Anisotropic Mumford-Shah Model and its approximation with Gamma-convergence

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Abstract

A new model is introduced for the segmentation problem of thin structures, like tubes or thin plates, in an image. The energy is based on the Mumford-Shah model and it introduces as a new variable a continuous and anisotropic perturbation of the Hausdorff measure. A relaxed formulation in the special space of functions with bounded variations is given and the existence of a solution is established. In order to get an energy more adapted for numerics, an approximation with Γ-convergence and its complete proof are given.

Introduction

This work is motivated by the problem of segmentation of sets strongly elongated in some directions as, for example, tubes or thin plates in an image of dimension $n \in \{2;3\}$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $g \in L^\infty(\Omega)$. We denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure. The model we introduce in this paper consists in minimizing

$$\mathcal{E}(u,K,\mathbf{M}) = \int_{\Omega\setminus K}(u-g)^2\mathrm{d}x + \int_{\Omega\setminus K}|\nabla u|^2\mathrm{d}x + \int_K \langle \mathbf{M}\nu,\nu\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{M}\|^{n+\alpha}\mathrm{d}x \quad (0.1)$$

where K is compact and $\mathcal{H}^{n-1}$-rectifiable with unitary normal vector ν, $\alpha > 0$ and $\mathbf{M}$ takes its values in a compact subset $\mathcal{G}$ of symmetric definite positive matrices. The associated minimizing problem is

$$(\mathcal{P}): \quad \min\{\mathcal{E}(u,K,\mathbf{M}) \colon K \text{compact and } \mathcal{H}^{n-1}\text{-rectifiable},\ u \in W^{1,2}(\Omega\setminus K),\ \mathbf{M} \in W^{1,n+\alpha}(\Omega;\mathcal{G})\}.$$

If we consider $\mathbf{M} \equiv \mathrm{Id}_n$, we recognize the well-known Mumford-Shah model (see [1] for the seminal paper). In this sense, our model is the anisotropic version of the Mumford-Shah energy.

At any point $x \in K$, the directions associated to the main eigenvalues of $\mathbf{M}(x)$ must represent the directions of elongation of the set K. For example, if we want to detect a tube in a two-dimensional image, we may consider $\mathcal{G}$ as the subset of symmetric definite positive matrices with fixed eigenvalues $\{1;\mu\}$, such that $1 \ll \mu$. For the detection of thin plates in a three dimensional image, we may consider the fixed eigenvalues $\{1;\mu;\mu\}$. By this way, the third term of (0.1) will force ν to be in the same direction as the first eigenvector and then K to be elongated in the orthogonal directions. Moreover, we set $\alpha > 0$ in the last term of (0.1) in order to force $\mathbf{M}$ to be at least continuous (α-Hölder), which corresponds to the assumption that the image g admits a local *geometrical coherence*.

This model does not consist in the detection of sets with codimension higher than one, as it has been done in [2] and [3] for vector-valued functions. In our case, the sets we want to detect are with (small) positive volume.

In order to prove that $(\mathcal{P})$ is a well posed problem, for $u \in \mathrm{SBV}(\Omega)$, we introduce

$$E(u,\mathbf{M}) = \int_\Omega (u-g)^2\mathrm{d}x + \int_\Omega |\nabla u|^2\mathrm{d}x + \int_{J_u}\langle \mathbf{M}\nu_u,\nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \|D\mathbf{M}\|^{n+\alpha}\mathrm{d}x \quad (0.2)$$

where ∇u is the derivative of u with respect to the Lebesgue measure, J_u is its jump set and ν_u is unitary and orthogonal vector to J_u. The associated minimizing problem is

$$(\mathcal{P}'): \quad \min\{E(u,\mathbf{M}) \colon u \in \mathrm{SBV}(\Omega), \mathbf{M} \in \mathrm{W}^{1,n+\alpha}(\Omega;\mathcal{G})\},$$

We verify that $(\mathcal{P}')$ admits a solution and, with a regularity result of the jump set of a minimizer, which has been established in a joint paper [4], we prove that a minimizer of $(\mathcal{P}')$ naturally provides a solution for $(\mathcal{P})$. In order to get a functional more adapted for a numerical implementation, we will approximate (0.2) by a family $(E_\varepsilon)_\varepsilon$ which only depends on the integration with respect to the Lebesgue measure. More precisely, we set

$$E_\varepsilon(u,z,\mathbf{M}) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \langle \mathbf{M}\nabla z, \nabla z\rangle + \frac{z^2}{4\varepsilon}\right) \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x.$$

As it has been done in [5] for the initial Mumford-Shah model, the function z takes its values in $[0;1]$ and plays the role of a control on the gradient of u. The approximation takes place for $\varepsilon \to 0^+$ in the sense of the Γ-convergence.

In section 1 we recall some results on spaces of functions with bounded variation and we mention a regularity result that we have proved in a previous article. In section 2, we prove that the problem admits a solution. Section 3 in completely devoted to the approximation process. We introduce the family of functionals $(E_\varepsilon)_\varepsilon$ with their domains and we give the complete proof of Γ-convergence to E when $\varepsilon \to 0^+$.

1 Functional framework and regularity result

We adopt the notations:

- $\langle \mathbf{v}_1, \mathbf{v}_2\rangle \in \mathbb{R}$ for the canonical scalar product of $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$,
- $|\mathbf{v}|$ for the euclidean norm of $\mathbf{v} \in \mathbb{R}^n$,
- $\|\mathbf{M}\|$ for the induced norm of $\mathbf{M} \in \mathrm{M}_n(\mathbb{R})$,
- $\bigwedge_{i=1}^{n-1} \mathbf{v}_i \in \mathbb{R}^n$ for the canonical vectorial product of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathbb{R}^n$,
- dist for the euclidean distance in $\mathbb{R}^n$,
- $\mathrm{S}_n^+(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ for the subset of symmetric definite positive matrices,
- $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ for the subset of invertible matrices,
- $\mathrm{O}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ for the subgroup of orthogonal matrices,
- $\mathcal{B}(\Omega)$ the class of Borelian subsets of Ω,
- $\mathbb{B}(\Omega)$ for the space of Borelian functions defined in Ω,
- $\mathcal{L}^n$ for the Lebesgue measure in $\mathbb{R}^n$,
- $\mathcal{H}^k$ for the k-dimensional Hausdorff measure,
- $\mathcal{M}(\Omega;\mathbb{R}^n)$ for the space of vectorial Radon measures defined in $\Omega \subset \mathbb{R}^n$,
- $\fint_A f(x)\mathrm{d}x = \dfrac{1}{\mathcal{L}^n(A)} \displaystyle\int_A f(x)\mathrm{d}x$, for $A \in \mathcal{B}(\Omega)$ and $\mathcal{L}^n(A) > 0$.

Let f be a function defined on open sets, we adopt the following vocabulary:

- f is superadditive if $f(A \cup B) \geq f(A) + f(B)$ for any disjoints sets A, B,
- f is non decreasing if $f(A) \leq f(B)$ for any sets A, B such that $A \subset B$.

1.1 Functional spaces

We assume throughout this paper that the following constraint is satisfieded by Ω which is obviously satisfied in the context of applications in *Image Processing* because the domain is a parallelepiped.

Definition 1.1. *We say that $\Omega \subset \mathbb{R}^n$ satisfies the reflexion condition (R) if Ω is an open and bounded domain with Lipschitz regular boundary $\partial\Omega$ such that there exists a neighborhood U of $\partial\Omega$ and a bi-Lipschitzian homeomorphism $\varphi : U \cap \Omega \to U \setminus \overline{\Omega}$ such that, for any $x \in \partial\Omega$, we have*

$$\lim_{y \to x} \varphi(y) = x.$$

For the classical definitions and results on BV and SBV we refer to [7]. In particular, for the definition of weak* convergence in BV, we refer to [7] (Definition 1.58). However, for the need of notations, we mention that, if $u \in \mathrm{BV}(\Omega)$, then its derivative Du belongs to the space $\mathcal{M}(\Omega;\mathbb{R}^n)$ of vectorial Radon measure. Moreover, if $u \in \mathrm{SBV}(\Omega)$ then the Cantor part of Du is null and we obtain

$$Du = \nabla u \cdot \mathcal{L}^n + (u^+ - u^-)\nu_u \cdot \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the density of Du with respect to the Lebesgue measure $\mathcal{L}^n$, u^+ (resp. u^-) is the approximate upper (resp. lower) limit and $\mathcal{H}^{n-1}\llcorner J_u$ is the restriction of $\mathcal{H}^{n-1}$ to its jump set J_u.

Now, we focus on some results which will be useful throughout the paper. First, we need the rectifiability of J_u.

Theorem 1.1. *Let u be a given function in $BV(\Omega)$. There exists a countable family $(C_i)_{i\in\mathbb{N}}$ of compact $\mathcal{C}^1$-hypersurfaces such that*

$$J_u = \mathcal{N} \cup \left(\bigcup_{i\in\mathbb{N}} C_i \right),$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$.

Then, we will need the following chain rule ([7], Theorem 3.99).

Theorem 1.2. *Let $u \in SBV(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then, $v = f \circ u$ belongs to $SBV(\Omega)$ and*

$$Dv = f'(u)\nabla u \cdot \mathcal{L}^n + (f(u^+) - f(u^-))\nu_u \cdot \mathcal{H}^{n-1}\llcorner J_u. \tag{1.1}$$

The following is a straightforward consequence of [7], Corollary 3.89.

Proposition 1.1. *Let Ω be an open and bounded domain satisfying (R) and $\Omega' = U \cup \Omega$. For $u \in BV(\Omega)$, we consider an extension in Ω' by the following way*

$$\forall x \in U \setminus \overline{\Omega}, \quad u(x) = u(\varphi^{-1}(x)),$$

Then, we have

$$\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0.$$

We also need *slicing* results.

Definition 1.2. *Let $\nu \in \mathbb{S}^{n-1}$ be fixed. We denote by Π_ν the hyperplane*

$$\{x \in \mathbb{R}^n : x \cdot \nu = 0\}.$$

If $x \in \Pi_\nu$, we set

$$\begin{aligned} \Omega_x &= \{t \in \mathbb{R} : x + t\nu \in \Omega\}, \\ \Omega_\nu &= \{x \in \Pi_\nu : \Omega_x \neq \emptyset\}. \end{aligned}$$

For any function u defined on Ω and any $x \in \Omega_\nu$, we set

$$\begin{aligned} (u)_x : \Omega_x &\to \mathbb{R} \\ t &\mapsto u(x + t\nu). \end{aligned}$$

The following Theorem is proved in [8].

Theorem 1.3. *Let $u \in L^\infty(\Omega)$ be a function such that, for all $\nu \in \mathbb{S}^{n-1}$,*

i) $(u)_x \in SBV(\Omega_x)$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Omega_\nu$,

ii) $\displaystyle\int_{\Omega_\nu} \left[\int_{\Omega_x} |\nabla(u)_x| dt + \mathcal{H}^0(J_{(u)_x}) \right] d\mathcal{H}^{n-1}(x) < +\infty;$

then, $u \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. Conversely, let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$. Then i) and ii) are satisfied. Moreover, we have

iii) $\langle \nabla u(x + t\nu), \nu \rangle = \nabla(u)_x(t)$, for a.e. $t \in \Omega_x$ and $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$,

iv) $\displaystyle\int_{J_u} \langle \nu_u, \nu \rangle d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(J_{(u)_x}) d\mathcal{H}^{n-1}(x).$

We need a generalization of the Coarea formula. For that, we introduce the following

Definition 1.3. *The function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ is sublinear with respect to the second variable,if*

i) $f(x, \mathbf{v}_1 + \mathbf{v}_2) \leq f(x, \mathbf{v}_1) + f(x, \mathbf{v}_2)$ *for any* $(x, \mathbf{v}_1, \mathbf{v}_2) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n$,

ii) $f(x, t\mathbf{v}) = tf(x, \mathbf{v})$ *for any* $(x, \mathbf{v}, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+$.

Suppose that μ_1 is a Radon measure and μ_2 is a vectorial Radon measure on Ω. According to Besicovitch derivation Theorem,

$$\lim_{r\to 0} \frac{\mu_2(B(x,r))}{\mu_1(B(x,r))}$$

exists and is finite for μ_1 almost every x, we denote by $\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}(x)$ this limit when it exists. We recall that μ_2 is absolutely continuous with respect to μ_1 if $\mu_2(A) = 0$ whenever $\mu_1(A) = 0$. When this holds, we write $\mu_2 \ll \mu_1$. We consider the convex functional defined on the space $\mathcal{M}(\Omega; \mathbb{R}^n)$ by

$$\Phi : \mu_2 \in \mathcal{M}(\Omega; \mathbb{R}^n) \mapsto \int_\Omega f\left(x, \frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}\right) \mathrm{d}\mu_1, \tag{1.2}$$

where μ_1 is a positive measure such that $\mu_2 \ll \mu_1$. It is shown in [9] that the integral in (1.2) does not depend on the choice of μ_1. For that reason, we will write it in the condensed form

$$\Phi(\mu_2) = \int_\Omega f\,(x, \mu_2).$$

We give a variant of the Coarea formula extended to the sublinear functionals which can be found in [10].

Proposition 1.2. *Let $\Phi(x, s, v)$ a Borel function of $\Omega \times \mathbb{R} \times \mathbb{R}^n$ which is sublinear in v. Let p be a Lipschitz continuous function on Ω and, for $t > 0$, we set $S_t = \{x \in \Omega; p(x) < t\}$. Then, for almost all $t \in \mathbb{R}$, $\mathbf{1}_{S_t}$ belongs to $BV(\Omega)$ and we have*

$$\int_\Omega \Phi(x, p, Dp)\,dx = \int_{\mathbb{R}} dt \int_\Omega \Phi(x, t, D\mathbf{1}_{S_t}).$$

1.2 Minkowski content and regularity result for the jump set

For $\mathbf{M} : \Omega \to \mathrm{S}_n^+(\mathbb{R})$ and $(x, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$, we set

$$\phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}(x)\mathbf{v}, \mathbf{v}\rangle^{1/2}.$$

This functional is a Riemannian metric, so, for $S \subset \Omega$ and $x, y \in \Omega$, we may define its associated distance as

$$\begin{array}{rcl} \mathrm{dist}_\phi(x, y) & = & \inf\left\{ \displaystyle\int_0^1 \phi\left(\gamma, \frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) \mathrm{d}t \colon \begin{array}{c} \gamma \in W^{1,1}([0;1]; \mathbb{R}^n), \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}, \\ \mathrm{dist}_\phi(x, S) & = & \inf\left\{\mathrm{dist}_\phi(x, y) \colon y \in S\right\}. \end{array}$$

The associated anisotropic Minkowski $(n-1)$-dimensional upper and lower contents are defined by the limits

$$\mathcal{M}^\star_{\mathbf{M}}(S) = \limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(\{x \colon \mathrm{dist}_\phi(x, S) < \rho\})}{2\rho}, \quad \mathcal{M}_{\star\mathbf{M}}(S) = \liminf_{\rho\to 0^+} \frac{\mathcal{L}^n(\{x \colon \mathrm{dist}_\phi(x, S) < \rho\})}{2\rho}. \tag{1.3}$$

If $\mathcal{M}^\star_{\mathbf{M}}(S) = \mathcal{M}_{\star\mathbf{M}}(S)$, we call their common value the $(n-1)$-dimensional anisotropic Minkowski content $\mathcal{M}_{\mathbf{M}}(S)$. In [4], we have proved the following

Theorem 1.4. *Let $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ be continuous, $h \in L^\infty(\Omega)$, $\gamma > 0$ and $u \in SBV(\Omega)$ a minimizer of*

$$\left\{ E^{\gamma,h,\mathbf{M}}(v) = \gamma \int_\Omega (v-h)^2\,dx + \int_\Omega |\nabla v|^2\,dx + \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v\rangle^{1/2}\,d\mathcal{H}^{n-1} \colon v \in SBV(\Omega) \right\}.$$

Then, we have

$$\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0, \qquad \mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v\rangle^{1/2}\,d\mathcal{H}^{n-1}.$$

2 Existence result for $(\mathcal{P})$

In order to prove that $(\mathcal{P})$ admits a solution, we prove that $(\mathcal{P}')$ is well posed and that it provides a minimizer for $(\mathcal{P})$.

2.1 Existence result for the relaxed problem $(\mathcal{P}')$

The main result of the section is the following

Theorem 2.1. *The problem $(\mathcal{P}')$ admits a solution.*

The proof is a straightforward consequence of Lemmas 2.1 and 2.2.

Lemma 2.1. *Let $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$ and $(\mathbf{M}_k)_k \subset W^{1,n+\alpha}(\Omega;\mathcal{G})$ be such that*

$$\sup_k E(u_k, \mathbf{M}_k) < \infty.$$

Then, there exist subsequences, still denoted by $(u_k)_k$ and $(\mathbf{M}_k)_k$, and $(u, \mathbf{M}) \in SBV(\Omega) \times W^{1,n+\alpha}(\Omega;\mathcal{G})$, such that $(u_k)_k$ is weakly convergent to u in $SBV(\Omega)$ and $(\mathbf{M}_k)_k$ is weakly convergent to $\mathbf{M}$ in $W^{1,n+\alpha}(\Omega)$.*

Proof. As $\mathcal{G} \subset \mathrm{S}_n^+(\mathbb{R})$ is compact, there exists $0 < \lambda < \Lambda$ such that, for any $(M, \mathbf{v}) \in \mathcal{G} \times \mathbb{R}^n$, the following ellipticity condition is satisfied

$$\lambda|\mathbf{v}|^2 \leq \langle M\mathbf{v}, \mathbf{v}\rangle \leq \Lambda|\mathbf{v}|^2. \tag{2.1}$$

It yields

$$\int_\Omega |\nabla u_k|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k}) \leq \max\left\{1; \lambda^{-1/2}\right\} E(u_k, \mathbf{M}_k). \tag{2.2}$$

As $E(u_k, \mathbf{M}_k)$ is bounded, Theorem 4.8. in [7] implies that there exists a subsequence $(u_k)_k$ weakly* convergent in $\mathrm{BV}(\Omega)$ to $u \in \mathrm{SBV}(\Omega)$. According to (2.1), we have

$$\|\mathbf{M}\|^{n+\alpha}_{W^{1,n+\alpha}(\Omega)} \leq \Lambda^{(n+\alpha)/2} + E(u_k, \mathbf{M}_k),$$

so $(\mathbf{M_k})_\mathbf{k}$ is a bounded sequence of $W^{1,n+\alpha}(\Omega)$ and then there exists a subsequence which weakly converges to $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$. It suffices to verify that $\mathbf{M}$ takes its values in $\mathcal{G}$, which is true because $W^{1,n+\alpha}(\Omega)$ is injected in a continuous way into the space of continuous function and $\mathcal{G}$ is closed.

□

Lemma 2.2. *Let $(u_k)_k \subset SBV(\Omega)$ be weakly* convergent to $u \in SBV(\Omega)$ and $(\mathbf{M}_k)_k$ be weakly convergent to $\mathbf{M}$ in $W^{1,n+\alpha}(\Omega)$. Then, we have*

$$E(u, \mathbf{M}) \leq \liminf_{k\to\infty} E(u_k, \mathbf{M}_k).$$

Proof. We may assume that $\liminf E(u_k, \mathbf{M}_k) < +\infty$, otherwise the result is ensured. Weak* convergence in $\mathrm{SBV}(\Omega)$ and weak convergence in $W^{1,n+\alpha}(\Omega)$ give

$$\liminf_{k\to\infty} \int_\Omega (u_k - g)^2 \mathrm{d}x + \int_\Omega \|D\mathbf{M}_k\|^{n+\alpha} \mathrm{d}x \leq \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x. \tag{2.3}$$

According to inequality (2.2), $\int_\Omega |\nabla u_k|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k})$ is bounded with respect to k. With [7], Theorem 4.7., we deduce that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 \mathrm{d}x, \qquad \mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}). \tag{2.4}$$

According to (2.3) and (2.4), it remains to prove that

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

This result is proved in [7], Theorem 5.2., for an homogeneous and fixed media (if $\mathbf{M}_k$ does not depend on $x \in \Omega$ nor in k). In order to generalize this result, we introduce a piecewise constant approximation of $\mathbf{M}$. For any vector $e_i \in \mathbb{S}^{n-1}$ of the canonical basis, we set

$$\begin{cases} \Pi_t^i = \{x \in \Omega \colon \langle x, e_i\rangle = t\}, \\ N_t^i = \left\{t \in \mathbb{R} \colon \mathcal{H}^{n-1}(J_u \cap \Pi_t^i) > 0\right\} \cup \left\{t \in \mathbb{R} \colon \exists k \in \mathbb{N}, \mathcal{H}^{n-1}(J_{u_k} \cap \Pi_t^i) > 0\right\}. \end{cases}$$

As $\mathcal{H}^{n-1}(J_u) < \infty$ (2.4) and $\mathcal{H}^{n-1}(J_{u_k}) < \infty$ for any $k \in \mathbb{N}$, then N_t^i is at most countable. We fix $\eta > 0$ arbitrary small. Intersecting Ω with disjoint cubes with edges orthogonal to the axes of the canonical basis, there exists $\mathcal{A}$ a finite partition of Ω which, for any $A \in \mathcal{A}$ and for any $k \in \mathbb{N}$, satisfies

$$\operatorname{diam}(A) < \eta, \qquad \mathcal{H}^{n-1}(J_u \cap \partial A) = 0, \qquad \mathcal{H}^{n-1}(J_{u_k} \cap \partial A) = 0. \tag{2.5}$$

For any $A \in \mathcal{A}$, we fix one point $x_A \in A$ and we set $\mathbf{M}^{\mathcal{A}}(x) = \mathbf{M}(x_A)$ for any $x \in A$. We have the decomposition

$$\begin{aligned} \int_{J_{u_k}} \langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \;=\; & \int_{J_{u_k}} \left(\langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k}\rangle^{1/2}\right) \mathrm{d}\mathcal{H}^{n-1} \\ & + \int_{J_{u_k}} \left(\langle \mathbf{M} \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k}\rangle^{1/2}\right) \mathrm{d}\mathcal{H}^{n-1} \\ & + \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{2.6}$$

In the three following *Claim*, we will estimate the limit of those three terms.

Claim 1:

$$\lim_{k\to\infty} \int_{J_{u_k}} \left(\langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k}\rangle^{1/2}\right) d\mathcal{H}^{n-1} = 0.$$

According to Ellipticity inequality (2.1), we have

$$\begin{aligned} |\langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k}\rangle^{1/2}| \;&\leq\; \frac{1}{2\sqrt{\lambda}} |\langle (\mathbf{M}_k - \mathbf{M}) \nu_{u_k}, \nu_{u_k}\rangle|, \\ &\leq\; \frac{1}{2\sqrt{\lambda}} \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \end{aligned}$$

and then

$$\int_{J_{u_k}} |\langle \mathbf{M}_k \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M} \nu_{u_k}, \nu_{u_k}\rangle^{1/2}| \mathrm{d}\mathcal{H}^{n-1} \leq \frac{1}{2\sqrt{\lambda}} \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \mathcal{H}^{n-1}(J_{u_k}).$$

As the inclusion $W^{1,n+\alpha}(\Omega) \subset L^\infty(\Omega)$ is compact and $(\mathbf{M}_k)_k$ weakly converges to $\mathbf{M}$, then $(\mathbf{M}_k)_k$ uniformly converges to $\mathbf{M}$. Moreover, $(\mathcal{H}^{n-1}(J_{u_k}))_k$ is bounded, it concludes the proof of the claim.

Claim 2: The sequence

$$\lim_{\eta\to 0^+} \int_{J_{u_k}} \left(\langle \mathbf{M} \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k}\rangle^{1/2}\right) d\mathcal{H}^{n-1} = 0$$

and the convergence takes place uniformly with respect to $k \in \mathbb{N}$.

For $A \in \mathcal{A}$ and $x \in A$, ellipticity inequality (2.1) gives

$$|\langle \mathbf{M}(x) \nu_{u_k}, \nu_{u_k}\rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}}(x) \nu_{u_k}, \nu_{u_k}\rangle^{1/2}| \leq \frac{1}{2\sqrt{\lambda}} \|\mathbf{M}(x) - \mathbf{M}(x_A)\|.$$

As $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$, there exist a constant $C > 0$ such that, for any $x \in A$, we have

$$\|\mathbf{M}(x) - \mathbf{M}(x_A)\| \leq C\eta^\alpha.$$

It yields

$$\left| \int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) \mathrm{d}\mathcal{H}^{n-1} \right| \leq \frac{C\eta^{\alpha}\mathcal{H}^{n-1}(J_{u_k})}{2\sqrt{\lambda}}.$$

As $(\mathcal{H}^{n-1}(J_{u_k}))_k$ is a bounded sequence, it concludes the proof of the claim.

Claim 3:

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u \rangle^{1/2} d\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} d\mathcal{H}^{n-1}.$$

We denote by $\overset{\circ}{A}$ the interior of the set A. According to [7], Theorem 5.2., we have

$$\int_{J_u \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k} \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

According to (2.5), the contribution of the boundaries is null, it gives

$$\int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$$

and then

$$\begin{aligned} \sum_{A\in\mathcal{A}} \int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} &\leq \sum_{A\in\mathcal{A}} \liminf_{k\to\infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \\ &\leq \liminf_{k\to\infty} \sum_{A\in\mathcal{A}} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

As $\mathcal{A}$ is a partition of Ω, it concludes the proof of *Claim 3*:

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

Let $\delta > 0$ be an arbitrary small number. With the same arguments as for *Claim 2*, we get

$$\left| \int_{J_u} \left(\langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u \rangle^{1/2} \right) \mathrm{d}\mathcal{H}^{n-1} \right| \leq \frac{C\eta^{\alpha}\mathcal{H}^{n-1}(J_u)}{2\sqrt{\lambda}}.$$

So, according to *Claim 2*, there exists a partition $\mathcal{A}$ which satisfies

$$\begin{cases} \limsup_{k\to\infty} \left| \int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \right) \mathrm{d}\mathcal{H}^{n-1} \right| \leq \delta, \\ \left| \int_{J_u} \left(\langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} - \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u \rangle^{1/2} \right) \mathrm{d}\mathcal{H}^{n-1} \right| \leq \delta. \end{cases}$$

According to (2.6), *Claim 1* and *Claim 3*, we have

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq 2\delta + \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}_k\nu_{u_k}, \nu_{u_k} \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

As $\delta > 0$ is arbitrary, it concludes the proof of Lemma 2.2. □

We now prove Theorem 2.1.

Proof. We denote by $(u_k, \mathbf{M}_k)_k \subset \mathrm{SBV}(\Omega)$ a minimizing sequence for E. As $g \in L^{\infty}(\Omega)$, we set

$$\varphi(t) = \begin{cases} -\|g\|_{L^{\infty}(\Omega)} & \text{if } t \leq -\|g\|_{L^{\infty}(\Omega)}, \\ t & \text{if } |t| \leq \|g\|_{L^{\infty}(\Omega)}, \\ \|g\|_{L^{\infty}(\Omega)} & \text{if } t \geq \|g\|_{L^{\infty}(\Omega)}. \end{cases} \tag{2.7}$$

We denote $v_k = \varphi \circ u_k$. As the function φ is 1-Lipshitz, we may apply Theorem 1.2 and then $v_k \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$ with the decomposition

$$Dv_k = \varphi'(u_k)\nabla u_k \cdot \mathcal{L}^n + (\varphi(u_k^+) - \varphi(u_k^-))\nu_{u_k} \cdot \mathcal{H}^{n-1} \llcorner J_{u_k}.$$

For any k, it is easy to check that $E(v_k, \mathbf{M}_k) \leq E(u_k, \mathbf{M}_k)$, so $(v_k, \mathbf{M}_k)_k$ is also a minimizing sequence for E. According to Theorem 2.1, there exists $v \in \mathrm{SBV}(\Omega)$ and a subsequence, still denoted $(v_k)_k$ weakly* convergent to v. With Theorem 2.2, we have $E(v, \mathbf{M}) \leq \liminf E(v_k, \mathbf{M}_k)$ and then $(v, \mathbf{M})$ is a minimizer of E.

□

2.2 Existence result for $(\mathcal{P})$

In this section we prove that the problems $(\mathcal{P})$ and $(\mathcal{P}')$ have common solutions and then $(\mathcal{P})$ is also well posed. First, we consider $(u, K, \mathbf{M})$ in the domain of $\mathcal{E}$. As in (2.7), we may define the truncated function $v = \varphi(u)$ and then we have $|v - g| \leq |u - g|$ and $|\nabla v| \leq |\nabla u|$. In particular, we get $\mathcal{E}(v, K, \mathbf{M}) \leq \mathcal{E}(u, K, \mathbf{M})$. Thus, for any $(u_\star, K_\star, \mathbf{M}_\star)$ in the domain of $\mathcal{E}$, we may assume that $u_\star \in L^\infty(\Omega \setminus K)$. On the other hand, in [7] (Proposition 4.4), is given the following

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $K \subset \mathbb{R}^n$ be closed, $\mathcal{H}^{n-1}(K \cap \Omega) < \infty$ and $u \in L^\infty(\Omega \setminus K) \cap W^{1,1}(\Omega \setminus K)$. Then, we have $u \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(S_u \setminus K) = 0$.*

As Ω is bounded and $u_\star \in W^{1,2}(\Omega \setminus K_\star)$, then we get $u_\star \in W^{1,1}(\Omega \setminus K_\star)$. As $\int_{K_\star} \langle \mathbf{M}_\star \nu, \nu \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} < +\infty$, then Ellipticity condition (2.1) gives $\mathcal{H}^{n-1}(K_\star \cap \Omega) < \infty$. According to Proposition 2.1, we deduce that $u_\star \in \mathrm{SBV}(\Omega)$ and $\mathcal{H}^{n-1}(S_{u_\star} \setminus K_\star) = 0$. It yields

$$E(u_\star, \mathbf{M}_\star) \leq \mathcal{E}(u_\star, K_\star, \mathbf{M}_\star),$$

and then $\min E \leq \min \mathcal{E}$. Conversely, according to Theorem 2.1, there exists a minimizer $(u^*, \mathbf{M}^*)$ of E. In particular, with the notations of Theorem 1.4, $u^\star$ is a minimizer of $E^{1,g,\mathbf{M}^\star}$ and then

$$\mathcal{H}^{n-1}(\overline{J_{u^\star}} \setminus J_{u^\star}) = 0.$$

So, we set $K^\star = \overline{J_{u^\star}}$ and then $K^\star$ is compact and $\mathcal{H}^{n-1}$-rectifiable, $\Omega \setminus K^\star$ is open, $u^\star \in W^{1,2}(\Omega \setminus K^\star)$ and

$$E(u^\star, \mathbf{M}^\star) = \mathcal{E}(u^\star, K^\star, \mathbf{M}^\star).$$

We may conclude that $\min E = \min \mathcal{E}$ and their minimizers coincide. Moreover, we have

Proposition 2.2. *Let $u \in SBV(\Omega)$ be a minimizer of $(\mathcal{P})$, then $u \in \mathcal{C}^1(\Omega \setminus \overline{J_u})$.*

Proof. Let $\overline{B}_r(x) \subset \Omega \setminus \overline{J_u}$; then $u \in W^{1,2}(B_r(x))$ and it is a minimizer of the functional

$$\mathcal{I}(v) = \int_{B_r(x)} (v - g)^2 \mathrm{d}x + \int_{B_r(x)} |\nabla v|^2 \mathrm{d}x$$

among the functions v in $u + W_0^{1,2}(B_r(x))$ and then classical regularity results give $u \in \mathcal{C}^1(B_r(x))$.

□

3 Γ-convergence result

This section is entirely devoted to the approximation process. In 3.1 we define the domain for E_ε and give the main Theorem of this paper. In 3.2 we prove, for $\varepsilon > 0$ fixed, that the minimization of E_ε admits a solution. In 3.3, the most technical part of the paper, we give the complete proof of Γ-convergence. Finally, in 3.4, we conclude the proof of the main Theorem.

3.1 The functionals, their domain and the main Theorem

Formally, we define the functional $E_\varepsilon(u, z, \mathbf{M})$ as

$$E_\varepsilon(u, z, \mathbf{M}) = \int_\Omega (u - g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1 - z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \langle \mathbf{M} \nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x.$$

As in [5], the function $z : \Omega \to [0;1]$ plays the role of *control* of the gradient of u. We need to introduce a domain for E_ε that ensures the existence of a minimizer. If $u, z \in \mathrm{W}^{1,2}(\Omega)$, then this functional is well defined. However, the coefficient $(1-z^2)^2$ removes the coercivity with respect to u and the existence result can not be achieved according to the Sobolev norm. If, by addition, u is bounded, we have

$$\begin{aligned} |\nabla(u(1-z^2))|^2 &= |\nabla u(1-z^2) - 2uz\nabla z|^2, \\ &\leq 2|\nabla u|^2(1-z^2)^2 + 4\|u\|_{L^\infty(\Omega)}|\nabla z|^2. \end{aligned}$$

According to ellipticity inequality (2.1), it gives

$$\int_\Omega |\nabla(u(1-z^2))|^2 \mathrm{d}x \leq \left(2 + \frac{4\|u\|_{L^\infty(\Omega)}}{\lambda\varepsilon}\right) E_\varepsilon(u,z,\mathbf{M}).$$

So, it is natural to set

$$\mathcal{D}_n(\Omega) = \left\{(u,z)\colon u \in \mathbb{B}(\Omega),\ z \in \mathrm{W}^{1,2}(\Omega;[0;1]),\ \forall N \in \mathbb{N} \quad \overline{u}^N(1-z^2) \in \mathrm{W}^{1,2}(\Omega)\right\},$$

where $\overline{u}^N$ is the truncated function defined, for any $x \in \Omega$, by

$$\overline{u}^N(x) = \begin{cases} -N & \text{if } u(x) \leq -N, \\ u(x) & \text{if } |u(x)| \leq N, \\ N & \text{if } u(x) \geq N. \end{cases} \tag{3.1}$$

Assuming $(u,z) \in \mathcal{D}_n(\Omega)$ does not ensure that $u \in \mathrm{W}^{1,2}(\Omega)$ and ∇u can not be defined as the gradient of u in the Sobolev sense. However, we can define ∇u in the following sense.

Definition 3.1. *Let $u \in L^1(\Omega)$ and $x \in \Omega$ a Lebesgue point of u; we say that u is approximately differentiable at x if there exists $L \in \mathbb{R}^n$ such that*

$$\lim_{r\to 0^+} \fint_{B(x,r)} \frac{|u(y) - u(x) - \langle L, y-x\rangle|}{r} dy = 0. \tag{3.2}$$

If u is approximately differentiable at x then L, uniquely determined by (3.2), is called the approximate differential of u at x.

The following ensures that $E_\varepsilon(u,z,\mathbf{M})$ is well defined for $(u,z) \in \mathcal{D}_n(\Omega)$.

Proposition 3.1. *If $(u,z) \in \mathcal{D}_n(\Omega)$, then u is approximately differentiable in $\{x \in \Omega\colon z(x) \neq 1\}$ and z is approximately differentiable in Ω.*

Proof. As Ω is open and bounded then $\mathrm{W}^{1,2}(\Omega) \subset \mathrm{BV}(\Omega)$. According to Calderon-Zygmund (see [7], Theorem 3.83), any function $u \in \mathrm{BV}(\Omega)$ is approximately differentiable at almost every point $x \in \Omega$. So, if $(u,z) \in \mathcal{D}_n(\Omega)$, then z and $\overline{u}^N(1-z^2)$ are approximately differentiable almost everywhere. The following properties are straightforward consequences of Definition 3.2

- if v_1, v_2 are approximately differentiable almost everywhere and $v_1 \in L^\infty(\Omega)$, then $v_1 v_2$ is approximately differentiable almost everywhere;
- if v_2 is approximately differentiable almost everywhere, then ${v_2}^{-1}$ is also approximately differentiable almost everywhere in $\{x\colon v_2(x) \neq 0\}$ (Proposition 3.71 in [7]).

We deduce that $\overline{u}^N$ is approximately differentiable almost everywhere in $\{x\colon z(x) \neq 1\}$. This is true for any $N \in \mathbb{N}$, so this is also true for u. □

The main result of the paper is the following

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain which satisfies reflexion condition (1.1), $\mathcal{G}$ be a compact subset of $S_n^+(\mathbb{R})$ and $\alpha > 0$ be fixed. We consider $H : \mathbb{B}(\Omega) \times W^{1,n+\alpha}(\Omega;\mathcal{G}) \to [0;+\infty]$ defined as*

$$H(u,\mathbf{M}) = \int_\Omega (u-g)^2\, dx + \int_\Omega \|D\mathbf{M}\|^{n+\alpha}\, dx,$$

$E : \mathbb{B}(\Omega) \times W^{1,n+\alpha}(\Omega;\mathcal{G}) \to [0;+\infty]$ defined as

$$E(u,\mathbf{M}) = \begin{cases} H(u,\mathbf{M}) + \displaystyle\int_\Omega |\nabla u|^2\, dx + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and $E_\varepsilon : \mathbb{B}(\Omega)\times\mathbb{B}(\Omega)\times W^{1,n+\alpha}(\Omega;\mathcal{G}) \to [0;+\infty]$ *defined as*

$$E_\varepsilon(u,z,\mathbf{M}) = \begin{cases} H(u,\mathbf{M}) + \displaystyle\int_\Omega |\nabla u|^2(1-z^2)^2\,dx + \int_\Omega \left(\varepsilon\langle \mathbf{M}\nabla z, \nabla z\rangle + \frac{z^2}{4\varepsilon}\right) dx & \text{if } (u,z)\in\mathcal{D}_n(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the following assertions are satisfied.

i) For any $\varepsilon>0$, E_ε *admits a minimizer, denoted by* $(u_\varepsilon, z_\varepsilon, \mathbf{M}_\varepsilon)$. *Moreover, we can assume that* $u_\varepsilon(x) = g(x)$ *on* $\{x\in\Omega\colon z_\varepsilon(x)=1\}$.

ii) For any $(\varepsilon_k)_k$ *converging to* 0^+, *there exists a subsequence, still denoted by* $(\varepsilon_k)_k$, *and* $u\in SBV(\Omega)$ *such that* $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$ *converges to* $(u,0)$ *almost everywhere,* $(\mathbf{M}_k)_k$ *converges weakly to* $\mathbf{M}$ *and* $(u,\mathbf{M})$ *is a minimizer of* E.

We set

$$(\mathcal{P}_\varepsilon): \qquad \mathrm{Min}\{E_\varepsilon(u,z,\mathbf{M})\colon (u,z,\mathbf{M})\in\mathbb{B}(\Omega)\times\mathbb{B}(\Omega)\times W^{1,n+\alpha}(\Omega;\mathcal{G})\},$$

Theorem 3.1 *i)* implies that, for $\varepsilon>0$ fixed, $(\mathcal{P}_\varepsilon)$ is a well posed problem. Theorem 3.1 *ii)* implies that, up to the extraction of a subsequence, the sequence of solutions of $(\mathcal{P}_{\varepsilon_k})$ converge to a solution of $(\mathcal{P}')$.

3.2 Existence result for $(\mathcal{P}_\varepsilon)$

Theorem 3.1 *i)* is a straightforward consequence of Propositions 3.2 and 3.3.

Proposition 3.2. *Let* $\varepsilon>0$ *be fixed. There exists* $(u_k,z_k,\mathbf{M}_k)_k$ *a minimizing sequence of* E_ε *such that* $(u_k)_k$ *is a bounded sequence of* $L^\infty(\Omega)$, $(u_k,z_k)_k$ *converges almost everywhere to* $(u,z)\in\mathcal{D}_n(\Omega)$, $u(x)=g(x)$ *on* $\{x\in\Omega\colon z(x)=1\}$ *and* $(\mathbf{M}_k)_k$ *weakly converges to* $\mathbf{M}\in W^{1,n+\alpha}(\Omega;\mathcal{G})$.

To prove it, we need the following Lemma which proof is given in Appendix 4.1.

Lemma 3.1. *For* $(u,z)\in\mathcal{D}_n(\Omega)$ *and* $\nu\in\mathbb{S}^{n-1}$ *fixed, we have* $(u_x,z_x)\in\mathcal{D}_1(\Omega_x)$ *for* $\mathcal{H}^{n-1}$*-almost every* $x\in\Omega_\nu$ *(see the notations of Definition 1.2), and*

$$\begin{aligned} \nabla u_x(t) &= \langle\nabla u(x+t\nu),\nu\rangle, \\ \nabla z_x(t) &= \langle\nabla z(x+t\nu),\nu\rangle, \end{aligned}$$

for almost every $t\in\Omega_x\setminus\{s\colon z(x+s\nu)=1\}$.

Now, we prove Proposition 3.2.

Proof. Let $(u_k,z_k,\mathbf{M}_k)_k$ be a minimizing sequence of E_ε. We fix $N\geq\|g\|_{L^\infty(\Omega)}$ and we consider the truncated functions $(\overline{u}_k^N)_k$ defined in (3.1). As $(u_k,z_k)\in\mathcal{D}_n(\Omega)$, we have $\overline{u}_k^N(1-z_k^2)\in W^{1,2}(\Omega)$. As Ω is bounded, then $\mathrm{W}^{1,2}(\Omega)\subset\mathrm{SBV}(\Omega)$. According to Calderón-Zygmund Theorem ([7], Theorem 3.83), $\overline{u}_k^N(1-z_k^2)$ is approximately differentiable almost everywhere. For the same reasons, $1-z_k^2$ is also approximately differentiable almost everywhere. According to Proposition 3.71 in [7], we deduce that $\overline{u}_k^N$ is approximately differentiable almost everywhere in $\{x\colon z_k(x)\neq 1\}$. Moreover, $\nabla\overline{u}_k^N(x)=0$ almost everywhere in $\{x\colon|\overline{u}_k^N(x)|=N\}$ and $\nabla\overline{u}_k^N(x)=\nabla u_k(x)$ almost everywhere in $\{x\colon|\overline{u}_k^N(x)|<N\}$ (Proposition 3.73 in [7]), it gives

$$\int_\Omega|\nabla\overline{u}_k^N|^2(1-z_k^2)^2\mathrm{d}x\leq\int_\Omega|\nabla u_k|^2(1-z_k^2)^2\mathrm{d}x. \tag{3.3}$$

so $E_\varepsilon(\overline{u}_k^N,z_k,\mathbf{M}_k)\leq E_\varepsilon(u_k,z_k,\mathbf{M}_k)$ and then $(\overline{u}_k^N,z_k,\mathbf{M}_k)_k$ is also a minimizing sequence. According to ellipticity inequality (2.1), we have

$$\int_\Omega|\nabla z_k|^2\mathrm{d}x+\int_\Omega z_k^2\mathrm{d}x\leq\left(\frac{1}{\lambda\varepsilon}+4\varepsilon\right)E_\varepsilon(u_k,z_k,\mathbf{M}_k),$$

and then $(z_k)_k$ is a bounded sequence of $\mathrm{W}^{1,2}(\Omega)$. So, there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to $z\in\mathrm{W}^{1,2}(\Omega)$. As $(z_k)_k$ takes its values almost everywhere in $[0;1]$, then z takes also its values in $[0;1]$. For $w_k=\overline{u}_k^N(1-z_k^2)$, we have

$$\int_\Omega|\nabla w_k|^2\mathrm{d}x+\int_\Omega w_k^2\mathrm{d}x\leq 2\int_\Omega|\nabla\overline{u}_k^N|^2(1-z_k^2)^2\mathrm{d}x+2N^2\int_\Omega|\nabla z_k|^2\mathrm{d}x+N^2\int_\Omega(1-z_k^2)^2\mathrm{d}x$$

and then $(w_k)_k$ is a bounded sequence of $\mathrm{W}^{1,2}(\Omega)$. So, there exists a subsequence, still denoted by $(w_k)_k$, which converges almost everywhere to $w \in \mathrm{W}^{1,2}(\Omega)$. In particular, $(\overline{u}_k^N(x))_k$ converges for almost every $x \in \{y\colon z(y) \neq 1\}$ to $u(x)$. We set $u(x) = g(x)$ for $x \in \{y\colon z(y) = 1\}$. This construction ensures that $(u, z) \in \mathcal{D}_n(\Omega)$. As $\mathcal{G} \subset \mathrm{S}_n^+(\mathbb{R})$ is compact, Ω is bounded and $\int_\Omega \|D\mathbf{M}_k\|\mathrm{d}x \leq E_\varepsilon(u_k, z_k, \mathbf{M}_k)$, then $(\mathbf{M}_k)_k$ is a bounded sequence of $W^{1,n+\alpha}(\Omega)$ and there exists a subsequence weakly convergent to $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$. As weakly convergence in $W^{1,n+\alpha}(\Omega)$ implies uniform convergence and $\mathcal{G}$ is closed, then $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$. □

Proposition 3.3. *If $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converges almost everywhere to $(u, z) \in \mathcal{D}_n(\Omega)$, $(u_k)_k$ is a bounded sequence of $L^\infty(\Omega)$ and $(\mathbf{M}_k)_k \subset W^{1,n+\alpha}(\Omega; \mathcal{G})$ weakly converges to $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$, then*

$$\liminf_{k\to\infty} E_\varepsilon(u_k, z_k, \mathbf{M}_k) \geq E_\varepsilon(u, z, \mathbf{M}).$$

Proof. Fatou Lemma and lower semi-continuity of the Sobolev semi-norm give

$$\liminf_{k\to\infty} \left(\int_\Omega (u_k - g)^2 \mathrm{d}x + \int_\Omega \frac{z_k^2}{4\varepsilon}\mathrm{d}x + \int_\Omega \|D\mathbf{M}_k\|^{n+\alpha}\mathrm{d}x \right) \geq \int_\Omega (u-g)^2\mathrm{d}x + \int_\Omega \frac{z^2}{4\varepsilon}\mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^{n+\alpha}\mathrm{d}x.$$

So, to show Proposition 3.3, it suffices to prove that

$$\liminf_{k\to\infty} \int_\Omega \varepsilon\langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x \geq \int_\Omega \varepsilon\langle \mathbf{M}\nabla z, \nabla z\rangle \mathrm{d}x \tag{3.4}$$

and

$$\liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}x \geq \int_\Omega |\nabla u|^2 (1 - z^2)^2 \mathrm{d}x. \tag{3.5}$$

Proof of (3.4)

If $\liminf_k \int_\Omega |\nabla z_k|^2 \mathrm{d}x = +\infty$, according to Ellipticity condition (2.1), we have

$$\liminf_k \int_\Omega \langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x = +\infty$$

and the result of Proposition 3.3 is ensured. So, we may assume that $\liminf \int_\Omega |\nabla z_k|^2 \mathrm{d}x < +\infty$. On the other hand, we have

$$\left| \int_\Omega \varepsilon\langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x - \int_\Omega \varepsilon\langle \mathbf{M}\nabla z_k, \nabla z_k\rangle \mathrm{d}x \right| \leq \varepsilon \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \int_\Omega |\nabla z_k|^2 \mathrm{d}x.$$

As the weakly convergence of $(\mathbf{M}_k)_k$ in $W^{1,n+\alpha}(\Omega)$ implies the uniform convergence, we may conclude that

$$\liminf_k \left(\int_\Omega \varepsilon\langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x - \int_\Omega \varepsilon\langle \mathbf{M}\nabla z_k, \nabla z_k\rangle \mathrm{d}x \right) = 0.$$

To prove inequality (3.4), it remains to verify that

$$\liminf_{k\to\infty} \int_\Omega \varepsilon\langle \mathbf{M}\nabla z_k, \nabla z_k\rangle \mathrm{d}x \geq \int_\Omega \varepsilon\langle \mathbf{M}\nabla z, \nabla z\rangle \mathrm{d}x.$$

As the application

$$\begin{array}{ccc} \mathrm{W}^{1,2}(\Omega) & \to & \mathrm{L}^2(\Omega; \mathbb{R}^n), \\ z & \mapsto & \nabla z \end{array}$$

is continuous for the strong topology, it remains to prove that the application

$$\begin{array}{ccc} \mathrm{L}^2(\Omega; \mathbb{R}^n) & \to & \mathbb{R}, \\ Z & \mapsto & \int_\Omega \langle \mathbf{M}Z, Z\rangle \mathrm{d}x \end{array}$$

is lower semi-continuous for the weak topology of $\mathrm{L}^2(\Omega; \mathbb{R}^n)$. Let $(Z_k)_k \subset \mathrm{L}^2(\Omega; \mathbb{R}^n)$ be weakly convergent to $Z \in \mathrm{L}^2(\Omega; \mathbb{R}^n)$. We set

$$\begin{array}{ccc} L : \mathrm{L}^2(\Omega; \mathbb{R}^n) & \to & \mathbb{R}, \\ U & \mapsto & \int_\Omega \langle \mathbf{M}Z, U\rangle \mathrm{d}x \end{array}$$

According to ellipticity inequality (2.1), $L \in (\mathrm{L}^2(\Omega;\mathbb{R}^n))'$ and then $(L(Z_k))_k$ converges to $L(Z)$. Moreover, for k fixed, the following polynomial function is positive

$$t \mapsto \int_\Omega \langle \mathbf{M}(Z+tZ_k), Z+tZ_k\rangle \mathrm{d}x.$$

Thus, its discriminant is negative and we deduce the following *anisotropic Cauchy-Schwarz inequality*

$$\int_\Omega \langle \mathbf{M}Z, Z_k\rangle \mathrm{d}x \leq \left(\int_\Omega \langle \mathbf{M}Z, Z\rangle \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \langle \mathbf{M}Z_k, Z_k\rangle \mathrm{d}x\right)^{\frac{1}{2}}.$$

As $(L(Z_k))_k$ converges to $L(Z)$, passing through the lim inf in the previous inequality yields

$$\int_\Omega \langle \mathbf{M}Z, Z\rangle \mathrm{d}x \leq \left(\int_\Omega \langle \mathbf{M}Z, Z\rangle \mathrm{d}x\right)^{\frac{1}{2}} \liminf_{k\to\infty} \left(\int_\Omega \langle \mathbf{M}Z_k, Z_k\rangle \mathrm{d}x\right)^{\frac{1}{2}}$$

and then we may conclude the *Proof of (3.4)* by taking $Z_k = \nabla z_k, Z = \nabla z$ in the previous inequality

$$\int_\Omega \langle \mathbf{M}\nabla z, \nabla z\rangle \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega \langle \mathbf{M}\nabla z_k, \nabla z_k\rangle \mathrm{d}x.$$

Proof of (3.5)

We first consider the one-dimensional case $n = 1$ and then by a slicing argument we get the lower semi-continuity for the general case $n \geq 1$. Let $A \subset \{x \in \Omega\colon z(x) < 1\}$ be an open and relatively compact subset of $\Omega \subset \mathbb{R}$. As $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, then $(z_k)_k$ uniformly converges to z. In particular, there exists $\delta > 0$ and $k_0 \in \mathbb{N}$

$$k \geq k_0 \quad \Rightarrow \quad A \subset \{x \in \Omega\colon z_k(x) \leq 1-\delta\}.$$

Thus, for any $k \geq k_0$, we have

$$\int_A |\nabla u_k|^2 \mathrm{d}x \leq \frac{1}{1-(1-\delta)^2} E_\varepsilon(u_k, z_k)$$

and then we deduce that $(u_k)_k$ is a bounded sequence of $\mathrm{W}^{1,2}(A)$. As $u_k(1-z_k^2)$ converges almost everywhere to $u(1-z^2)$ in Ω, there exists a subsequence, still denoted by $(u_k)_k$, which weakly converges to u in $\mathrm{W}^{1,2}(A)$. In particular, $(\nabla u_k)_k$ weakly converges to ∇u in $\mathrm{L}^2(A)$. For $\xi \in \mathrm{L}^2(\Omega)$, we decompose

$$\int_A \xi\left[\nabla u_k(1-z_k^2) - \nabla u(1-z^2)\right]\mathrm{d}x = \int_A \xi\nabla u_k(z^2-z_k^2)\mathrm{d}x + \int_A \xi\nabla u_k(1-z^2)\mathrm{d}x + \int_A \xi\nabla u(z^2-z_k^2)\mathrm{d}x. \tag{3.6}$$

As $(1-z^2)\xi \in \mathrm{L}^2(A)$ and $(\nabla u_k)_k$ weakly converges to ∇u in $\mathrm{L}^2(A)$, then we have

$$\int_A \xi\nabla u_k(1-z^2)\mathrm{d}x \to \int_A \xi\nabla u(1-z^2)\mathrm{d}x. \tag{3.7}$$

Moreover, we have

$$\int_A \xi\nabla u_k(z-z_k)\mathrm{d}x \leq \|\xi\|_{L^2(A)}\|\nabla u_k\|_{L^2(A)}\|z^2-z_k^2\|_{L^\infty(A)}$$

and

$$\int_A \xi\nabla u(z-z_k)\mathrm{d}x \leq \|\xi\|_{L^2(A)}\|\nabla u\|_{L^2(A)}\|z^2-z_k^2\|_{L^\infty(A)}.$$

As a weakly convergent sequence is bounded, then $(\nabla u_k)_k$ is bounded in $\mathrm{L}^2(A)$ and we deduce that

$$\int_A \xi\nabla u_k(z^2-z_k^2)\mathrm{d}x \to 0, \qquad \int_A \xi\nabla u(z^2-z_k^2)\mathrm{d}x \to 0. \tag{3.8}$$

According to (3.6), (3.7) and (3.8), we get

$$\int_A \xi \nabla u_k (1-z_k^2) \mathrm{d}x \to \int_A \xi \nabla u (1-z^2) \mathrm{d}x$$

and then $(\nabla u_k(1-z_k^2))_k$ weakly converges to $\nabla u(1-z^2)$ in $\mathrm{L}^2(A)$. As the norm is lower semi-continuous, we deduce

$$\begin{aligned} \int_A |\nabla u|^2 (1-z^2)^2 \mathrm{d}x &\leq \liminf_{k\to\infty} \int_A |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x, \\ &\leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x. \end{aligned}$$

Passing to the limit $A \uparrow \{x \in \Omega \colon z(x) < 1\}$ gives

$$\int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x.$$

We generalize this result to the dimension $n \geq 1$. With the notation u_x introduced in (1.2), using the previous result obtained in dimension 1, Lemma 3.1 and Fatou Lemma, give

$$\begin{aligned} \int_A |\langle \nabla u, \nu \rangle|^2 (1-z^2)^2 \mathrm{d}x &= \int_{A_\nu} \int_{A_x} |\nabla u_x(t)|^2 (1-z_x(t)^2)^2 \mathrm{d}t\mathrm{d}x, \\ &\leq \int_{A_\nu} \liminf_{k\to\infty} \int_{A_x} |\nabla (u_k)_x(t)|^2 (1-(z_k)_x(t)^2)^2 \mathrm{d}t\mathrm{d}x, \\ &\leq \liminf_{k\to\infty} \int_{A_\nu} \int_{A_x} |\nabla (u_k)_x(t)|^2 (1-(z_k)_x(t)^2)^2 \mathrm{d}t\mathrm{d}x, \\ &\leq \liminf_{k\to\infty} \int_{A_\nu} \int_{A_x} |\langle \nabla u_k(x+t\nu), \nu \rangle|^2 (1-z_k(x+t\nu)^2)^2 \mathrm{d}t\mathrm{d}x, \\ &\leq \liminf_{k\to\infty} \int_A |\langle \nabla u_k, \nu \rangle|^2 (1-z_k^2)^2 \mathrm{d}x, \\ &\leq \liminf_{k\to\infty} \int_A |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x, \end{aligned}$$

for any open set $A \subset \Omega$ and every $\nu \in \mathbb{S}^{n-1}$. The function $x \to \frac{\nabla u(x)}{|\nabla u(x)|}$ is measurable in $U = \{x \in \Omega \colon z(x) \neq 1, \nabla u(x) \neq 0\}$. According to Lusin Theorem (1.45 of [7]), there exists an increasing sequence of compacts $(K_l)_l \subset U$ such that

$$\begin{cases} \mathcal{L}^n(U \setminus K_l) \leq \frac{1}{l}, \\ x \to \frac{\nabla u(x)}{|\nabla u(x)|} \text{ is continuous in } K_l. \end{cases}$$

Thus, for any $x \in K_l$, there exists $r > 0$ such that

$$y \in B(x,r) \quad \Rightarrow \quad \left| \frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|} \right| \leq \frac{1}{l}, \tag{3.9}$$

As a consequence of Besicovitch Covering Theorem (2.18 of [7]), there exists a countable, pairwise disjoint collection of balls $(B_i)_{i\in I}$ satisfying (3.9) such that

$$B_i \subset \Omega, \qquad \mathcal{L}^n \left(K_l \setminus \bigcup_{i\in I} B_i \right) = 0, \text{ for any } \forall i \in I.$$

For any $i \in I$, we fix $x_i \in B_i$ and we set $\nu_i = \frac{\nabla u(x_i)}{|\nabla u(x_i)|}$; then

$$\int_{B_i} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_{B_i} |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x.$$

As $(B_i)_i$ is pairwise dijoint, we deduce

$$\begin{aligned}\int_{\cup_i B_i} |\langle \nabla u, \nu_i\rangle|^2(1-z^2)^2 \mathrm{d}x &= \sum_{i\in I} \int_{B_i} |\langle \nabla u, \nu_i\rangle|^2(1-z^2)^2 \mathrm{d}x,\\ &\leq \sum_{i\in I} \liminf_{k\to\infty} \int_{B_i} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x,\\ &\leq \liminf_{k\to\infty} \sum_{i\in I} \int_{B_i} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x,\\ &\leq \liminf_{k\to\infty} \int_{\cup_i B_i} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x,\\ &\leq \liminf_{k\to\infty} \int_{\Omega} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\forall x \in B_i \cap K_l, \quad \big||\nabla u|^2 - |\langle \nabla u, \nu_i\rangle|^2\big| &\leq \left|\langle \nabla u, \frac{\nabla u}{|\nabla u|}\rangle^2 - \langle \nabla u, \nu_i\rangle^2\right|,\\ &\leq \left|\langle \nabla u, \frac{\nabla u}{|\nabla u|} - \nu_i\rangle \langle \nabla u, \frac{\nabla u}{|\nabla u|} + \nu_i\rangle\right|,\\ &\leq \frac{2}{l}|\nabla u|^2.\end{aligned}$$

It gives

$$\int_{B_i\cap K_l} |\nabla u|^2(1-z^2)^2 \mathrm{d}x \leq \frac{l}{l-2} \int_{B_i\cap K_l} |\langle \nabla u, \nu_i\rangle|^2(1-z^2)^2 \mathrm{d}x.$$

As $\mathcal{L}^n(K_l \setminus \cup_i B_i) = 0$ and $(B_i)_i$ is pairwise disjoint, we get

$$\begin{aligned}\int_{K_l} |\nabla u|^2(1-z^2)^2 \mathrm{d}x &= \sum_i \int_{B_i\cap K_l} |\nabla u|^2(1-z^2)^2 \mathrm{d}x,\\ &\leq \frac{l}{l-2} \sum_i \int_{B_i\cap K_l} |\langle \nabla u, \nu_i\rangle|^2(1-z^2)^2 \mathrm{d}x,\\ &\leq \frac{l}{l-2} \int_{\cup_i B_i\cap K_l} |\langle \nabla u, \nu_i\rangle|^2(1-z^2)^2 \mathrm{d}x,\\ &\leq \frac{l}{l-2} \int_{\cup_i B_i} |\langle \nabla u, \nu_i\rangle|^2(1-z^2)^2 \mathrm{d}x,\\ &\leq \frac{l}{l-2} \liminf_{k\to\infty} \int_{\Omega} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x.\end{aligned}$$

As $(K_l)_l$ is an increasing sequence such that $\mathcal{L}^n(U \setminus K_l) \to 0$, passing to the limit $l \to \infty$ gives

$$\int_U |\nabla u|^2(1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_{\Omega} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x$$

and we may conclude

$$\int_{\Omega} |\nabla u|^2(1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_{\Omega} |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x.$$

□

3.3 Γ-convergence result for $\varepsilon \to 0^+$

This section is dedicated to the proof of Theorem 3.1 *ii*). For that, we will prove the following Γ-convergence result.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied. Let $(\varepsilon_k)_k$ be a sequence which converges to 0^+, we have*

i) if $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$, $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$, $(u_k)_k$ *is bounded in* $L^\infty(\Omega)$, $(u_k, z_k)_k$ *converges to* $(u, 0)$ *almost everywhere and* $(\mathbf{M}_k)_k$ *weakly converges to* $\mathbf{M}$, *then*

$$\liminf_{k\to\infty} E_{\varepsilon_k}(u_k, z_k, \mathbf{M}_k) \geq E(u, \mathbf{M}); \tag{3.10}$$

ii) for any $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ *and* $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$, *there exists a sequence* $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ *such that* $(u_k)_k$ *is bounded in* $L^\infty(\Omega)$, $(u_k, z_k)_k$ *converges to* $(u, 0)$ *almost everywhere and*

$$\limsup_{k\to\infty} E_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq E(u, \mathbf{M}). \tag{3.11}$$

3.3.1 The inequality for the *lower* Γ-limit

We now prove the first inequality of Γ-convergence (3.10). Let $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$, $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere and $(\mathbf{M}_k)_k$ weakly converges to $\mathbf{M}$. In the sequel, we emphasize on the domain of the function: for U an open subset of Ω, we adopt the following notation

$$\begin{aligned} F(u, \mathbf{M}; U) &= \int_U |\nabla u|^2 \mathrm{d}x + \int_{J_u \cap U} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \\ F_{\varepsilon_k}(u_k, z_k, \mathbf{M}_k; U) &= \int_U |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}x + \int_U \left(\varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle + \frac{z_k^2}{4\varepsilon_k} \right) \mathrm{d}x, \end{aligned}$$

Fatou Lemma and lower continuity for the Sobolev semi-norm yields

$$\liminf_{k\to\infty} \int_\Omega (u_k - g)^2 \mathrm{d}x + \int_\Omega \|D\mathbf{M}_k\|^{n+\alpha} \mathrm{d}x \geq \int_\Omega (u - g)^2 \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^{n+\alpha} \mathrm{d}x$$

and then it suffices to prove that $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}_k; \Omega) \geq F(u, \mathbf{M}; \Omega)$. Moreover, if

$$\liminf_k \int_\Omega \varepsilon_k |\nabla z_k|^2 \mathrm{d}x = +\infty,$$

according to ellipticity condition (2.1), we have

$$\liminf_k \int_\Omega \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x = +\infty$$

and (3.10) is ensured. So, we may assume that $\liminf_k \int_\Omega \varepsilon_k |\nabla z_k|^2 \mathrm{d}x < +\infty$. On the other hand, we have

$$\left| \int_\Omega \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x - \int_\Omega \varepsilon_k \langle \mathbf{M} \nabla z_k, \nabla z_k\rangle \mathrm{d}x \right| \leq \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \varepsilon_k \int_\Omega |\nabla z_k|^2 \mathrm{d}x.$$

As the weakly convergence of $(\mathbf{M}_k)_k$ to $\mathbf{M}$ in $W^{1,n+\alpha}(\Omega)$ implies the uniform convergence, we may conclude that

$$\liminf_k \left(\int_\Omega \varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle \mathrm{d}x - \int_\Omega \varepsilon_k \langle \mathbf{M} \nabla z_k, \nabla z_k\rangle \mathrm{d}x \right) = 0.$$

To prove inequality (3.10), it remains to verify that

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; U).$$

We perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension $n \geq 2$.

The one-dimensional case

In this section, we assume that $\Omega = I$ is an open interval and the metric $\mathbf{M}$ is simply a constant $m > 0$. We assume in this section that m is fixed. To avoid confusion, we denote the approximating functional by

$$G_\varepsilon(u, z; I) = \int_I |\nabla u(t)|^2 (1 - z(t)^2)^2 \mathrm{d}t + \int_I \left(m\varepsilon |\nabla z(t)|^2 + \frac{z(t)^2}{4\varepsilon} \right) \mathrm{d}t, \tag{3.12}$$

where the domain is

$$\mathcal{D}_1(I) = \{(u,z)\colon u \in \mathbb{B}(I), z \in \mathrm{W}^{1,2}(I;[0;1]), \forall N \in \mathbb{N} \quad \overline{u}^N(1-z^2) \in \mathrm{W}^{1,2}(I)\}.$$

We denote the lower Γ-limit, by

$$G_-(u;I) = \inf\left\{\liminf_{k\to\infty} G_{\varepsilon_k}(u_k,z_k;I)\right\},$$

where the inf is taken over all sequence $(u_k,z_k)_k \subset \mathcal{D}_1(I)$ such that (u_k,z_k) converges almost everywhere to $(u,0)$ in I. We need the following Lemma which proof is given in appendix 4.2.

Lemma 3.2. *Let $I \subset \mathbb{R}$ be an open interval, $J \subset I$ be a set with finite cardinal. We have*

$$u \in W^{1,2}(I \setminus J) \Rightarrow u \in SBV(I),\ J_u \subset J.$$

The main result of this subsection is given by the following.

Proposition 3.4. *Let $I \subset \mathbb{R}$ be an open interval and $u \in \mathbb{B}(I)$. If $G_-(u;I) < \infty$, then $u \in SBV(I)$ and*

$$\int_I |\nabla u(t)|^2\,dt + m^{\frac{1}{2}}\mathcal{H}^0(J_u \cap I) \le G_-(u;I).$$

The proof of this Proposition consists in showing the two following Lemmas.

Lemma 3.3. *If $u \in W^{1,2}(B_\eta(x))$, then we have*

$$G_-(u;B_\eta(x)) \ge \int_{B_\eta(x)} |\nabla u(t)|^2\,dt.$$

Lemma 3.4. *If $u \notin W^{1,2}(B_\rho(x))$ for any $\rho \in]0;\eta[$, then for any $\rho \in]0;\eta[$ we have*

$$G_-(u;B_\rho(x)) \ge m^{1/2}.$$

Suppose that Lemma 3.3 and 3.4 are proved, we deduce Proposition 3.4.

Proof. We set

$$J = \left\{x \in I\colon \forall \rho > 0,\ u \notin \mathrm{W}^{1,2}(B_\rho(x))\right\}.$$

Let $\{x_1,\dots,x_N\} \subset J$ and $\rho > 0$ be such that $\{B_\rho(x_i)\colon i = 1,\dots,N\}$ is pairwise disjoint. According to Lemma 3.4 we have

$$\forall i \in \{1,\dots,N\}, \quad G_-(u;B_\rho(x_i)) \ge m^{1/2}$$

and then

$$\sum_{i=1}^N G_-(u;B_\rho(x_i)) \ge N m^{1/2}.$$

As $G_-(u;\cdot)$ is superadditive, we have

$$G_-(u;\cup_{i=1}^N B_\rho(x_i)) \ge N m^{1/2}.$$

and $G_-(u;\cdot)$ is non decreasing, it gives

$$G_-(u;I) \ge N m^{1/2}.$$

As $G_-(u;\cdot) < +\infty$, the set J is finite. So, there exists $\rho > 0$ such that $\{B_\rho(x)\colon x \in J\}$ is pairwise disjoint. As $G_-(u;\ \cdot\ )$ is superadditive and non decreasing, we have

$$\sum_{x\in J} G_-(u;B_\rho(x)) + G_-(u;I \setminus \cup_{x\in J}\overline{B_\rho(x)}) \le G_-(u;I).$$

According to Lemma 3.3 and 3.4, it gives

$$\mathcal{H}^0(J)m^{\frac{1}{2}} + \int_{I\setminus\cup_{x\in J}\overline{B_\rho(x)}} |\nabla u(t)|^2 \mathrm{d}t \le G_-(u;I).$$

Taking the limit $\rho \to 0^+$ yields

$$\mathcal{H}^0(J)m^{\frac{1}{2}} + \int_{I\setminus J} |\nabla u(t)|^2 \mathrm{d}t \leq G_-(u;I).$$

In particular $u \in \mathrm{W}^{1,2}(I \setminus J)$ and, according to Lemma 3.2, we get $u \in \mathrm{SBV}(I)$, $J_u \subset J$ and then

$$\mathcal{H}^0(J)m^{\frac{1}{2}} + \int_{I} |\nabla u(t)|^2 \mathrm{d}t \leq G_-(u;I).$$

□

Now, we prove lemma 3.3.

Proof. We can assume that $G_-(u;B_\eta(x)) < +\infty$, otherwise the result is ensured. By a diagonal extraction, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_1(B_\rho(x))$ converging almost everywhere to $(u, 0)$ and

$$G_{\varepsilon_k}(u_k, z_k; B_\rho(x)) \to G_-(u; B_\rho(x)).$$

As $G_-(u;B_\eta(x))$ is finite, there exists $C > 0$ such that

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} \left(\varepsilon_k |\nabla z_k|^2 + \frac{z_k^2}{4\varepsilon_k}\right) \mathrm{d}t \leq C. \tag{3.13}$$

Applying the inequality $2ab \leq a^2 + b^2$ with $a^2 = \varepsilon_k |\nabla z_k|^2$ and $b^2 = \frac{z_k^2}{4\varepsilon_k}$ gives

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} |\nabla z_k| z_k \mathrm{d}t \leq C. \tag{3.14}$$

We set $c_k = 1 - z_k^2$. As $z_k \in \mathrm{W}^{1,2}(B_\eta(x))$, then $c_k \in BV(B_\eta(x))$ and (3.14) is

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} |\nabla c_k| \mathrm{d}t \leq 2C.$$

Coarea formula (see [7]) yields

$$\forall k \in \mathbb{N}, \quad \int_0^1 \mathcal{H}^0(\{y \in B_\eta(x) \colon c_k(y) = t\}) \mathrm{d}t \leq 2C. \tag{3.15}$$

Let $\sigma < 1$ in an arbitrary neighborhood of 1 and $\delta \in]0;\sigma[$ be fixed numbers. According to (3.15) and mean value Theorem, there exits $\delta_k \in]\delta;\sigma[$ such that

$$\forall k \in \mathbb{N}, \quad \mathcal{H}^0(\{y \in B_\eta(x) \colon c_k(y) = \delta_k\}) \leq \frac{2C}{\sigma - \delta}. \tag{3.16}$$

We set $A_k = \{y \in B_\eta(x) \colon c_k(y) \geq \delta_k\}$. As $(\varepsilon_k)_k$ converges to 0, inequality (3.13) implies that $(z_k)_k$ converges to 0 and c_k to 1 almost everywhere. As $\delta_k < \sigma$ and $\sigma < 1$ then $(\mathcal{L}^1(A_k))_k$ converges to $\mathcal{L}^1(B_\eta(x))$.

Sobolev embedding Theorem ensures that $\mathrm{W}^{1,2}(B_\eta(x)) \subset \mathcal{C}(B_\eta(x))$, so c_k is continuous and A_k is a countable union of closed intervals of $B_\eta(x)$. According to (3.16), this union is finite and its cardinality is uniformly bounded by N. For any k, there exits a disjoint family of closed intervals $(I_k^i)_{i=1\ldots N}$ such that

$$\begin{cases} A_k = \bigcup_{i=1}^N I_k^i, \\ \forall i \in \{1, \ldots, N-1\}, \max(I_k^i) < \min(I_k^{i+1}). \end{cases}$$

There exists a subsequence, still denoted by $(I_k^i)_{i=1\ldots N}$, such that $(\min(I_k^i))_k$ and $(\max(I_k^i))_k$ converge for any $i \in \{1, \ldots, N\}$. We set a_∞^i and b_∞^i the previous limits, $I_\infty^i =]a_\infty^i; b_\infty^i[$ and $A = \bigcup_{i=1}^N I_\infty^i$. As $(\mathcal{L}^1(A_k))_k$ converges to $\mathcal{L}^1(B_\eta(x))$, then A is a subset of full measure in $B_\eta(x)$.

Let O be an open subset such that $\overline{O} \subset A$. For k with a sufficiently large value, we have $O \subset A_k$ and then

$$\liminf_{k\to\infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}t \geq \liminf_{k\to\infty} \int_O |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}t.$$

As z_k takes its values in $[0;1]$, we get

$$\begin{aligned}\liminf_{k\to\infty}\int_{B_\eta(x)}|\nabla u_k|^2(1-z_k^2)^2\mathrm{d}t &\geq \liminf_{k\to\infty}\int_O|\nabla u_k|^2\delta_k^2\mathrm{d}t,\\ &\geq \delta^2\liminf_{k\to\infty}\int_O|\nabla u_k|^2\mathrm{d}t.\end{aligned}$$

Moreover, there exists k_0 such that: $k\geq k_0\Rightarrow O\subset A_k$, then we have

$$\forall x\in O\quad 1-z_k^2\geq\delta.$$

As $u_k(1-z_k^2)\in W^{1,2}(\Omega)$ and $\sqrt{\delta}>0$, we get $u_k\in \mathrm{W}^{1,2}(O)$ and the lower semi-continuity property of the Sobolev norm gives

$$\liminf_{k\to\infty}\int_O|\nabla u_k|^2\mathrm{d}t\geq\int_O|\nabla u|^2\mathrm{d}t.$$

As O is chosen arbitrary in A and A is of full measure in $B_\eta(x)$, it gives

$$\liminf_{k\to\infty}\int_{B_\eta(x)}|\nabla u_k|^2(1-z_k^2)^2\mathrm{d}t\geq\delta^2\int_{B_\eta(x)}|\nabla u|^2\mathrm{d}t.$$

Letting δ to 1^-, it concludes the proof of lemma 3.3.

□

We prove lemma 3.4.

Proof. We can assume that $G_-(u;B_\rho(x))<+\infty$ for any $\rho\in]0;\eta[$, otherwise the result is ensured. As $u\notin \mathrm{W}^{1,2}(B_\rho(x))$, there exists three sequences $(y_k^1)_{k\in\mathbb{N}}$, $(y_k^2)_{k\in\mathbb{N}}$ and $(y_k^3)_{k\in\mathbb{N}}$ such that:

$$\begin{cases} y_k^1\to x,\ z_k(y_k^1)\to 0,\\ y_k^2\to x,\ z_k(y_k^2)\to 1,\\ y_k^3\to x,\ z_k(y_k^3)\to 0,\\ \forall k\in\mathbb{N},\ y_k^1<y_k^2<y_k^3.\end{cases}$$

We have

$$G(u_k,z_k;B_\rho(x))\geq\int_{x-\rho}^{x+\rho}\left(\varepsilon_k m|\nabla z_k|^2+\frac{z_k^2}{4\varepsilon_k}\right)\mathrm{d}t.$$

The inequality $a^2+b^2\geq 2ab$ gives:

$$G(u_k,z_k;B_\rho(x))\geq\int_{x-\rho}^{x+\rho}m^{1/2}|\nabla z_k|z_k\,\mathrm{d}t.$$

As $[y_k^1,y_k^2]\subset B_\rho(x)$, we obtain:

$$G(u_k,z_k;B_\rho(x))\geq\int_{y_k^1}^{y_k^3}m^{1/2}|\nabla z_k|z_k\,\mathrm{d}t.$$

We have

$$G(u_k,z_k;B_\rho(x))\geq m^{1/2}\int_{y_k^1}^{y_k^2}|\nabla z_k(t)|z_k(t)\,\mathrm{d}t+m^{1/2}\int_{y_k^2}^{y_k^3}|\nabla z_k(t)|z_k(t)\,\mathrm{d}t.$$

Since $z_k\in \mathrm{W}^{1,2}(B_\eta(x))$, we may use the change of variable $s=z_k(t)$. This yields:

$$\begin{aligned}(\star)_k^2 &\geq m^{1/2}\int_{z_k(y_k^1)}^{z_k(y_k^2)}s\mathrm{d}s+m^{1/2}\int_{z_k(y_k^2)}^{z_k(y_k^3)}s\mathrm{d}s,\\ &\geq m^{1/2}\left(\frac{z_k^2(y_k^2)-z_k^2(y_k^1)}{2}+\frac{z_k^2(y_k^2)-z_k^2(y_k^3)}{2}\right)\end{aligned}$$

By assumption, we have $z_k(y_k^1)\to 0$, $z_k(y_k^2)\to 1$ and $z_k(y_k^3)\to 0$, so that we deduce:

$$\frac{z_k^2(y_k^2)-z_k^2(y_k^1)}{2}+\frac{z_k^2(y_k^2)-z_k^2(y_k^3)}{2}\to 1.$$

We can conclude :

$$\liminf_{k\to\infty}G(u_k,z_k;B_\rho(x))\geq m^{1/2}.$$

□

Generalization to dimension $n \geq 2$

We give the proof of the first inequality of Γ-convergence (3.10) for $n \geq 2$.

Proof. Let $u \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converging almost everywhere to $(u, 0)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$. We have to prove

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq F(u, \mathbf{M}; \Omega). \tag{3.17}$$

We assume that $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega)$ is finite, otherwise the result is ensured. As $\mathbf{M} \in W^{1,n+\alpha}(\Omega)$, then $\mathbf{M}$ is α-Hölder regular. So, there exists $\theta \geq 0$ such that, for any $x, y \in \Omega$, we have

$$\|\mathbf{M}(x) - \mathbf{M}(y)\| \leq \theta |x - y|^\alpha. \tag{3.18}$$

Claim: There exists $C(\delta)$ such that

i) $\lim_{\delta\to 0^+} C(\delta) = 1$,

ii) for $A \subset \Omega$ open, $a \in A$, $\mathrm{diam}(A) \leq \delta$ and $\nu \in \mathbb{S}^{n-1}$, we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A \langle \nabla u, \omega\rangle^2 \; dx + C(\delta) \int_{J_u \cap A} \frac{|\mathbf{M}(a)\nu|}{\langle \mathbf{M}(a)\nu, \nu\rangle^{1/2}} \langle \omega, \nu_u\rangle \; d\mathcal{H}^{n-1},$$

where $\omega = \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}$.

We denote by A an arbitrary open subset of Ω such that $\mathrm{diam}(A) \leq \delta$ and we fix $a \in A$. Let $\nu \in \mathbb{S}^{n-1}$ be fixed. According to Hölder regularity (3.18) and ellipticity inequality (2.1), we have

$$\begin{aligned} \forall (x, \mathbf{v}) \in A \times \mathbb{R}^n, \quad |\langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle| &\leq \theta\delta^\alpha |\mathbf{v}|^2, \\ &\leq \theta\delta^\alpha \lambda^{-1} \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle. \end{aligned}$$

Then, we get

$$\forall (x, \mathbf{v}) \in A \times \mathbb{R}^n, \quad \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \geq (1 - \theta\delta^\alpha\lambda^{-1}) \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle$$

We set $C(\delta) = 1 - \theta\delta^\alpha\lambda^{-1}$. Then, we may write

$$\forall x \in A, \quad \langle \mathbf{M}(x)\nabla z_k(x), \nabla z_k(x)\rangle^2 \geq C(\delta) \langle \mathbf{M}(a)\nabla z_k(x), \nabla z_k(x)\rangle^2.$$

As $\mathbf{M}(a)$ is a symmetric definite positive matrix, Cauchy-Schwartz inequality gives

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{M}(a)\nu, \nu\rangle \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle \geq \langle \mathbf{M}(a)\nu, \mathbf{v}\rangle^2,$$

which is equivalent to

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle \geq \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle} \left\langle \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}, \mathbf{v} \right\rangle^2. \tag{3.19}$$

We set $\omega = \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}$. If we apply inequality (3.19) to $F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A)$, we have

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A \left(|\nabla u_k|^2 (1 - z_k^2)^2 + C(\delta) \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle} \varepsilon_k \langle \omega, \nabla z_k\rangle^2 + \frac{z_k^2}{4\varepsilon_k} \right) \mathrm{d}x.$$

With the notation introduced in (1.2), $(v)_y$ is the function defined on A_ω^y as $(v)_y(t) = v(y + t\omega)$. According to Lemma 3.1, we have $\nabla(u_k)_y(t) = \langle \nabla u(y + t\omega), \omega\rangle$ and $\nabla(z_k)_y(t) = \langle \nabla z(y + t\omega), \omega\rangle$, so Fubini Theorem gives

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} \int_{A_\omega^y} \left(|\nabla(u_k)_y|^2 (1 - ((z_k)_y)^2)^2 + C(\delta) \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle} \varepsilon_k |\nabla(z_k)_y|^2 + \frac{((z_k)_y)^2}{4\varepsilon_k} \right) \mathrm{d}t \; \mathrm{d}\mathcal{H}^{n-1}(y).$$

With the one-dimensional notations (3.12), it gives

$$F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} G_{\varepsilon_k}((u_k)_y, (z_k)_y; A_\omega^y) \; \mathrm{d}\mathcal{H}^{n-1}(y),$$

where $m = C(\delta)\frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu,\nu\rangle}$ for any $x \in A$. Fatou lemma yields

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} \liminf_{k\to\infty} G_{\varepsilon_k}((u_k)_y, (z_k)_y; A^y_\omega)\ \mathrm{d}\mathcal{H}^{n-1}(y)$$

and then

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} G_-((u)_y; A^y_\omega)\ \mathrm{d}\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A)$ is finite, we deduce that $G_-((u)_y; A^y_\omega)$ is finite for $\mathcal{H}^{n-1}$ almost every $y \in A_\omega$. We may apply Proposition 3.4 with $I = A^y_\omega$ and $u = (u)_y$, it gives that $(u)^y_\omega \in \mathrm{SBV}(A^y_\omega)$ for $\mathcal{H}^{n-1}$ almost every $y \in A_\omega$ and we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_{A_\omega} \left[\int_{A^y_\omega} |\nabla(u)_y|^2 \mathrm{d}t + \mathcal{H}^0(J_{(u)_y} \cap A^y_\omega) m^{\frac{1}{2}}\right]\ \mathrm{d}\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A)$ is finite, Theorem 1.3 implies

$$\int_{A_\omega} \left[\int_{A^y_\omega} |\nabla(u)_y|^2 \mathrm{d}t + \mathcal{H}^0(J_{(u)_y} \cap A^y_\omega) m^{1/2}\right]\ \mathrm{d}\mathcal{H}^{n-1}(y) = \int_\Omega |\langle \nabla u, \omega\rangle|^2 \mathrm{d}x + \int_{J_u\cap A} m^{1/2}\langle \omega, \nu_u\rangle\ \mathrm{d}\mathcal{H}^{n-1}.$$

We deduce

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A |\langle \nabla u, \omega\rangle|^2 \mathrm{d}x + \int_{J_u\cap A} m^{1/2}\langle \omega, \nu_u\rangle\ \mathrm{d}\mathcal{H}^{n-1}.$$

If we replace m and ω by their values, it gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; A) \geq \int_A \langle \nabla u, \omega\rangle^2 \mathrm{d}x + C(\delta)\int_{J_u\cap A} \frac{\langle \mathbf{M}(a)\nu, \nu_u\rangle}{\langle \mathbf{M}(a)\nu, \nu\rangle^{1/2}}\ \mathrm{d}\mathcal{H}^{n-1}$$

and the *Claim* is proved.

The function $x \to \frac{\nabla u(x)}{|\nabla u(x)|}$ is measurable in $U = \{x \in \Omega : \nabla u(x) \neq 0\}$. According to Lusin Theorem (1.45 of [7]), there exists an increasing sequence of compacts $(K_l)_l \subset U$ such that

$$\begin{cases} \mathcal{L}^n(U \setminus K_l) \leq \frac{1}{l}, \\ x \to \frac{\nabla u(x)}{|\nabla u(x)|} \text{ is continuous in } K_l. \end{cases}$$

Thus, for any $x \in K_l$, there exists $r > 0$ such that

$$y \in B(x, r) \quad \Rightarrow \left|\frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|}\right| \leq \frac{1}{l}, \tag{3.20}$$

As a consequence of Besicovitch Covering Theorem (2.18 of [7]), there exists a countable, pairwise disjoint collection of balls $(B_i)_{i\in I}$ satisfying (3.20) such that

$$\forall i \in I, B_i \subset \Omega, \mathrm{diam}(B_i) \leq \delta, \qquad \mathcal{L}^n\left(K_l \setminus \bigcup_{i\in I} B_i\right) = 0.$$

For any $i \in I$, we fix $x_i \in B_i$ and we set $\nu_i = \frac{(\mathbf{M}(a))^{-1}\nabla u(x_i)}{|(\mathbf{M}(a))^{-1}\nabla u(x_i)|}$. According to *First Step*, with $A = B_i$, $a = x_i$ and $\nu = \nu_i$, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; B_i) \geq \int_{B_i} \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2\ \mathrm{d}x.$$

Moreover, we have

$$\begin{aligned} \forall x \in B_i \cap K_l, \quad \left||\nabla u|^2 - \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2\right| &\leq \left|\langle \nabla u, \frac{\nabla u}{|\nabla u|}\rangle^2 - \langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\rangle^2\right|, \\ &\leq \left|\langle \nabla u, \frac{\nabla u}{|\nabla u|} - \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\rangle\langle \nabla u, \frac{\nabla u}{|\nabla u|} + \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\rangle\right|, \\ &\leq \frac{2}{l}|\nabla u|^2. \end{aligned}$$

It gives

$$\int_{B_i\cap K_l}\left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2 \,\mathrm{d}x \geq \frac{l}{l+2}\int_{B_i\cap K_l}|\nabla u|^2\mathrm{d}x.$$

As $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M};\,\cdot\,)$ is supperadditive and non decreasing, we have

$$\begin{aligned}
\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) &\geq \sum_{i\in I}\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; B_i),\\
&\geq \sum_{i\in I}\int_{B_i}\left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2 \,\mathrm{d}x,\\
&\geq \sum_{i\in I}\int_{B_i\cap K_l}\left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2 \,\mathrm{d}x,\\
&\geq \frac{l}{l+2}\sum_{i\in I}\int_{B_i\cap K_l}|\nabla u|^2\,\mathrm{d}x,\\
&\geq \frac{l}{l+2}\int_{\cup_i B_i\cap K_l}|\nabla u|^2\,\mathrm{d}x,
\end{aligned}$$

As $\mathcal{L}^n(K_l\setminus\cup_i B_i)=0$, we deduce

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq \frac{l}{l+2}\int_{K_l}|\nabla u|^2\,\mathrm{d}x$$

and taking the limit $l\to\infty$ gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq \int_U |\nabla u|^2\,\mathrm{d}x = \int_\Omega |\nabla u|^2\,\mathrm{d}x.$$

In particular, $\int_\Omega|\nabla u|^2\,\mathrm{d}x$ is finite. As u belongs to $\mathrm{SBV}(\Omega)$, according to Theorem 1.1, there exists a pairwise disjoint family $(C_i)_{i\in\mathbb{N}}$ of $\mathcal{C}^1$ compact manifolds and $M\in\Omega$ such that:

$$J_u = \mathcal{N}\cup\left(\bigcup_{i\in\mathbb{N}} C_i\right), \qquad \mathcal{H}^{n-1}(\mathcal{N})=0.$$

As $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega)$ is finite, *First Step* and Theorem 1.3 imply that $\mathcal{H}^{n-1}(J_u)$ is also finite. According to ellipticity inequality (2.1), we deduce that $\int_{J_u}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\,\mathrm{d}\mathcal{H}^{n-1}$ is finite. Then, for a fixed $\delta>0$, there exists $N\in\mathbb{N}$ such that

$$\int_{J_u\setminus\bigcup_{i=1}^N C_i}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\,\mathrm{d}\mathcal{H}^{n-1}\leq\delta. \tag{3.21}$$

We set $K=\bigcup_{i=1}^N C_i$ and $K_\tau=\{x\in\Omega\colon \mathrm{dist}(x,K)<\tau\}$. As $\int_\Omega|\nabla u|^2\,\mathrm{d}x$ is finite, there exists $\tau>0$ such that

$$\int_{\Omega\setminus K_\tau}|\nabla u|^2\,\mathrm{d}x\leq\delta. \tag{3.22}$$

With the same arguments as before, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega\setminus\overline{K_\tau}) \geq \int_{\Omega\setminus\overline{K_\tau}}|\nabla u|^2\,\mathrm{d}x. \tag{3.23}$$

As $x\to\frac{\mathbf{M}(x)\nu_u(x)}{\langle\mathbf{M}(x)\nu_u(x),\nu_u(x)\rangle^{1/2}}$ is continuous in K, for any $x\in K$ there exists $r>0$ such that

$$y\in B(x,r)\cap K \quad\Rightarrow \left|\frac{\mathbf{M}(x)\nu_u(x)}{\langle\mathbf{M}(x)\nu_u(x),\nu_u(x)\rangle^{1/2}} - \frac{\mathbf{M}(y)\nu_u(y)}{\langle\mathbf{M}(y)\nu_u(y),\nu_u(y)\rangle^{1/2}}\right|\leq\delta. \tag{3.24}$$

As a consequence of Besicovitch Covering Theorem (2.18 of [7]), there exists a countable, pairwise disjoint, collection of balls $(\widetilde{B_j})_{j\in\tilde{I}}$ satisfying (3.24) such that, for any $j\in\tilde{I}$, it satisfies

$$\widetilde{B_j}\subset K_\tau, \quad \mathrm{diam}(\widetilde{B_j})\leq\delta, \quad \mathcal{H}^{n-1}\left(K\setminus\bigcup_{j\in\tilde{I}}\widetilde{B_j}\right)=0.$$

For any $j \in \tilde{I}$, we fix $\tilde{x}_j \in \widetilde{B_j}$. According to *First Step*, with $A = \widetilde{B_j}$, $a = \tilde{x}_j$ and $\nu = \nu_u(\tilde{x}_j)$, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \widetilde{B_j}) \geq C(\delta) \int_{J_u \cap \widetilde{B_j}} \frac{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u\rangle}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{1/2}} \, \mathrm{d}\mathcal{H}^{n-1}.$$

For any $x \in \widetilde{B_j} \cap K$, we have

$$\begin{aligned} \left| \frac{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(x)\rangle}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{1/2}} - \langle \mathbf{M}(x)\nu_u(x), \nu_u(x)\rangle^{1/2} \right| &\leq \left| \frac{\mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j)}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{1/2}} - \frac{\mathbf{M}(x)\nu_u(x)}{\langle \mathbf{M}(x)\nu_u(x), \nu_u(x)\rangle^{1/2}} \right|, \\ &\leq \delta. \end{aligned}$$

It gives

$$\int_{\widetilde{B_j}\cap K} \frac{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u\rangle}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{1/2}} \, \mathrm{d}\mathcal{H}^{n-1} \geq \int_{\widetilde{B_j}\cap K} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\widetilde{B_j} \cap K).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \,\cdot\,)$ is supperaddditive and non decreasing, we have

$$\begin{aligned} \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; K_\tau) &\geq \sum_{j\in\tilde{I}} \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \widetilde{B_j}), \\ &\geq \sum_{j\in\tilde{I}} C(\delta) \left(\int_{\widetilde{B_j}\cap K} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\widetilde{B_j} \cap K) \right), \\ &\geq C(\delta) \left(\int_{\cup_j \widetilde{B_j}\cap K} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\cup_j \widetilde{B_j} \cap K) \right). \end{aligned}$$

As $\mathcal{L}^n(K \setminus \widetilde{B_j}) = 0$, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; K_\tau) \geq C(\delta) \left(\int_K \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(K) \right). \tag{3.25}$$

According to (3.23) and (3.25), we deduce

$$\begin{aligned} \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) &\geq \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; K_\tau) + \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega \setminus \overline{K_\tau}), \\ &\geq C(\delta) \left(\int_K \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(K) \right) + \int_{\Omega\setminus\overline{K_\tau}} |\nabla u|^2 \, \mathrm{d}x. \end{aligned}$$

According to (3.22) and (3.21), we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq C(\delta) \left(\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} - \delta - \delta\mathcal{H}^{n-1}(K) \right) + \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \delta.$$

Letting $\delta \to 0^+$ concludes the proof

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}; \Omega) \geq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \, \mathrm{d}\mathcal{H}^{n-1} + \int_\Omega |\nabla u|^2 \, \mathrm{d}x.$$

□

3.3.2 The inequality for the *higher* Γ-limit

In this section we prove the *upper* inequality of Γ-convergence, that is ii) of Theorem 3.2. Let $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{M} \in W^{1,n+\alpha}(\Omega; \mathcal{G})$ be fixed. It is sufficient to prove that there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere, $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and

$$\limsup_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon_k \langle \mathbf{M}_k \nabla z_k, \nabla z_k\rangle + \frac{z_k^2}{4\varepsilon_k} \right) \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

We first prove a weaker result, where $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ is replaced by its approximation with a Minkowski content. Then, with Theorem 1.4, we extend this result to the general setting.

Approximation with anisotropic Minkowski content

We set

$$F_{\varepsilon_k}(u,z,\mathbf{M}) = \int_\Omega |\nabla u|^2(1-z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon_k \langle \mathbf{M}\nabla z, \nabla z\rangle + \frac{z^2}{4\varepsilon_k}\right) \mathrm{d}x$$

and we prove the following

Proposition 3.5. *Assume that the conditions of Theorem 3.1 are satisfied. For $u \in SBV(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{M} \in W^{1,n+\alpha}(\Omega;\mathcal{G})$, there exists a sequence $(u_k,z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k,z_k)_k$ converges to $(u,0)$ almost everywhere $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and*

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k,z_k,\mathbf{M}) \leq \int_\Omega |\nabla u|^2 dx + \mathcal{M}^\star_{\mathbf{M}}(J_u),$$

where $\mathcal{M}^\star_{\mathbf{M}}$ is defined in (1.3).

Proof. Without loss of generality, we may assume that $|\nabla u| \in \mathrm{L}^2(\Omega)$ and $\mathcal{M}^\star_{\mathbf{M}}(J_u) < +\infty$, otherwise the result is obvious. If $u \in \mathrm{W}^{1,2}(\Omega)$, then $J_u = \emptyset$ and the stationary sequence $u_k = u$, $z_k = 0$ is a solution. If $u \notin \mathrm{W}^{1,2}(\Omega)$, then $J_u \neq \emptyset$ and $(1-z_k^2)^2$ has to be infinitesimal near of J_u. For $\rho > 0$, we set

$$(J_u)_\rho = \{x\colon \mathrm{d}_\phi^{J_u}(x) < \rho\}.$$

We separate Ω in three parts:

$$(J_u)_{b_k}, \quad (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}, \quad \Omega \setminus (J_u)_{a_k+b_k}$$

with

$$\left\{ \begin{array}{rcl} a_k &=& -4\varepsilon_k \ln(\varepsilon_k), \\ b_k &=& \varepsilon_k^2. \end{array} \right. \tag{3.26}$$

Let $\Psi_k \in \mathcal{C}_0^\infty(\Omega)$ such that $\Psi_k = 1$ in $(J_u)_{\frac{b_k}{2}}$ and $\Psi_k = 0$ in $\Omega \setminus (J_u)_{b_k}$. We set $u_k = (1-\Psi_k)u$ and then $u_k = u$ in $\Omega \setminus (J_u)_{b_k}$. As $(b_k)_k$ converges to 0 then u_k converges to u almost everywhere. We set $z_k = 1$ in $(J_u)_{b_k}$ and $z_k = \varepsilon_k^2$ in $\Omega \setminus (J_u)_{a_k+b_k}$. In $(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$ we adopt the following construction: we introduce

$$\theta_k(t) = \varepsilon_k^2 \exp\left(\frac{t}{2\varepsilon_k}\right)$$

and we set

$$\tilde{z}_k(t) = \left\{ \begin{array}{ll} 1 & \forall t \in [0;b_k], \\ \theta_k(a_k+b_k-t) & \forall t \in]b_k;a_k+b_k], \\ \varepsilon_k^2 & \forall t \in]a_k+b_k;+\infty[. \end{array} \right. \tag{3.27}$$

This is a continuous and decreasing function defined on $[0;+\infty[$, moreover, for any $t \in]b_k;a_k+b_k[$, it satisfies

$$\varepsilon_k(\tilde{z}_k'(t))^2 = \frac{(\tilde{z}_k(t))^2}{4\varepsilon_k}. \tag{3.28}$$

We set $z_k = \tilde{z}_k \circ \mathrm{d}_\phi^{J_u}$. As z_k is constant in $(J_u)_{b_k} \cup (\Omega \setminus (J_u)_{a_k+b_k})$, we have

$$\begin{aligned} F_{\varepsilon_k}(u_k,z_k,\mathbf{M}) &= \int_{\Omega\setminus(J_u)_{a_k+b_k}} |\nabla u|^2(1-\varepsilon_k^4)^2 \mathrm{d}x + \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} |\nabla u|^2(1-z_k^2)^2 \mathrm{d}x \\ &+ \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} \left(\varepsilon_k\langle \mathbf{M}\nabla z_k, \nabla z_k\rangle + \frac{z_k^2}{4\varepsilon_k}\right) \mathrm{d}x \\ &+ \frac{\varepsilon_k^3}{4}\mathcal{L}^n(\Omega \setminus (J_u)_{a_k+b_k}) + \frac{1}{4\varepsilon_k}\mathcal{L}^n((J_u)_{b_k}) \end{aligned} \tag{3.29}$$

As $|\nabla u| \in \mathrm{L}^2(\Omega)$ and $(a_k+b_k)_k$ converges to 0, the first term of (3.29) converges to $\int_\Omega |\nabla u|^2 \mathrm{d}x$. As $\|z_k\|_{L^\infty} \leq 1$, the second term converges to 0. As Ω is a bounded domain, the fourth term converges to 0. As $\mathcal{M}^\star_{\mathbf{M}}(J_u) < +\infty$, there exists $(\omega_k)_k$ a sequence which converges to 0^+ such that

$$\mathcal{L}^n((J_u)_{b_k}) \leq 2b_k(\mathcal{M}^\star_{\mathbf{M}}(J_u) + \omega_k) \tag{3.30}$$

and then the fifth term is lower than $\frac{1}{2}\varepsilon_k(\mathcal{M}^\star_{\mathbf{M}}(J_u)+\omega_k)$. So, the fifth term converges to 0. To compute the limit of $(F_{\varepsilon_k}(u_k,z_k,\mathbf{M}))_k$, it remains to study the convergence of

$$A_k(z_k)=\int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}}\left(\varepsilon_k\langle\mathbf{M}\nabla z_k,\nabla z_k\rangle+\frac{z_k^2}{4\varepsilon_k}\right)\mathrm{d}x.$$

Ellipticity inequality (2.1) yields

$$\begin{aligned}|\mathrm{d}_\phi^{J_u}(x)-\mathrm{d}_\phi^{J_u}(y)| &\leq \mathrm{d}_\phi(x,y),\\ &\leq \lambda^{-\frac{1}{2}}|x-y|.\end{aligned}$$

So, $\mathrm{d}_\phi^{J_u}$ is Lipschitzian and Rademacher Theorem ensures that $\mathrm{d}_\phi^{J_u}$ exists for almost every $x\in\Omega$, in the sense of the approximate differentiability 3.2. Thus, for almost every $x\in(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}$, we have

$$\nabla z_k=\tilde{z}_k^{'}\circ\mathrm{d}_\phi^{J_u}\ \nabla\mathrm{d}_\phi^{J_u}.$$

It gives

$$A_k(z_k)=\int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}}\left(\varepsilon_k(\tilde{z}_k^{'}\circ\mathrm{d}_\phi^{J_u})^2\langle\mathbf{M}\nabla\mathrm{d}_\phi^{J_u},\nabla\mathrm{d}_\phi^{J_u}\rangle+\frac{(\tilde{z}_k\circ\mathrm{d}_\phi^{J_u})^2}{4\varepsilon_k}\right)\mathrm{d}x.$$

In [12], Theorem 3.2, it is proved that $\langle\mathbf{M}(x)\nabla\mathrm{d}_\phi^{J_u}(x),\nabla\mathrm{d}_\phi^{J_u}(x)\rangle=1$ for almost every x. So, we may write

$$A_k(z_k)=\int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}}\left(\varepsilon_k(\tilde{z}_k^{'}\circ\mathrm{d}_\phi^{J_u})^2+\frac{(\tilde{z}_k\circ\mathrm{d}_\phi^{J_u})^2}{4\varepsilon_k}\right)\langle\mathbf{M}\nabla\mathrm{d}_\phi^{J_u},\nabla\mathrm{d}_\phi^{J_u}\rangle^{1/2}\mathrm{d}x.$$

We may apply Proposition 1.2 with $\Phi=\phi$ and $p=\mathrm{d}_\phi^{J_u}$, it gives

$$A_k(z_k)=\int_{b_k}^{a_k+b_k}\left(\varepsilon_k\tilde{z}_k^{'}(t)^2+\frac{\tilde{z}_k(t)^2}{4\varepsilon_k}\right)\left[\int_\Omega\langle\mathbf{M}D\mathbf{1}_{(J_u)_t},D\mathbf{1}_{(J_u)_t}\rangle^{1/2}\right]\mathrm{d}t. \tag{3.31}$$

We set

$$\begin{cases}\mathcal{H}_{\mathbf{M}}(t) &= \displaystyle\int_\Omega\langle\mathbf{M}D\mathbf{1}_{(J_u)_t},D\mathbf{1}_{(J_u)_t}\rangle^{1/2},\\ \mathcal{A}_{\mathbf{M}}(s) &= \displaystyle\int_0^s\mathcal{H}_{\mathbf{M}}(t)\mathrm{d}t.\end{cases}$$

Applying another time Proposition 1.2 gives

$$\begin{aligned}\mathcal{A}_{\mathbf{M}}(s_2)-\mathcal{A}_{\mathbf{M}}(s_1) &= \int_{s_1}^{s_2}\left[\int_\Omega\langle\mathbf{M}D\mathbf{1}_{(J_u)_t},D\mathbf{1}_{(J_u)_t}\rangle^{1/2}\right]\mathrm{d}t,\\ &= \int_{(J_u)_{s_2}\setminus(J_u)_{s_1}}\langle\mathbf{M}\nabla\mathrm{d}_\phi^{J_u},\nabla\mathrm{d}_\phi^{J_u}\rangle^{1/2}\mathrm{d}x,\\ &= \mathcal{L}^n((J_u)_{s_2}\setminus(J_u)_{s_1}).\end{aligned}$$

So, $\mathcal{A}_{\mathbf{M}}\in\mathrm{W}^{1,1}_{\mathrm{loc}}(]0;+\infty[)$ and $\nabla\mathcal{A}_{\mathbf{M}}=\mathcal{H}_{\mathbf{M}}$ almost everywhere. Using equality (3.28) and then integrating by parts (3.31) gives

$$\begin{aligned}A_k(z_k) &= \int_{b_k}^{a_k+b_k}\left(\varepsilon_k\tilde{z}_k^{'}(t)^2+\frac{\tilde{z}_k(t)^2}{4\varepsilon_k}\right)\mathcal{H}_{\mathbf{M}}(t)\mathrm{d}t,\\ &= \int_{b_k}^{a_k+b_k}\frac{\tilde{z}_k(t)^2}{2\varepsilon_k}\mathcal{H}_{\mathbf{M}}(t)\mathrm{d}t,\\ &= \frac{(a_k+b_k)^2}{2\varepsilon_k}\mathcal{A}_{\mathbf{M}}(a_k+b_k)-\frac{b_k}{2\varepsilon_k}\mathcal{A}_{\mathbf{M}}(b_k)-\frac{1}{\varepsilon_k}\int_{b_k}^{a_k+b_k}\tilde{z}_k^{'}(t)\tilde{z}_k(t)\mathcal{A}_{\mathbf{M}}(t).\end{aligned}$$

The first term obviously converges to 0. As for (3.30), we have

$$\mathcal{A}_{\mathbf{M}}(b_k)\leq 2b_k(\mathcal{M}^\star_{\mathbf{M}}(J_u)+\omega_k)$$

and then the second term converges to 0 too. As $s \to \mathcal{A}_{\mathbf{M}}(s)$ is non decreasing, then

$$\mathcal{A}_{\mathbf{M}}(t) \leq 2t(\mathcal{M}^{\star}_{\mathbf{M}}(J_u) + \omega_k)$$

for any $t \in [b_k; a_k + b_k]$. For the last term, we apply another time this inequality, it gives

$$-\frac{1}{\varepsilon_k}\int_{b_k}^{a_k+b_k} \tilde{z}_k'(t)\tilde{z}_k(t)\mathcal{A}_{\mathbf{M}}(t)\mathrm{d}t \leq -\frac{(\mathcal{M}^{\star}_{\mathbf{M}}(J_u)+\omega_k)}{\varepsilon_k}\int_{b_k}^{a_k+b_k} 2t\tilde{z}_k'(t)\tilde{z}_k(t)\mathrm{d}t. \tag{3.32}$$

Integrating by parts yields

$$\int_{b_k}^{a_k+b_k} 2t\tilde{z}_k'(t)\tilde{z}_k(t)\mathrm{d}t = (a_k+b_k)\tilde{z}_k(a_k+b_k)^2 - b_k\tilde{z}_k(b_k)^2 - \int_{b_k}^{a_k+b_k}\tilde{z}_k(t)^2\mathrm{d}t. \tag{3.33}$$

According to the definitions of (a_k, b_k, z_k) (3.26) and (3.27), we have

$$(a_k+b_k)\tilde{z}_k(a_k+b_k)^2 - b_k\tilde{z}_k(b_k)^2 = o(\varepsilon_k) \tag{3.34}$$

and equation (3.28) gives

$$\begin{aligned}\int_{b_k}^{a_k+b_k}\tilde{z}_k(t)^2\mathrm{d}t &= 2\varepsilon_k\int_{b_k}^{a_k+b_k}|\tilde{z}_k'(t)|\tilde{z}_k(t)\mathrm{d}t,\\ &= \varepsilon_k(1-\varepsilon_k^2).\end{aligned} \tag{3.35}$$

From (3.32), (3.33), (3.34) and (3.35) we deduce that $\limsup_k A_k(z_k) \leq \mathcal{M}^{\star}_{\mathbf{M}}(J_u)$ and, according to the decomposition (3.29), we have

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq \int_{\Omega}|\nabla u|^2 + \mathcal{M}^{\star}_{\mathbf{M}}(J_u).$$

To conclude the proof, it suffices to notice that $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$.

□

Approximation in the general setting

The goal of this section is to replace $\mathcal{M}^{\star}_{\mathbf{M}}(J_u)$ by $\int_{J_u}\langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1}$ in Proposition 3.5.

Proposition 3.6. *Assume that the conditions of Theorem 3.1 are satisfied. For $u \in SBV(\Omega)\cap L^{\infty}(\Omega)$ and $\mathbf{M} \in W^{1,n+\alpha}(\Omega;\mathcal{G})$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere $(u_k)_k$ is bounded in $L^{\infty}(\Omega)$ and*

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq F(u, \mathbf{M}).$$

To prove this result, we need to introduce the following

Definition 3.2. *Let $\mathcal{F}(\Omega)$ be the set of functions $u \in SBV(\Omega)$ for which, if $F(u,\mathbf{M}) < +\infty$, then there exists a sequence $(u_k)_k \subset SBV(\Omega)\cap L^{\infty}(\Omega)$ converging almost everywhere to u, $\lim_{k\to\infty} F(u_k, \mathbf{M}) = F(u, \mathbf{M})$ and*

$$\forall k \in \mathbb{N}, \quad \mathcal{M}_{\mathbf{M}}(J_{u_k}) = \int_{J_{u_k}}\langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle d\mathcal{H}^{n-1}.$$

Proof. Assume $\mathcal{F}(\Omega) = \mathrm{SBV}(\Omega)$. According to Proposition 3.5, by a diagonal extraction we may exhibit a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere and

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k, \mathbf{M}) \leq F(u, \mathbf{M}).$$

So, to prove the *upper* inequality of Γ-convergence, it suffices to show that $\mathcal{F}(\Omega) = \mathrm{SBV}(\Omega)$. We divide the proof in three *Claims*.

By a direct diagonal extraction process, we have the following.

Claim 1: If $u \in SBV(\Omega)$ and $(u_k)_k \subset SBV(\Omega)$ satisfy

i) $(u_k)_k \subset \mathcal{F}(\Omega)$,

ii) $\lim_{k\to\infty} F(u_k, \mathbf{M}) = F(u, \mathbf{M})$ *and* $F(u, \mathbf{M}) < \infty$,

iii) $(u_k)_k$ *converges to* u *almost everywhere,*

then $u \in \mathcal{F}(\Omega)$.

Claim 2: It suffices to prove that $SBV(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}(\Omega)$.

For $u \in \mathrm{SBV}(\Omega)$ and $N > 0$, we denote by $\overline{u}^N$ the truncated function defined in (3.1). So, $(\overline{u}^N)_N$ converges to u almost everywhere for $N \to \infty$. Moreover, Theorem 1.2 gives

$$D\overline{u}^N = \mathbf{1}_{-N\le u\le N}\nabla u\mathcal{L}^n + ((\overline{u}^N)^+ - (\overline{u}^N)^-)\nu_u\mathcal{H}^{n-1}\llcorner J_u.$$

and then we deduce $\lim_{N\to\infty} F(\overline{u}^N) = F(u, \mathbf{M})$. According to *Claim 1*, it suffices to prove that $\mathrm{SBV}(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}(\Omega)$.

Claim 3: Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$, *we have* $u \in \mathcal{F}(\Omega)$.

We may assume that $F(u, \mathbf{M}) < +\infty$, otherwise the result is ensured. Let us extend $\mathbf{M}$ and u in $\Omega' = \Omega \cup U$ as in Proposition 1.1, so we have

$$\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0.$$

With the same arguments as for Theorem 2.1, we may prove that there exists $v_k \in \mathrm{SBV}(\Omega')$ a minimizer of the following functional:

$$E^{u,k}(v) = k\int_{\Omega'}(v-u)^2\mathrm{d}x + \int_{\Omega'}|\nabla v|^2\mathrm{d}x + \int_{J_v}\langle \mathbf{M}\nu_v, \nu_v\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

In particular, $E^{u,k}(v_k) \le E^{u,k}(u)$ for any $k \in \mathbb{N}$, gives

$$k\int_{\Omega'}(v_k-u)^2\mathrm{d}x \le F(u, \mathbf{M})$$

and then $(v_k)_k$ converges to u almost everywhere. As $u \in L^\infty(\Omega)$, Theorem 1.4 gives

$$\mathcal{M}^\star_{\mathbf{M}}(J_{v_k}) = \int_{J_{v_k}}\langle \mathbf{M}\nu_{v_k}, \nu_{v_k}\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

We introduce the sequence of positive Radon measures $(\mu_k)_k$ and μ defined for any $B \in \mathcal{B}(\Omega')$ by

$$\begin{aligned}\mu_k(B) &= \int_B |\nabla v_k|^2\mathrm{d}x + \int_{J_{v_k}\cap B}\langle \mathbf{M}\nu_{v_k}, \nu_{v_k}\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1},\\ \mu(B) &= \int_B |\nabla u|^2\mathrm{d}x + \int_{J_u\cap B}\langle \mathbf{M}\nu_u, \nu_u\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

With the same arguments as for Theorem 2.1, F is lower semi-continuous in SBV, for any open $A \subset \Omega'$, it gives

$$\liminf_{k\to\infty}\mu_k(A) \ge \mu(A).$$

The inequality

$$\limsup_{k\to\infty}\mu_k(\Omega') \le \mu(\Omega')$$

follows by the definition of v_k. According to [7], Proposition 1.80, the measures $(\mu_k)_k$ weakly converge to μ. Thus, $(\mu_k(B))_k$ converges to $\mu(B)$ if $\mu(\partial B) = 0$, and then $(\mu_k(\Omega))_k$ converges to $\mu(\Omega)$, that is $(F(v_k))_k$ converges to $F(u, \mathbf{M})$. According to *Claim 1*, we deduce that $u \in \mathcal{F}(\Omega)$.

□

3.4 Proof of Theorem 3.1 *ii)*

Proof. With the same arguments as for Theorem 3.1 *i)*, for any $\varepsilon > 0$, there exists $(u_\varepsilon, z_\varepsilon)$ a minimizer of $E_\varepsilon(\cdot,\cdot,\mathbf{M})$. According to (3.3), with $N \geq \|g\|_{L^\infty(\Omega)}$, we have

$$\mathcal{L}^n(\{x \in \Omega\colon |u_\varepsilon(x)| > N\}) > 0 \quad \Rightarrow \quad E_\varepsilon(\overline{u}_\varepsilon^N, z_\varepsilon, \mathbf{M}) < E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}).$$

We deduce that $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq N$ for any $\varepsilon > 0$. For $\omega_\varepsilon = u_\varepsilon(1 - z_\varepsilon^2)$, we get

$$\nabla\omega_\varepsilon = \nabla u_\varepsilon(1 - z_\varepsilon^2) - 2u_\varepsilon z_\varepsilon \nabla z_\varepsilon.$$

It yields

$$\int_\Omega |\nabla\omega_\varepsilon| \mathrm{d}x \leq \mathcal{L}^n(\Omega)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_\varepsilon|^2 (1 - z_\varepsilon^2)^2 \mathrm{d}x \right)^{\frac{1}{2}} + 2N \int_\Omega |\nabla z_\varepsilon| z_\varepsilon \mathrm{d}x. \tag{3.36}$$

Applying the inequality $2ab \leq a^2 + b^2$ with $a = \frac{z_\varepsilon^2}{2\varepsilon^{\frac{1}{2}}}$ and $b = \varepsilon^{\frac{1}{2}} |\nabla z_\varepsilon|$ gives

$$\int_\Omega |\nabla z_\varepsilon| z_\varepsilon \mathrm{d}x \leq \int_\Omega \varepsilon |\nabla z_\varepsilon|^2 \mathrm{d}x + \int_\Omega \frac{z_\varepsilon^2}{4\varepsilon} \mathrm{d}x. \tag{3.37}$$

According to ellipticity inequality (2.1), we get

$$\int_\Omega \varepsilon |\nabla z_\varepsilon|^2 \mathrm{d}x \leq \frac{1}{\lambda} E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}). \tag{3.38}$$

By (3.36), (3.37) and (3.38), we deduce

$$\int_\Omega |\nabla\omega_\varepsilon| \mathrm{d}x \leq \mathcal{L}^n(\Omega)^{\frac{1}{2}} \left(E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}) \right)^{\frac{1}{2}} + \left(1 + \frac{1}{\lambda}\right) E_\varepsilon(u_\varepsilon, z_\varepsilon, \mathbf{M}).$$

According to Proposition 3.6, we deduce that $(E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M}))_k$ is a bounded sequence. So, $(\omega_{\varepsilon_k})_k$ is bounded in $\mathrm{BV}(\Omega)$ and there exists a subsequence, still denoted by $(\omega_{\varepsilon_k})_k$ which converges almost everywhere to $\omega \in \mathrm{BV}(\Omega)$. As $\int_\Omega z_k^2 \, \mathrm{d}x \leq \varepsilon_k E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M})$, then $(z_k)_k$ converges to 0 in $\mathrm{L}^2(\Omega)$ and there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to 0. As $\omega_{\varepsilon_k} = u_{\varepsilon_k}(1 - z_{\varepsilon_k}^2)$, then $(u_{\varepsilon_k})_k$ converges almost everywhere to $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$.

With the same arguments as for Theorem 2.1, $E(\cdot, \mathbf{M})$ admits a minimizer $v \in \mathrm{SBV}(\Omega)$ and $v \in L^\infty(\Omega)$. According to Theorem 3.2 *ii)*, there exists $(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k})_k \subset \mathcal{D}_n(\Omega)$ such that $(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k})_k$ converges to $(v, 0)$ almost everywhere and

$$\limsup_{k\to\infty} E_{\varepsilon_k}(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k}, \mathbf{M}) \leq E(v, \mathbf{M}).$$

According to Theorem 3.2, *i)*, we get

$$\liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M}) \geq E(u, \mathbf{M}).$$

As $(u_{\varepsilon_k}, z_{\varepsilon_k})$ is a minimizer of E_{ε_k}, we have

$$\forall k \in \mathbb{N}, \quad E_{\varepsilon_k}(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k}, \mathbf{M}) \geq E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}, \mathbf{M}).$$

We conclude that $E(v, \mathbf{M}) \geq E(u, \mathbf{M})$ and then $(u, \mathbf{M})$ is also a minimizer of E.

□

4 Appendix

4.1 Proof of Lemma 3.1

Proof. As Ω is bounded, then $\mathrm{W}^{1,2}(\Omega) \subset \mathrm{SBV}(\Omega)$ and then, according to Calderón-Zygmund Theorem (3.83 in [7]), the derivative in the Sobolev sense is equal to the approximate differential for almost every point in Ω. Moreover, according to Theorem 3.107 of [7], for $v \in \mathrm{SBV}(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$, we have

$$\langle \nabla v(x + t\nu), \nu \rangle = \nabla v_x(t) \text{ a.e. } t \in \Omega_x$$

for almost every $x \in \Omega_\nu$. So, we have

$$\begin{aligned} v \in \mathrm{W}^{1,2}(\Omega) &\Rightarrow \int_\Omega \left(v^2 + |\nabla v|^2\right) \mathrm{d}x < \infty, \\ &\Rightarrow \int_{\Omega_\nu} \int_{\Omega_x} \left(v_x^2 + \langle \nabla v(x+t\nu), \nu\rangle^2\right) \mathrm{d}t\mathrm{d}x < \infty, \\ &\Rightarrow \int_{\Omega_\nu} \int_{\Omega_x} \left(v_x^2 + |\nabla v_x(t)|^2\right) \mathrm{d}t\mathrm{d}x < \infty, \\ &\Rightarrow v_x \in \mathrm{W}^{1,2}(\Omega_x) \text{ for a.e. } x \in \Omega. \end{aligned}$$

Applying this property with $v = z$ and $v = u(1 - z^2)$ gives the result of Lemma 3.1. □

4.2 Proof of Lemma 3.2

Proof. For $]a;b[\subset I \setminus J$, according to Theorem 2.8. of [7], there exists a unique function $\tilde{u} \in \mathcal{C}([a;b])$ such that $u(x) \in \tilde{u}(x)$ for $\mathcal{L}^1$-a.e. $x \in I$ and then

$$u(x^-) := \lim_{y \to x^-} u(y) \quad \text{and} \quad u(x^+) := \lim_{y \to x^+} u(y)$$

exist for any $x \in I$. As J is finite, then $\mathrm{W}^{1,2}(I \setminus J) \subset \mathrm{W}^{1,1}(I \setminus J)$. According to the jump formula, it gives that $Du \in \mathcal{M}(I)$ and we have the decomposition

$$Du = \nabla u \mathcal{L}^1 + (u(x^+) - u(x^-))\mathcal{H}^0 \llcorner J.$$

So, we can conclude that $u \in \mathrm{SBV}(I)$ and $J_u \subset J$.

□

References

[1] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variationnal problems. *Communications on Pure and Applied Mathematics*, XLVII(5):577–685, 1989.

[2] G. Alberti, S. Baldo, and G. Orlandi. Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.*, 54(5):1411–1472, 2005.

[3] F. Ghiraldin. Variational approximation of a functional of Mumford-Shah type in codimension higher than one. *ESIAM: COCV*, 2013.

[4] D. Vicente. Anisotropic Minkowski content and application to minimizers of free discontinuity problems. *HAL Archives Ouvertes*, 2015.

[5] L. Ambrosio and V. M. Tortorelli. Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Communications on Pure and Applied Mathematics*, XLIII:999–1036, 1990.

[6] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Science Publications, 2000.

[7] L. Ambrosio. A compactness theorem for a new class of functions of bounded variation. *Bollettino U.M.I.*, 3-B:8857–881, 1989.

[8] C. Goffman and J. Serrin. Sublinear functions of measures and variational integrals. *Duke Math. J.*, 1964.

[9] G. Dal Maso. Integral representation on bounded variation spaces of Γ-limits of variational integrals. *Manuscripta Mathematica*, 1980.

[10] G. Bellettini, M. Paolini, and S. Venturini. Some results on surface measures in calculus of variations. *Annali di matematica pura ed applicata (IV)*, CLXX:329–359, 1996.