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# On diffeomorphism extension

Pauline Bernard, Laurent Praly\* and Vincent Andrieu<sup>†</sup>

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## Abstract

We state and prove a result concerning the fact that a diffeomorphism can be “extended” in such a way that its image is  $\mathbb{R}^n$ .

Before stating the theorem, we introduce the following property :

**Definition 1 (Conditions (C))** *An open subset  $E$  of  $\mathbb{R}^n$  is said to verify conditions (C) if there exist a  $C^1$  function  $h$ , a bounded<sup>1</sup>  $C^1$  vector field  $f$ , and a point  $P$  in  $E$  such that:*

- $E = \{x \in \mathbb{R}^n, h(x) < 0\}$
- $P$  is globally asymptotically stable for  $f^2$
- we have the following transversality assumption:

$$\frac{\partial h}{\partial x}(x)f(x) < 0, \quad \text{for all } x \text{ such that } h(x) = 0.$$

We now state the main theorem of this note :

**Theorem 1** *Let a diffeomorphism  $\psi: \mathcal{D} \rightarrow \psi(\mathcal{D}) \subsetneq \mathbb{R}^n$ . If  $\mathcal{D}$  or  $\psi(\mathcal{D})$  verifies conditions (C) then for any compact  $K$  in  $\mathcal{D}$  there exists a diffeomorphism  $\psi_e: \mathcal{D} \rightarrow \mathbb{R}^n$  such that :*

- $\psi_e(\mathcal{D}) = \mathbb{R}^n$
- $\psi_e(x) = \psi(x)$  for all  $x$  in  $K$ .

The proof of this theorem, given in Section 2, relies on two preliminary lemmas presented in Section 1.

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<sup>1</sup>If not replace  $f$  by  $\frac{f}{\sqrt{1+|f|^2}}$ .

<sup>2</sup>This assumption can be weakened, see Remark 2

# 1 Technical lemmas

## 1.1 Construction of a diffeomorphism from an open set to $\mathbb{R}^n$

In this section, we give sufficient conditions to build a diffeomorphism from  $\mathbb{R}^n$  to an open subset  $E$  which leaves  $E_\varepsilon \subset E$  unchanged. This construction is made explicit in the proof.

The complementary, closure, interior and boundary of a set  $S$  are denoted  $S^c$ ,  $\overline{S}$ ,  $\overset{\circ}{S}$  and  $\partial S$ , respectively, with  $\partial S = \overline{S} \setminus \overset{\circ}{S}$ .

The Hausdorff distance  $d_H$  between two sets  $A$  and  $B$  is defined by :

$$d_H(A, B) = \max \left\{ \sup_{x_A \in A} \inf_{x_B \in B} |x_A - x_B|, \sup_{x_B \in B} \inf_{x_A \in A} |x_A - x_B| \right\}$$

With  $X(x, t)$  we denote the (unique) solution to

$$\dot{x} = f(x)$$

at time  $t$ , going through  $x$  at time 0.

**Lemma 1** *Let  $E$  be an open strict subset of  $\mathbb{R}^n$  verifying (C). Then, for any strictly positive real number  $\varepsilon$ , there exists a diffeomorphism  $\phi: \mathbb{R}^n \rightarrow E$ , such that  $\phi = \text{Id}$  on  $E_\varepsilon = E \cap \Sigma^c$ , with*

$$\Sigma = \bigcup_{t \in [0, \varepsilon]} X(\partial E, t)$$

*Besides,  $E_\varepsilon$  may be rendered as close to  $E$  as needed by taking  $\varepsilon$  sufficiently small, in so far as :*

$$d_H(\partial E_\varepsilon, \partial E) \leq F \varepsilon .$$

*where  $F$  is a bound for  $f$ .*

**Proof :**

- **Forward-invariance of  $E$**

Pick  $x$  arbitrary in  $E$ . By definition,  $h(x) < 0$ . Let

$$S = \{t > 0, h(X(x, t)) \geq 0\} .$$

Suppose this set is not empty. Let

$$t^* = \min_{t \in S} t .$$

Since  $t \mapsto h \circ X(x, t)$  is continuous and  $h(X(x, 0)) = h(x) < 0$ ,  $t^*$  is strictly positive and satisfies

$$h(X(x, t^*)) = 0 .$$

Thus the transversality assumption gives  $\dot{h}(X(x, t^*)) < 0$ . The continuity of  $t \mapsto \dot{h}(X(x, t))$  implies the existence of  $\delta > 0$  such that  $\dot{h}(X(x, t)) < 0$  for all  $t \in [t^* - \delta, t^* + \delta]$ . This yields

$$h(X(x, t^* - \delta)) > h(X(x, t^*)) = 0 ,$$

which is impossible according to the definition of  $t^*$ . Consequently,  $S$  is empty and  $E$  is forward-invariant by  $f$ .

- **Properties of  $\Sigma$  and  $E_\varepsilon$**

$\Sigma$  is closed. Indeed, take a sequence  $(x_n)$  of points in  $\Sigma$  converging to  $x^*$ . By definition, there exists a sequence  $(t_n)$ , such that :

$$t_n \in [0, \varepsilon] \quad \text{and} \quad X(x_n, -t_n) \in \partial E \quad \forall n \in \mathbb{N} .$$

Since  $[0, \varepsilon]$  is compact, one can extract a subsequence  $(t_{\sigma(n)})$  converging to  $t^*$  in  $[0, \varepsilon]$ , and by continuity of the function  $(x, t) \mapsto X(x, -t)$ ,  $(X(x_{\sigma(n)}, t_{\sigma(n)}))$  tends to  $X(x^*, -t^*)$  which is in  $\partial E$ , since  $\partial E$  is closed. Finally, because  $t^*$  is in  $[0, \varepsilon]$ ,  $x^*$  is in  $\Sigma$  by definition.

$\Sigma \subset \overline{E}$ . Indeed,  $E$  is forward invariant by  $f$ , and thus  $\overline{E}$  too (see [4, Theorem 16.3]). This implies

$$\partial E \subset \Sigma = \bigcup_{t \in [0, \varepsilon]} X(\partial E, t) \subset \overline{E} = E \cup \partial E .$$

It may help the reader to note that, because  $E$  is open, the following set properties follow :

$$\Sigma \cap E = \Sigma \setminus \partial E$$

which yields :

$$E \setminus E_\varepsilon = (E_\varepsilon)^c \cap E = (E^c \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \setminus \partial E, \quad (1)$$

and :

$$E = E_\varepsilon \cup_{\neq} (\Sigma \setminus \partial E) .$$

- **Definition of  $t_x$  and  $\theta_x$**

The assumptions of global asymptotic stability of  $P$  which is in  $E$ , transversality of  $f$  and  $\partial E$ , and forward-invariance of  $E$ , imply that for all  $x$  in  $E^c$  there exists a unique  $t_x$  which is non negative and such that:

$$h(X(x, t_x)) = 0.$$

Notice that  $P$  being an equilibrium point in  $E$  and  $f$  being locally Lipschitz, it cannot be reached in finite time by a solution initialized in  $\partial E$ . We conclude that  $P$  is not in  $\Sigma$ . Besides, the same arguments in reverse time allow us to see that for all  $x$  in  $\Sigma$ ,  $t_x$  exists, is unique and in  $[-\varepsilon, 0]$ .

Next, for all  $x$  in  $(E_\varepsilon)^c$ , we define

$$\theta_x = X(x, t_x).$$

Through the Implicit Function Theorem, and thanks to the transversality assumption, the functions  $x \mapsto t_x$  and  $x \mapsto \theta_x$  are  $C^1$  on  $(E_\varepsilon)^c$ .

**Remark 1**  $h$  having constant rank 1 in a neighborhood of  $\partial E$ , this set is a closed, regular submanifold of  $\mathbb{R}^n$ . The arguments above show that  $x \mapsto (\theta_x, t_x)$  is a diffeomorphism between  $E^c$  and  $\partial E \times [0, +\infty[$ . Since  $\partial E$  is a deformation retract of  $E^c$  and the open unit ball is diffeomorphic to  $\mathbb{R}^n$  [3], if  $E$  were bounded,  $E^c$  could be seen as a  $h$ -cobordism between  $\partial E$  and the unit sphere  $\mathbb{S}^{n-1}$  and  $t_x$  as a Morse function with no critical point in  $E^c$ . See [5] for instance.

• **Characterization of  $\partial\Sigma$  and  $\partial E_\varepsilon$**

Pick  $x$  arbitrary in  $\partial\Sigma$ .  $\Sigma$  being closed,  $x$  is in  $\Sigma$ . Assume its corresponding  $t_x$  is in  $] -\varepsilon, 0[$ . The Implicit Function Theorem shows that  $x \mapsto t_x$  and  $x \mapsto \theta_x$  are defined and continuous on a neighborhood of  $x$ . Therefore, there exists a strictly positive real number  $r$  such that, with  $\mathcal{B}(x, r)$  denoting the open ball centered at  $x$  and with radius  $r$ ,

$$\forall y \in \mathcal{B}(x, r), \exists t_y \in ] -\varepsilon, 0[ : X(y, t_y) \in \partial E .$$

This implies that the neighborhood  $\mathcal{B}(x, r)$  of  $x$  is contained in  $\Sigma$ , in contradiction with the fact that  $x$  is on the boundary of  $\Sigma$ .

This shows that on  $\partial\Sigma$ ,  $t_x$  is either 0 or  $-\varepsilon$ . We write this as :

$$\partial\Sigma = \partial E \cup (\partial\Sigma)_i ,$$

with the notation

$$(\partial\Sigma)_i = \left\{ x \in \Sigma : t_x = -\varepsilon \right\} .$$

Now we want to prove  $\partial E_\varepsilon \subset (\partial\Sigma)_i$ . To obtain this result, we will successively show that :

$$\partial E_\varepsilon \cap \partial E = \emptyset \quad \text{and} \quad \partial E_\varepsilon \subset \partial\Sigma .$$

Suppose the existence of  $x$  in  $\partial E_\varepsilon \cap \partial E$ .  $x$  being in  $\partial E$ , it has a corresponding  $t_x$  which is 0. By the Implicit Function Theorem, there exists a strictly positive real number  $r$  such that,

$$\forall y \in \mathcal{B}(x, r), \exists t_y \in ] -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[ : X(y, t_y) \in \partial E .$$

But, by definition, any  $y$  for which there exists  $t_y$  in  $] -\frac{\varepsilon}{2}, 0]$  is in  $\Sigma$ . If instead  $t_y$  is strictly positive, then necessarily  $y$  is in  $E^c$ , because  $E$  is forward-invariant by  $f$  and a flow starting in  $E$  cannot reach  $\partial E$  in positive finite time. We have obtained :

$$\mathcal{B}(x, r) \subset \Sigma \cup E^c = (E_\varepsilon)^c .$$

$\mathcal{B}(x, r)$  being a neighborhood of  $x$ , this contradicts the fact that  $x$  is in the boundary of  $E_\varepsilon$ .

At this point, we have proved that  $\partial E_\varepsilon \cap \partial E = \emptyset$ , and, because  $E_\varepsilon$  is contained in  $E$ , this implies :

$$\partial E_\varepsilon \subset E .$$

With this, our result will be established by proving that we have

$$\partial E_\varepsilon \subset \partial\Sigma .$$

So let  $x$  be arbitrary in  $\partial E_\varepsilon$ . We now know that  $x$  is in  $E$  which is open. So there exists a strictly positive real number  $r$  such that we have :

$$\begin{aligned} \mathcal{B}(x, r) \cap E_\varepsilon &= \mathcal{B}(x, r) \cap (E \cap \Sigma^c) \neq \emptyset , \\ \mathcal{B}(x, r) \cap E_\varepsilon^c &= \mathcal{B}(x, r) \cap (E^c \cup \Sigma) \neq \emptyset , \\ &\mathcal{B}(x, r) \subset E . \end{aligned}$$

This implies

$$\begin{aligned}\mathcal{B}(x, r) \cap \Sigma^c &\neq \emptyset, \\ \mathcal{B}(x, r) \cap \Sigma &\neq \emptyset.\end{aligned}$$

Therefore  $x$  is in  $\partial\Sigma$ .

We have established :

$$\partial E_\varepsilon \cap \partial E = \emptyset, \quad \partial E_\varepsilon \subset \partial\Sigma, \quad \partial\Sigma = \partial E \cup (\partial\Sigma)_i$$

This does imply :

$$\partial E_\varepsilon \subset (\partial\Sigma)_i = \{x \in E : t_x = -\varepsilon\}. \quad (2)$$

• **Diffeomorphism construction**

We define a function  $\phi: \mathbb{R}^n \rightarrow E$  by:

$$\phi(x) = \begin{cases} X(x, t_x + \tau(t_x)), & \text{if } x \in (E_\varepsilon)^c, \\ x, & \text{if } x \in E_\varepsilon, \end{cases} \quad (3)$$

where  $\tau$  is a  $C^1$  and strictly decreasing function defined on  $\mathbb{R}$  such that:

$$\tau(t) = -t, \quad \forall t \leq -\varepsilon, \quad \lim_{t \rightarrow +\infty} \tau(t) = 0.$$

Since we have :

$$\begin{aligned}E_\varepsilon &\subset E, \\ t_x + \tau(t_x) &> t_x \quad \forall x \in E_\varepsilon^c, \\ X(x, t_x) &\in \partial E, \\ X(x, t) &\in E \quad \forall (x, t) \in \partial E \times \mathbb{R}_{>0},\end{aligned}$$

we do get that the image of  $\phi$  is contained in  $E$ .

The continuity of the functions  $(x, t) \in \mathbb{R}^n \times \mathbb{R} \mapsto X(x, t) \in \mathbb{R}$  and  $x \in E_\varepsilon^c \mapsto t_x \in [-\varepsilon, +\infty[$  implies that this function  $\phi$  is continuous at least on  $\mathbb{R}^n \setminus \partial E_\varepsilon$ . Also, for any  $x$  in  $\partial E_\varepsilon$ ,  $t_x$  is defined and equal to  $-\varepsilon$  (see (2)). For any strictly positive real number  $\eta$ , there exists a real number  $r$  such that :

$$\begin{aligned}|t_y + \varepsilon| &\leq \eta & \forall y \in \mathcal{B}(x, r), \\ \tau(t_y) + \varepsilon &\leq \eta & \forall y \in \mathcal{B}(x, r), \\ \phi(y) &= y & \forall y \in \mathcal{B}(x, r) \cap E_\varepsilon, \\ \phi(y) &= X(y, t_y + \tau(t_y)) & \forall y \in \mathcal{B}(x, r) \cap E_\varepsilon^c.\end{aligned}$$

Since we have :

$$\phi(x) = X(x, t_x + \tau(t_x)) = X(x, -\varepsilon + \tau(-\varepsilon)) = x,$$

we conclude that  $\phi$  is also continuous at  $x$ .

By differentiating, , we obtain :

– at any interior point  $x$  of  $(E_\varepsilon)^c$

$$\frac{\partial \phi}{\partial x}(x) = \frac{\partial X}{\partial x}(x, t_x + \tau(t_x)) + f(X(x, t_x + \tau(t_x))) \frac{\partial t_x}{\partial x}(x)(1 + \tau'(t_x)) ;$$

– at any  $x$  in  $E_\varepsilon$  (which is open) :

$$\frac{\partial \phi}{\partial x}(x) = I .$$

Also, for any  $x$  in  $\partial E_\varepsilon$ , we have :

$$\begin{aligned} \frac{\partial X}{\partial x}(x, t_x + \tau(t_x)) + f(X(x, t_x + \tau(t_x))) \frac{\partial t_x}{\partial x}(x)(1 + \tau'(t_x)) \\ = \frac{\partial X}{\partial x}(x, 0) + f(X(x, 0)) \frac{\partial t_x}{\partial x}(x)(1 - 1) , \\ = I . \end{aligned}$$

This implies that  $\phi$  is  $C^1$  on  $\mathbb{R}^n$ .

We now show that  $\phi$  is invertible. Let  $y$  be arbitrary in  $E \cap E_\varepsilon^c = E \cap \Sigma$ . There exists  $t_y$  in  $[-\varepsilon, 0[$ . The function  $\tau$  being strictly monotonic,  $\tau^{-1}(t_y)$  exists and is in  $[-\varepsilon, +\infty[$ . This allows us to define properly  $\phi^{-1}$  as :

$$\phi^{-1}(y) = \begin{cases} X(y, t_y - \tau^{-1}(-t_y)), & \text{if } y \in E \setminus E_\varepsilon \\ y, & \text{if } y \in E_\varepsilon \end{cases} \quad (4)$$

This function is an inverse of  $\phi$  as can be seen by reverting the flow induced by  $f$  when needed. Also, with the same arguments as before, we can prove that it is also  $C^1$ .

This implies that  $\phi$  is a diffeomorphism from  $\mathbb{R}^n$  to  $E$ .

- **Choice of  $\varepsilon$**

For any point  $x_\varepsilon$  in  $\partial E_\varepsilon$ , there exists a point  $x$  in  $\partial E$  satisfying :

$$|x_\varepsilon - x| = \left| \int_0^\varepsilon f(X(x, s)) ds \right| \leq F\varepsilon .$$

And conversely, for any  $x$  in  $\partial E$ , there exist  $x_\varepsilon$  in  $\partial E_\varepsilon$  satisfying :

$$|x_\varepsilon - x| = \left| \int_0^\varepsilon f(X(x, s)) ds \right| \leq F\varepsilon .$$

It follows that

$$d_H(\partial E_\varepsilon, \partial E) \leq F\varepsilon \quad (5)$$

and  $\varepsilon$  may be chosen as small as needed.

□

**Remark 2** The global asymptotic stability of  $P$  is sufficient, but not necessary. It is already sufficient to know that any solution initialized in a neighborhood of  $E^c$  reaches  $\partial E$  in finite positive or negative time. Note also that since (C) implies that  $E$  is diffeomorphic to  $\mathbb{R}^n$ ,  $E$  must be contractible.

Two direct consequences of Lemma 1 are :

**Corollary 1** *Let  $\psi : E \rightarrow \psi(E)$  be a diffeomorphism, with  $E$  satisfying (C). For any  $\varepsilon$  strictly positive, there exist an open subset  $E_\varepsilon$  of  $E$  and a diffeomorphism  $\psi_\varepsilon : \mathbb{R}^n \rightarrow \psi(E)$  satisfying :*

$$\begin{aligned} d_H(\partial E_\varepsilon, \partial E) &\leq \varepsilon , \\ \psi_\varepsilon(x) &= \psi(x) \quad \forall x \in E_\varepsilon . \end{aligned}$$

**Proof :** With  $\phi$  given by Lemma 1, we pick  $\psi_\varepsilon = \psi \circ \phi$ . □

**Corollary 2** *Let  $\psi : E \rightarrow \psi(E)$  be a diffeomorphism, with  $\psi(E)$  satisfying (C). For any  $\varepsilon$  strictly positive, there exist an open subset  $\psi(E)_\varepsilon$  of  $\psi(E)$  and a diffeomorphism  $\psi_\varepsilon : E \rightarrow \mathbb{R}^n$  satisfying :*

$$\begin{aligned} d_H(\partial(\psi(E)_\varepsilon), \partial\psi(E)) &\leq \varepsilon , \\ \psi_\varepsilon(x) &= \psi(x) \quad \forall x \in \psi^{-1}(\psi(E)_\varepsilon) . \end{aligned}$$

**Proof :** With  $\phi$  given by Lemma 1 from  $\mathbb{R}^n$  to  $\psi(E)$ , we pick  $\psi_\varepsilon = \phi^{-1} \circ \psi$ . □

**Remark 3** In Corollary 2,  $\psi$  being a diffeomorphism on an open set  $E$ , we know that the image of any compact subset  $K$  of  $E$  is a compact subset of  $\psi(E)$  which is also open by Brouwer's invariance theorem. Therefore, with (5), we can find  $\varepsilon$  such that  $\psi(K) \subset (\psi(E))_\varepsilon$  and thus,  $\psi_\varepsilon(x) = \psi(x)$  for all  $x$  in  $K$ .

## 1.2 Diffeomorphism extension from a ball

Let  $R > 0$ . We denote  $\mathcal{B}_R$  the open ball in  $\mathbb{R}^n$  of radius  $R$  and centered at 0.

**Lemma 2** *Consider a  $C^2$  diffeomorphism  $\psi : \mathcal{B}_R \rightarrow \psi(\mathcal{B}_R) \subset \mathbb{R}^n$ . For any  $\varepsilon$  strictly positive, there exists a diffeomorphism  $\psi_\varepsilon : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\psi_\varepsilon(x) = \psi(x)$  for all  $x$  in  $\text{cl}(\mathcal{B}_{R-\varepsilon})$ .*

**Proof :** Without loss of generality we may assume that  $\psi(0) = 0$ . Consider the mapping  $\varphi : \mathcal{B}_R \times [0, 1] \rightarrow \mathbb{R}^n$  defined as

$$\varphi(x, t) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\psi(xt)}{t} , \quad \varphi(x, 0) = x .$$



Note that for all  $t$  the mapping  $\varphi_t(x) = \varphi(x, t)$  is a diffeomorphism from  $\mathcal{B}_R$  toward  $\varphi_t(\mathcal{B}_R)$ . Indeed, given  $x_a$  and  $x_b$  such that  $\varphi(x_a, t) = \varphi(x_b, t)$  it yields  $\psi(x_a t) = \psi(x_b t)$ . Note that the couple  $(x_a t, x_b t)$  is in  $\mathcal{B}_R$ . The mapping  $\psi$  being injective on this set, it yields  $x_a = x_b$ . Moreover,

$$\frac{\partial \varphi_t}{\partial x}(x) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\partial \psi}{\partial x}(xt), \quad t > 0, \quad \frac{\partial \varphi_0}{\partial x}(x) = \text{Id}$$

Hence, this mapping is full rank in  $\mathcal{B}_R$ . Consequently, for all  $t$  in  $[0, 1]$ , this mapping is a diffeomorphism  $\mathcal{B}_R$  toward  $\varphi_t(\mathcal{B}_R)$ . Consequently, for all  $t$  in  $[0, 1]$  we can introduce  $\varphi_t^{-1}$  its inverse map.

Note moreover that

$$\overline{\varphi(x, t)} = \frac{\partial \varphi}{\partial t}(x, t) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(x, t)$$

where  $\rho$  is the function given as

$$\rho(x, t) = \frac{1}{t^2} \left[ \frac{\partial \psi}{\partial x}(xt)xt - \psi(xt) \right], \quad \rho(x, 0) = \frac{1}{2} x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

The mapping  $\psi$  being  $C^2$ , and  $\psi(0) = 0$  it yields,

$$\psi(xt) = \frac{\partial \psi}{\partial x}(0)xt + x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x \frac{t^2}{2} + o(t^2)$$

and,

$$\lim_{t \rightarrow 0} \frac{\frac{\partial \psi}{\partial x}(xt)x - \frac{\partial \psi}{\partial x}(0)x}{t} = x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

Hence, the function  $\rho$  is well defined and locally Lipschitz. Consequently,  $\varphi(x, t)$  is (the unique) solution of the time varying system defined for  $(z, t)$  in  $\varphi_t(\mathcal{B}_R) \times [0, 1]$  as

$$\dot{z} = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t)$$

This time varying system can be extended to  $\mathbb{R}^n$  as

$$\dot{z} = \begin{cases} 0, & z \notin \varphi_t(\mathcal{B}_R) \\ \chi(\varphi_t^{-1}(z)) \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t), & z \in \varphi_t(\mathcal{B}_R) \end{cases} \quad (6)$$

where  $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^+$  is a  $C^1$  function such that

$$\chi(x) = \begin{cases} 0, & x \notin \mathcal{B}_R \\ 1, & x \in \mathcal{B}_{R-\varepsilon} \end{cases}$$

Notice that the vector field is zero outside  $S = \bigcup_{\substack{z \in \mathcal{B}_R \\ t \in [0, 1]}} \varphi_t(z)$ , which is a compact set. Indeed, if

it is not, one can construct  $(x_n)$  and  $(t_n)$ , such that for all  $n \in \mathbb{N}$ ,

$$\frac{|\psi(x_n t_n)|}{t_n} \geq n.$$

Since  $\text{cl}(\mathcal{B}_R)$  is compact, we extract  $x_{\sigma(n)}t_{\sigma(n)}$  tending towards  $x^* \in \text{cl}(\mathcal{B}_R)$ . Necessarily,  $t_{\sigma(n)}$  tends to 0 and  $x^* = 0$  since  $(x_n)$  is bounded. But this is impossible because,  $\frac{\psi(x_n t_n)}{t_n}$  is equivalent to  $\frac{\partial \psi}{\partial x}(0)x_n$  around zero.

Therefore, the maximal solutions to this system are defined (at least) in  $[0, 1]$  for all initial conditions, and through backward integration, we obtain that  $\varphi(\mathbb{R}^n, 1) = \mathbb{R}^n$ .

We finally consider

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0)Z(x, 1)$$

where  $Z(x, 1)$  is the solution of the system (6) evaluated at time 1 and initiated from  $x$  at time 0. This mapping being a linear transformation of a (time varying) flow, it is a diffeomorphism, and  $\psi_e(\mathbb{R}^n) = \mathbb{R}^n$ . Note moreover that for all  $x \in \mathcal{B}_{R-\varepsilon}$ , we have

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0)\varphi(x, 1) = \psi(x) .$$

□

## 2 Proof of Theorem 1

Consider a diffeomorphism  $\psi: \mathcal{D} \rightarrow \psi(\mathcal{D}) \subsetneq \mathbb{R}^n$ . We want to extend the image of  $\psi$  to  $\mathbb{R}^n$ , i-e find a diffeomorphism  $\psi_e: \mathcal{D} \rightarrow \mathbb{R}^n$  such that:

- $\psi_e(\mathcal{D}) = \mathbb{R}^n$
- $\psi_e(x) = \psi(x)$  for all  $x$  in  $\mathcal{D}_\varepsilon \subset \mathcal{D}$ , where  $\mathcal{D}_\varepsilon$  can be rendered as close to  $\mathcal{D}$  as needed.

Let us successively study the following two cases :

- **First case:  $\psi(\mathcal{D})$  satisfies (C)**

According to Corollary 2, for any  $\varepsilon$  strictly positive, we build a diffeomorphism  $\psi_e: \mathcal{D} \rightarrow \mathbb{R}^n$ , such that  $\psi_e = \psi$  in  $\psi^{-1}(\psi(\mathcal{D})_\varepsilon) = \mathcal{D}_\varepsilon$ . Besides, as seen in Remark 3, for any compact  $K$  in  $\mathcal{D}$ , we can choose  $\varepsilon$  so that  $K \subset \mathcal{D}_\varepsilon$  and thus  $\psi_e(x) = \psi(x)$  for all  $x$  in  $K$ .

In practice, the reader may find an explicit construction of such an extension via the proofs of Lemma 1 and Corollary 2.

- **Second case:  $\mathcal{D}$  satisfies (C)**

We know via Lemma 1 that  $\mathcal{D}$  is diffeomorphic to  $\mathbb{R}^n$ , i-e one can build  $\phi_1: \mathcal{D} \rightarrow \mathbb{R}^n$ .

Let  $R > 0$  and consider an open ball  $\mathcal{B}_R$ .  $\mathcal{B}_R$  satisfies (C) too, and still via Lemma 1, one constructs  $\phi_2: \mathbb{R}^n \rightarrow \mathcal{B}_R$  such that  $\phi_2 = \text{Id}$  in  $\mathcal{B}_{R-\varepsilon}$ , for any  $\varepsilon$  strictly positive.

At this point, we have obtained a diffeomorphism  $\phi = \phi_2^{-1} \circ \phi_1: \mathcal{D} \rightarrow \mathcal{B}_R$ . Consider  $\lambda = \psi \circ \phi^{-1}: \mathcal{B}_R \rightarrow \psi(\mathcal{D})$ . According to Lemma 2, one can extend  $\lambda$  to  $\lambda_e: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\lambda_e = \psi \circ \phi^{-1}$  on  $\mathcal{B}_{R-\varepsilon}$ .

Finally, consider  $\psi_e = \lambda_e \circ \phi_1: \mathcal{D} \rightarrow \mathbb{R}^n$  and  $\mathcal{D}_\varepsilon = \phi^{-1}(\mathcal{B}_{R-\varepsilon})$ . Noting that by construction of  $\phi_2$ ,  $\phi = \phi_1$  on  $\mathcal{D}_\varepsilon$ , it yields  $\psi_e = \psi$  on  $\mathcal{D}_\varepsilon$ .

Besides, for any compact  $K$  of  $\mathcal{D}$ ,  $\phi(K)$  is included in  $\mathcal{B}_R$ , and we may choose  $\varepsilon$  sufficiently small such that  $\mathcal{B}_{R-\varepsilon}$  contains  $\phi(K)$ . Then,  $K \subset \mathcal{D}_\varepsilon$  and  $\psi_e = \psi$  in  $K$ .

**Remark 4** Note that the second construction is more complex than the first one, for it contains several extensions and in particular that of Lemma 2 which is difficult to implement. For this reason, one may prefer applying the first case whenever it is possible.

## References

- [1] P. Bernard, L. Praly, V. Andrieu, Tools for observers based on dynamic extension, Conference on Decision and Control, Under review
- [2] V. Andrieu, J.-B. Eytard, L. Praly, Dynamic extension without inversion for observers, Conference on Decision and Control, 2014
- [3] S. Gonnord, N. Tosel. Calcul Diffrentiel. Ellipses, 1998.
- [4] W. Hahn : *Stability of Motion*. Springer-Verlag, 1967
- [5] J. Milnor, Lectures on the  $h$ -cobordism Theorem. Notes by L. Siebenmann and J. Sondow. Princeton University Press, 1965.