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On diffeomorphism extension

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Abstract

We state and prove a result concerning the fact that a diffeomorphism can be “extended” in such a way that its image is \( \mathbb{R}^n \).

Before stating the theorem, we introduce the following property :

Definition 1 (Conditions (C)) An open subset \( E \) of \( \mathbb{R}^m \) is said to verify condition (C) if there exist a \( C^1 \) function \( \kappa : \mathbb{R}^m \to \mathbb{R} \), a bounded\(^1 \) \( C^1 \) vector field \( \chi \), and a closed set \( K_0 \) contained in \( E \) such that:

1. \( E = \{ z \in \mathbb{R}^n, \kappa(z) < 0 \} \)
2. \( K_0 \) is globally attractive for \( \chi \)
3. we have the following transversality assumption:

\[
\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \quad \forall z \in \mathbb{R}^m : \kappa(z) = 0.
\]

We now state the main theorem of this note :

Theorem 1 (Image extension) Let \( \psi : \mathcal{D} \subset \mathbb{R}^m \to \psi(\mathcal{D}) \subset \mathbb{R}^m \) be a diffeomorphism. If \( \psi(\mathcal{D}) \) verifies condition (C) or \( \mathcal{D} \) is \( C^2 \)-diffeomorphic to \( \mathbb{R}^m \) and \( \psi \) is \( C^2 \), then for any compact set \( K \) in \( \mathcal{D} \) there exists a diffeomorphism \( \psi_e : \mathcal{D} \to \mathbb{R}^m \) satisfying :

\[
\psi_e(\mathcal{D}) = \mathbb{R}^m, \quad \psi_e(z) = \psi(z) \quad \forall z \in K.
\]

The proof of this theorem, given in Section 2, relies on two preliminary lemmas presented in Section 1.

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\(^1\)If not replace \( \chi \) by \( \frac{\chi}{\sqrt{1+|\chi|^2}} \).
1 Technical lemmas

1.1 Construction of a diffeomorphism from an open set to $\mathbb{R}^m$

In this section, we give sufficient conditions to build a diffeomorphism from $\mathbb{R}^m$ to an open subset $E$ which leaves $E_{\varepsilon} \subset E$ unchanged. This construction is made explicit in the proof.

The complementary, closure, interior and boundary of a set $S$ are denoted $S^c$, $\overline{S}$, $\overset{\circ}{S}$ and $\partial S$, respectively, with $\partial S = \overline{S}\setminus \overset{\circ}{S}$.

The Hausdorff distance $d_H$ between two sets $A$ and $B$ is defined by:

$$d_H(A,B) = \max \left\{ \sup_{x_A \in A} \inf_{x_B \in B} |x_A - x_B|, \sup_{x_B \in B} \inf_{x_A \in A} |x_A - x_B| \right\}$$

With $Z(z,t)$ we denote the (unique) solution, at time $t$, to $\dot{z} = \chi(z)$ going trough $z$ at time $0$.

**Lemma 1** Let $E$ be an open strict subset of $\mathbb{R}^m$ verifying (C), with a $C^s$ vector field $\chi$. Then, for any strictly positive real number $\varepsilon$, there exists a $C^s$-diffeomorphism $\phi: \mathbb{R}^m \to E$, such that, with

$$\Sigma = \bigcup_{t \in [0,\varepsilon]} Z(\partial E, t),$$

we have $\phi(z) = z$ for all $z \in E_{\varepsilon} = E \cap \Sigma^c$ and $d_H(\partial E_{\varepsilon}, \partial E) \leq \varepsilon \sup_z |\chi(z)|$.

**Proof** : We start by establishing some properties.

- $E$ is forward invariant by $\chi$. This is a direct consequence of points 1 and 3 of the condition (C).

- $\Sigma$ is closed. Take a sequence $(z_k)$ of points in $\Sigma$ converging to $z^*$. By definition, there exists a sequence $(t_k)$, such that:

$$t_k \in [0,\varepsilon] \quad \text{and} \quad Z(z_k, -t_k) \in \partial E \quad \forall k \in \mathbb{N}.$$

Since $[0,\varepsilon]$ is compact, one can extract a subsequence $(t_{\sigma(k)})$ converging to $t^*$ in $[0,\varepsilon]$, and by continuity of the function $(z,t) \mapsto Z(z, -t)$, $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$ tends to $Z(z^*, -t^*)$ which is in $\partial E$, since $\partial E$ is closed. Finally, because $t^*$ is in $[0,\varepsilon]$, $z^*$ is in $\Sigma$ by definition.

- $\Sigma$ is contained in $c1(E)$. Since, $E$ is forward invariant by $\chi$, and so is $c1(E)$ (see [4, Theorem 16.3]). This implies

$$\partial E \subset \Sigma = \bigcup_{t \in [0,\varepsilon]} Z(\partial E, t) \subset c1(E) = E \cup \partial E.$$

At this point, it is useful to note that, because $\Sigma$ is a closed subset of the open set $E$, we have $\Sigma \cap E = \Sigma \setminus \partial E$. This implies:

$$E \setminus E_{\varepsilon} = (E_{\varepsilon})^c \cap E = (E^c \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \setminus \partial E,$$

and $E = E_{\varepsilon} \cup (\Sigma \setminus \partial E)$.

With all these properties at hand, we define now two functions $t_z$ and $\theta_z$. The assumptions of global attractiveness of the closed set $K_0$ contained in $E$ open, of transversality of $\chi$ to $\partial E$,
and the property of forward-invariance of \( E \), imply that, for all \( z \) in \( E^\varepsilon \), there exists a unique non negative real number \( t_z \) satisfying:

\[
\kappa (Z(z, t_z)) = 0 \iff Z(z, t_z) \in \partial E.
\]

The same arguments in reverse time allow us to see that, for all \( z \) in \( \Sigma \), \( t_z \) exists, is unique and in \([ -\varepsilon, 0] \). This way, the function \( z \to t_z \) is defined on \((E_\varepsilon)^\varepsilon \). Next, for all \( z \) in \((E_\varepsilon)^\varepsilon \), we define :

\[
\theta_z = Z(z, t_z).
\]

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions \( z \mapsto t_z \) and \( z \mapsto \theta_z \) are \( C^s \) on \((E_\varepsilon)^\varepsilon \).

**Remark 1** \( \kappa \) having constant rank 1 in a neighborhood of \( \partial E \), this set is a closed, regular submanifold of \( \mathbb{R}^m \). The arguments above show that \( z \mapsto (\theta_z, t_z) \) is a diffeomorphism between \( E^\varepsilon \) and \( \partial E \times [0, +\infty[ \). Since \( \partial E \) is a deformation retraction of \( E^\varepsilon \) and the open unit ball is diffeomorphic to \( \mathbb{R}^m \) [?], if \( E \) were bounded, \( E^\varepsilon \) could be seen as a \( h \)-cobordism between \( \partial E \) and the unit sphere \( S^{m-1} \) and \( t_z \) as a Morse function with no critical point in \( E^\varepsilon \). See [5] for instance.

Now we evaluate \( t_z \) for \( z \) in \( \partial \Sigma \). Let \( z \) be arbitrary in \( \partial \Sigma \) and therefore in \( \Sigma \) which is closed. Assume its corresponding \( t_z \) is in \( ]-\varepsilon, 0[ \). The Implicit Function Theorem shows that \( z \mapsto t_z \) and \( z \mapsto \theta_z \) are defined and continuous on a neighborhood of \( z \). Therefore, there exists a strictly positive real number \( r \) satisfying

\[
\forall y \in B_r(z), \exists t_y \in ]-\varepsilon, 0[ : Z(y, t_y) \in \partial E.
\]

This implies that the neighborhood \( B_r(z) \) of \( z \) is contained in \( \Sigma \), in contradiction with the fact that \( z \) is on the boundary of \( \Sigma \).

This shows that, for all \( z \) in \( \partial \Sigma \), \( t_z \) is either 0 or \(-\varepsilon \). We write this as \( \partial \Sigma = \partial E \cup (\partial \Sigma)_i \), with the notation \( (\partial \Sigma)_i = \{ z \in \Sigma : t_z = -\varepsilon \} \).

Now we want to prove \( \partial E_\varepsilon \subset (\partial \Sigma)_i \). To obtain this result, we start by showing :

\[
\partial E_\varepsilon \cap \partial E = \emptyset \quad \text{and} \quad \partial E_\varepsilon \subset \partial \Sigma.
\] (2)

Suppose the existence of \( z \) in \( \partial E_\varepsilon \cap \partial E \). \( z \) being in \( \partial E \), its corresponding \( t_z \) is 0. By the Implicit Function Theorem, there exists a strictly positive real number \( r \) such that,

\[
\forall y \in B_r(z), \exists t_y \in ]-\varepsilon, 0[ : Z(y, t_y) \in \partial E.
\]

But, by definition, any \( y \), for which there exists \( t_y \) in \( ]-\varepsilon, 0[ \), is in \( \Sigma \). If instead \( t_y \) is strictly positive, then necessarily \( y \) is in \( E^\varepsilon \), because \( E \) is forward-invariant by \( \chi \) and a solution starting in \( E \) cannot reach \( \partial E \) in positive finite time. We have obtained : \( B_r(z) \subset \Sigma \cup E^\varepsilon = (E_\varepsilon)^\varepsilon \). \( B_r(z) \) being a neighborhood of \( z \), this contradicts the fact that \( z \) is in the boundary of \( E_\varepsilon \).

At this point, we have proved that \( \partial E_\varepsilon \cap \partial E = \emptyset \), and, because \( E_\varepsilon \) is contained in \( E \), this implies \( \partial E_\varepsilon \subset E \). With this, (2) will be established by proving that we have \( \partial E_\varepsilon \subset \partial \Sigma \). Let \( z \) be arbitrary in \( \partial E_\varepsilon \) and therefore in \( E \) which is open. There exists a strictly positive real number \( r \) such that we have :

\[
B_r(z) \cap E_\varepsilon = B_r(z) \cap (E \cap \Sigma^c) \neq \emptyset, \quad B_r(z) \cap E_\varepsilon^c = B_r(z) \cap (E^c \cup \Sigma) \neq \emptyset, \quad B_r(z) \subset E.
\]
This implies $\mathcal{B}_r(z) \cap \Sigma^c \neq \emptyset$ and $\mathcal{B}_r(z) \cap \Sigma \neq \emptyset$ and therefore that $z$ is in $\partial \Sigma$.

We have established $\partial E_\epsilon \cap \partial E = \emptyset$, $\partial E_\epsilon \subset \partial \Sigma$ and $\partial \Sigma = \partial E \cup (\partial \Sigma)_t$. This does imply:

$$\partial E_\epsilon \subset (\partial \Sigma)_t = \{ z \in E : t_z = -\epsilon \} . \tag{3}$$

This allows us to extend by continuity the definition of $t_z$ to $\mathbb{R}^m$ by letting $t_z = -\epsilon$ for all $z \in E_\epsilon$.

Thanks to all these preparatory steps, we are finally ready to define a function $\phi : \mathbb{R}^m \to E$ as:

$$\phi(z) = \begin{cases} Z(z, t_z + \nu(t_z)), & \text{if } z \in (E_\epsilon)^c, \\ z, & \text{if } z \in E_\epsilon, \end{cases} \tag{4}$$

where $\nu$ is an arbitrary $C^s$ and strictly decreasing function defined on $\mathbb{R}$ satisfying:

$$\nu(t) = -t \quad \forall t \leq -\epsilon, \quad \lim_{t \to +\infty} \nu(t) = 0.$$

The image of $\phi$ is contained in $E$ since we have $E_\epsilon \subset E$ and:

$$Z(z, t_z) \in \partial E,$$
$$Z(z, t) \in E \quad \forall (z, t) \in \partial E \times \mathbb{R}_{> 0}.$$

The continuity of the functions $(z, t) \in \mathbb{R}^m \times \mathbb{R} \mapsto Z(z, t) \in \mathbb{R}$ and $z \in E_\epsilon \mapsto t_z \in [-\epsilon, +\infty[$ implies that this function $\phi$ is continuous at least on $\mathbb{R}^m \setminus \partial E_\epsilon$. Also, for any $z$ in $\partial E_\epsilon$, $t_z$ is defined and equal to $-\epsilon$ (see (3)). So, for any strictly positive real number $\eta$, there exists a real number $r$ such that:

$$|t_{y} + \epsilon| \leq \eta \quad \forall y \in B_r(z),$$
$$\nu(t_y) + \epsilon \leq \eta \quad \forall y \in B_r(z),$$
$$\phi(y) = y \quad \forall y \in B_r(z) \cap E_\epsilon,$$
$$\phi(y) = Z(y, t_y + \nu(t_y)) \quad \forall y \in B_r(z) \cap E_\epsilon.$$

Since we have:

$$\phi(z) = Z(z, t_z + \nu(t_z)) = Z(z, -\epsilon + \nu(-\epsilon)) = z,$$

we conclude that $\phi$ is also continuous at $z$.

By differentiating, we obtain:

- at any interior point $z$ of $(E_\epsilon)^c$

$$\frac{\partial \phi}{\partial z}(z) = \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z));$$

- at any $z$ in $E_\epsilon$ (which is open) $\frac{\partial \phi}{\partial z}(z) = I$. Also, for any $z$ in $\partial E_\epsilon$, we have:

$$\frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) = \frac{\partial Z}{\partial z}(z, 0) + \chi(Z(z, 0)) \frac{\partial t_z}{\partial z}(z)(1 - 1),$$

$$= I.$$

This implies that $\phi$ is $C^1$ on $\mathbb{R}^m$. 
We now show that $\phi$ is invertible. Let $y$ be arbitrary in $E \cap E_\epsilon^c = E \cap \Sigma$. There exists $t_y$ in $[-\epsilon, 0]$. The function $\nu$ being strictly monotonic, $\nu^{-1}(t_y)$ exists and is in $[-\epsilon, +\infty]$. This allows us to define properly $\phi^{-1}$ as:

$$
\phi^{-1}(y) = \begin{cases} 
Z(y, t_y - \nu^{-1}(-t_y)), & \text{if } y \in E \setminus E_\epsilon \\
y, & \text{if } y \in E_\epsilon
\end{cases}
$$

This function is an inverse of $\phi$ as can be seen be reverting the flow induced by $\chi$ when needed. Also, with the same arguments as before, we can prove that it is $C^1$.

This implies that $\phi$ is a diffeomorphism from $\mathbb{R}^m$ to $E$.

Besides, the functions $z \mapsto Z(z, t)$ for $t > 0$, $z \mapsto t_z$ and $\nu$ being $C^s$, $\phi$ is $C^s$ at any interior point of $(E_\epsilon)^c$. By continuity of $\nu(r)$ for $r \leq s$, it can be verified that $\phi$ is also $C^s$ on the boundary $\partial E_\epsilon$. So, $\phi$ is a $C^s$-diffeomorphism from $\mathbb{R}^m$ to $E$.

Finally, we note that, for any point $z_\epsilon$ in $\partial E_\epsilon$, there exists a point $z$ in $\partial E$ satisfying:

$$
|z_\epsilon - z| = \left| \int_0^\epsilon \chi(Z(z, s))ds \right| \leq \epsilon \sup_{\zeta} |\chi(\zeta)|.
$$

And conversely, for any $z$ in $\partial E$, there exist $z_\epsilon$ in $\partial E_\epsilon$ satisfying:

$$
|z_\epsilon - z| = \left| \int_0^\epsilon \chi(Z(z, s))ds \right| \leq \epsilon \sup_{\zeta} |\chi(\zeta)|.
$$

It follows that

$$
d_H(\partial E_\epsilon, \partial E) \leq \epsilon \sup_{\zeta} |\chi(\zeta)| \tag{6}
$$

and $\epsilon$ may be chosen as small as needed. \hfill \square

Two direct consequences of Lemma 1 are:

**Corollary 1** Let $\psi : E \to \psi(E)$ be a diffeomorphism, with $E$ satisfying $(C)$. For any $\epsilon$ strictly positive, there exist an open subset $E_\epsilon$ of $E$ and a diffeomorphism $\psi_\epsilon : \mathbb{R}^m \to \psi(E)$ satisfying:

$$
d_H(\partial E_\epsilon, \partial E) \leq \epsilon, \\
\psi_\epsilon(z) = \psi(z) \quad \forall z \in E_\epsilon.
$$

**Proof**: With $\phi$ given by Lemma 1, we pick $\psi_\epsilon = \psi \circ \phi$. \hfill \square

**Corollary 2** Let $\psi : E \to \psi(E)$ be a diffeomorphism, with $\psi(E)$ satisfying $(C)$. For any $\epsilon$ strictly positive, there exist an open subset $\psi(E)_\epsilon$ of $\psi(E)$ and a diffeomorphism $\psi_\epsilon : E \to \mathbb{R}^n$ satisfying:

$$
d_H(\partial (\psi(E)_\epsilon), \partial \psi(E)) \leq \epsilon, \\
\psi_\epsilon(z) = \psi(z) \quad \forall z \in \psi^{-1}(\psi(E)_\epsilon).
$$
Proof: With $\phi$ given by Lemma 1 from $\mathbb{R}^n$ to $\psi(E)$, we pick $\psi_e = \phi^{-1} \circ \psi$. \qed

Remark 2 In Corollary 2, $\psi$ being a diffeomorphism on an open set $E$, we know that the image of any compact subset $K$ of $E$ is a compact subset of $\psi(E)$ which is also open by Brouwer’s invariance theorem. Therefore, with (6), we can find $\varepsilon$ such that $\psi(K) \subset (\psi(E))_\varepsilon$ and thus, $\psi_e(z) = \psi(z)$ for all $x$ in $K$.

1.2 Diffeomorphism extension from a ball

Let $R > 0$. We denote $B_R$ the open ball in $\mathbb{R}^n$ of radius $R$ and centered at 0.

Lemma 2 Consider a $C^2$ diffeomorphism $\psi : B_R \rightarrow \psi(B_R) \subset \mathbb{R}^n$. For any $\varepsilon$ strictly positive, there exists a diffeomorphism $\psi_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\psi_e(x) = \psi(x)$ for all $x$ in $\text{cl}(B_R - \varepsilon)$.

Proof: Without loss of generality we may assume that $\psi(0) = 0$. Consider the mapping $\varphi : B_R \times [0,1] \rightarrow \mathbb{R}^n$ defined as

$$\varphi(x,t) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\psi(xt)}{t}, \quad \varphi(x,0) = x.$$

Note that for all $t$ the mapping $\varphi_t(x) = \varphi(x,t)$ is a diffeomorphism from $B_R$ toward $\varphi_t(B_R)$. Indeed, given $x_a$ and $x_b$ such that $\varphi(x_a,t) = \varphi(x_b,t)$ it yields $\psi(x_at) = \psi(x_bt)$. Note that the couple $(x_at,x.bt)$ is in $B_R$. The mapping $\psi$ being injective on this set, it yields $x_a = x_b$. Moreover,

$$\frac{\partial \varphi_t}{\partial x}(x) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\partial \psi}{\partial x}(xt), \quad t > 0, \quad \frac{\partial \varphi_0}{\partial x}(x) = \text{Id}$$

Hence, this mapping is full rank in $B_R$. Consequently, for all $t$ in $[0,1]$, this mapping is a diffeomorphism $B_R$ toward $\varphi_t(B_R)$. Consequently, for all $t$ in $[0,1]$ we can introduce $\varphi_t^{-1}$ its inverse map.

Note moreover that

$$\overline{\varphi(x,t)} = \frac{\partial \varphi}{\partial t}(x,t) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(x,t)$$

where $\rho$ is the function given as

$$\rho(x,t) = \frac{1}{t^2} \left[ \frac{\partial \psi}{\partial x}(xt)(xt - \psi(xt)) \right], \quad \rho(x,0) = \frac{1}{2} x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

The mapping $\psi$ being $C^2$, and $\psi(0) = 0$ it yields,

$$\psi(xt) = \frac{\partial \psi}{\partial x}(0)(xt) + x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) \frac{t^2}{2} + o(t^2),$$

and,

$$\lim_{t \rightarrow 0} \frac{\frac{\partial \psi}{\partial x}(xt) - \frac{\partial \psi}{\partial x}(0)x}{t} = x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$
Hence, the function $\rho$ is well defined and locally Lipschitz. Consequently, $\varphi(x, t)$ is (the unique) solution of the time varying system defined for $(z, t)$ in $\varphi_t(B_R) \times [0, 1]$ as

$$\dot{z} = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho \left( \varphi^{-1}_t(z), t \right)$$

This time varying system can be extended to $\mathbb{R}^n$ as

$$\dot{z} = \begin{cases} 0, & z \notin \varphi_t(B_R) \\ \chi \left( \varphi^{-1}_t(z) \right) \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho \left( \varphi^{-1}_t(z), t \right), & z \in \varphi_t(B_R) \end{cases} \quad (7)$$

where $\chi : \mathbb{R}^n \to \mathbb{R}^+$ is a $C^1$ function such that

$$\chi(x) = \begin{cases} 0, & x \notin B_R \\ 1, & x \in B_{R-\varepsilon} \end{cases}$$

Notice that the vector field is zero outside $S = \bigcup_{z \in B_R} \varphi_t(z)$, which is a compact set. Indeed, if it is not, one can construct $(x_n)$ and $(t_n)$, such that for all $n \in \mathbb{N}$,

$$\frac{|\psi(x_n t_n)|}{t_n} \geq n.$$ 

Since $cl(B_R)$ is compact, we extract $x_{\sigma(n)} t_{\sigma(n)}$ tending towards $x^* \in cl(B_R)$. Necessarily, $t_{\sigma(n)}$ tends to 0 and $x^* = 0$ since $(x_n)$ is bounded. But this is impossible because, $\frac{\psi(x_n t_n)}{t_n}$ is equivalent to $\frac{\partial \psi(0) x_n}{t_n}$ around zero.

Therefore, the maximal solutions to this system are defined (at least) in $[0, 1]$ for all initial conditions, and through backward integration, we obtain that $\varphi(\mathbb{R}^n, 1) = \mathbb{R}^n$.

We finally consider

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) Z(x, 1)$$

where $Z(x, 1)$ is the solution of the system (7) evaluated at time 1 and initiated from $x$ at time 0. This mapping being a linear transformation of a (time varying) flow, it is a diffeomorphism, and $\psi_e(\mathbb{R}^n) = \mathbb{R}^n$. Note moreover that for all $x \in B_{R-\varepsilon}$, we have

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) \varphi(x, 1) = \psi(x) .$$ 

\[ \square \]

### 2 Proof of Theorem 1

Consider a diffeomorphism $\psi : D \to \psi(D) \subset \mathbb{R}^m$. We want to extend the image of $\psi$ to $\mathbb{R}^m$, i.e. find a diffeomorphism $\psi_e : D \to \mathbb{R}^m$ such that:

- $\psi_e(D) = \mathbb{R}^m$
ψ_e(z) = ψ(z) for all z in K ⊂ D, where K is any compact subset of D.

Let us successively study the following two cases:

- **First case:** ψ(D) satisfies (C) The result follows directly from Corollary 2 and Remark 2. In practice, the reader may find an explicit construction of such an extension in the proof of Lemma 1.

- **Second case:** D is $C^2$-diffeomorphic to $\mathbb{R}^m$ and ψ is $C^2$:

  Let $\phi_1 : D \to \mathbb{R}^m$ denote the corresponding diffeomorphism. Let $R_1$ be a strictly positive real number such that the open ball $B_{R_1}(0)$ contains $\phi_1(K)$. Let $R_2$ be a real number strictly larger than $R_1$. With Lemma 1 again, and since $B_{R_2}(0)$ verifies condition (C), there exists a $C^2$-diffeomorphism $\phi_2 : B_{R_2}(0) \to \mathbb{R}^m$ satisfying $\phi_2(z) = z \forall z \in B_{R_1}(0)$.

  At this point, we have obtained a $C^2$-diffeomorphism $\phi = \phi_2^{-1} \circ \phi_1 : D \to B_{R_2}(0)$. Consider $\lambda = \psi \circ \phi^{-1} : B_{R_2}(0) \to \psi(D)$. According to Lemma 2, we can extend $\lambda$ to $\lambda_e : \mathbb{R}^m \to \mathbb{R}^m$ such that $\lambda_e = \psi \circ \phi^{-1}$ on $B_{R_2}(0)$. Finally, consider $\psi_e = \lambda_e \circ \phi_1 : D \to \mathbb{R}^m$. Since, by construction of $\phi_2$, $\phi = \phi_1$ on $\phi_2^{-1}(B_{R_2}(0))$ which contains $K$, we have $\psi_e = \psi$ on $K$.

**Remark 3** Note that the second construction is more complex than the first one, for it contains several extensions and in particular that of Lemma 2 which is difficult to implement. For this reason, one may prefer applying the first case whenever it is possible.

**References**


