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# Anisotropic Mumford-Shah Model and its Approximation with $\Gamma$ -convergence

David Vicente

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## Abstract

A new model is introduced for the segmentation problem of thin structures, like tubes or thin plates, in an image. The energy is based on the Mumford-Shah model and it introduces as a new variable a continuous and anisotropic perturbation of the Hausdorff measure. A relaxed formulation in the special space of functions with bounded variations is given and the existence of a solution is established. In order to get an energy more adapted for numerics, an approximation with  $\Gamma$ -convergence and its complete proof are given.

## Introduction

This work is motivated by the problem of segmentation of sets strongly elongated in some directions as, for example, tubes or thin plates in an image of dimension  $n \in \{2; 3\}$ . Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain and  $g \in L^\infty(\Omega)$ . We denote by  $\mathcal{H}^{n-1}$  the  $(n-1)$ -dimensional Hausdorff measure. The model we introduce in this paper consists in minimizing

$$\mathcal{E}(u, K, \mathbf{M}) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_K \langle \mathbf{M}\nu, \nu \rangle^{1/2} d\mathcal{H}^{n-1} + \int_{\Omega} \|D\mathbf{M}\|^{n+\alpha} dx \quad (0.1)$$

where  $K$  is compact and  $\mathcal{H}^{n-1}$ -rectifiable with unitary normal vector  $\nu$ ,  $\alpha > 0$  and  $\mathbf{M}$  takes its values in a compact subset  $\mathcal{G}$  of symmetric definite positive matrices. The associated minimizing problem is

$$(\mathcal{P}) : \quad \min\{\mathcal{E}(u, K, \mathbf{M}) : K \text{ compact and } \mathcal{H}^{n-1}\text{-rectifiable, } u \in W^{1,2}(\Omega \setminus K), \mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})\}.$$

If we consider  $\mathbf{M} \equiv \text{Id}_n$ , we recognize the well-known Mumford-Shah model (see [1] for the seminal paper). In this sense, our model is the anisotropic version of the Mumford-Shah energy.

At any point  $x \in K$ , the directions associated to the main eigenvalues of  $\mathbf{M}(x)$  must represent the directions of elongation of the set  $K$ . For example, if we want to detect a tube in a two-dimensional image, we may consider  $\mathcal{G}$  as the subset of symmetric definite positive matrices with fixed eigenvalues  $\{1; \mu\}$ , such that  $1 \ll \mu$ . For the detection of thin plates in a three dimensional image, we may consider the fixed eigenvalues  $\{1; \mu; \mu\}$ . By this way, the third term of (0.1) will force  $\nu$  to be in the same direction as the first eigenvector and then  $K$  to be elongated in the orthogonal directions. Moreover, we set  $\alpha > 0$  in the last term of (0.1) in order to force  $\mathbf{M}$  to be at least continuous ( $\alpha$ -Hölder), which corresponds to the assumption that the image  $g$  admits a local *geometrical coherence*.

This model does not consist in the detection of sets with codimension higher than one, as it has been done in [2] and [3] for vector-valued functions. In our case, the sets we want to detect are with (small) positive volume.

In order to prove that  $(\mathcal{P})$  is a well posed problem, for  $u \in \text{SBV}(\Omega)$ , we introduce

$$E(u, \mathbf{M}) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} + \int_{\Omega} \|D\mathbf{M}\|^{n+\alpha} dx \quad (0.2)$$

where  $\nabla u$  is the derivative of  $u$  with respect to the Lebesgue measure,  $J_u$  is jump set and  $\nu_u$  is unitary and orthogonal vector to  $J_u$ . The associated minimizing problem is

$$(\mathcal{P}') : \quad \min\{E(u, \mathbf{M}) : u \in \text{SBV}(\Omega), \mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})\},$$

We verify that  $(\mathcal{P}')$  admits a solution and, with a regularity result of the jump set of a minimizer, which has been established in a joint paper [4], we prove that a minimizer of  $(\mathcal{P}')$  naturally provides a solution for  $(\mathcal{P})$ . In order to get a functional more adapted for a numerical implementation, we will approximate (0.2) by a family  $(E_\varepsilon)_\varepsilon$  which only depends on the integration with respect to the Lebesgue measure. More precisely, we set

$$E_\varepsilon(u, z, \mathbf{M}) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left( \varepsilon \langle \mathbf{M} \nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) dx + \int_{\Omega} \|D\mathbf{M}\|^{n+\alpha} dx.$$

As it has been done in [5] for the initial Mumford-Shah model, the function  $z$  takes its values in  $[0; 1]$  and plays the role of a control on the gradient of  $u$ . The approximation takes place for  $\varepsilon \rightarrow 0^+$  in the sense of the  $\Gamma$ -convergence.

In section 1 we recall some results on spaces of functions with bounded variation and we mention a regularity result that we have proved in a previous article. In section 2, we prove that the problem admits a solution. Section 3 is completely devoted to the approximation process. We introduce the family of functionals  $(E_\varepsilon)_\varepsilon$  with their domains and we give the complete proof of  $\Gamma$ -convergence to  $E$  when  $\varepsilon \rightarrow 0^+$ .

## 1 Functional framework and regularity result

We adopt the notations:

- $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \in \mathbb{R}$  for the canonical scalar product of  $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ ,
- $|\mathbf{v}|$  for the euclidean norm of  $\mathbf{v} \in \mathbb{R}^n$ ,
- $\|\mathbf{M}\|$  for the induced norm of  $\mathbf{M} \in M_n(\mathbb{R})$ ,
- $\bigwedge_{i=1}^{n-1} \mathbf{v}_i \in \mathbb{R}^n$  for the canonical vectorial product of  $\mathbf{v}_1, \dots, \mathbf{v}_{n-1} \in \mathbb{R}^n$ ,
- $\text{dist}$  for the euclidean distance in  $\mathbb{R}^n$ ,
- $S_n^+(\mathbb{R}) \subset M_n(\mathbb{R})$  for the subset of symmetric definite positive matrices,
- $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$  for the subset of invertible matrices,
- $O_n(\mathbb{R}) \subset GL_n(\mathbb{R})$  for the subgroup of orthogonal matrices,
- $\mathcal{B}(\Omega)$  the class of Borelian subsets of  $\Omega$ ,
- $\mathbb{B}(\Omega)$  for the space of Borelian functions defined in  $\Omega$ ,
- $\mathcal{L}^n$  for the Lebesgue measure in  $\mathbb{R}^n$ ,
- $\mathcal{H}^k$  for the  $k$ -dimensional Hausdorff measure,
- $\mathcal{M}(\Omega; \mathbb{R}^n)$  for the space of vectorial Radon measures defined in  $\Omega \subset \mathbb{R}^n$ ,
- $\oint_A f(x) dx = \frac{1}{\mathcal{L}^n(A)} \int_A f(x) dx$ , for  $A \in \mathcal{B}(\Omega)$  and  $\mathcal{L}^n(A) > 0$ .

Let  $f$  be a function defined on open sets, we adopt the following vocabulary:

- $f$  is superadditive if  $f(A \cup B) \geq f(A) + f(B)$  for any disjoint sets  $A, B$ ,
- $f$  is non decreasing if  $f(A) \leq f(B)$  for any sets  $A, B$  such that  $A \subset B$ .

### 1.1 Functional spaces

We assume throughout this paper that the following constraint is satisfied by  $\Omega$  which is obviously satisfied in the context of applications in *Image Processing* because the domain is a parallelepiped.

**Definition 1.1.** *We say that  $\Omega \subset \mathbb{R}^n$  satisfies the reflexion condition (R) if  $\Omega$  is an open and bounded domain with Lipschitz regular boundary  $\partial\Omega$  such that there exists a neighborhood  $U$  of  $\partial\Omega$  and a bi-Lipschitzian homeomorphism  $\varphi : U \cap \Omega \rightarrow U \setminus \overline{\Omega}$  such that, for any  $x \in \partial\Omega$ , we have*

$$\lim_{y \rightarrow x} \varphi(y) = x.$$

For the classical definitions and results on BV and SBV we refer to [7]. In particular, for the definition of weak\* convergence in BV, we refer to [7] (Definition 1.58). However, for the need of notations, we mention that, if  $u \in BV(\Omega)$ , then its derivative  $Du$  belongs to the space  $\mathcal{M}(\Omega; \mathbb{R}^n)$  of vectorial Radon measure. Moreover, if  $u \in SBV(\Omega)$  then the Cantor part of  $Du$  is null and we obtain

$$Du = \nabla u \cdot \mathcal{L}^n + (u^+ - u^-)\nu_u \cdot \mathcal{H}^{n-1} \llcorner J_u,$$

where  $\nabla u$  is the density of  $Du$  with respect to the Lebesgue measure  $\mathcal{L}^n$ ,  $u^+$  (resp.  $u^-$ ) is the approximate upper (resp. lower) limit and  $\mathcal{H}^{n-1} \llcorner J_u$  is the restriction of  $\mathcal{H}^{n-1}$  to its jump set  $J_u$ .

Now, we focus on some results which will be useful throughout the paper. First, we need the rectifiability of  $J_u$ .

**Theorem 1.1.** *Let  $u$  be a given function in  $BV(\Omega)$ . There exists a countable family  $(C_i)_{i \in \mathbb{N}}$  of compact  $\mathcal{C}^1$ -hypersurfaces such that*

$$J_u = \mathcal{N} \cup \left( \bigcup_{i \in \mathbb{N}} C_i \right),$$

where  $\mathcal{H}^{n-1}(\mathcal{N}) = 0$ .

Then, we will need the following chain rule ([7], Theorem 3.99).

**Theorem 1.2.** *Let  $u \in SBV(\Omega)$  and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz function. Then,  $v = f \circ u$  belongs to  $SBV(\Omega)$  and*

$$Dv = f'(u)\nabla u \cdot \mathcal{L}^n + (f(u^+) - f(u^-))\nu_u \cdot \mathcal{H}^{n-1} \llcorner J_u. \quad (1.1)$$

The following is a straightforward consequence of [7], Corollary 3.89.

**Proposition 1.1.** *Let  $\Omega$  be an open and bounded domain satisfying (R) and  $\Omega' = U \cup \Omega$ . For  $u \in BV(\Omega)$ , we consider an extension in  $\Omega'$  by the following way*

$$\forall x \in U \setminus \bar{\Omega}, \quad u(x) = u(\varphi^{-1}(x)),$$

Then, we have

$$\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0.$$

We also need *slicing* results.

**Definition 1.2.** *Let  $\nu \in \mathbb{S}^{n-1}$  be fixed. We denote by  $\Pi_\nu$  the hyperplane*

$$\{x \in \mathbb{R}^n : x \cdot \nu = 0\}.$$

If  $x \in \Pi_\nu$ , we set

$$\begin{aligned} \Omega_x &= \{t \in \mathbb{R} : x + t\nu \in \Omega\}, \\ \Omega_\nu &= \{x \in \Pi_\nu : \Omega_x \neq \emptyset\}. \end{aligned}$$

For any function  $u$  defined on  $\Omega$  and any  $x \in \Omega_\nu$ , we set

$$\begin{aligned} (u)_x : \Omega_x &\rightarrow \mathbb{R} \\ t &\mapsto u(x + t\nu). \end{aligned}$$

The following Theorem is proved in [8].

**Theorem 1.3.** *Let  $u \in L^\infty(\Omega)$  be a function such that, for all  $\nu \in \mathbb{S}^{n-1}$ ,*

- i)  $(u)_x \in SBV(\Omega_x)$  for  $\mathcal{H}^{n-1}$  a.e.  $x \in \Omega_\nu$ ,
- ii)  $\int_{\Omega_\nu} \left[ \int_{\Omega_x} |\nabla(u)_x| dt + \mathcal{H}^0(J_{(u)_x}) \right] d\mathcal{H}^{n-1}(x) < +\infty$ ;

then,  $u \in SBV(\Omega)$  and  $\mathcal{H}^{n-1}(J_u) < +\infty$ . Conversely, let  $u \in SBV(\Omega) \cap L^\infty(\Omega)$  be such that  $\mathcal{H}^{n-1}(J_u) < +\infty$ . Then i) and ii) are satisfied. Moreover, we have

- iii)  $\langle \nabla u(x + t\nu), \nu \rangle = \nabla(u)_x(t)$ , for a.e.  $t \in \Omega_x$  and  $\mathcal{H}^{n-1}$ -a.e.  $x \in \Omega_\nu$ ,
- iv)  $\int_{J_u} \langle \nu_u, \nu \rangle d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(J_{(u)_x}) d\mathcal{H}^{n-1}(x)$ .

We need a generalization of the Coarea formula. For that, we introduce the following

**Definition 1.3.** *The function  $f : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^+$  is sublinear with respect to the second variable, if*

- i)  $f(x, \mathbf{v}_1 + \mathbf{v}_2) \leq f(x, \mathbf{v}_1) + f(x, \mathbf{v}_2)$  for any  $(x, \mathbf{v}_1, \mathbf{v}_2) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n$ ,*
- ii)  $f(x, t\mathbf{v}) = tf(x, \mathbf{v})$  for any  $(x, \mathbf{v}, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+$ .*

Suppose that  $\mu_1$  is a Radon measure and  $\mu_2$  is a vectorial Radon measure on  $\Omega$ . According to Besicovitch derivation Theorem,

$$\lim_{r \rightarrow 0} \frac{\mu_2(B(x, r))}{\mu_1(B(x, r))}$$

exists and is finite for  $\mu_1$  almost every  $x$ , we denote by  $\frac{d\mu_2}{d\mu_1}(x)$  this limit when it exists. We recall that  $\mu_2$  is absolutely continuous with respect to  $\mu_1$  if  $\mu_2(A) = 0$  whenever  $\mu_1(A) = 0$ . When this holds, we write  $\mu_2 \ll \mu_1$ . We consider the convex functional defined on the space  $\mathcal{M}(\Omega; \mathbb{R}^n)$  by

$$\Phi : \mu_2 \in \mathcal{M}(\Omega; \mathbb{R}^n) \mapsto \int_{\Omega} f\left(x, \frac{d\mu_2}{d\mu_1}\right) d\mu_1, \quad (1.2)$$

where  $\mu_1$  is a positive measure such that  $\mu_2 \ll \mu_1$ . It is shown in [9] that the integral in (1.2) does not depend on the choice of  $\mu_1$ . For that reason, we will write it in the condensed form

$$\Phi(\mu_2) = \int_{\Omega} f(x, \mu_2).$$

We give a variant of the Coarea formula extended to the sublinear functionals which can be found in [10].

**Proposition 1.2.** *Let  $\Phi(x, s, v)$  a Borel function of  $\Omega \times \mathbb{R} \times \mathbb{R}^n$  which is sublinear in  $v$ . Let  $p$  be a Lipschitz continuous function on  $\Omega$  and, for  $t > 0$ , we set  $S_t = \{x \in \Omega; p(x) < t\}$ . Then, for almost all  $t \in \mathbb{R}$ ,  $\mathbf{1}_{S_t}$  belongs to  $BV(\Omega)$  and we have*

$$\int_{\Omega} \Phi(x, p, Dp) dx = \int_{\mathbb{R}} dt \int_{\Omega} \Phi(x, t, D\mathbf{1}_{S_t}).$$

## 1.2 Minkowski content and regularity result for the jump set

For  $\mathbf{M} : \Omega \rightarrow S_n^+(\mathbb{R})$  and  $(x, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ , we set

$$\phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}(x)\mathbf{v}, \mathbf{v} \rangle^{1/2}.$$

This functional is a Riemannian metric, so, for  $S \subset \Omega$  and  $x, y \in \Omega$ , we may define its associated distance as

$$\begin{aligned} \text{dist}_{\phi}(x, y) &= \inf \left\{ \int_0^1 \phi\left(\gamma, \frac{d\gamma}{dt}\right) dt : \gamma \in W^{1,1}([0; 1]; \mathbb{R}^n), \right. \\ &\quad \left. \text{dist}_{\phi}(x, S) = \inf \{ \text{dist}_{\phi}(x, y) : y \in S \}. \right. \\ &\quad \left. \gamma(0) = x, \gamma(1) = y \right\}, \end{aligned}$$

The associated anisotropic Minkowski  $(n - 1)$ -dimensional upper and lower contents are defined by the limits

$$\mathcal{M}_{\mathbf{M}}^*(S) = \limsup_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{x : \text{dist}_{\phi}(x, S) < \rho\})}{2\rho}, \quad \mathcal{M}_{*\mathbf{M}}(S) = \liminf_{\rho \rightarrow 0^+} \frac{\mathcal{L}^n(\{x : \text{dist}_{\phi}(x, S) < \rho\})}{2\rho}. \quad (1.3)$$

If  $\mathcal{M}_{\mathbf{M}}^*(S) = \mathcal{M}_{*\mathbf{M}}(S)$ , we call their common value the  $(n - 1)$ -dimensional anisotropic Minkowski content  $\mathcal{M}_{\mathbf{M}}(S)$ . In [4], we have proved the following

**Theorem 1.4.** *Let  $\mathbf{M} : \Omega \rightarrow S_n^+(\mathbb{R})$  be continuous,  $h \in L^{\infty}(\Omega)$ ,  $\gamma > 0$  and  $u \in SBV(\Omega)$  a minimizer of*

$$\left\{ E^{\gamma, h, \mathbf{M}}(v) = \gamma \int_{\Omega} (v - h)^2 dx + \int_{\Omega} |\nabla v|^2 dx + \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v \rangle^{1/2} d\mathcal{H}^{n-1} : v \in SBV(\Omega) \right\}.$$

Then, we have

$$\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0, \quad \mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M}\nu_v, \nu_v \rangle^{1/2} d\mathcal{H}^{n-1}.$$

## 2 Existence result for $(\mathcal{P})$

In order to prove that  $(\mathcal{P})$  admits a solution, we prove that  $(\mathcal{P}')$  is well posed and that it provides a minimizer for  $(\mathcal{P})$ .

### 2.1 Existence result for the relaxed problem $(\mathcal{P}')$

The main result of the section is the following

**Theorem 2.1.** *The problem  $(\mathcal{P}')$  admits a solution.*

The proof is a straightforward consequence of Lemmas 2.1 and 2.2.

**Lemma 2.1.** *Let  $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$  and  $(\mathbf{M}_k)_k \subset W^{1,n+\alpha}(\Omega; \mathcal{G})$  be such that*

$$\sup_k E(u_k, \mathbf{M}_k) < \infty.$$

*Then, there exist subsequences, still denoted by  $(u_k)_k$  and  $(\mathbf{M}_k)_k$ , and  $(u, \mathbf{M}) \in SBV(\Omega) \times W^{1,n+\alpha}(\Omega; \mathcal{G})$ , such that  $(u_k)_k$  is weakly\* convergent to  $u$  in  $SBV(\Omega)$  and  $(\mathbf{M}_k)_k$  is weakly convergent to  $\mathbf{M}$  in  $W^{1,n+\alpha}(\Omega)$ .*

*Proof.* As  $\mathcal{G} \subset S_n^+(\mathbb{R})$  is compact, there exists  $0 < \lambda < \Lambda$  such that, for any  $(M, \mathbf{v}) \in \mathcal{G} \times \mathbb{R}^n$ , the following ellipticity condition is satisfied

$$\lambda|\mathbf{v}|^2 \leq \langle M\mathbf{v}, \mathbf{v} \rangle \leq \Lambda|\mathbf{v}|^2. \quad (2.1)$$

It yields

$$\int_{\Omega} |\nabla u_k|^2 dx + \mathcal{H}^{n-1}(J_{u_k}) \leq \max\{1, \lambda^{-1/2}\} E(u_k, \mathbf{M}_k). \quad (2.2)$$

As  $E(u_k, \mathbf{M}_k)$  is bounded, Theorem 4.8. in [7] implies that there exists a subsequence  $(u_k)_k$  weakly\* convergent in  $BV(\Omega)$  to  $u \in SBV(\Omega)$ . According to (2.1), we have

$$\|\mathbf{M}\|_{W^{1,n+\alpha}(\Omega)}^{n+\alpha} \leq \Lambda^{(n+\alpha)/2} + E(u_k, \mathbf{M}_k),$$

so  $(\mathbf{M}_k)_k$  is a bounded sequence of  $W^{1,n+\alpha}(\Omega)$  and then there exists a subsequence which weakly converges to  $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$ . It suffices to verify that  $\mathbf{M}$  takes its values in  $\mathcal{G}$ , which is true because  $W^{1,n+\alpha}(\Omega)$  is injected in a continuous way into the space of continuous function and  $\mathcal{G}$  is closed. □

**Lemma 2.2.** *Let  $(u_k)_k \subset SBV(\Omega)$  be weakly\* convergent to  $u \in SBV(\Omega)$  and  $(\mathbf{M}_k)_k$  be weakly convergent to  $\mathbf{M}$  in  $W^{1,n+\alpha}(\Omega)$ . Then, we have*

$$E(u, \mathbf{M}) \leq \liminf_{k \rightarrow \infty} E(u_k, \mathbf{M}_k).$$

*Proof.* We may assume that  $\liminf E(u_k, \mathbf{M}_k) < +\infty$ , otherwise the result is ensured. Weak\* convergence in  $SBV(\Omega)$  and weak convergence in  $W^{1,n+\alpha}(\Omega)$  give

$$\liminf_{k \rightarrow \infty} \int_{\Omega} (u_k - g)^2 dx + \int_{\Omega} \|D\mathbf{M}_k\|^{n+\alpha} dx \leq \int_{\Omega} (u - g)^2 dx + \int_{\Omega} \|D\mathbf{M}\|^{n+\alpha} dx. \quad (2.3)$$

According to inequality (2.2),  $\int_{\Omega} |\nabla u_k|^2 dx + \mathcal{H}^{n-1}(J_{u_k})$  is bounded with respect to  $k$ . With [7], Theorem 4.7., we deduce that

$$\int_{\Omega} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 dx, \quad \mathcal{H}^{n-1}(J_u) \leq \liminf_{k \rightarrow \infty} \mathcal{H}^{n-1}(J_{u_k}). \quad (2.4)$$

According to (2.3) and (2.4), it remains to prove that

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

This result is proved in [7], Theorem 5.2., for an homogeneous and fixed media (if  $\mathbf{M}_k$  does not depend on  $x \in \Omega$  nor in  $k$ ). In order to generalize this result, we introduce a piecewise constant approximation of  $\mathbf{M}$ . For any vector  $e_i \in \mathbb{S}^{n-1}$  of the canonical basis, we set

$$\begin{cases} \Pi_t^i = \{x \in \Omega : \langle x, e_i \rangle = t\}, \\ N_t^i = \{t \in \mathbb{R} : \mathcal{H}^{n-1}(J_u \cap \Pi_t^i) > 0\} \cup \{t \in \mathbb{R} : \exists k \in \mathbb{N}, \mathcal{H}^{n-1}(J_{u_k} \cap \Pi_t^i) > 0\}. \end{cases}$$

As  $\mathcal{H}^{n-1}(J_u) < \infty$  (2.4) and  $\mathcal{H}^{n-1}(J_{u_k}) < \infty$  for any  $k \in \mathbb{N}$ , then  $N_t^i$  is at most countable. We fix  $\eta > 0$  arbitrary small. Intersecting  $\Omega$  with disjoint cubes with edges orthogonal to the axes of the canonical basis, there exists  $\mathcal{A}$  a finite partition of  $\Omega$  which, for any  $A \in \mathcal{A}$  and for any  $k \in \mathbb{N}$ , satisfies

$$\text{diam}(A) < \eta, \quad \mathcal{H}^{n-1}(J_u \cap \partial A) = 0, \quad \mathcal{H}^{n-1}(J_{u_k} \cap \partial A) = 0. \quad (2.5)$$

For any  $A \in \mathcal{A}$ , we fix one point  $x_A \in A$  and we set  $\mathbf{M}^{\mathcal{A}}(x) = \mathbf{M}(x_A)$  for any  $x \in A$ . We have the decomposition

$$\begin{aligned} \int_{J_{u_k}} \langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1} &= \int_{J_{u_k}} \left( \langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) d\mathcal{H}^{n-1} \\ &\quad + \int_{J_{u_k}} \left( \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) d\mathcal{H}^{n-1} \\ &\quad + \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}. \end{aligned} \quad (2.6)$$

In the three following *Claim*, we will estimate the limit of those three terms.

*Claim 1:*

$$\lim_{k \rightarrow \infty} \int_{J_{u_k}} \left( \langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) d\mathcal{H}^{n-1} = 0.$$

According to Ellipticity inequality (2.1), we have

$$\begin{aligned} |\langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{1/2}| &\leq \frac{1}{2\sqrt{\lambda}} |\langle (\mathbf{M}_k - \mathbf{M}) \nu_{u_k}, \nu_{u_k} \rangle|, \\ &\leq \frac{1}{2\sqrt{\lambda}} \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \end{aligned}$$

and then

$$\int_{J_{u_k}} |\langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{1/2}| d\mathcal{H}^{n-1} \leq \frac{1}{2\sqrt{\lambda}} \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \mathcal{H}^{n-1}(J_{u_k}).$$

As the inclusion  $W^{1,n+\alpha}(\Omega) \subset L^\infty(\Omega)$  is compact and  $(\mathbf{M}_k)_k$  weakly converges to  $\mathbf{M}$ , then  $(\mathbf{M}_k)_k$  uniformly converges to  $\mathbf{M}$ . Moreover,  $(\mathcal{H}^{n-1}(J_{u_k}))_k$  is bounded, it concludes the proof of the claim.

*Claim 2: The sequence*

$$\lim_{\eta \rightarrow 0^+} \int_{J_{u_k}} \left( \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) d\mathcal{H}^{n-1} = 0$$

and the convergence takes place uniformly with respect to  $k \in \mathbb{N}$ .

For  $A \in \mathcal{A}$  and  $x \in A$ , ellipticity inequality (2.1) gives

$$|\langle \mathbf{M}(x) \nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}}(x) \nu_{u_k}, \nu_{u_k} \rangle^{1/2}| \leq \frac{1}{2\sqrt{\lambda}} \|\mathbf{M}(x) - \mathbf{M}(x_A)\|.$$

As  $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$ , there exist a constant  $C > 0$  such that, for any  $x \in A$ , we have

$$\|\mathbf{M}(x) - \mathbf{M}(x_A)\| \leq C\eta^\alpha.$$

It yields

$$\left| \int_{J_{u_k}} \left( \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) d\mathcal{H}^{n-1} \right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_{u_k})}{2\sqrt{\lambda}}.$$

As  $(\mathcal{H}^{n-1}(J_{u_k}))_k$  is a bounded sequence, it concludes the proof of the claim.

*Claim 3:*

$$\int_{J_u} \langle \mathbf{M}^A \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

We denote by  $\overset{\circ}{A}$  the interior of the set  $A$ . According to [7], Theorem 5.2., we have

$$\int_{J_u \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A) \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A) \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

According to (2.5), the contribution of the boundaries is null, it gives

$$\int_{J_u \cap A} \langle \mathbf{M}(x_A) \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A) \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}$$

and then

$$\begin{aligned} \sum_{A \in \mathcal{A}} \int_{J_u \cap A} \langle \mathbf{M}(x_A) \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} &\leq \sum_{A \in \mathcal{A}} \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A) \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}, \\ &\leq \liminf_{k \rightarrow \infty} \sum_{A \in \mathcal{A}} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A) \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}. \end{aligned}$$

As  $\mathcal{A}$  is a partition of  $\Omega$ , it concludes the proof of *Claim 3*:

$$\int_{J_u} \langle \mathbf{M}^A \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

Let  $\delta > 0$  be an arbitrary small number. With the same arguments as for *Claim 2*, we get

$$\left| \int_{J_u} \left( \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} - \langle \mathbf{M}^A \nu_u, \nu_u \rangle^{1/2} \right) d\mathcal{H}^{n-1} \right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_u)}{2\sqrt{\lambda}}.$$

So, according to *Claim 2*, there exists a partition  $\mathcal{A}$  which satisfies

$$\begin{cases} \limsup_{k \rightarrow \infty} \left| \int_{J_{u_k}} \left( \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) d\mathcal{H}^{n-1} \right| \leq \delta, \\ \left| \int_{J_u} \left( \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} - \langle \mathbf{M}^A \nu_u, \nu_u \rangle^{1/2} \right) d\mathcal{H}^{n-1} \right| \leq \delta. \end{cases}$$

According to (2.6), *Claim 1* and *Claim 3*, we have

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq 2\delta + \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

As  $\delta > 0$  is arbitrary, it concludes the proof of Lemma 2.2.  $\square$

We now prove Theorem 2.1.

*Proof.* We denote by  $(u_k, \mathbf{M}_k)_k \subset \text{SBV}(\Omega)$  a minimizing sequence for  $E$ . As  $g \in L^\infty(\Omega)$ , we set

$$\varphi(t) = \begin{cases} -\|g\|_{L^\infty(\Omega)} & \text{if } t \leq -\|g\|_{L^\infty(\Omega)}, \\ t & \text{if } |t| \leq \|g\|_{L^\infty(\Omega)}, \\ \|g\|_{L^\infty(\Omega)} & \text{if } t \geq \|g\|_{L^\infty(\Omega)}. \end{cases} \quad (2.7)$$



We denote  $v_k = \varphi \circ u_k$ . As the function  $\varphi$  is 1-Lipshitz, we may apply Theorem 1.2 and then  $v_k \in \text{SBV}(\Omega) \cap L^\infty(\Omega)$  with the decomposition

$$Dv_k = \varphi'(u_k) \nabla u_k \cdot \mathcal{L}^n + (\varphi(u_k^+) - \varphi(u_k^-)) \nu_{u_k} \cdot \mathcal{H}^{n-1} \llcorner J_{u_k}.$$

For any  $k$ , it is easy to check that  $E(v_k, \mathbf{M}_k) \leq E(u_k, \mathbf{M}_k)$ , so  $(v_k, \mathbf{M}_k)_k$  is also a minimizing sequence for  $E$ . According to Theorem 2.1, there exists  $v \in \text{SBV}(\Omega)$  and a subsequence, still denoted  $(v_k)_k$  weakly\* convergent to  $v$ . With Theorem 2.2, we have  $E(v, \mathbf{M}) \leq \liminf E(v_k, \mathbf{M}_k)$  and then  $(v, \mathbf{M})$  is a minimizer of  $E$ .  $\square$

## 2.2 Existence result for $(\mathcal{P})$

In this section we prove that the problems  $(\mathcal{P})$  and  $(\mathcal{P}')$  have common solutions and then  $(\mathcal{P})$  is also well posed. First, we consider  $(u, K, \mathbf{M})$  in the domain of  $\mathcal{E}$ . As in (2.7), we may define the truncated function  $v = \varphi(u)$  and then we have  $|v - g| \leq |u - g|$  and  $|\nabla v| \leq |\nabla u|$ . In particular, we get  $\mathcal{E}(v, K, \mathbf{M}) \leq \mathcal{E}(u, K, \mathbf{M})$ . Thus, for any  $(u_*, K_*, \mathbf{M}_*)$  in the domain of  $\mathcal{E}$ , we may assume that  $u_* \in L^\infty(\Omega \setminus K)$ . On the other hand, in [7] (Proposition 4.4), is given the following

**Proposition 2.1.** *Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $K \subset \mathbb{R}^n$  be closed,  $\mathcal{H}^{n-1}(K \cap \Omega) < \infty$  and  $u \in L^\infty(\Omega \setminus K) \cap W^{1,1}(\Omega \setminus K)$ . Then, we have  $u \in \text{SBV}(\Omega)$  and  $\mathcal{H}^{n-1}(S_u \setminus K) = 0$ .*

As  $\Omega$  is bounded and  $u_* \in W^{1,2}(\Omega \setminus K_*)$ , then we get  $u_* \in W^{1,1}(\Omega \setminus K_*)$ . As  $\int_{K_*} \langle \mathbf{M}_* \nu, \nu \rangle^{1/2} d\mathcal{H}^{n-1} < +\infty$ , then Ellipticity condition (2.1) gives  $\mathcal{H}^{n-1}(K_* \cap \Omega) < \infty$ . According to Proposition 2.1, we deduce that  $u_* \in \text{SBV}(\Omega)$  and  $\mathcal{H}^{n-1}(S_{u_*} \setminus K_*) = 0$ . It yields

$$E(u_*, \mathbf{M}_*) \leq \mathcal{E}(u_*, K_*, \mathbf{M}_*),$$

and then  $\min E \leq \min \mathcal{E}$ . Conversely, according to Theorem 2.1, there exists a minimizer  $(u^*, \mathbf{M}^*)$  of  $E$ . In particular, with the notations of Theorem 1.4,  $u^*$  is a minimizer of  $E^{1,g,\mathbf{M}^*}$  and then

$$\mathcal{H}^{n-1}(\overline{J_{u^*}} \setminus J_{u^*}) = 0.$$

So, we set  $K^* = \overline{J_{u^*}}$  and then  $K^*$  is compact and  $\mathcal{H}^{n-1}$ -rectifiable,  $\Omega \setminus K^*$  is open,  $u^* \in W^{1,2}(\Omega \setminus K^*)$  and

$$E(u^*, \mathbf{M}^*) = \mathcal{E}(u^*, K^*, \mathbf{M}^*).$$

We may conclude that  $\min E = \min \mathcal{E}$  and their minimizers coincide. Moreover, we have

**Proposition 2.2.** *Let  $u \in \text{SBV}(\Omega)$  be a minimizer of  $(\mathcal{P})$ , then  $u \in \mathcal{C}^1(\Omega \setminus \overline{J_u})$ .*

*Proof.* Let  $\overline{B_r}(x) \subset \Omega \setminus \overline{J_u}$ ; then  $u \in W^{1,2}(B_r(x))$  and it is a minimizer of the functional

$$\mathcal{I}(v) = \int_{B_r(x)} (v - g)^2 dx + \int_{B_r(x)} |\nabla v|^2 dx$$

among the functions  $v$  in  $u + W_0^{1,2}(B_r(x))$  and then classical regularity results give  $u \in \mathcal{C}^1(B_r(x))$ .  $\square$

## 3 $\Gamma$ -convergence result

This section is entirely devoted to the approximation process. In 3.1 we define the domain for  $E_\varepsilon$  and give the main Theorem of this paper. In 3.2 we prove, for  $\varepsilon > 0$  fixed, that the minimization of  $E_\varepsilon$  admits a solution. In 3.3, the most technical part of the paper, we give the complete proof of  $\Gamma$ -convergence. Finally, in 3.4, we conclude the proof of the main Theorem.

### 3.1 The functionals, their domain and the main Theorem

Formally, we define the functional  $E_\varepsilon(u, z, \mathbf{M})$  as

$$E_\varepsilon(u, z, \mathbf{M}) = \int_\Omega (u - g)^2 dx + \int_\Omega |\nabla u|^2 (1 - z^2)^2 dx + \int_\Omega \left( \varepsilon \langle \mathbf{M} \nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) dx + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} dx.$$

As in [5], the function  $z : \Omega \rightarrow [0; 1]$  plays the role of *control* of the gradient of  $u$ . We need to introduce a domain for  $E_\varepsilon$  that ensures the existence of a minimizer. If  $u, z \in W^{1,2}(\Omega)$ , then this functional is well defined. However, the coefficient  $(1 - z^2)^2$  removes the coercivity with respect to  $u$  and the existence result can not be achieved according to the Sobolev norm. If, by addition,  $u$  is bounded, we have

$$\begin{aligned} |\nabla(u(1 - z^2))|^2 &= |\nabla u(1 - z^2) - 2uz\nabla z|^2, \\ &\leq 2|\nabla u|^2(1 - z^2)^2 + 4\|u\|_{L^\infty(\Omega)}|\nabla z|^2. \end{aligned}$$

According to ellipticity inequality (2.1), it gives

$$\int_{\Omega} |\nabla(u(1 - z^2))|^2 dx \leq \left(2 + \frac{4\|u\|_{L^\infty(\Omega)}}{\lambda\varepsilon}\right) E_\varepsilon(u, z, \mathbf{M}).$$

So, it is natural to set

$$\mathcal{D}_n(\Omega) = \{(u, z) : u \in \mathbb{B}(\Omega), z \in W^{1,2}(\Omega; [0; 1]), \forall N \in \mathbb{N} \quad \bar{u}^N(1 - z^2) \in W^{1,2}(\Omega)\},$$

where  $\bar{u}^N$  is the truncated function defined, for any  $x \in \Omega$ , by

$$\bar{u}^N(x) = \begin{cases} -N & \text{if } u(x) \leq -N, \\ u(x) & \text{if } |u(x)| \leq N, \\ N & \text{if } u(x) \geq N. \end{cases} \quad (3.1)$$

Assuming  $(u, z) \in \mathcal{D}_n(\Omega)$  does not ensure that  $u \in W^{1,2}(\Omega)$  and  $\nabla u$  can not be defined as the gradient of  $u$  in the Sobolev sense. However, we can define  $\nabla u$  in the following sense.

**Definition 3.1.** Let  $u \in L^1(\Omega)$  and  $x \in \Omega$  a Lebesgue point of  $u$ ; we say that  $u$  is approximately differentiable at  $x$  if there exists  $L \in \mathbb{R}^n$  such that

$$\lim_{r \rightarrow 0^+} \oint_{B(x,r)} \frac{|u(y) - u(x) - \langle L, y - x \rangle|}{r} dy = 0. \quad (3.2)$$

If  $u$  is approximately differentiable at  $x$  then  $L$ , uniquely determined by (3.2), is called the approximate differential of  $u$  at  $x$ .

The following ensures that  $E_\varepsilon(u, z, \mathbf{M})$  is well defined for  $(u, z) \in \mathcal{D}_n(\Omega)$ .

**Proposition 3.1.** If  $(u, z) \in \mathcal{D}_n(\Omega)$ , then  $u$  is approximately differentiable in  $\{x \in \Omega : z(x) \neq 1\}$  and  $z$  is approximately differentiable in  $\Omega$ .

*Proof.* As  $\Omega$  is open and bounded then  $W^{1,2}(\Omega) \subset \text{BV}(\Omega)$ . According to Calderon-Zygmund (see [7], Theorem 3.83), any function  $u \in \text{BV}(\Omega)$  is approximately differentiable at almost every point  $x \in \Omega$ . So, if  $(u, z) \in \mathcal{D}_n(\Omega)$ , then  $z$  and  $\bar{u}^N(1 - z^2)$  are approximately differentiable almost everywhere. The following properties are straightforward consequences of Definition 3.2

- if  $v_1, v_2$  are approximately differentiable almost everywhere and  $v_1 \in L^\infty(\Omega)$ , then  $v_1 v_2$  is approximately differentiable almost everywhere;
- if  $v_2$  is approximately differentiable almost everywhere, then  $v_2^{-1}$  is also approximately differentiable almost everywhere in  $\{x : v_2(x) \neq 0\}$  (Proposition 3.71 in [7]).

We deduce that  $\bar{u}^N$  is approximately differentiable almost everywhere in  $\{x : z(x) \neq 1\}$ . This is true for any  $N \in \mathbb{N}$ , so this is also true for  $u$ .  $\square$

The main result of the paper is the following

**Theorem 3.1.** Let  $\Omega \subset \mathbb{R}^n$  be an open and bounded domain which satisfies reflexion condition (1.1),  $\mathcal{G}$  be a compact subset of  $S_n^+(\mathbb{R})$  and  $\alpha > 0$  be fixed. We consider  $H : \mathbb{B}(\Omega) \times W^{1,n+\alpha}(\Omega; \mathcal{G}) \rightarrow [0; +\infty]$  defined as

$$H(u, \mathbf{M}) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} \|D\mathbf{M}\|^{n+\alpha} dx,$$

$E : \mathbb{B}(\Omega) \times W^{1,n+\alpha}(\Omega; \mathcal{G}) \rightarrow [0; +\infty]$  defined as

$$E(u, \mathbf{M}) = \begin{cases} H(u, \mathbf{M}) + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} & \text{if } u \in \text{SBV}(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and  $E_\varepsilon : \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \times W^{1,n+\alpha}(\Omega; \mathcal{G}) \rightarrow [0; +\infty]$  defined as

$$E_\varepsilon(u, z, \mathbf{M}) = \begin{cases} H(u, \mathbf{M}) + \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left( \varepsilon \langle \mathbf{M} \nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) dx & \text{if } (u, z) \in \mathcal{D}_n(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the following assertions are satisfied.

- i) For any  $\varepsilon > 0$ ,  $E_\varepsilon$  admits a minimizer, denoted by  $(u_\varepsilon, z_\varepsilon, \mathbf{M}_\varepsilon)$ . Moreover, we can assume that  $u_\varepsilon(x) = g(x)$  on  $\{x \in \Omega : z_\varepsilon(x) = 1\}$ .
- ii) For any  $(\varepsilon_k)_k$  converging to  $0^+$ , there exists a subsequence, still denoted by  $(\varepsilon_k)_k$ , and  $u \in \text{SBV}(\Omega)$  such that  $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$  converges to  $(u, 0)$  almost everywhere,  $(\mathbf{M}_k)_k$  converges weakly to  $\mathbf{M}$  and  $(u, \mathbf{M})$  is a minimizer of  $E$ .

We set

$$(\mathcal{P}_\varepsilon) : \quad \text{Min}\{E_\varepsilon(u, z, \mathbf{M}) : (u, z, \mathbf{M}) \in \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \times W^{1,n+\alpha}(\Omega; \mathcal{G})\},$$

Theorem 3.1 i) implies that, for  $\varepsilon > 0$  fixed,  $(\mathcal{P}_\varepsilon)$  is a well posed problem. Theorem 3.1 ii) implies that, up to the extraction of a subsequence, the sequence of solutions of  $(\mathcal{P}_{\varepsilon_k})$  converge to a solution of  $(\mathcal{P}')$ .

### 3.2 Existence result for $(\mathcal{P}_\varepsilon)$

Theorem 3.1 i) is a straightforward consequence of Propositions 3.2 and 3.3.

**Proposition 3.2.** *Let  $\varepsilon > 0$  be fixed. There exists  $(u_k, z_k, \mathbf{M}_k)_k$  a minimizing sequence of  $E_\varepsilon$  such that  $(u_k)_k$  is a bounded sequence of  $L^\infty(\Omega)$ ,  $(u_k, z_k)_k$  converges almost everywhere to  $(u, z) \in \mathcal{D}_n(\Omega)$ ,  $u(x) = g(x)$  on  $\{x \in \Omega : z(x) = 1\}$  and  $(\mathbf{M}_k)_k$  weakly converges to  $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$ .*

To prove it, we need the following Lemma which proof is given in Appendix 4.1.

**Lemma 3.1.** *For  $(u, z) \in \mathcal{D}_n(\Omega)$  and  $\nu \in \mathbb{S}^{n-1}$  fixed, we have  $(u_x, z_x) \in \mathcal{D}_1(\Omega_x)$  for  $\mathcal{H}^{n-1}$ -almost every  $x \in \Omega_\nu$  (see the notations of Definition 1.2), and*

$$\begin{aligned} \nabla u_x(t) &= \langle \nabla u(x + t\nu), \nu \rangle, \\ \nabla z_x(t) &= \langle \nabla z(x + t\nu), \nu \rangle, \end{aligned}$$

for almost every  $t \in \Omega_x \setminus \{s : z(x + s\nu) = 1\}$ .

Now, we prove Proposition 3.2.

*Proof.* Let  $(u_k, z_k, \mathbf{M}_k)_k$  be a minimizing sequence of  $E_\varepsilon$ . We fix  $N \geq \|g\|_{L^\infty(\Omega)}$  and we consider the truncated functions  $(\bar{u}_k^N)_k$  defined in (3.1). As  $(u_k, z_k) \in \mathcal{D}_n(\Omega)$ , we have  $\bar{u}_k^N(1 - z_k^2) \in W^{1,2}(\Omega)$ . As  $\Omega$  is bounded, then  $W^{1,2}(\Omega) \subset \text{SBV}(\Omega)$ . According to Calderón-Zygmund Theorem ([7], Theorem 3.83),  $\bar{u}_k^N(1 - z_k^2)$  is approximately differentiable almost everywhere. For the same reasons,  $1 - z_k^2$  is also approximately differentiable almost everywhere. According to Proposition 3.71 in [7], we deduce that  $\bar{u}_k^N$  is approximately differentiable almost everywhere in  $\{x : z_k(x) \neq 1\}$ . Moreover,  $\nabla \bar{u}_k^N(x) = 0$  almost everywhere in  $\{x : |\bar{u}_k^N(x)| = N\}$  and  $\nabla \bar{u}_k^N(x) = \nabla u_k(x)$  almost everywhere in  $\{x : |\bar{u}_k^N(x)| < N\}$  (Proposition 3.73 in [7]), it gives

$$\int_{\Omega} |\nabla \bar{u}_k^N|^2 (1 - z_k^2)^2 dx \leq \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx. \quad (3.3)$$

so  $E_\varepsilon(\bar{u}_k^N, z_k, \mathbf{M}_k) \leq E_\varepsilon(u_k, z_k, \mathbf{M}_k)$  and then  $(\bar{u}_k^N, z_k, \mathbf{M}_k)_k$  is also a minimizing sequence. According to ellipticity inequality (2.1), we have

$$\int_{\Omega} |\nabla z_k|^2 dx + \int_{\Omega} z_k^2 dx \leq \left( \frac{1}{\lambda\varepsilon} + 4\varepsilon \right) E_\varepsilon(u_k, z_k, \mathbf{M}_k),$$

and then  $(z_k)_k$  is a bounded sequence of  $W^{1,2}(\Omega)$ . So, there exists a subsequence, still denoted by  $(z_k)_k$ , which converges almost everywhere to  $z \in W^{1,2}(\Omega)$ . As  $(z_k)_k$  takes its values almost everywhere in  $[0; 1]$ , then  $z$  takes also its values in  $[0; 1]$ . For  $w_k = \bar{u}_k^N(1 - z_k^2)$ , we have

$$\int_{\Omega} |\nabla w_k|^2 dx + \int_{\Omega} w_k^2 dx \leq 2 \int_{\Omega} |\nabla \bar{u}_k^N|^2 (1 - z_k^2)^2 dx + 2N^2 \int_{\Omega} |\nabla z_k|^2 dx + N^2 \int_{\Omega} (1 - z_k^2)^2 dx$$

and then  $(w_k)_k$  is a bounded sequence of  $W^{1,2}(\Omega)$ . So, there exists a subsequence, still denoted by  $(w_k)_k$ , which converges almost everywhere to  $w \in W^{1,2}(\Omega)$ . In particular,  $(\bar{u}_k^N(x))_k$  converges for almost every  $x \in \{y: z(y) \neq 1\}$  to  $u(x)$ . We set  $u(x) = g(x)$  for  $x \in \{y: z(y) = 1\}$ . This construction ensures that  $(u, z) \in \mathcal{D}_n(\Omega)$ . As  $\mathcal{G} \subset S_n^+(\mathbb{R})$  is compact,  $\Omega$  is bounded and  $\int_{\Omega} \|D\mathbf{M}_k\| dx \leq E_{\varepsilon}(u_k, z_k, \mathbf{M}_k)$ , then  $(\mathbf{M}_k)_k$  is a bounded sequence of  $W^{1,n+\alpha}(\Omega)$  and there exists a subsequence weakly convergent to  $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$ . As weakly convergence in  $W^{1,n+\alpha}(\Omega)$  implies uniform convergence and  $\mathcal{G}$  is closed, then  $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$ .  $\square$

**Proposition 3.3.** *If  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  converges almost everywhere to  $(u, z) \in \mathcal{D}_n(\Omega)$ ,  $(u_k)_k$  is a bounded sequence of  $L^{\infty}(\Omega)$  and  $(\mathbf{M}_k)_k \subset W^{1,n+\alpha}(\Omega; \mathcal{G})$  weakly converges to  $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$ , then*

$$\liminf_{k \rightarrow \infty} E_{\varepsilon}(u_k, z_k, \mathbf{M}_k) \geq E_{\varepsilon}(u, z, \mathbf{M}).$$

*Proof.* Fatou Lemma and lower semi-continuity of the Sobolev semi-norm give

$$\liminf_{k \rightarrow \infty} \left( \int_{\Omega} (u_k - g)^2 dx + \int_{\Omega} \frac{z_k^2}{4\varepsilon} dx + \int_{\Omega} \|D\mathbf{M}_k\|^{n+\alpha} dx \right) \geq \int_{\Omega} (u - g)^2 dx + \int_{\Omega} \frac{z^2}{4\varepsilon} dx + \int_{\Omega} \|D\mathbf{M}\|^{n+\alpha} dx.$$

So, to show Proposition 3.3, it suffices to prove that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \varepsilon \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx \geq \int_{\Omega} \varepsilon \langle \mathbf{M} \nabla z, \nabla z \rangle dx \quad (3.4)$$

and

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx \geq \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx. \quad (3.5)$$

*Proof of (3.4)*

If  $\liminf_k \int_{\Omega} |\nabla z_k|^2 dx = +\infty$ , according to Ellipticity condition (2.1), we have

$$\liminf_k \int_{\Omega} \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx = +\infty$$

and the result of Proposition 3.3 is ensured. So, we may assume that  $\liminf \int_{\Omega} |\nabla z_k|^2 dx < +\infty$ . On the other hand, we have

$$\left| \int_{\Omega} \varepsilon \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx - \int_{\Omega} \varepsilon \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle dx \right| \leq \varepsilon \|\mathbf{M}_k - \mathbf{M}\|_{L^{\infty}} \int_{\Omega} |\nabla z_k|^2 dx.$$

As the weakly convergence of  $(\mathbf{M}_k)_k$  in  $W^{1,n+\alpha}(\Omega)$  implies the uniform convergence, we may conclude that

$$\liminf_k \left( \int_{\Omega} \varepsilon \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx - \int_{\Omega} \varepsilon \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle dx \right) = 0.$$

To prove inequality (3.4), it remains to verify that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \varepsilon \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle dx \geq \int_{\Omega} \varepsilon \langle \mathbf{M} \nabla z, \nabla z \rangle dx.$$

As the application

$$\begin{array}{ccc} W^{1,2}(\Omega) & \rightarrow & L^2(\Omega; \mathbb{R}^n), \\ z & \mapsto & \nabla z \end{array}$$

is continuous for the strong topology, it remains to prove that the application

$$\begin{array}{ccc} L^2(\Omega; \mathbb{R}^n) & \rightarrow & \mathbb{R}, \\ Z & \mapsto & \int_{\Omega} \langle \mathbf{M} Z, Z \rangle dx \end{array}$$

is lower semi-continuous for the weak topology of  $L^2(\Omega; \mathbb{R}^n)$ . Let  $(Z_k)_k \subset L^2(\Omega; \mathbb{R}^n)$  be weakly convergent to  $Z \in L^2(\Omega; \mathbb{R}^n)$ . We set

$$\begin{array}{ccc} L : L^2(\Omega; \mathbb{R}^n) & \rightarrow & \mathbb{R}, \\ U & \mapsto & \int_{\Omega} \langle \mathbf{M} Z, U \rangle dx \end{array}$$

According to ellipticity inequality (2.1),  $L \in (L^2(\Omega; \mathbb{R}^n))'$  and then  $(L(Z_k))_k$  converges to  $L(Z)$ . Moreover, for  $k$  fixed, the following polynomial function is positive

$$t \mapsto \int_{\Omega} \langle \mathbf{M}(Z + tZ_k), Z + tZ_k \rangle dx.$$

Thus, its discriminant is negative and we deduce the following *anisotropic Cauchy-Schwarz inequality*

$$\int_{\Omega} \langle \mathbf{M}Z, Z_k \rangle dx \leq \left( \int_{\Omega} \langle \mathbf{M}Z, Z \rangle dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \langle \mathbf{M}Z_k, Z_k \rangle dx \right)^{\frac{1}{2}}.$$

As  $(L(Z_k))_k$  converges to  $L(Z)$ , passing through the  $\liminf$  in the previous inequality yields

$$\int_{\Omega} \langle \mathbf{M}Z, Z \rangle dx \leq \left( \int_{\Omega} \langle \mathbf{M}Z, Z \rangle dx \right)^{\frac{1}{2}} \liminf_{k \rightarrow \infty} \left( \int_{\Omega} \langle \mathbf{M}Z_k, Z_k \rangle dx \right)^{\frac{1}{2}}$$

and then we may conclude the *Proof of (3.4)* by taking  $Z_k = \nabla z_k, Z = \nabla z$  in the previous inequality

$$\int_{\Omega} \langle \mathbf{M}\nabla z, \nabla z \rangle dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \langle \mathbf{M}\nabla z_k, \nabla z_k \rangle dx.$$

*Proof of (3.5)*

We first consider the one-dimensional case  $n = 1$  and then by a slicing argument we get the lower semi-continuity for the general case  $n \geq 1$ . Let  $A \subset \{x \in \Omega : z(x) < 1\}$  be an open and relatively compact subset of  $\Omega \subset \mathbb{R}$ . As  $(z_k)_k$  weakly converges to  $z$  in  $W^{1,2}(\Omega)$ , then  $(z_k)_k$  uniformly converges to  $z$ . In particular, there exists  $\delta > 0$  and  $k_0 \in \mathbb{N}$

$$k \geq k_0 \quad \Rightarrow \quad A \subset \{x \in \Omega : z_k(x) \leq 1 - \delta\}.$$

Thus, for any  $k \geq k_0$ , we have

$$\int_A |\nabla u_k|^2 dx \leq \frac{1}{1 - (1 - \delta)^2} E_{\varepsilon}(u_k, z_k)$$

and then we deduce that  $(u_k)_k$  is a bounded sequence of  $W^{1,2}(A)$ . As  $u_k(1 - z_k^2)$  converges almost everywhere to  $u(1 - z^2)$  in  $\Omega$ , there exists a subsequence, still denoted by  $(u_k)_k$ , which weakly converges to  $u$  in  $W^{1,2}(A)$ . In particular,  $(\nabla u_k)_k$  weakly converges to  $\nabla u$  in  $L^2(A)$ . For  $\xi \in L^2(\Omega)$ , we decompose

$$\int_A \xi [\nabla u_k(1 - z_k^2) - \nabla u(1 - z^2)] dx = \int_A \xi \nabla u_k(z^2 - z_k^2) dx + \int_A \xi \nabla u_k(1 - z^2) dx + \int_A \xi \nabla u(z^2 - z_k^2) dx. \quad (3.6)$$

As  $(1 - z^2)\xi \in L^2(A)$  and  $(\nabla u_k)_k$  weakly converges to  $\nabla u$  in  $L^2(A)$ , then we have

$$\int_A \xi \nabla u_k(1 - z^2) dx \rightarrow \int_A \xi \nabla u(1 - z^2) dx. \quad (3.7)$$

Moreover, we have

$$\int_A \xi \nabla u_k(z - z_k) dx \leq \|\xi\|_{L^2(A)} \|\nabla u_k\|_{L^2(A)} \|z^2 - z_k^2\|_{L^\infty(A)}$$

and

$$\int_A \xi \nabla u(z - z_k) dx \leq \|\xi\|_{L^2(A)} \|\nabla u\|_{L^2(A)} \|z^2 - z_k^2\|_{L^\infty(A)}.$$

As a weakly convergent sequence is bounded, then  $(\nabla u_k)_k$  is bounded in  $L^2(A)$  and we deduce that

$$\int_A \xi \nabla u_k(z^2 - z_k^2) dx \rightarrow 0, \quad \int_A \xi \nabla u(z^2 - z_k^2) dx \rightarrow 0. \quad (3.8)$$

According to (3.6), (3.7) and (3.8), we get

$$\int_A \xi \nabla u_k (1 - z_k^2) dx \rightarrow \int_A \xi \nabla u (1 - z^2) dx$$

and then  $(\nabla u_k (1 - z_k^2))_k$  weakly converges to  $\nabla u (1 - z^2)$  in  $L^2(A)$ . As the norm is lower semi-continuous, we deduce

$$\begin{aligned} \int_A |\nabla u|^2 (1 - z^2)^2 dx &\leq \liminf_{k \rightarrow \infty} \int_A |\nabla u_k|^2 (1 - z_k^2)^2 dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 dx. \end{aligned}$$

Passing to the limit  $A \uparrow \{x \in \Omega: z(x) < 1\}$  gives

$$\int_\Omega |\nabla u|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

We generalize this result to the dimension  $n \geq 1$ . With the notation  $u_x$  introduced in (1.2), using the previous result obtained in dimension 1, Lemma 3.1 and Fatou Lemma, give

$$\begin{aligned} \int_A |\langle \nabla u, \nu \rangle|^2 (1 - z^2)^2 dx &= \int_{A_\nu} \int_{A_x} |\nabla u_x(t)|^2 (1 - z_x(t))^2 dt dx, \\ &\leq \int_{A_\nu} \liminf_{k \rightarrow \infty} \int_{A_x} |\nabla (u_k)_x(t)|^2 (1 - (z_k)_x(t))^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_\nu} \int_{A_x} |\nabla (u_k)_x(t)|^2 (1 - (z_k)_x(t))^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_{A_\nu} \int_{A_x} |\langle \nabla u_k(x + t\nu), \nu \rangle|^2 (1 - z_k(x + t\nu))^2 dt dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_A |\langle \nabla u_k, \nu \rangle|^2 (1 - z_k^2)^2 dx, \\ &\leq \liminf_{k \rightarrow \infty} \int_A |\nabla u_k|^2 (1 - z_k^2)^2 dx, \end{aligned}$$

for any open set  $A \subset \Omega$  and every  $\nu \in \mathbb{S}^{n-1}$ . The function  $x \rightarrow \frac{\nabla u(x)}{|\nabla u(x)|}$  is measurable in  $U = \{x \in \Omega: z(x) \neq 1, \nabla u(x) \neq 0\}$ . According to Lusin Theorem (1.45 of [7]), there exists an increasing sequence of compacts  $(K_l)_l \subset U$  such that

$$\begin{cases} \mathcal{L}^n(U \setminus K_l) \leq \frac{1}{l}, \\ x \rightarrow \frac{\nabla u(x)}{|\nabla u(x)|} \text{ is continuous in } K_l. \end{cases}$$

Thus, for any  $x \in K_l$ , there exists  $r > 0$  such that

$$y \in B(x, r) \quad \Rightarrow \quad \left| \frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|} \right| \leq \frac{1}{l}, \quad (3.9)$$

As a consequence of Besicovitch Covering Theorem (2.18 of [7]), there exists a countable, pairwise disjoint collection of balls  $(B_i)_{i \in I}$  satisfying (3.9) such that

$$B_i \subset \Omega, \quad \mathcal{L}^n \left( K_l \setminus \bigcup_{i \in I} B_i \right) = 0, \text{ for any } \forall i \in I.$$

For any  $i \in I$ , we fix  $x_i \in B_i$  and we set  $\nu_i = \frac{\nabla u(x_i)}{|\nabla u(x_i)|}$ ; then

$$\int_{B_i} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{B_i} |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

As  $(B_i)_i$  is pairwise disjoint, we deduce

$$\begin{aligned}
\int_{\cup_i B_i} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx &= \sum_{i \in I} \int_{B_i} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx, \\
&\leq \sum_{i \in I} \liminf_{k \rightarrow \infty} \int_{B_i} |\nabla u_k|^2 (1 - z_k^2)^2 dx, \\
&\leq \liminf_{k \rightarrow \infty} \sum_{i \in I} \int_{B_i} |\nabla u_k|^2 (1 - z_k^2)^2 dx, \\
&\leq \liminf_{k \rightarrow \infty} \int_{\cup_i B_i} |\nabla u_k|^2 (1 - z_k^2)^2 dx, \\
&\leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\forall x \in B_i \cap K_l, \quad \left| |\nabla u|^2 - |\langle \nabla u, \nu_i \rangle|^2 \right| &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} \rangle^2 - \langle \nabla u, \nu_i \rangle^2 \right|, \\
&\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} - \nu_i \rangle \langle \nabla u, \frac{\nabla u}{|\nabla u|} + \nu_i \rangle \right|, \\
&\leq \frac{2}{l} |\nabla u|^2.
\end{aligned}$$

It gives

$$\int_{B_i \cap K_l} |\nabla u|^2 (1 - z^2)^2 dx \leq \frac{l}{l-2} \int_{B_i \cap K_l} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx.$$

As  $\mathcal{L}^n(K_l \setminus \cup_i B_i) = 0$  and  $(B_i)_i$  is pairwise disjoint, we get

$$\begin{aligned}
\int_{K_l} |\nabla u|^2 (1 - z^2)^2 dx &= \sum_i \int_{B_i \cap K_l} |\nabla u|^2 (1 - z^2)^2 dx, \\
&\leq \frac{l}{l-2} \sum_i \int_{B_i \cap K_l} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx, \\
&\leq \frac{l}{l-2} \int_{\cup_i B_i \cap K_l} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx, \\
&\leq \frac{l}{l-2} \int_{\cup_i B_i} |\langle \nabla u, \nu_i \rangle|^2 (1 - z^2)^2 dx, \\
&\leq \frac{l}{l-2} \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx.
\end{aligned}$$

As  $(K_l)_l$  is an increasing sequence such that  $\mathcal{L}^n(U \setminus K_l) \rightarrow 0$ , passing to the limit  $l \rightarrow \infty$  gives

$$\int_U |\nabla u|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx$$

and we may conclude

$$\int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx.$$

□

### 3.3 $\Gamma$ -convergence result for $\varepsilon \rightarrow 0^+$

This section is dedicated to the proof of Theorem 3.1 ii). For that, we will prove the following  $\Gamma$ -convergence result.

**Theorem 3.2.** *Assume that the conditions of Theorem 3.1 are satisfied. Let  $(\varepsilon_k)_k$  be a sequence which converges to  $0^+$ , we have*

i) if  $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ ,  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ ,  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$ ,  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere and  $(\mathbf{M}_k)_k$  weakly converges to  $\mathbf{M}$ , then

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(u_k, z_k, \mathbf{M}_k) \geq E(u, \mathbf{M}); \quad (3.10)$$

ii) for any  $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$  and  $\mathbf{M} \in W^{1, n+\alpha}(\Omega; \mathcal{G})$ , there exists a sequence  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  such that  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$ ,  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere and

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq E(u, \mathbf{M}). \quad (3.11)$$

### 3.3.1 The inequality for the lower $\Gamma$ -limit

We now prove the first inequality of  $\Gamma$ -convergence (3.10). Let  $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$  and  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  such that  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$ ,  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere and  $(\mathbf{M}_k)_k$  weakly converges to  $\mathbf{M}$ . In the sequel, we emphasize on the domain of the function: for  $U$  an open subset of  $\Omega$ , we adopt the following notation

$$F(u, \mathbf{M}; U) = \int_U |\nabla u|^2 dx + \int_{J_u \cap U} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1},$$

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}_k; U) = \int_U |\nabla u_k|^2 (1 - z_k^2)^2 dx + \int_U \left( \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle + \frac{z_k^2}{4\varepsilon_k} \right) dx,$$

Fatou Lemma and lower continuity for the Sobolev semi-norm yields

$$\liminf_{k \rightarrow \infty} \int_\Omega (u_k - g)^2 dx + \int_\Omega \|D\mathbf{M}_k\|^{n+\alpha} dx \geq \int_\Omega (u - g)^2 dx + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} dx$$

and then it suffices to prove that  $\liminf_k F_{\varepsilon_k}(u_k, z_k, \mathbf{M}_k; \Omega) \geq F(u, \mathbf{M}; \Omega)$ . Moreover, if

$$\liminf_k \int_\Omega \varepsilon_k |\nabla z_k|^2 dx = +\infty,$$

according to ellipticity condition (2.1), we have

$$\liminf_k \int_\Omega \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx = +\infty$$

and (3.10) is ensured. So, we may assume that  $\liminf_k \int_\Omega \varepsilon_k |\nabla z_k|^2 dx < +\infty$ . On the other hand, we have

$$\left| \int_\Omega \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx - \int_\Omega \varepsilon_k \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle dx \right| \leq \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \varepsilon_k \int_\Omega |\nabla z_k|^2 dx.$$

As the weakly convergence of  $(\mathbf{M}_k)_k$  to  $\mathbf{M}$  in  $W^{1, n+\alpha}(\Omega)$  implies the uniform convergence, we may conclude that

$$\liminf_k \left( \int_\Omega \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle dx - \int_\Omega \varepsilon_k \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle dx \right) = 0.$$

To prove inequality (3.10), it remains to verify that

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; U).$$

We perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension  $n \geq 2$ .

#### The one-dimensional case

In this section, we assume that  $\Omega = I$  is an open interval and the metric  $\mathbf{M}$  is simply a constant  $m > 0$ . We assume in this section that  $m$  is fixed. To avoid confusion, we denote the approximating functional by

$$G_\varepsilon(u, z; I) = \int_I |\nabla u(t)|^2 (1 - z(t)^2)^2 dt + \int_I \left( m\varepsilon |\nabla z(t)|^2 + \frac{z(t)^2}{4\varepsilon} \right) dt, \quad (3.12)$$



where the domain is

$$\mathcal{D}_1(I) = \{(u, z) : u \in \mathbb{B}(I), z \in W^{1,2}(I; [0; 1]), \forall N \in \mathbb{N} \quad \bar{u}^N(1 - z^2) \in W^{1,2}(I)\}.$$

We denote the lower  $\Gamma$ -limit, by

$$G_-(u; I) = \inf \left\{ \liminf_{k \rightarrow \infty} G_{\varepsilon_k}(u_k, z_k; I) \right\},$$

where the inf is taken over all sequence  $(u_k, z_k)_k \subset \mathcal{D}_1(I)$  such that  $(u_k, z_k)$  converges almost everywhere to  $(u, 0)$  in  $I$ . We need the following Lemma which proof is given in appendix 4.2.

**Lemma 3.2.** *Let  $I \subset \mathbb{R}$  be an open interval,  $J \subset I$  be a set with finite cardinal. We have*

$$u \in W^{1,2}(I \setminus J) \Rightarrow u \in SBV(I), \quad J_u \subset J.$$

The main result of this subsection is given by the following.

**Proposition 3.4.** *Let  $I \subset \mathbb{R}$  be an open interval and  $u \in \mathbb{B}(I)$ . If  $G_-(u; I) < \infty$ , then  $u \in SBV(I)$  and*

$$\int_I |\nabla u(t)|^2 dt + m^{\frac{1}{2}} \mathcal{H}^0(J_u \cap I) \leq G_-(u; I).$$

The proof of this Proposition consists in showing the two following Lemmas.

**Lemma 3.3.** *If  $u \in W^{1,2}(B_\eta(x))$ , then we have*

$$G_-(u; B_\eta(x)) \geq \int_{B_\eta(x)} |\nabla u(t)|^2 dt.$$

**Lemma 3.4.** *If  $u \notin W^{1,2}(B_\rho(x))$  for any  $\rho \in ]0; \eta[$ , then for any  $\rho \in ]0; \eta[$  we have*

$$G_-(u; B_\rho(x)) \geq m^{1/2}.$$

Suppose that Lemma 3.3 and 3.4 are proved, we deduce Proposition 3.4.

*Proof.* We set

$$J = \{x \in I : \forall \rho > 0, u \notin W^{1,2}(B_\rho(x))\}.$$

Let  $\{x_1, \dots, x_N\} \subset J$  and  $\rho > 0$  be such that  $\{B_\rho(x_i) : i = 1, \dots, N\}$  is pairwise disjoint. According to Lemma 3.4 we have

$$\forall i \in \{1, \dots, N\}, \quad G_-(u; B_\rho(x_i)) \geq m^{1/2}$$

and then

$$\sum_{i=1}^N G_-(u; B_\rho(x_i)) \geq Nm^{1/2}.$$

As  $G_-(u; \cdot)$  is superadditive, we have

$$G_-(u; \cup_{i=1}^N B_\rho(x_i)) \geq Nm^{1/2}.$$

and  $G_-(u; \cdot)$  is non decreasing, it gives

$$G_-(u; I) \geq Nm^{1/2}.$$

As  $G_-(u; \cdot) < +\infty$ , the set  $J$  is finite. So, there exists  $\rho > 0$  such that  $\{B_\rho(x) : x \in J\}$  is pairwise disjoint. As  $G_-(u; \cdot)$  is superadditive and non decreasing, we have

$$\sum_{x \in J} G_-(u; B_\rho(x)) + G_-(u; I \setminus \cup_{x \in J} \overline{B_\rho(x)}) \leq G_-(u; I).$$

According to Lemma 3.3 and 3.4, it gives

$$\mathcal{H}^0(J)m^{\frac{1}{2}} + \int_{I \setminus \cup_{x \in J} \overline{B_\rho(x)}} |\nabla u(t)|^2 dt \leq G_-(u; I).$$

Taking the limit  $\rho \rightarrow 0^+$  yields

$$\mathcal{H}^0(J)m^{\frac{1}{2}} + \int_{I \setminus J} |\nabla u(t)|^2 dt \leq G_-(u; I).$$

In particular  $u \in W^{1,2}(I \setminus J)$  and, according to Lemma 3.2, we get  $u \in \text{SBV}(I)$ ,  $J_u \subset J$  and then

$$\mathcal{H}^0(J)m^{\frac{1}{2}} + \int_I |\nabla u(t)|^2 dt \leq G_-(u; I).$$

□

Now, we prove lemma 3.3.

*Proof.* We can assume that  $G_-(u; B_\eta(x)) < +\infty$ , otherwise the result is ensured. By a diagonal extraction, there exists a sequence  $(u_k, z_k)_k \subset \mathcal{D}_1(B_\rho(x))$  converging almost everywhere to  $(u, 0)$  and

$$G_{\varepsilon_k}(u_k, z_k; B_\rho(x)) \rightarrow G_-(u; B_\rho(x)).$$

As  $G_-(u; B_\eta(x))$  is finite, there exists  $C > 0$  such that

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} \left( \varepsilon_k |\nabla z_k|^2 + \frac{z_k^2}{4\varepsilon_k} \right) dt \leq C. \quad (3.13)$$

Applying the inequality  $2ab \leq a^2 + b^2$  with  $a^2 = \varepsilon_k |\nabla z_k|^2$  and  $b^2 = \frac{z_k^2}{4\varepsilon_k}$  gives

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} |\nabla z_k| z_k dt \leq C. \quad (3.14)$$

We set  $c_k = 1 - z_k^2$ . As  $z_k \in W^{1,2}(B_\eta(x))$ , then  $c_k \in BV(B_\eta(x))$  and (3.14) is

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} |\nabla c_k| dt \leq 2C.$$

Coarea formula (see [7]) yields

$$\forall k \in \mathbb{N}, \quad \int_0^1 \mathcal{H}^0(\{y \in B_\eta(x) : c_k(y) = t\}) dt \leq 2C. \quad (3.15)$$

Let  $\sigma < 1$  in an arbitrary neighborhood of 1 and  $\delta \in ]0; \sigma[$  be fixed numbers. According to (3.15) and mean value Theorem, there exists  $\delta_k \in ]\delta; \sigma[$  such that

$$\forall k \in \mathbb{N}, \quad \mathcal{H}^0(\{y \in B_\eta(x) : c_k(y) = \delta_k\}) \leq \frac{2C}{\sigma - \delta}. \quad (3.16)$$

We set  $A_k = \{y \in B_\eta(x) : c_k(y) \geq \delta_k\}$ . As  $(\varepsilon_k)_k$  converges to 0, inequality (3.13) implies that  $(z_k)_k$  converges to 0 and  $c_k$  to 1 almost everywhere. As  $\delta_k < \sigma$  and  $\sigma < 1$  then  $(\mathcal{L}^1(A_k))_k$  converges to  $\mathcal{L}^1(B_\eta(x))$ .

Sobolev embedding Theorem ensures that  $W^{1,2}(B_\eta(x)) \subset \mathcal{C}(B_\eta(x))$ , so  $c_k$  is continuous and  $A_k$  is a countable union of closed intervals of  $B_\eta(x)$ . According to (3.16), this union is finite and its cardinality is uniformly bounded by  $N$ . For any  $k$ , there exists a disjoint family of closed intervals  $(I_k^i)_{i=1 \dots N}$  such that

$$\begin{cases} A_k = \bigcup_{i=1}^N I_k^i, \\ \forall i \in \{1, \dots, N-1\}, \max(I_k^i) < \min(I_k^{i+1}). \end{cases}$$

There exists a subsequence, still denoted by  $(I_k^i)_{i=1 \dots N}$ , such that  $(\min(I_k^i))_k$  and  $(\max(I_k^i))_k$  converge for any  $i \in \{1, \dots, N\}$ . We set  $a_\infty^i$  and  $b_\infty^i$  the previous limits,  $I_\infty^i = ]a_\infty^i; b_\infty^i[$  and  $A = \bigcup_{i=1}^N I_\infty^i$ . As  $(\mathcal{L}^1(A_k))_k$  converges to  $\mathcal{L}^1(B_\eta(x))$ , then  $A$  is a subset of full measure in  $B_\eta(x)$ .

Let  $O$  be an open subset such that  $\overline{O} \subset A$ . For  $k$  with a sufficiently large value, we have  $O \subset A_k$  and then

$$\liminf_{k \rightarrow \infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 dt \geq \liminf_{k \rightarrow \infty} \int_O |\nabla u_k|^2 (1 - z_k^2)^2 dt.$$

As  $z_k$  takes its values in  $[0; 1]$ , we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 dt &\geq \liminf_{k \rightarrow \infty} \int_O |\nabla u_k|^2 \delta_k^2 dt, \\ &\geq \delta^2 \liminf_{k \rightarrow \infty} \int_O |\nabla u_k|^2 dt. \end{aligned}$$

Moreover, there exists  $k_0$  such that:  $k \geq k_0 \Rightarrow O \subset A_k$ , then we have

$$\forall x \in O \quad 1 - z_k^2 \geq \delta.$$

As  $u_k(1 - z_k^2) \in W^{1,2}(\Omega)$  and  $\sqrt{\delta} > 0$ , we get  $u_k \in W^{1,2}(O)$  and the lower semi-continuity property of the Sobolev norm gives

$$\liminf_{k \rightarrow \infty} \int_O |\nabla u_k|^2 dt \geq \int_O |\nabla u|^2 dt.$$

As  $O$  is chosen arbitrary in  $A$  and  $A$  is of full measure in  $B_\eta(x)$ , it gives

$$\liminf_{k \rightarrow \infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 dt \geq \delta^2 \int_{B_\eta(x)} |\nabla u|^2 dt.$$

Letting  $\delta$  to  $1^-$ , it concludes the proof of lemma 3.3. □

We prove lemma 3.4.

*Proof.* We can assume that  $G_-(u; B_\rho(x)) < +\infty$  for any  $\rho \in ]0; \eta[$ , otherwise the result is ensured. As  $u \notin W^{1,2}(B_\rho(x))$ , there exists three sequences  $(y_k^1)_{k \in \mathbb{N}}$ ,  $(y_k^2)_{k \in \mathbb{N}}$  and  $(y_k^3)_{k \in \mathbb{N}}$  such that:

$$\begin{cases} y_k^1 \rightarrow x, & z_k(y_k^1) \rightarrow 0, \\ y_k^2 \rightarrow x, & z_k(y_k^2) \rightarrow 1, \\ y_k^3 \rightarrow x, & z_k(y_k^3) \rightarrow 0, \\ \forall k \in \mathbb{N}, & y_k^1 < y_k^2 < y_k^3. \end{cases}$$

We have

$$G(u_k, z_k; B_\rho(x)) \geq \int_{x-\rho}^{x+\rho} \left( \varepsilon_k m |\nabla z_k|^2 + \frac{z_k^2}{4\varepsilon_k} \right) dt.$$

The inequality  $a^2 + b^2 \geq 2ab$  gives:

$$G(u_k, z_k; B_\rho(x)) \geq \int_{x-\rho}^{x+\rho} m^{1/2} |\nabla z_k| z_k dt.$$

As  $[y_k^1, y_k^2] \subset B_\rho(x)$ , we obtain:

$$G(u_k, z_k; B_\rho(x)) \geq \int_{y_k^1}^{y_k^3} m^{1/2} |\nabla z_k| z_k dt.$$

We have

$$G(u_k, z_k; B_\rho(x)) \geq m^{1/2} \int_{y_k^1}^{y_k^2} |\nabla z_k(t)| z_k(t) dt + m^{1/2} \int_{y_k^2}^{y_k^3} |\nabla z_k(t)| z_k(t) dt.$$

Since  $z_k \in W^{1,2}(B_\eta(x))$ , we may use the change of variable  $s = z_k(t)$ . This yields:

$$\begin{aligned} (\star)_k^2 &\geq m^{1/2} \int_{z_k(y_k^1)}^{z_k(y_k^2)} s ds + m^{1/2} \int_{z_k(y_k^2)}^{z_k(y_k^3)} s ds, \\ &\geq m^{1/2} \left( \frac{z_k^2(y_k^2) - z_k^2(y_k^1)}{2} + \frac{z_k^2(y_k^2) - z_k^2(y_k^3)}{2} \right) \end{aligned}$$

By assumption, we have  $z_k(y_k^1) \rightarrow 0$ ,  $z_k(y_k^2) \rightarrow 1$  and  $z_k(y_k^3) \rightarrow 0$ , so that we deduce:

$$\frac{z_k^2(y_k^2) - z_k^2(y_k^1)}{2} + \frac{z_k^2(y_k^2) - z_k^2(y_k^3)}{2} \rightarrow 1.$$

We can conclude :

$$\liminf_{k \rightarrow \infty} G(u_k, z_k; B_\rho(x)) \geq m^{1/2}. \quad \square$$

## Generalization to dimension $n \geq 2$

We give the proof of the first inequality of  $\Gamma$ -convergence (3.10) for  $n \geq 2$ .

*Proof.* Let  $u \in \text{SBV}(\Omega) \cap L^\infty(\Omega)$  and  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  converging almost everywhere to  $(u, 0)$  such that  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$ . We have to prove

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq F(u, \mathbf{M}; \Omega). \quad (3.17)$$

We assume that  $\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega)$  is finite, otherwise the result is ensured. As  $\mathbf{M} \in W^{1, n+\alpha}(\Omega)$ , then  $\mathbf{M}$  is  $\alpha$ -Hölder regular. So, there exists  $\theta \geq 0$  such that, for any  $x, y \in \Omega$ , we have

$$\|\mathbf{M}(x) - \mathbf{M}(y)\| \leq \theta |x - y|^\alpha. \quad (3.18)$$

*Claim:* There exists  $C(\delta)$  such that

i)  $\lim_{\delta \rightarrow 0^+} C(\delta) = 1$ ,

ii) for  $A \subset \Omega$  open,  $a \in A$ ,  $\text{diam}(A) \leq \delta$  and  $\nu \in \mathbb{S}^{n-1}$ , we have

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A \langle \nabla u, \omega \rangle^2 dx + C(\delta) \int_{J_u \cap A} \frac{|\mathbf{M}(a)\nu|}{\langle \mathbf{M}(a)\nu, \nu \rangle^{1/2}} \langle \omega, \nu_u \rangle d\mathcal{H}^{n-1},$$

where  $\omega = \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}$ .

We denote by  $A$  an arbitrary open subset of  $\Omega$  such that  $\text{diam}(A) \leq \delta$  and we fix  $a \in A$ . Let  $\nu \in \mathbb{S}^{n-1}$  be fixed. According to Hölder regularity (3.18) and ellipticity inequality (2.1), we have

$$\begin{aligned} \forall (x, \mathbf{v}) \in A \times \mathbb{R}^n, \quad |\langle \mathbf{M}(x)\mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v} \rangle| &\leq \theta \delta^\alpha |\mathbf{v}|^2, \\ &\leq \theta \delta^\alpha \lambda^{-1} \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

Then, we get

$$\forall (x, \mathbf{v}) \in A \times \mathbb{R}^n, \quad \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v} \rangle \geq (1 - \theta \delta^\alpha \lambda^{-1}) \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v} \rangle$$

We set  $C(\delta) = 1 - \theta \delta^\alpha \lambda^{-1}$ . Then, we may write

$$\forall x \in A, \quad \langle \mathbf{M}(x)\nabla z_k(x), \nabla z_k(x) \rangle^2 \geq C(\delta) \langle \mathbf{M}(a)\nabla z_k(x), \nabla z_k(x) \rangle^2.$$

As  $\mathbf{M}(a)$  is a symmetric definite positive matrix, Cauchy-Schwartz inequality gives

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{M}(a)\nu, \nu \rangle \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v} \rangle \geq \langle \mathbf{M}(a)\nu, \mathbf{v} \rangle^2,$$

which is equivalent to

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v} \rangle \geq \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu \rangle} \left\langle \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}, \mathbf{v} \right\rangle^2. \quad (3.19)$$

We set  $\omega = \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}$ . If we apply inequality (3.19) to  $F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A)$ , we have

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A \left( |\nabla u_k|^2 (1 - z_k^2)^2 + C(\delta) \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu \rangle} \varepsilon_k \langle \omega, \nabla z_k \rangle^2 + \frac{z_k^2}{4\varepsilon_k} \right) dx.$$

With the notation introduced in (1.2),  $(v)_y$  is the function defined on  $A_\omega^y$  as  $(v)_y(t) = v(y + t\omega)$ . According to Lemma 3.1, we have  $\nabla(u_k)_y(t) = \langle \nabla u(y + t\omega), \omega \rangle$  and  $\nabla(z_k)_y(t) = \langle \nabla z(y + t\omega), \omega \rangle$ , so Fubini Theorem gives

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} \int_{A_\omega^y} \left( |\nabla(u_k)_y|^2 (1 - ((z_k)_y)^2)^2 + C(\delta) \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu \rangle} \varepsilon_k |\nabla(z_k)_y|^2 + \frac{((z_k)_y)^2}{4\varepsilon_k} \right) dt d\mathcal{H}^{n-1}(y).$$

With the one-dimensional notations (3.12), it gives

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} G_{\varepsilon_k}((u_k)_y, (z_k)_y; A_\omega^y) d\mathcal{H}^{n-1}(y),$$

where  $m = C(\delta) \frac{\langle \mathbf{M}(a)\nu \rangle^2}{\langle \mathbf{M}(a)\nu, \nu \rangle}$  for any  $x \in A$ . Fatou lemma yields

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} \liminf_{k \rightarrow \infty} G_{\varepsilon_k}((u_k)_y, (z_k)_y; A_\omega^y) \, d\mathcal{H}^{n-1}(y)$$

and then

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} G_-((u)_y; A_\omega^y) \, d\mathcal{H}^{n-1}(y).$$

As  $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A)$  is finite, we deduce that  $G_-((u)_y; A_\omega^y)$  is finite for  $\mathcal{H}^{n-1}$  almost every  $y \in A_\omega$ . We may apply Proposition 3.4 with  $I = A_\omega^y$  and  $u = (u)_y$ , it gives that  $(u)_\omega^y \in \text{SBV}(A_\omega^y)$  for  $\mathcal{H}^{n-1}$  almost every  $y \in A_\omega$  and we have

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} \left[ \int_{A_\omega^y} |\nabla(u)_y|^2 dt + \mathcal{H}^0(J_{(u)_y} \cap A_\omega^y) m^{\frac{1}{2}} \right] \, d\mathcal{H}^{n-1}(y).$$

As  $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A)$  is finite, Theorem 1.3 implies

$$\int_{A_\omega} \left[ \int_{A_\omega^y} |\nabla(u)_y|^2 dt + \mathcal{H}^0(J_{(u)_y} \cap A_\omega^y) m^{1/2} \right] \, d\mathcal{H}^{n-1}(y) = \int_\Omega |\langle \nabla u, \omega \rangle|^2 dx + \int_{J_u \cap A} m^{1/2} \langle \omega, \nu_u \rangle \, d\mathcal{H}^{n-1}.$$

We deduce

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A |\langle \nabla u, \omega \rangle|^2 dx + \int_{J_u \cap A} m^{1/2} \langle \omega, \nu_u \rangle \, d\mathcal{H}^{n-1}.$$

If we replace  $m$  and  $\omega$  by their values, it gives

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A \langle \nabla u, \omega \rangle^2 dx + C(\delta) \int_{J_u \cap A} \frac{\langle \mathbf{M}(a)\nu, \nu_u \rangle}{\langle \mathbf{M}(a)\nu, \nu \rangle^{1/2}} \, d\mathcal{H}^{n-1}$$

and the *Claim* is proved.

The function  $x \rightarrow \frac{\nabla u(x)}{|\nabla u(x)|}$  is measurable in  $U = \{x \in \Omega : \nabla u(x) \neq 0\}$ . According to Lusin Theorem (1.45 of [7]), there exists an increasing sequence of compacts  $(K_l)_l \subset U$  such that

$$\begin{cases} \mathcal{L}^n(U \setminus K_l) \leq \frac{1}{l}, \\ x \rightarrow \frac{\nabla u(x)}{|\nabla u(x)|} \text{ is continuous in } K_l. \end{cases}$$

Thus, for any  $x \in K_l$ , there exists  $r > 0$  such that

$$y \in B(x, r) \quad \Rightarrow \quad \left| \frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|} \right| \leq \frac{1}{l}, \quad (3.20)$$

As a consequence of Besicovitch Covering Theorem (2.18 of [7]), there exists a countable, pairwise disjoint collection of balls  $(B_i)_{i \in I}$  satisfying (3.20) such that

$$\forall i \in I, B_i \subset \Omega, \text{diam}(B_i) \leq \delta, \quad \mathcal{L}^n \left( K_l \setminus \bigcup_{i \in I} B_i \right) = 0.$$

For any  $i \in I$ , we fix  $x_i \in B_i$  and we set  $\nu_i = \frac{(\mathbf{M}(a))^{-1} \nabla u(x_i)}{|\mathbf{M}(a)|^{-1} |\nabla u(x_i)|}$ . According to *First Step*, with  $A = B_i$ ,  $a = x_i$  and  $\nu = \nu_i$ , we get

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; B_i) \geq \int_{B_i} \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 \, dx.$$

Moreover, we have

$$\begin{aligned} \forall x \in B_i \cap K_l, \quad \left| |\nabla u|^2 - \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 \right| &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} \rangle^2 - \langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \rangle^2 \right| \\ &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} - \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \rangle \langle \nabla u, \frac{\nabla u}{|\nabla u|} + \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \rangle \right| \\ &\leq \frac{2}{l} |\nabla u|^2. \end{aligned}$$

It gives

$$\int_{B_i \cap K_l} \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 dx \geq \frac{l}{l+2} \int_{B_i \cap K_l} |\nabla u|^2 dx.$$

As  $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \cdot)$  is superadditive and non decreasing, we have

$$\begin{aligned} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) &\geq \sum_{i \in I} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; B_i), \\ &\geq \sum_{i \in I} \int_{B_i} \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 dx, \\ &\geq \sum_{i \in I} \int_{B_i \cap K_l} \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 dx, \\ &\geq \frac{l}{l+2} \sum_{i \in I} \int_{B_i \cap K_l} |\nabla u|^2 dx, \\ &\geq \frac{l}{l+2} \int_{\cup_i B_i \cap K_l} |\nabla u|^2 dx, \end{aligned}$$

As  $\mathcal{L}^n(K_l \setminus \cup_i B_i) = 0$ , we deduce

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq \frac{l}{l+2} \int_{K_l} |\nabla u|^2 dx$$

and taking the limit  $l \rightarrow \infty$  gives

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq \int_U |\nabla u|^2 dx = \int_{\Omega} |\nabla u|^2 dx.$$

In particular,  $\int_{\Omega} |\nabla u|^2 dx$  is finite. As  $u$  belongs to  $\text{SBV}(\Omega)$ , according to Theorem 1.1, there exists a pairwise disjoint family  $(C_i)_{i \in \mathbb{N}}$  of  $\mathcal{C}^1$  compact manifolds and  $M \in \Omega$  such that:

$$J_u = \mathcal{N} \cup \left( \bigcup_{i \in \mathbb{N}} C_i \right), \quad \mathcal{H}^{n-1}(\mathcal{N}) = 0.$$

As  $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega)$  is finite, *First Step* and Theorem 1.3 imply that  $\mathcal{H}^{n-1}(J_u)$  is also finite. According to ellipticity inequality (2.1), we deduce that  $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}$  is finite. Then, for a fixed  $\delta > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\int_{J_u \setminus \bigcup_{i=1}^N C_i} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \delta. \quad (3.21)$$

We set  $K = \bigcup_{i=1}^N C_i$  and  $K_\tau = \{x \in \Omega : \text{dist}(x, K) < \tau\}$ . As  $\int_{\Omega} |\nabla u|^2 dx$  is finite, there exists  $\tau > 0$  such that

$$\int_{\Omega \setminus K_\tau} |\nabla u|^2 dx \leq \delta. \quad (3.22)$$

With the same arguments as before, we get

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega \setminus \overline{K_\tau}) \geq \int_{\Omega \setminus \overline{K_\tau}} |\nabla u|^2 dx. \quad (3.23)$$

As  $x \rightarrow \frac{\mathbf{M}(x)\nu_u(x)}{\langle \mathbf{M}(x)\nu_u(x), \nu_u(x) \rangle^{1/2}}$  is continuous in  $K$ , for any  $x \in K$  there exists  $r > 0$  such that

$$y \in B(x, r) \cap K \Rightarrow \left| \frac{\mathbf{M}(x)\nu_u(x)}{\langle \mathbf{M}(x)\nu_u(x), \nu_u(x) \rangle^{1/2}} - \frac{\mathbf{M}(y)\nu_u(y)}{\langle \mathbf{M}(y)\nu_u(y), \nu_u(y) \rangle^{1/2}} \right| \leq \delta. \quad (3.24)$$

As a consequence of Besicovitch Covering Theorem (2.18 of [7]), there exists a countable, pairwise disjoint, collection of balls  $(\widetilde{B}_j)_{j \in \tilde{I}}$  satisfying (3.24) such that, for any  $j \in \tilde{I}$ , it satisfies

$$\widetilde{B}_j \subset K_\tau, \quad \text{diam}(\widetilde{B}_j) \leq \delta, \quad \mathcal{H}^{n-1} \left( K \setminus \bigcup_{j \in \tilde{I}} \widetilde{B}_j \right) = 0.$$

For any  $j \in \tilde{I}$ , we fix  $\tilde{x}_j \in \widetilde{B}_j$ . According to *First Step*, with  $A = \widetilde{B}_j$ ,  $a = \tilde{x}_j$  and  $\nu = \nu_u(\tilde{x}_j)$ , we get

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \widetilde{B}_j) \geq C(\delta) \int_{J_u \cap \widetilde{B}_j} \frac{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u \rangle}{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j) \rangle^{1/2}} d\mathcal{H}^{n-1}.$$

For any  $x \in \widetilde{B}_j \cap K$ , we have

$$\left| \frac{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u(x) \rangle}{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j) \rangle^{1/2}} - \langle \mathbf{M}(x) \nu_u(x), \nu_u(x) \rangle^{1/2} \right| \leq \left| \frac{\mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j)}{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j) \rangle^{1/2}} - \frac{\mathbf{M}(x) \nu_u(x)}{\langle \mathbf{M}(x) \nu_u(x), \nu_u(x) \rangle^{1/2}} \right| \leq \delta.$$

It gives

$$\int_{\widetilde{B}_j \cap K} \frac{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u \rangle}{\langle \mathbf{M}(\tilde{x}_j) \nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j) \rangle^{1/2}} d\mathcal{H}^{n-1} \geq \int_{\widetilde{B}_j \cap K} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} - \delta \mathcal{H}^{n-1}(\widetilde{B}_j \cap K).$$

As  $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \cdot)$  is superadditive and non decreasing, we have

$$\begin{aligned} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; K_\tau) &\geq \sum_{j \in \tilde{I}} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \widetilde{B}_j), \\ &\geq \sum_{j \in \tilde{I}} C(\delta) \left( \int_{\widetilde{B}_j \cap K} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} - \delta \mathcal{H}^{n-1}(\widetilde{B}_j \cap K) \right), \\ &\geq C(\delta) \left( \int_{\cup_j \widetilde{B}_j \cap K} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} - \delta \mathcal{H}^{n-1}(\cup_j \widetilde{B}_j \cap K) \right). \end{aligned}$$

As  $\mathcal{L}^n(K \setminus \widetilde{B}_j) = 0$ , we get

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; K_\tau) \geq C(\delta) \left( \int_K \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} - \delta \mathcal{H}^{n-1}(K) \right). \quad (3.25)$$

According to (3.23) and (3.25), we deduce

$$\begin{aligned} \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) &\geq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; K_\tau) + \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega \setminus \overline{K_\tau}), \\ &\geq C(\delta) \left( \int_K \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} - \delta \mathcal{H}^{n-1}(K) \right) + \int_{\Omega \setminus \overline{K_\tau}} |\nabla u|^2 dx. \end{aligned}$$

According to (3.22) and (3.21), we have

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq C(\delta) \left( \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} - \delta - \delta \mathcal{H}^{n-1}(K) \right) + \int_{\Omega} |\nabla u|^2 dx - \delta.$$

Letting  $\delta \rightarrow 0^+$  concludes the proof

$$\liminf_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} + \int_{\Omega} |\nabla u|^2 dx.$$

□

### 3.3.2 The inequality for the *higher* $\Gamma$ -limit

In this section we prove the *upper* inequality of  $\Gamma$ -convergence, that is ii) of Theorem 3.2. Let  $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$  and  $\mathbf{M} \in W^{1, n+\alpha}(\Omega; \mathcal{G})$  be fixed. It is sufficient to prove that there exists a sequence  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  such that  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere,  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$  and

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 (1 - z_k^2)^2 dx + \int_{\Omega} \left( \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k \rangle + \frac{z_k^2}{4\varepsilon_k} \right) dx \leq \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}.$$

We first prove a weaker result, where  $\int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}$  is replaced by its approximation with a Minkowski content. Then, with Theorem 1.4, we extend this result to the general setting.

## Approximation with anisotropic Minkowski content

We set

$$F_{\varepsilon_k}(u, z, \mathbf{M}) = \int_{\Omega} |\nabla u|^2 (1 - z^2)^2 dx + \int_{\Omega} \left( \varepsilon_k \langle \mathbf{M} \nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon_k} \right) dx$$

and we prove the following

**Proposition 3.5.** *Assume that the conditions of Theorem 3.1 are satisfied. For  $u \in SBV(\Omega) \cap L^\infty(\Omega)$  and  $\mathbf{M} \in W^{1, n+\alpha}(\Omega; \mathcal{G})$ , there exists a sequence  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  such that  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$  and*

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq \int_{\Omega} |\nabla u|^2 dx + \mathcal{M}_{\mathbf{M}}^*(J_u),$$

where  $\mathcal{M}_{\mathbf{M}}^*$  is defined in (1.3).

*Proof.* Without loss of generality, we may assume that  $|\nabla u| \in L^2(\Omega)$  and  $\mathcal{M}_{\mathbf{M}}^*(J_u) < +\infty$ , otherwise the result is obvious. If  $u \in W^{1,2}(\Omega)$ , then  $J_u = \emptyset$  and the stationary sequence  $u_k = u, z_k = 0$  is a solution. If  $u \notin W^{1,2}(\Omega)$ , then  $J_u \neq \emptyset$  and  $(1 - z_k^2)^2$  has to be infinitesimal near of  $J_u$ . For  $\rho > 0$ , we set

$$(J_u)_\rho = \{x: d_\phi^{J_u}(x) < \rho\}.$$

We separate  $\Omega$  in three parts:

$$(J_u)_{b_k}, \quad (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}, \quad \Omega \setminus (J_u)_{a_k+b_k}$$

with

$$\begin{cases} a_k &= -4\varepsilon_k \ln(\varepsilon_k), \\ b_k &= \varepsilon_k^2. \end{cases} \quad (3.26)$$

Let  $\Psi_k \in \mathcal{C}_0^\infty(\Omega)$  such that  $\Psi_k = 1$  in  $(J_u)_{\frac{b_k}{2}}$  and  $\Psi_k = 0$  in  $\Omega \setminus (J_u)_{b_k}$ . We set  $u_k = (1 - \Psi_k)u$  and then  $u_k = u$  in  $\Omega \setminus (J_u)_{b_k}$ . As  $(b_k)_k$  converges to 0 then  $u_k$  converges to  $u$  almost everywhere. We set  $z_k = 1$  in  $(J_u)_{b_k}$  and  $z_k = \varepsilon_k^2$  in  $\Omega \setminus (J_u)_{a_k+b_k}$ . In  $(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$  we adopt the following construction: we introduce

$$\theta_k(t) = \varepsilon_k^2 \exp\left(\frac{t}{2\varepsilon_k}\right)$$

and we set

$$\tilde{z}_k(t) = \begin{cases} 1 & \forall t \in [0; b_k], \\ \theta_k(a_k + b_k - t) & \forall t \in ]b_k; a_k + b_k], \\ \varepsilon_k^2 & \forall t \in ]a_k + b_k; +\infty[. \end{cases} \quad (3.27)$$

This is a continuous and decreasing function defined on  $[0; +\infty[$ , moreover, for any  $t \in ]b_k; a_k + b_k[$ , it satisfies

$$\varepsilon_k (\tilde{z}'_k(t))^2 = \frac{(\tilde{z}_k(t))^2}{4\varepsilon_k}. \quad (3.28)$$

We set  $z_k = \tilde{z}_k \circ d_\phi^{J_u}$ . As  $z_k$  is constant in  $(J_u)_{b_k} \cup (\Omega \setminus (J_u)_{a_k+b_k})$ , we have

$$\begin{aligned} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) &= \int_{\Omega \setminus (J_u)_{a_k+b_k}} |\nabla u|^2 (1 - \varepsilon_k^4)^2 dx + \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} |\nabla u|^2 (1 - z_k^2)^2 dx \\ &+ \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left( \varepsilon_k \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle + \frac{z_k^2}{4\varepsilon_k} \right) dx \\ &+ \frac{\varepsilon_k^3}{4} \mathcal{L}^n(\Omega \setminus (J_u)_{a_k+b_k}) + \frac{1}{4\varepsilon_k} \mathcal{L}^n((J_u)_{b_k}) \end{aligned} \quad (3.29)$$

As  $|\nabla u| \in L^2(\Omega)$  and  $(a_k + b_k)_k$  converges to 0, the first term of (3.29) converges to  $\int_{\Omega} |\nabla u|^2 dx$ . As  $\|z_k\|_{L^\infty} \leq 1$ , the second term converges to 0. As  $\Omega$  is a bounded domain, the fourth term converges to 0. As  $\mathcal{M}_{\mathbf{M}}^*(J_u) < +\infty$ , there exists  $(\omega_k)_k$  a sequence which converges to  $0^+$  such that

$$\mathcal{L}^n((J_u)_{b_k}) \leq 2b_k(\mathcal{M}_{\mathbf{M}}^*(J_u) + \omega_k) \quad (3.30)$$



and then the fifth term is lower than  $\frac{1}{2}\varepsilon_k(\mathcal{M}_{\mathbf{M}}^*(J_u) + \omega_k)$ . So, the fifth term converges to 0. To compute the limit of  $(F_{\varepsilon_k}(u_k, z_k, \mathbf{M}))_k$ , it remains to study the convergence of

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left( \varepsilon_k \langle \mathbf{M} \nabla z_k, \nabla z_k \rangle + \frac{z_k^2}{4\varepsilon_k} \right) dx.$$

Ellipticity inequality (2.1) yields

$$\begin{aligned} |d_\phi^{J_u}(x) - d_\phi^{J_u}(y)| &\leq d_\phi(x, y), \\ &\leq \lambda^{-\frac{1}{2}} |x - y|. \end{aligned}$$

So,  $d_\phi^{J_u}$  is Lipschitzian and Rademacher Theorem ensures that  $d_\phi^{J_u}$  exists for almost every  $x \in \Omega$ , in the sense of the approximate differentiability 3.2. Thus, for almost every  $x \in (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$ , we have

$$\nabla z_k = \tilde{z}'_k \circ d_\phi^{J_u} \nabla d_\phi^{J_u}.$$

It gives

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left( \varepsilon_k (\tilde{z}'_k \circ d_\phi^{J_u})^2 \langle \mathbf{M} \nabla d_\phi^{J_u}, \nabla d_\phi^{J_u} \rangle + \frac{(\tilde{z}_k \circ d_\phi^{J_u})^2}{4\varepsilon_k} \right) dx.$$

In [12], Theorem 3.2, it is proved that  $\langle \mathbf{M}(x) \nabla d_\phi^{J_u}(x), \nabla d_\phi^{J_u}(x) \rangle = 1$  for almost every  $x$ . So, we may write

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}} \left( \varepsilon_k (\tilde{z}'_k \circ d_\phi^{J_u})^2 + \frac{(\tilde{z}_k \circ d_\phi^{J_u})^2}{4\varepsilon_k} \right) \langle \mathbf{M} \nabla d_\phi^{J_u}, \nabla d_\phi^{J_u} \rangle^{1/2} dx.$$

We may apply Proposition 1.2 with  $\Phi = \phi$  and  $p = d_\phi^{J_u}$ , it gives

$$A_k(z_k) = \int_{b_k}^{a_k+b_k} \left( \varepsilon_k \tilde{z}'_k(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \left[ \int_{\Omega} \langle \mathbf{M} D\mathbf{1}_{(J_u)_t}, D\mathbf{1}_{(J_u)_t} \rangle^{1/2} \right] dt. \quad (3.31)$$

We set

$$\begin{cases} \mathcal{H}_{\mathbf{M}}(t) &= \int_{\Omega} \langle \mathbf{M} D\mathbf{1}_{(J_u)_t}, D\mathbf{1}_{(J_u)_t} \rangle^{1/2}, \\ \mathcal{A}_{\mathbf{M}}(s) &= \int_0^s \mathcal{H}_{\mathbf{M}}(t) dt. \end{cases}$$

Applying another time Proposition 1.2 gives

$$\begin{aligned} \mathcal{A}_{\mathbf{M}}(s_2) - \mathcal{A}_{\mathbf{M}}(s_1) &= \int_{s_1}^{s_2} \left[ \int_{\Omega} \langle \mathbf{M} D\mathbf{1}_{(J_u)_t}, D\mathbf{1}_{(J_u)_t} \rangle^{1/2} \right] dt, \\ &= \int_{(J_u)_{s_2} \setminus (J_u)_{s_1}} \langle \mathbf{M} \nabla d_\phi^{J_u}, \nabla d_\phi^{J_u} \rangle^{1/2} dx, \\ &= \mathcal{L}^n((J_u)_{s_2} \setminus (J_u)_{s_1}). \end{aligned}$$

So,  $\mathcal{A}_{\mathbf{M}} \in W_{\text{loc}}^{1,1}([0; +\infty[)$  and  $\nabla \mathcal{A}_{\mathbf{M}} = \mathcal{H}_{\mathbf{M}}$  almost everywhere. Using equality (3.28) and then integrating by parts (3.31) gives

$$\begin{aligned} A_k(z_k) &= \int_{b_k}^{a_k+b_k} \left( \varepsilon_k \tilde{z}'_k(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \mathcal{H}_{\mathbf{M}}(t) dt, \\ &= \int_{b_k}^{a_k+b_k} \frac{\tilde{z}_k(t)^2}{2\varepsilon_k} \mathcal{H}_{\mathbf{M}}(t) dt, \\ &= \frac{(a_k + b_k)^2}{2\varepsilon_k} \mathcal{A}_{\mathbf{M}}(a_k + b_k) - \frac{b_k}{2\varepsilon_k} \mathcal{A}_{\mathbf{M}}(b_k) - \frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}'_k(t) \tilde{z}_k(t) \mathcal{A}_{\mathbf{M}}(t). \end{aligned}$$

The first term obviously converges to 0. As for (3.30), we have

$$\mathcal{A}_{\mathbf{M}}(b_k) \leq 2b_k(\mathcal{M}_{\mathbf{M}}^*(J_u) + \omega_k)$$

and then the second term converges to 0 too. As  $s \rightarrow \mathcal{A}_{\mathbf{M}}(s)$  is non decreasing, then

$$\mathcal{A}_{\mathbf{M}}(t) \leq 2t(\mathcal{M}_{\mathbf{M}}^*(J_u) + \omega_k)$$

for any  $t \in [b_k; a_k + b_k]$ . For the last term, we apply another time this inequality, it gives

$$-\frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}'_k(t) \tilde{z}_k(t) \mathcal{A}_{\mathbf{M}}(t) dt \leq -\frac{(\mathcal{M}_{\mathbf{M}}^*(J_u) + \omega_k)}{\varepsilon_k} \int_{b_k}^{a_k+b_k} 2t \tilde{z}'_k(t) \tilde{z}_k(t) dt. \quad (3.32)$$

Integrating by parts yields

$$\int_{b_k}^{a_k+b_k} 2t \tilde{z}'_k(t) \tilde{z}_k(t) dt = (a_k + b_k) \tilde{z}_k(a_k + b_k)^2 - b_k \tilde{z}_k(b_k)^2 - \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2 dt. \quad (3.33)$$

According to the definitions of  $(a_k, b_k, z_k)$  (3.26) and (3.27), we have

$$(a_k + b_k) \tilde{z}_k(a_k + b_k)^2 - b_k \tilde{z}_k(b_k)^2 = o(\varepsilon_k) \quad (3.34)$$

and equation (3.28) gives

$$\begin{aligned} \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2 dt &= 2\varepsilon_k \int_{b_k}^{a_k+b_k} |\tilde{z}'_k(t)| \tilde{z}_k(t) dt, \\ &= \varepsilon_k (1 - \varepsilon_k^2). \end{aligned} \quad (3.35)$$

From (3.32), (3.33), (3.34) and (3.35) we deduce that  $\limsup_k A_k(z_k) \leq \mathcal{M}_{\mathbf{M}}^*(J_u)$  and, according to the decomposition (3.29), we have

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq \int_{\Omega} |\nabla u|^2 + \mathcal{M}_{\mathbf{M}}^*(J_u).$$

To conclude the proof, it suffices to notice that  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ . □

### Approximation in the general setting

The goal of this section is to replace  $\mathcal{M}_{\mathbf{M}}^*(J_u)$  by  $\int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}$  in Proposition 3.5.

**Proposition 3.6.** *Assume that the conditions of Theorem 3.1 are satisfied. For  $u \in SBV(\Omega) \cap L^\infty(\Omega)$  and  $\mathbf{M} \in W^{1, n+\alpha}(\Omega; \mathcal{G})$ , there exists a sequence  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  such that  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere  $(u_k)_k$  is bounded in  $L^\infty(\Omega)$  and*

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq F(u, \mathbf{M}).$$

To prove this result, we need to introduce the following

**Definition 3.2.** *Let  $\mathcal{F}(\Omega)$  be the set of functions  $u \in SBV(\Omega)$  for which, if  $F(u, \mathbf{M}) < +\infty$ , then there exists a sequence  $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$  converging almost everywhere to  $u$ ,  $\lim_{k \rightarrow \infty} F(u_k, \mathbf{M}) = F(u, \mathbf{M})$  and*

$$\forall k \in \mathbb{N}, \quad \mathcal{M}_{\mathbf{M}}(J_{u_k}) = \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle d\mathcal{H}^{n-1}.$$

*Proof.* Assume  $\mathcal{F}(\Omega) = SBV(\Omega)$ . According to Proposition 3.5, by a diagonal extraction we may exhibit a sequence  $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$  such that  $(u_k, z_k)_k$  converges to  $(u, 0)$  almost everywhere and

$$\limsup_{k \rightarrow \infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq F(u, \mathbf{M}).$$

So, to prove the upper inequality of  $\Gamma$ -convergence, it suffices to show that  $\mathcal{F}(\Omega) = SBV(\Omega)$ . We divide the proof in three *Claims*.

By a direct diagonal extraction process, we have the following.

*Claim 1:* *If  $u \in SBV(\Omega)$  and  $(u_k)_k \subset SBV(\Omega)$  satisfy*

*i)  $(u_k)_k \subset \mathcal{F}(\Omega)$ ,*

ii)  $\lim_{k \rightarrow \infty} F(u_k, \mathbf{M}) = F(u, \mathbf{M})$  and  $F(u, \mathbf{M}) < \infty$ ,  
iii)  $(u_k)_k$  converges to  $u$  almost everywhere,  
then  $u \in \mathcal{F}(\Omega)$ .

*Claim 2: It suffices to prove that  $SBV(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}(\Omega)$ .*

For  $u \in SBV(\Omega)$  and  $N > 0$ , we denote by  $\bar{u}^N$  the truncated function defined in (3.1). So,  $(\bar{u}^N)_N$  converges to  $u$  almost everywhere for  $N \rightarrow \infty$ . Moreover, Theorem 1.2 gives

$$D\bar{u}^N = \mathbf{1}_{-N \leq u \leq N} \nabla u \mathcal{L}^n + ((\bar{u}^N)^+ - (\bar{u}^N)^-) \nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

and then we deduce  $\lim_{N \rightarrow \infty} F(\bar{u}^N) = F(u, \mathbf{M})$ . According to *Claim 1*, it suffices to prove that  $SBV(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}(\Omega)$ .

*Claim 3: Let  $u \in SBV(\Omega) \cap L^\infty(\Omega)$ , we have  $u \in \mathcal{F}(\Omega)$ .*

We may assume that  $F(u, \mathbf{M}) < +\infty$ , otherwise the result is ensured. Let us extend  $\mathbf{M}$  and  $u$  in  $\Omega' = \Omega \cup U$  as in Proposition 1.1, so we have

$$\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0.$$

With the same arguments as for Theorem 2.1, we may prove that there exists  $v_k \in SBV(\Omega')$  a minimizer of the following functional:

$$E^{u,k}(v) = k \int_{\Omega'} (v - u)^2 dx + \int_{\Omega'} |\nabla v|^2 dx + \int_{J_v} \langle \mathbf{M} \nu_v, \nu_v \rangle^{1/2} d\mathcal{H}^{n-1}.$$

In particular,  $E^{u,k}(v_k) \leq E^{u,k}(u)$  for any  $k \in \mathbb{N}$ , gives

$$k \int_{\Omega'} (v_k - u)^2 dx \leq F(u, \mathbf{M})$$

and then  $(v_k)_k$  converges to  $u$  almost everywhere. As  $u \in L^\infty(\Omega)$ , Theorem 1.4 gives

$$\mathcal{M}_{\mathbf{M}}^*(J_{v_k}) = \int_{J_{v_k}} \langle \mathbf{M} \nu_{v_k}, \nu_{v_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

We introduce the sequence of positive Radon measures  $(\mu_k)_k$  and  $\mu$  defined for any  $B \in \mathcal{B}(\Omega')$  by

$$\begin{aligned} \mu_k(B) &= \int_B |\nabla v_k|^2 dx + \int_{J_{v_k} \cap B} \langle \mathbf{M} \nu_{v_k}, \nu_{v_k} \rangle^{1/2} d\mathcal{H}^{n-1}, \\ \mu(B) &= \int_B |\nabla u|^2 dx + \int_{J_u \cap B} \langle \mathbf{M} \nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1}. \end{aligned}$$

With the same arguments as for Theorem 2.1,  $F$  is lower semi-continuous in SBV, for any open  $A \subset \Omega'$ , it gives

$$\liminf_{k \rightarrow \infty} \mu_k(A) \geq \mu(A).$$

The inequality

$$\limsup_{k \rightarrow \infty} \mu_k(\Omega') \leq \mu(\Omega')$$

follows by the definition of  $v_k$ . According to [7], Proposition 1.80, the measures  $(\mu_k)_k$  weakly converge to  $\mu$ . Thus,  $(\mu_k(B))_k$  converges to  $\mu(B)$  if  $\mu(\partial B) = 0$ , and then  $(\mu_k(\Omega))_k$  converges to  $\mu(\Omega)$ , that is  $(F(v_k))_k$  converges to  $F(u, \mathbf{M})$ . According to *Claim 1*, we deduce that  $u \in \mathcal{F}(\Omega)$ .  $\square$

### 3.4 Proof of Theorem 3.1 ii)

*Proof.* With the same arguments as for Theorem 3.1 i), for any  $\varepsilon > 0$ , there exists  $(u_\varepsilon, z_\varepsilon)$  a minimizer of  $E_\varepsilon(\cdot, \cdot, \mathbf{M})$ . According to (3.3), with  $N \geq \|g\|_{L^\infty(\Omega)}$ , we have

$$\mathcal{L}^n(\{x \in \Omega: |u_\varepsilon(x)| > N\}) > 0 \quad \Rightarrow \quad E_\varepsilon(\bar{u}_\varepsilon^N, z_\varepsilon, \mathbf{M}) < E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}).$$

We deduce that  $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq N$  for any  $\varepsilon > 0$ . For  $\omega_\varepsilon = u_\varepsilon(1 - z_\varepsilon^2)$ , we get

$$\nabla \omega_\varepsilon = \nabla u_\varepsilon(1 - z_\varepsilon^2) - 2u_\varepsilon z_\varepsilon \nabla z_\varepsilon.$$

It yields

$$\int_\Omega |\nabla \omega_\varepsilon| dx \leq \mathcal{L}^n(\Omega)^{\frac{1}{2}} \left( \int_\Omega |\nabla u_\varepsilon|^2 (1 - z_\varepsilon^2)^2 dx \right)^{\frac{1}{2}} + 2N \int_\Omega |\nabla z_\varepsilon| z_\varepsilon dx. \quad (3.36)$$

Applying the inequality  $2ab \leq a^2 + b^2$  with  $a = \frac{z_\varepsilon^2}{2\varepsilon^{\frac{1}{2}}}$  and  $b = \varepsilon^{\frac{1}{2}} |\nabla z_\varepsilon|$  gives

$$\int_\Omega |\nabla z_\varepsilon| z_\varepsilon dx \leq \int_\Omega \varepsilon |\nabla z_\varepsilon|^2 dx + \int_\Omega \frac{z_\varepsilon^2}{4\varepsilon} dx. \quad (3.37)$$

According to ellipticity inequality (2.1), we get

$$\int_\Omega \varepsilon |\nabla z_\varepsilon|^2 dx \leq \frac{1}{\lambda} E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}). \quad (3.38)$$

By (3.36), (3.37) and (3.38), we deduce

$$\int_\Omega |\nabla \omega_\varepsilon| dx \leq \mathcal{L}^n(\Omega)^{\frac{1}{2}} (E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}))^{\frac{1}{2}} + \left(1 + \frac{1}{\lambda}\right) E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}).$$

According to Proposition 3.6, we deduce that  $(E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M}))_k$  is a bounded sequence. So,  $(\omega_{\varepsilon_k})_k$  is bounded in  $BV(\Omega)$  and there exists a subsequence, still denoted by  $(\omega_{\varepsilon_k})_k$  which converges almost everywhere to  $\omega \in BV(\Omega)$ . As  $\int_\Omega z_k^2 dx \leq \varepsilon_k E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M})$ , then  $(z_k)_k$  converges to 0 in  $L^2(\Omega)$  and there exists a subsequence, still denoted by  $(z_k)_k$ , which converges almost everywhere to 0. As  $\omega_{\varepsilon_k} = u_{\varepsilon_k}(1 - z_{\varepsilon_k}^2)$ , then  $(u_{\varepsilon_k})_k$  converges almost everywhere to  $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ .

With the same arguments as for Theorem 2.1,  $E(\cdot, \mathbf{M})$  admits a minimizer  $v \in SBV(\Omega)$  and  $v \in L^\infty(\Omega)$ . According to Theorem 3.2 ii), there exists  $(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k})_k \subset \mathcal{D}_n(\Omega)$  such that  $(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k})_k$  converges to  $(v, 0)$  almost everywhere and

$$\limsup_{k \rightarrow \infty} E_{\varepsilon_k}(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k}, \mathbf{M}) \leq E(v, \mathbf{M}).$$

According to Theorem 3.2, i), we get

$$\liminf_{k \rightarrow \infty} E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M}) \geq E(u, \mathbf{M}).$$

As  $(u_{\varepsilon_k}, z_{\varepsilon_k})$  is a minimizer of  $E_{\varepsilon_k}$ , we have

$$\forall k \in \mathbb{N}, \quad E_{\varepsilon_k}(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k}, \mathbf{M}) \geq E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M}).$$

We conclude that  $E(v, \mathbf{M}) \geq E(u, \mathbf{M})$  and then  $(u, \mathbf{M})$  is also a minimizer of  $E$ .  $\square$

## 4 Appendix

### 4.1 Proof of Lemma 3.1

*Proof.* As  $\Omega$  is bounded, then  $W^{1,2}(\Omega) \subset SBV(\Omega)$  and then, according to Calderón-Zygmund Theorem (3.83 in [7]), the derivative in the Sobolev sense is equal to the approximate differential for almost every point in  $\Omega$ . Moreover, according to Theorem 3.107 of [7], for  $v \in SBV(\Omega)$  and  $\nu \in \mathbb{S}^{n-1}$ , we have

$$\langle \nabla v(x + t\nu), \nu \rangle = \nabla v_x(t) \text{ a.e. } t \in \Omega_x$$

for almost every  $x \in \Omega_\nu$ . So, we have

$$\begin{aligned}
v \in W^{1,2}(\Omega) &\Rightarrow \int_{\Omega} (v^2 + |\nabla v|^2) dx < \infty, \\
&\Rightarrow \int_{\Omega_\nu} \int_{\Omega_x} (v_x^2 + \langle \nabla v(x + t\nu), \nu \rangle^2) dt dx < \infty, \\
&\Rightarrow \int_{\Omega_\nu} \int_{\Omega_x} (v_x^2 + |\nabla v_x(t)|^2) dt dx < \infty, \\
&\Rightarrow v_x \in W^{1,2}(\Omega_x) \text{ for a.e. } x \in \Omega.
\end{aligned}$$

Applying this property with  $v = z$  and  $v = u(1 - z^2)$  gives the result of Lemma 3.1.  $\square$

## 4.2 Proof of Lemma 3.2

*Proof.* For  $a, b \subset I \setminus J$ , according to Theorem 2.8. of [7], there exists a unique function  $\tilde{u} \in \mathcal{C}([a; b])$  such that  $u(x) \in \tilde{u}(x)$  for  $\mathcal{L}^1$ -a.e.  $x \in I$  and then

$$u(x^-) := \lim_{y \rightarrow x^-} u(y) \quad \text{and} \quad u(x^+) := \lim_{y \rightarrow x^+} u(y)$$

exist for any  $x \in I$ . As  $J$  is finite, then  $W^{1,2}(I \setminus J) \subset W^{1,1}(I \setminus J)$ . According to the jump formula, it gives that  $Du \in \mathcal{M}(I)$  and we have the decomposition

$$Du = \nabla u \mathcal{L}^1 + (u(x^+) - u(x^-)) \mathcal{H}^0 \llcorner J.$$

So, we can conclude that  $u \in \text{SBV}(I)$  and  $J_u \subset J$ .  $\square$

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