z) , \\ \phi(y) & = & y & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon , \\ \phi(y) & = & Z(y, t_y + \nu(t_y)) & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon^{\mathsf{c}} . \end{array}$$

Since we have :

$$\phi(z) = Z\left(z, t_z + \nu(t_z)\right) = Z\left(z, -\varepsilon + \nu(-\varepsilon)\right) = z ,$$

we conclude that ϕ is also continuous at z.

By differentiating, we obtain :

– at any interior point z of $(E_\varepsilon)^{\mathsf{c}}$

$$\frac{\partial\phi}{\partial z}(z) = \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z)))\frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) ;$$

– at any z in E_ε (which is open) $\frac{\partial\phi}{\partial z}(z) = I$. Also, for any z in ∂E_ε, we have :

$$\begin{aligned} \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z)))\frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) &= \frac{\partial Z}{\partial z}(z, 0) + \chi(Z(z, 0))\frac{\partial t_z}{\partial z}(z)(1 - 1) , \\ &= I . \end{aligned}$$

This implies that ϕ is C^1 on $\mathbb{R}^m$.

We now show that ϕ is invertible. Let y be arbitrary in $E \cap E_\varepsilon^{\mathsf{c}} = E \cap \Sigma$. There exists t_y in $[-\varepsilon, 0[$. The function ν being strictly monotonic, $\nu^{-1}(t_y)$ exists and is in $[-\varepsilon, +\infty[$. This allows us to define properly ϕ^{-1} as :

$$\phi^{-1}(y) = \begin{cases} Z\left(y, t_y - \nu^{-1}(-t_y)\right), & \text{if } y \in E \backslash E_\varepsilon \\ y, & \text{if } y \in E_\varepsilon \end{cases} \tag{5}$$

This function is an inverse of ϕ as can be seen be reverting the flow induced by χ when needed. Also, with the same arguments as before, we can prove that it is C^1.

This implies that ϕ is a diffeomorphism from $\mathbb{R}^m$ to E.

Besides, the functions $z \mapsto Z(z,t)$ for $t > 0$, $z \mapsto t_z$ and ν being C^s, ϕ is C^s at any interior point of $(E_\varepsilon)^{\mathsf{c}}$. By continuity of $\nu^{(r)}$ for $r \leq s$, it can be verified that ϕ is also C^s on the boundary ∂E_ε. So, ϕ is a C^s-diffeomorphism from $\mathbb{R}^m$ to E.

Finally, we note that, for any point z_ε in ∂E_ε, there exists a point z in ∂E satisfying :

$$|z_\varepsilon - z| \;=\; \left| \int_0^\varepsilon \chi(Z(z,s))ds \right| \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \;.$$

And conversely, for any z in ∂E, there exist z_ε in ∂E_ε satisfying :

$$|z_\varepsilon - z| \;=\; \left| \int_0^\varepsilon \chi(Z(z,s))ds \right| \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \;.$$

It follows that

$$d_H(\partial E_\varepsilon, \partial E) \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \tag{6}$$

and ε may be chosen as small as needed. □

Two direct consequences of Lemma 1 are :

Corollary 1 *Let $\psi : E \to \psi(E)$ be a diffeomorphism, with E satisfying $(\mathcal{C})$. For any ε strictly positive, there exist an open subset E_ε of E and a diffeomorphism ψ_e: $\mathbb{R}^m \to \psi(E)$ satisfying :*

$$\begin{aligned} d_H(\partial E_\varepsilon, \partial E) \;&\leq\; \varepsilon \;, \\ \psi_e(z) = \psi(z) \qquad & \forall z \in E_\varepsilon \;. \end{aligned}$$

Proof : With ϕ given by Lemma 1, we pick $\psi_e \;=\; \psi \circ \phi$. □

Corollary 2 *Let $\psi : E \to \psi(E)$ be a diffeomorphism, with $\psi(E)$ satisfying $(\mathcal{C})$. For any ε strictly positive, there exist an open subset $\psi(E)_\varepsilon$ of $\psi(E)$ and a diffeomorphism ψ_e: $E \to \mathbb{R}^n$ satisfying :*

$$\begin{aligned} d_H(\partial\left(\psi(E)_\varepsilon\right), \partial\psi(E)) \;&\leq\; \varepsilon \;, \\ \psi_e(z) = \psi(z) \qquad & \forall z \in \psi^{-1}\left(\psi(E)_\varepsilon\right) \;. \end{aligned}$$

Proof : With ϕ given by Lemma 1 from $\mathbb{R}^n$ to $\psi(E)$, we pick $\psi_e = \phi^{-1} \circ \psi$. $\square$

Remark 2 In Corollary 2, ψ being a diffeomorphism on an open set E, we know that the image of any compact subset K of E is a compact subset of $\psi(E)$ which is also open by Brouwer's invariance theorem. Therefore, with (6), we can find ε such that $\psi(K) \subset (\psi(E))_\varepsilon$ and thus, $\psi_e(z) = \psi(z)$ for all x in K.

1.2 Diffeomorphism extension from a ball

Let $R > 0$. We denote $\mathcal{B}_R$ the open ball in $\mathbb{R}^n$ of radius R and centered at 0.

Lemma 2 *Consider a C^2 diffeomorphism $\psi : \mathcal{B}_R \to \psi(\mathcal{B}_R) \subset \mathbb{R}^n$. For any ε strictly positive, there exists a diffeomorphism $\psi_e : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi_e(x) = \psi(x)$ for all x in $\mathtt{cl}(\mathcal{B}_{R-\varepsilon})$.*

Proof : Without loss of generality we may assume that $\psi(0) = 0$. Consider the mapping $\varphi : \mathcal{B}_R \times [0,1] \to \mathbb{R}^n$ defined as

$$\varphi(x,t) = \left(\frac{\partial \psi}{\partial x}(0)\right)^{-1} \frac{\psi(xt)}{t} \ , \ \varphi(x,0) = x \ .$$

Note that for all t the mapping $\varphi_t(x) = \varphi(x,t)$ is a diffeomorphism from $\mathcal{B}_R$ toward $\varphi_t(\mathcal{B}_R)$. Indeed, given x_a and x_b such that $\varphi(x_a,t) = \varphi(x_b,t)$ it yields $\psi(x_a t) = \psi(x_b t)$. Note that the couple $(x_a t, x_b t)$ is in $\mathcal{B}_R$. The mapping ψ being injective on this set, it yields $x_a = x_b$. Moreover,

$$\frac{\partial \varphi_t}{\partial x}(x) = \left(\frac{\partial \psi}{\partial x}(0)\right)^{-1} \frac{\partial \psi}{\partial x}(xt) \ , \ t > 0 \ , \ \frac{\partial \varphi_0}{\partial x}(x) = \mathrm{Id}$$

Hence, this mapping is full rank in $\mathcal{B}_R$. Consequently, for all t in $[0,1]$, this mapping is a diffeomorphism $\mathcal{B}_R$ toward $\varphi_t(\mathcal{B}_R)$. Consequently, for all t in $[0,1]$ we can introduce φ_t^{-1} its inverse map.

Note moreover that

$$\dot{\overbrace{\varphi(x,t)}} = \frac{\partial \varphi}{\partial t}(x,t) = \left(\frac{\partial \psi}{\partial x}(0)\right)^{-1} \rho(x,t)$$

where ρ is the function given as

$$\rho(x,t) = \frac{1}{t^2}\left[\frac{\partial \psi}{\partial x}(xt)xt - \psi(xt)\right] \ , \ \rho(x,0) = \frac{1}{2}x'\left(\frac{\partial^2 \psi}{\partial x \partial x}(0)\right)x$$

The mapping ψ being C^2, and $\psi(0) = 0$ it yields,

$$\psi(xt) = \frac{\partial \psi}{\partial x}(0)xt + x'\left(\frac{\partial^2 \psi}{\partial x \partial x}(0)\right)x\frac{t^2}{2} + \circ(t^2)$$

and,

$$\lim_{t \to 0} \frac{\frac{\partial \psi}{\partial x}(xt)x - \frac{\partial \psi}{\partial x}(0)x}{t} = x'\left(\frac{\partial^2 \psi}{\partial x \partial x}(0)\right)x$$

Hence, the function ρ is well defined and locally Lipschitz. Consequently, $\varphi(x,t)$ is (the unique) solution of the time varying system defined for (z,t) in $\varphi_t(\mathcal{B}_R)\times[0,1]$ as

$$\dot{z} = \left(\frac{\partial\psi}{\partial x}(0)\right)^{-1} \rho\left(\varphi_t^{-1}(z),t\right)$$

This time varying system can be extended to $\mathbb{R}^n$ as

$$\dot{z} = \begin{cases} 0\ ,\ z\notin\varphi_t\left(\mathcal{B}_R\right) \\ \chi\left(\varphi_t^{-1}(z)\right)\left(\frac{\partial\psi}{\partial x}(0)\right)^{-1}\rho\left(\varphi_t^{-1}(z),t\right)\ ,\ z\in\varphi_t\left(\mathcal{B}_R\right) \end{cases} \tag{7}$$

where $\chi:\mathbb{R}^n\to\mathbb{R}^+$ is a C^1 function such that

$$\chi(x) = \begin{cases} 0\ ,\ x\notin\mathcal{B}_R \\ 1\ ,\ x\in\mathcal{B}_{R-\varepsilon} \end{cases}$$

Notice that the vector field is zero outside $S = \bigcup\limits_{\substack{z\in\mathcal{B}_R\\ t\in[0,1]}}\varphi_t(z)$, which is a compact set. Indeed, if it is not, one can construct (x_n) and (t_n), such that for all $n\in\mathbb{N}$,

$$\frac{|\psi(x_n t_n)|}{t_n}\geq n.$$

Since $\mathtt{cl}(\mathcal{B}_R)$ is compact, we extract $x_{\sigma(n)}t_{\sigma(n)}$ tending towards $x^*\in\mathtt{cl}(\mathcal{B}_R)$. Necessarily, $t_{\sigma(n)}$ tends to 0 and $x*=0$ since (x_n) is bounded. But this is impossible because, $\frac{\psi(x_n t_n)}{t_n}$ is equivalent to $\frac{\partial\psi}{\partial x}(0)x_n$ around zero.
Therefore,the maximal solutions to this system are defined (at least) in $[0,1]$ for all initial conditions, and through backward integration, we obtain that $\varphi(\mathbb{R}^n,1)=\mathbb{R}^n$.

We finally consider

$$\psi_e(x) = \frac{\partial\psi}{\partial x}(0)Z(x,1)$$

where $Z(x,1)$ is the solution of the system (7) evaluated at time 1 and initiated from x at time 0. This mapping being a linear transformation of a (time varying) flow, it is a diffeomorphism, and $\psi_e(\mathbb{R}^n)=\mathbb{R}^n$. Note moreover that for all $x\in\mathcal{B}_{R-\varepsilon}$, we have

$$\psi_e(x) = \frac{\partial\psi}{\partial x}(0)\varphi(x,1) = \psi(x)\ .$$

□

2 Proof of Theorem 1

Consider a diffeomorphism ψ: $\mathcal{D}\to\psi(\mathcal{D})\subsetneq\mathbb{R}^m$. We want to extend the image of ψ to $\mathbb{R}^m$, i-e find a diffeomorphism $\psi_e:\mathcal{D}\to\mathbb{R}^m$ such that:

- $\psi_e(\mathcal{D})=\mathbb{R}^m$

- $\psi_e(z) = \psi(z)$ for all z in $K \subset \mathcal{D}$, where K is any compact subset of $\mathcal{D}$.

Let us successively study the following two cases :

- **First case: $\psi(\mathcal{D})$ satisfies $(\mathcal{C})$** The result follows directly from Corollary 2 and Remark 2. In practice, the reader may find an explicit construction of such an extension in the proof of Lemma 1.

- **Second case: $\mathcal{D}$ is C^2-diffeomorphic to $\mathbb{R}^m$ and ψ is C^2:**

  Let $\phi_1 : \mathcal{D} \to \mathbb{R}^m$ denote the corresponding diffeomorphism. Let R_1 be a strictly positive real number such that the open ball $\mathcal{B}_{R_1}(0)$ contains $\phi_1(K)$. Let R_2 be a real number strictly larger than R_1. With Lemma 1 again, and since $\mathcal{B}_{R_2}(0)$ verifies condition $(\mathcal{C})$, there exists of C^2-diffeomorphism $\phi_2 : \mathcal{B}_{R_2}(0) \to \mathbb{R}^m$ satisfying

  $$\phi_2(z) \;=\; z \qquad \forall z \in \mathcal{B}_{R_1}(0)\ .$$

  At this point, we have obtained a C^2-diffeomorphism $\phi = \phi_2^{-1} \circ \phi_1 : \mathcal{D} \to \mathcal{B}_{R_2}(0)$. Consider $\lambda = \psi \circ \phi^{-1} : \mathcal{B}_{R_2}(0) \to \psi(\mathcal{D})$. According to Lemma 2, we can extend λ to $\lambda_e : \mathbb{R}^m \to \mathbb{R}^m$ such that $\lambda_e = \psi \circ \phi^{-1}$ on $\mathcal{B}_{R_1}(0)$. Finally, consider $\psi_e = \lambda_e \circ \phi_1 : \mathcal{D} \to \mathbb{R}^m$. Since, by construction of ϕ_2, $\phi = \phi_1$ on $\phi_1^{-1}(\mathcal{B}_{R_1}(0))$ which contains K, we have $\psi_e = \psi$ on K.

Remark 3 Note that the second construction is more complex than the first one, for it contains several extensions and in particular that of Lemma 2 which is difficult to implement. For this reason, one may prefer applying the first case whenever it is possible.

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