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# On diffeomorphism extension

Pauline Bernard, Laurent Praly\* and Vincent Andrieu<sup>†</sup>

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## Abstract

We state and prove a result concerning the fact that a diffeomorphism can be “extended” in such a way that its image is  $\mathbb{R}^n$ .

Before stating the theorem, we introduce the following property :

**Definition 1 (Conditions (C))** *An open subset  $E$  of  $\mathbb{R}^m$  is said to verify condition (C) if there exist a  $C^1$  function  $\kappa : \mathbb{R}^m \rightarrow \mathbb{R}$ , a bounded<sup>1</sup>  $C^1$  vector field  $\chi$ , and a closed set  $K_0$  contained in  $E$  such that:*

1.  $E = \{z \in \mathbb{R}^n, \kappa(z) < 0\}$
2.  $K_0$  is globally attractive for  $\chi$
3. we have the following transversality assumption:

$$\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \quad \forall z \in \mathbb{R}^m : \kappa(z) = 0.$$

We now state the main theorem of this note :

**Theorem 1 (Image extension)** *Let  $\psi : \mathcal{D} \subset \mathbb{R}^m \rightarrow \psi(\mathcal{D}) \subset \mathbb{R}^m$  be a diffeomorphism. If  $\psi(\mathcal{D})$  verifies condition (C) or  $\mathcal{D}$  is  $C^2$ -diffeomorphic to  $\mathbb{R}^m$  and  $\psi$  is  $C^2$ , then for any compact set  $K$  in  $\mathcal{D}$  there exists a diffeomorphism  $\psi_e : \mathcal{D} \rightarrow \mathbb{R}^m$  satisfying :*

$$\psi_e(\mathcal{D}) = \mathbb{R}^m \quad , \quad \psi_e(z) = \psi(z) \quad \forall z \in K.$$

The proof of this theorem, given in Section 2, relies on two preliminary lemmas presented in Section 1.

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<sup>1</sup>If not replace  $\chi$  by  $\frac{\chi}{\sqrt{1+|\chi|^2}}$ .

# 1 Technical lemmas

## 1.1 Construction of a diffeomorphism from an open set to $\mathbb{R}^m$

In this section, we give sufficient conditions to build a diffeomorphism from  $\mathbb{R}^m$  to an open subset  $E$  which leaves  $E_\varepsilon \subset E$  unchanged. This construction is made explicit in the proof.

The complementary, closure, interior and boundary of a set  $S$  are denoted  $S^c$ ,  $\overline{S}$ ,  $\overset{\circ}{S}$  and  $\partial S$ , respectively, with  $\partial S = \overline{S} \setminus \overset{\circ}{S}$ .

The Hausdorff distance  $d_H$  between two sets  $A$  and  $B$  is defined by :

$$d_H(A, B) = \max \left\{ \sup_{x_A \in A} \inf_{x_B \in B} |x_A - x_B|, \sup_{x_B \in B} \inf_{x_A \in A} |x_A - x_B| \right\}$$

With  $Z(z, t)$  we denote the (unique) solution, at time  $t$ , to  $\dot{z} = \chi(z)$  going through  $z$  at time 0.

**Lemma 1** *Let  $E$  be an open strict subset of  $\mathbb{R}^m$  verifying (C), with a  $C^s$  vector field  $\chi$ . Then, for any strictly positive real number  $\varepsilon$ , there exists a  $C^s$ -diffeomorphism  $\phi: \mathbb{R}^m \rightarrow E$ , such that, with*

$$\Sigma = \bigcup_{t \in [0, \varepsilon]} Z(\partial E, t),$$

*we have  $\phi(z) = z$  for all  $z \in E_\varepsilon = E \cap \Sigma^c$  and  $d_H(\partial E_\varepsilon, \partial E) \leq \varepsilon \sup_z |\chi(z)|$ .*

**Proof :** We start by establishing some properties.

- $E$  is forward invariant by  $\chi$ . This is a direct consequence of points 1 and 3 of the condition (C).
- $\Sigma$  is closed. Take a sequence  $(z_k)$  of points in  $\Sigma$  converging to  $z^*$ . By definition, there exists a sequence  $(t_k)$ , such that :

$$t_k \in [0, \varepsilon] \quad \text{and} \quad Z(z_k, -t_k) \in \partial E \quad \forall k \in \mathbb{N}.$$

Since  $[0, \varepsilon]$  is compact, one can extract a subsequence  $(t_{\sigma(k)})$  converging to  $t^*$  in  $[0, \varepsilon]$ , and by continuity of the function  $(z, t) \mapsto Z(z, -t)$ ,  $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$  tends to  $Z(z^*, -t^*)$  which is in  $\partial E$ , since  $\partial E$  is closed. Finally, because  $t^*$  is in  $[0, \varepsilon]$ ,  $z^*$  is in  $\Sigma$  by definition.

–  $\Sigma$  is contained in  $\text{cl}(E)$ . Since,  $E$  is forward invariant by  $\chi$ , and so is  $\text{cl}(E)$  (see [4, Theorem 16.3]). This implies

$$\partial E \subset \Sigma = \bigcup_{t \in [0, \varepsilon]} Z(\partial E, t) \subset \text{cl}(E) = E \cup \partial E.$$

At this point, it is useful to note that, because  $\Sigma$  is a closed subset of the open set  $E$ , we have  $\Sigma \cap E = \Sigma \setminus \partial E$ . This implies :

$$E \setminus E_\varepsilon = (E_\varepsilon)^c \cap E = (E^c \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \setminus \partial E, \tag{1}$$

and  $E = E_\varepsilon \cup (\Sigma \setminus \partial E)$ .

With all these properties at hand, we define now two functions  $t_z$  and  $\theta_z$ . The assumptions of global attractiveness of the closed set  $K_0$  contained in  $E$  open, of transversality of  $\chi$  to  $\partial E$ ,

and the property of forward-invariance of  $E$ , imply that, for all  $z$  in  $E^c$ , there exists a unique non negative real number  $t_z$  satisfying:

$$\kappa(Z(z, t_z)) = 0 \iff Z(z, t_z) \in \partial E.$$

The same arguments in reverse time allow us to see that, for all  $z$  in  $\Sigma$ ,  $t_z$  exists, is unique and in  $[-\varepsilon, 0]$ . This way, the function  $z \rightarrow t_z$  is defined on  $(E_\varepsilon)^c$ . Next, for all  $z$  in  $(E_\varepsilon)^c$ , we define :

$$\theta_z = Z(z, t_z).$$

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions  $z \mapsto t_z$  and  $z \mapsto \theta_z$  are  $C^s$  on  $(E_\varepsilon)^c$ .

**Remark 1**  $\kappa$  having constant rank 1 in a neighborhood of  $\partial E$ , this set is a closed, regular submanifold of  $\mathbb{R}^m$ . The arguments above show that  $z \mapsto (\theta_z, t_z)$  is a diffeomorphism between  $E^c$  and  $\partial E \times [0, +\infty[$ . Since  $\partial E$  is a deformation retract of  $E^c$  and the open unit ball is diffeomorphic to  $\mathbb{R}^m$  [?], if  $E$  were bounded,  $E^c$  could be seen as a  $h$ -cobordism between  $\partial E$  and the unit sphere  $\mathbb{S}^{m-1}$  and  $t_z$  as a Morse function with no critical point in  $E^c$ . See [5] for instance.

Now we evaluate  $t_z$  for  $z$  in  $\partial\Sigma$ . Let  $z$  be arbitrary in  $\partial\Sigma$  and therefore in  $\Sigma$  which is closed. Assume its corresponding  $t_z$  is in  $] -\varepsilon, 0[$ . The Implicit Function Theorem shows that  $z \mapsto t_z$  and  $z \mapsto \theta_z$  are defined and continuous on a neighborhood of  $z$ . Therefore, there exists a strictly positive real number  $r$  satisfying

$$\forall y \in \mathcal{B}_r(z), \exists t_y \in ] -\varepsilon, 0[ : Z(y, t_y) \in \partial E .$$

This implies that the neighborhood  $\mathcal{B}_r(z)$  of  $z$  is contained in  $\Sigma$ , in contradiction with the fact that  $z$  is on the boundary of  $\Sigma$ .

This shows that, for all  $z$  in  $\partial\Sigma$ ,  $t_z$  is either 0 or  $-\varepsilon$ . We write this as  $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$ , with the notation  $(\partial\Sigma)_i = \left\{ z \in \Sigma : t_z = -\varepsilon \right\}$ .

Now we want to prove  $\partial E_\varepsilon \subset (\partial\Sigma)_i$ . To obtain this result, we start by showing :

$$\partial E_\varepsilon \cap \partial E = \emptyset \quad \text{and} \quad \partial E_\varepsilon \subset \partial\Sigma . \tag{2}$$

Suppose the existence of  $z$  in  $\partial E_\varepsilon \cap \partial E$ .  $z$  being in  $\partial E$ , its corresponding  $t_z$  is 0. By the Implicit Function Theorem, there exists a strictly positive real number  $r$  such that,

$$\forall y \in \mathcal{B}_r(z), \exists t_y \in ] -\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[ : Z(y, t_y) \in \partial E .$$

But, by definition, any  $y$ , for which there exists  $t_y$  in  $] -\frac{\varepsilon}{2}, 0]$ , is in  $\Sigma$ . If instead  $t_y$  is strictly positive, then necessarily  $y$  is in  $E^c$ , because  $E$  is forward-invariant by  $\chi$  and a solution starting in  $E$  cannot reach  $\partial E$  in positive finite time. We have obtained :  $\mathcal{B}_r(z) \subset \Sigma \cup E^c = (E_\varepsilon)^c$ .  $\mathcal{B}_r(z)$  being a neighborhood of  $z$ , this contradicts the fact that  $z$  is in the boundary of  $E_\varepsilon$ .

At this point, we have proved that  $\partial E_\varepsilon \cap \partial E = \emptyset$ , and, because  $E_\varepsilon$  is contained in  $E$ , this implies  $\partial E_\varepsilon \subset E$ . With this, (2) will be established by proving that we have  $\partial E_\varepsilon \subset \partial\Sigma$ . Let  $z$  be arbitrary in  $\partial E_\varepsilon$  and therefore in  $E$  which is open. There exists a strictly positive real number  $r$  such that we have :

$$\mathcal{B}_r(z) \cap E_\varepsilon = \mathcal{B}_r(z) \cap (E \cap \Sigma^c) \neq \emptyset, \quad \mathcal{B}_r(z) \cap E_\varepsilon^c = \mathcal{B}_r(z) \cap (E^c \cup \Sigma) \neq \emptyset, \quad \mathcal{B}_r(z) \subset E .$$

This implies  $\mathcal{B}_r(z) \cap \Sigma^c \neq \emptyset$  and  $\mathcal{B}_r(z) \cap \Sigma \neq \emptyset$  and therefore that  $z$  is in  $\partial\Sigma$ .

We have established  $\partial E_\varepsilon \cap \partial E = \emptyset$ ,  $\partial E_\varepsilon \subset \partial\Sigma$  and  $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$ . This does imply :

$$\partial E_\varepsilon \subset (\partial\Sigma)_i = \{z \in E : t_z = -\varepsilon\} . \quad (3)$$

This allows us to extend by continuity the definition of  $t_z$  to  $\mathbb{R}^m$  by letting  $t_z = -\varepsilon$  for all  $z \in E_\varepsilon$ .

Thanks to all these preparatory steps, we are finally ready to define a function  $\phi : \mathbb{R}^m \rightarrow E$  as:

$$\phi(z) = \begin{cases} Z(z, t_z + \nu(t_z)), & \text{if } z \in (E_\varepsilon)^c , \\ z, & \text{if } z \in E_\varepsilon , \end{cases} \quad (4)$$

where  $\nu$  is an arbitrary  $C^s$  and strictly decreasing function defined on  $\mathbb{R}$  satisfying:

$$\nu(t) = -t \quad \forall t \leq -\varepsilon \quad , \quad \lim_{t \rightarrow +\infty} \nu(t) = 0 .$$

The image of  $\phi$  is contained in  $E$  since we have  $E_\varepsilon \subset E$  and :

$$\begin{aligned} t_z + \nu(t_z) &> t_z \quad \forall z \in E_\varepsilon^c , \\ Z(z, t_z) &\in \partial E , \\ Z(z, t) &\in E \quad \forall (z, t) \in \partial E \times \mathbb{R}_{>0} . \end{aligned}$$

The continuity of the functions  $(z, t) \in \mathbb{R}^m \times \mathbb{R} \mapsto Z(z, t) \in \mathbb{R}$  and  $z \in E_\varepsilon^c \mapsto t_z \in [-\varepsilon, +\infty[$  implies that this function  $\phi$  is continuous at least on  $\mathbb{R}^m \setminus \partial E_\varepsilon$ . Also, for any  $z$  in  $\partial E_\varepsilon$ ,  $t_z$  is defined and equal to  $-\varepsilon$  (see (3)). So, for any strictly positive real number  $\eta$ , there exists a real number  $r$  such that :

$$\begin{aligned} |t_y + \varepsilon| &\leq \eta & \forall y \in \mathcal{B}_r(z) , \\ \nu(t_y) + \varepsilon &\leq \eta & \forall y \in \mathcal{B}_r(z) , \\ \phi(y) &= y & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon , \\ \phi(y) &= Z(y, t_y + \nu(t_y)) & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon^c . \end{aligned}$$

Since we have :

$$\phi(z) = Z(z, t_z + \nu(t_z)) = Z(z, -\varepsilon + \nu(-\varepsilon)) = z ,$$

we conclude that  $\phi$  is also continuous at  $z$ .

By differentiating, we obtain :

– at any interior point  $z$  of  $(E_\varepsilon)^c$

$$\frac{\partial \phi}{\partial z}(z) = \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) ;$$

– at any  $z$  in  $E_\varepsilon$  (which is open)  $\frac{\partial \phi}{\partial z}(z) = I$ . Also, for any  $z$  in  $\partial E_\varepsilon$ , we have :

$$\begin{aligned} \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z))) \frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) &= \frac{\partial Z}{\partial z}(z, 0) + \chi(Z(z, 0)) \frac{\partial t_z}{\partial z}(z)(1 - 1) , \\ &= I . \end{aligned}$$

This implies that  $\phi$  is  $C^1$  on  $\mathbb{R}^m$ .

We now show that  $\phi$  is invertible. Let  $y$  be arbitrary in  $E \cap E_\varepsilon^c = E \cap \Sigma$ . There exists  $t_y$  in  $[-\varepsilon, 0[$ . The function  $\nu$  being strictly monotonic,  $\nu^{-1}(t_y)$  exists and is in  $[-\varepsilon, +\infty[$ . This allows us to define properly  $\phi^{-1}$  as :

$$\phi^{-1}(y) = \begin{cases} Z(y, t_y - \nu^{-1}(-t_y)), & \text{if } y \in E \setminus E_\varepsilon \\ y, & \text{if } y \in E_\varepsilon \end{cases} \quad (5)$$

This function is an inverse of  $\phi$  as can be seen by reverting the flow induced by  $\chi$  when needed. Also, with the same arguments as before, we can prove that it is  $C^1$ .

This implies that  $\phi$  is a diffeomorphism from  $\mathbb{R}^m$  to  $E$ .

Besides, the functions  $z \mapsto Z(z, t)$  for  $t > 0$ ,  $z \mapsto t_z$  and  $\nu$  being  $C^s$ ,  $\phi$  is  $C^s$  at any interior point of  $(E_\varepsilon)^c$ . By continuity of  $\nu^{(r)}$  for  $r \leq s$ , it can be verified that  $\phi$  is also  $C^s$  on the boundary  $\partial E_\varepsilon$ . So,  $\phi$  is a  $C^s$ -diffeomorphism from  $\mathbb{R}^m$  to  $E$ .

Finally, we note that, for any point  $z_\varepsilon$  in  $\partial E_\varepsilon$ , there exists a point  $z$  in  $\partial E$  satisfying :

$$|z_\varepsilon - z| = \left| \int_0^\varepsilon \chi(Z(z, s)) ds \right| \leq \varepsilon \sup_\zeta |\chi(\zeta)| .$$

And conversely, for any  $z$  in  $\partial E$ , there exist  $z_\varepsilon$  in  $\partial E_\varepsilon$  satisfying :

$$|z_\varepsilon - z| = \left| \int_0^\varepsilon \chi(Z(z, s)) ds \right| \leq \varepsilon \sup_\zeta |\chi(\zeta)| .$$

It follows that

$$d_H(\partial E_\varepsilon, \partial E) \leq \varepsilon \sup_\zeta |\chi(\zeta)| \quad (6)$$

and  $\varepsilon$  may be chosen as small as needed.  $\square$

Two direct consequences of Lemma 1 are :

**Corollary 1** *Let  $\psi : E \rightarrow \psi(E)$  be a diffeomorphism, with  $E$  satisfying (C). For any  $\varepsilon$  strictly positive, there exist an open subset  $E_\varepsilon$  of  $E$  and a diffeomorphism  $\psi_\varepsilon : \mathbb{R}^m \rightarrow \psi(E)$  satisfying :*

$$\begin{aligned} d_H(\partial E_\varepsilon, \partial E) &\leq \varepsilon , \\ \psi_\varepsilon(z) &= \psi(z) \quad \forall z \in E_\varepsilon . \end{aligned}$$

**Proof :** With  $\phi$  given by Lemma 1, we pick  $\psi_\varepsilon = \psi \circ \phi$ .  $\square$

**Corollary 2** *Let  $\psi : E \rightarrow \psi(E)$  be a diffeomorphism, with  $\psi(E)$  satisfying (C). For any  $\varepsilon$  strictly positive, there exist an open subset  $\psi(E)_\varepsilon$  of  $\psi(E)$  and a diffeomorphism  $\psi_\varepsilon : E \rightarrow \mathbb{R}^n$  satisfying :*

$$\begin{aligned} d_H(\partial(\psi(E)_\varepsilon), \partial\psi(E)) &\leq \varepsilon , \\ \psi_\varepsilon(z) &= \psi(z) \quad \forall z \in \psi^{-1}(\psi(E)_\varepsilon) . \end{aligned}$$

**Proof :** With  $\phi$  given by Lemma 1 from  $\mathbb{R}^n$  to  $\psi(E)$ , we pick  $\psi_e = \phi^{-1} \circ \psi$ .  $\square$

**Remark 2** In Corollary 2,  $\psi$  being a diffeomorphism on an open set  $E$ , we know that the image of any compact subset  $K$  of  $E$  is a compact subset of  $\psi(E)$  which is also open by Brouwer's invariance theorem. Therefore, with (6), we can find  $\varepsilon$  such that  $\psi(K) \subset (\psi(E))_\varepsilon$  and thus,  $\psi_e(z) = \psi(z)$  for all  $x$  in  $K$ .

## 1.2 Diffeomorphism extension from a ball

Let  $R > 0$ . We denote  $\mathcal{B}_R$  the open ball in  $\mathbb{R}^n$  of radius  $R$  and centered at 0.

**Lemma 2** Consider a  $C^2$  diffeomorphism  $\psi : \mathcal{B}_R \rightarrow \psi(\mathcal{B}_R) \subset \mathbb{R}^n$ . For any  $\varepsilon$  strictly positive, there exists a diffeomorphism  $\psi_e : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\psi_e(x) = \psi(x)$  for all  $x$  in  $\mathcal{c}\mathcal{L}(\mathcal{B}_{R-\varepsilon})$ .

**Proof :** Without loss of generality we may assume that  $\psi(0) = 0$ . Consider the mapping  $\varphi : \mathcal{B}_R \times [0, 1] \rightarrow \mathbb{R}^n$  defined as

$$\varphi(x, t) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\psi(xt)}{t}, \quad \varphi(x, 0) = x.$$

Note that for all  $t$  the mapping  $\varphi_t(x) = \varphi(x, t)$  is a diffeomorphism from  $\mathcal{B}_R$  toward  $\varphi_t(\mathcal{B}_R)$ . Indeed, given  $x_a$  and  $x_b$  such that  $\varphi(x_a, t) = \varphi(x_b, t)$  it yields  $\psi(x_a t) = \psi(x_b t)$ . Note that the couple  $(x_a t, x_b t)$  is in  $\mathcal{B}_R$ . The mapping  $\psi$  being injective on this set, it yields  $x_a = x_b$ . Moreover,

$$\frac{\partial \varphi_t}{\partial x}(x) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \frac{\partial \psi}{\partial x}(xt), \quad t > 0, \quad \frac{\partial \varphi_0}{\partial x}(x) = \text{Id}$$

Hence, this mapping is full rank in  $\mathcal{B}_R$ . Consequently, for all  $t$  in  $[0, 1]$ , this mapping is a diffeomorphism  $\mathcal{B}_R$  toward  $\varphi_t(\mathcal{B}_R)$ . Consequently, for all  $t$  in  $[0, 1]$  we can introduce  $\varphi_t^{-1}$  its inverse map.

Note moreover that

$$\overline{\varphi(x, t)} = \frac{\partial \varphi}{\partial t}(x, t) = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(x, t)$$

where  $\rho$  is the function given as

$$\rho(x, t) = \frac{1}{t^2} \left[ \frac{\partial \psi}{\partial x}(xt)xt - \psi(xt) \right], \quad \rho(x, 0) = \frac{1}{2} x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

The mapping  $\psi$  being  $C^2$ , and  $\psi(0) = 0$  it yields,

$$\psi(xt) = \frac{\partial \psi}{\partial x}(0)xt + x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x \frac{t^2}{2} + o(t^2)$$

and,

$$\lim_{t \rightarrow 0} \frac{\frac{\partial \psi}{\partial x}(xt)x - \frac{\partial \psi}{\partial x}(0)x}{t} = x' \left( \frac{\partial^2 \psi}{\partial x \partial x}(0) \right) x$$

Hence, the function  $\rho$  is well defined and locally Lipschitz. Consequently,  $\varphi(x, t)$  is (the unique) solution of the time varying system defined for  $(z, t)$  in  $\varphi_t(\mathcal{B}_R) \times [0, 1]$  as

$$\dot{z} = \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t)$$

This time varying system can be extended to  $\mathbb{R}^n$  as

$$\dot{z} = \begin{cases} 0, & z \notin \varphi_t(\mathcal{B}_R) \\ \chi(\varphi_t^{-1}(z)) \left( \frac{\partial \psi}{\partial x}(0) \right)^{-1} \rho(\varphi_t^{-1}(z), t), & z \in \varphi_t(\mathcal{B}_R) \end{cases} \quad (7)$$

where  $\chi : \mathbb{R}^n \rightarrow \mathbb{R}^+$  is a  $C^1$  function such that

$$\chi(x) = \begin{cases} 0, & x \notin \mathcal{B}_R \\ 1, & x \in \mathcal{B}_{R-\varepsilon} \end{cases}$$

Notice that the vector field is zero outside  $S = \bigcup_{\substack{z \in \mathcal{B}_R \\ t \in [0, 1]}} \varphi_t(z)$ , which is a compact set. Indeed, if it is not, one can construct  $(x_n)$  and  $(t_n)$ , such that for all  $n \in \mathbb{N}$ ,

$$\frac{|\psi(x_n t_n)|}{t_n} \geq n.$$

Since  $\text{cl}(\mathcal{B}_R)$  is compact, we extract  $x_{\sigma(n)} t_{\sigma(n)}$  tending towards  $x^* \in \text{cl}(\mathcal{B}_R)$ . Necessarily,  $t_{\sigma(n)}$  tends to 0 and  $x^* = 0$  since  $(x_n)$  is bounded. But this is impossible because,  $\frac{\psi(x_n t_n)}{t_n}$  is equivalent to  $\frac{\partial \psi}{\partial x}(0) x_n$  around zero.

Therefore, the maximal solutions to this system are defined (at least) in  $[0, 1]$  for all initial conditions, and through backward integration, we obtain that  $\varphi(\mathbb{R}^n, 1) = \mathbb{R}^n$ .

We finally consider

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) Z(x, 1)$$

where  $Z(x, 1)$  is the solution of the system (7) evaluated at time 1 and initiated from  $x$  at time 0. This mapping being a linear transformation of a (time varying) flow, it is a diffeomorphism, and  $\psi_e(\mathbb{R}^n) = \mathbb{R}^n$ . Note moreover that for all  $x \in \mathcal{B}_{R-\varepsilon}$ , we have

$$\psi_e(x) = \frac{\partial \psi}{\partial x}(0) \varphi(x, 1) = \psi(x).$$

□

## 2 Proof of Theorem 1

Consider a diffeomorphism  $\psi : \mathcal{D} \rightarrow \psi(\mathcal{D}) \subsetneq \mathbb{R}^m$ . We want to extend the image of  $\psi$  to  $\mathbb{R}^m$ , i-e find a diffeomorphism  $\psi_e : \mathcal{D} \rightarrow \mathbb{R}^m$  such that:

$$- \psi_e(\mathcal{D}) = \mathbb{R}^m$$

-  $\psi_e(z) = \psi(z)$  for all  $z$  in  $K \subset \mathcal{D}$ , where  $K$  is any compact subset of  $\mathcal{D}$ .

Let us successively study the following two cases :

- **First case:**  $\psi(\mathcal{D})$  satisfies (C) The result follows directly from Corollary 2 and Remark 2. In practice, the reader may find an explicit construction of such an extension in the proof of Lemma 1.
- **Second case:**  $\mathcal{D}$  is  $C^2$ -diffeomorphic to  $\mathbb{R}^m$  and  $\psi$  is  $C^2$ :

Let  $\phi_1 : \mathcal{D} \rightarrow \mathbb{R}^m$  denote the corresponding diffeomorphism. Let  $R_1$  be a strictly positive real number such that the open ball  $\mathcal{B}_{R_1}(0)$  contains  $\phi_1(K)$ . Let  $R_2$  be a real number strictly larger than  $R_1$ . With Lemma 1 again, and since  $\mathcal{B}_{R_2}(0)$  verifies condition (C), there exists of  $C^2$ -diffeomorphism  $\phi_2 : \mathcal{B}_{R_2}(0) \rightarrow \mathbb{R}^m$  satisfying

$$\phi_2(z) = z \quad \forall z \in \mathcal{B}_{R_1}(0) .$$

At this point, we have obtained a  $C^2$ -diffeomorphism  $\phi = \phi_2^{-1} \circ \phi_1 : \mathcal{D} \rightarrow \mathcal{B}_{R_2}(0)$ . Consider  $\lambda = \psi \circ \phi^{-1} : \mathcal{B}_{R_2}(0) \rightarrow \psi(\mathcal{D})$ . According to Lemma 2, we can extend  $\lambda$  to  $\lambda_e : \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that  $\lambda_e = \psi \circ \phi^{-1}$  on  $\mathcal{B}_{R_1}(0)$ . Finally, consider  $\psi_e = \lambda_e \circ \phi_1 : \mathcal{D} \rightarrow \mathbb{R}^m$ . Since, by construction of  $\phi_2$ ,  $\phi = \phi_1$  on  $\phi_1^{-1}(\mathcal{B}_{R_1}(0))$  which contains  $K$ , we have  $\psi_e = \psi$  on  $K$ .

**Remark 3** Note that the second construction is more complex than the first one, for it contains several extensions and in particular that of Lemma 2 which is difficult to implement. For this reason, one may prefer applying the first case whenever it is possible.

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