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An Anisotropic Mumford-Shah Model

David Vicente

March 26, 2015

Abstract

We introduce an anisotropic Mumford-Shah functional in dimension $n = 2, 3$. To detect the thin tubular structures of an image, the classical Hausdorff measure in the original model is replaced by an anisotropic surface measure depending on a riemannian metric \mathbf{M} . We then consider a relaxation of this energy in the set of *SBV* functions and we prove that the minimizing problem admits solution under suitable conditions. We also prove that a relaxed solution provides in fact a regular solution to the initial problem.

Introduction

This work is a contribution to the problem of detection of thin structures, namely tubes, in a digital image with dimension $n = 2$ or $n = 3$. In a previous work [1], we have introduced an energy in the binary context. More precisely, we assumed that the image histogram was bimodal. In this paper, we remove this assumption and generalize our previous results. To solve this problem, we modify the so-called Mumford-Shah model [2] by introducing a geometric prior which favors tubes. The domain of the image is an open and bounded set $\Omega \subset \mathbb{R}^n$, $g : \Omega \rightarrow \mathbb{R}$ is a given image with normalized gray level in $[0; 1]$ and the well-known Mumford-Shah energy associated to this image is defined as

$$E(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \mathcal{H}^{n-1}(K),$$

where K a compact subset of Ω and $u \in W^{1,2}(\Omega \setminus K)$. To introduce a geometric prior, we consider \mathbf{M} a riemannian metric associated to g . Formally, \mathbf{M} is a function defined on Ω with values in the set $\mathcal{S}_n^+(\mathbb{R})$ of symmetric positive definite matrices. To favor the detection of a set T which has a *tubular* geometry, at each point of the tube, the unit ball of the metric must be elongated in the tube direction.

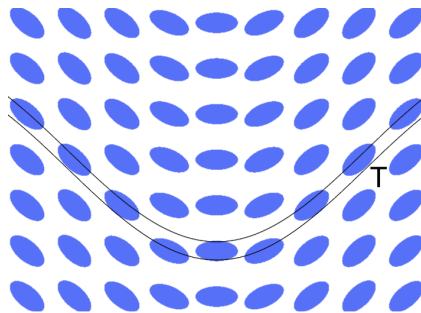


Figure 0.1: Unit Balls for \mathbf{M} adapted to a tube T

The energy associated to this metric can be defined as

$$E_{\mathbf{M}}(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_K \langle \mathbf{M}\nu, \nu \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \quad (0.1)$$

where K is a compact \mathcal{C}^1 -hypersurface, $\nu : K \rightarrow \mathbb{S}^{n-1}$ an unitary, normal vector to K , and $u \in W^{1,2}(\Omega \setminus K)$.

In section 1, we introduce a relaxed formulation of this problem and prove that it admits a solution. In section 2, we show a regularity result and prove that it provides a solution to the initial unrelaxed problem. In section 3, we present various techniques to construct the metric \mathbf{M} .

1 Relaxed problem

In order to prove that the minimization of $E_{\mathbf{M}}$ is a well-posed problem, we introduce a relaxed formulation and prove that the new relaxed problem admits a solution.

1.1 Functional framework

The following definitions and results are taken from [4], chapters 3 and 4. A function $u \in L^1(\Omega)$ is said with *bounded variation*, denoted $u \in BV(\Omega)$, if its derivative, in the sense of the distribution, is a Radon measure.

We are interested by the property for this space to allow functions with *jump* discontinuities. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in \mathbb{R}^n and introduce

$$\begin{cases} B_r^+(x, \nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle > 0\}, \\ B_r^-(x, \nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle < 0\}, \end{cases}$$

for the two half balls contained in the ball $B_r(x) \subset \mathbb{R}^n$ determined by $\nu \in \mathbb{S}^{n-1}$.

Definition 1.1. *Let $u \in L^1(\Omega)$ and $x \in \Omega$. We say that x is an approximate jump point of u if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and*

$$\lim_{r \rightarrow 0^+} \oint_{B_r^+(x, \nu)} |u(y) - a| dy = 0, \quad \lim_{r \rightarrow 0^+} \oint_{B_r^-(x, \nu)} |u(y) - b| dy = 0.$$

The set of approximate jump points is denoted by J_u . The triplet (a, b, ν) , uniquely determined up to a permutation of (a, b) and a change of sign of ν , is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

The set J_u inherits the following structure theorem.

Theorem 1.1. *Let u be a given function in $BV(\Omega)$. Then, J_u is countably $(n - 1)$ -rectifiable. There exists a countable family $(K_i)_i$ of compact \mathcal{C}^1 -hypersurfaces such that $J_u = N \cup (\bigcup_i K_i)$, where $\mathcal{H}^{n-1}(N) = 0$.*

We say that $u \in BV(\Omega)$ is a *special function with bounded variation* and we write $u \in SBV(\Omega)$, if the Cantor part of its derivative is zero, we obtain:

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the density of Du with respect to the Lebesgue measure \mathcal{L}^n , ν_u the normal of the jump set J_u and $\mathcal{H}^{n-1} \llcorner J_u$ the restriction of the Hausdorff measure to the jump set.

1.2 Existence result

For $u \in SBV(\Omega)$, replacing K by J_u in (0.1), we define the relaxed energy by

$$\tilde{E}_{\mathbf{M}}(u) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \quad (1.1)$$

where ∇u , J_u and ν_u are defined in the sense of $SBV(\Omega)$. We denote by $(\tilde{\mathcal{P}}_{\mathbf{M}})$ the relaxed problem

$$(\tilde{\mathcal{P}}_{\mathbf{M}}) : \quad \min\{\tilde{E}_{\mathbf{M}}(u) : u \in SBV(\Omega)\}.$$

We introduce the following constraints on \mathbf{M} :

i) ellipticity:

$$(H_1) : \quad \exists \lambda > 0, \exists \Lambda > 0, \forall (x, v) \in \Omega \times \mathbb{R}^n, \quad \lambda |v|^2 \leq \langle \mathbf{M}(x)v, v \rangle \leq \Lambda |v|^2,$$

ii) Hölder-regularity:

$$(H_2) : \quad \exists \alpha > 0, \exists C \geq 0, \forall (x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq C|x - y|^\alpha.$$

Condition (H_1) is equivalent to the inclusion of \mathbf{M} spectrum in $[\lambda; \Lambda]$. If $\mathbf{M} \in W^{1,r}(\Omega)$ and $p > n$ then, according to Sobolev embedding theorem (see [3], chapter 5), condition (H_2) is satisfied with $\alpha = 1 - \frac{n}{p}$.

In this section, we will prove the following result.

Theorem 1.2. *Let $\tilde{E}_{\mathbf{M}}$ be defined as in (1.1) and \mathbf{M} a metric which satisfies (H_1) and (H_2) . Then, the problem $(\tilde{\mathcal{P}}_{\mathbf{M}})$ admits at least one solution.*

To prove this result, we will use the *direct method* of calculus of variation. The key tools are Theorem 4.8. (compactness) Theorem 4.7. (lower semi-continuity) of [4] in the context of a constant and homogeneous metric (that is $\mathbf{M} \equiv \text{Id}$). Our result is a generalization: \mathbf{M} is not necessary the identity matrix (anisotropy) and may depend on $x \in \Omega$ (non homogeneity).

In the sequel we assume that the hypothesis of Theorem 1.2 are satisfied. We use the weak*-convergence which is defined in [4], definition 3.11.

Lemma 1.1 (Compactness). *Let $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$ be such that $(\tilde{E}_{\mathbf{M}}(u_k))_k$ is bounded. Then, there exists a sequence weakly* convergent to $u \in SBV(\Omega)$.*

Proof. According to ellipticity condition (H_1) , we have

$$\int_{\Omega} |\nabla u_k|^2 dx + \mathcal{H}^{n-1}(J_{u_k}) \leq \max\{1; \lambda^{-\frac{1}{2}}\} \tilde{E}_{\mathbf{M}}(u_k). \quad (1.2)$$

In [4] (Theorem 4.8.), it is proved that the boundedness of the left hand side of (1.2) ensures the existence of a subsequence $(u_k)_k$ weakly* converging in $BV(\Omega)$ to $u \in SBV(\Omega)$. \square

Lemma 1.2 (Lower semicontinuity). *Let $(u_k)_k \subset SBV(\Omega)$ be a weakly* convergent sequence to $u \in SBV(\Omega)$. Then, we have*

$$\tilde{E}_{\mathbf{M}}(u) \leq \liminf_{k \rightarrow \infty} \tilde{E}_{\mathbf{M}}(u_k).$$

Proof. As $(u_k)_k$ weakly* converges to u then it converges in $L^1(\Omega)$ and

$$\lim_{k \rightarrow \infty} \int_{\Omega} (u_k - g)^2 dx = \int_{\Omega} (u - g)^2 dx. \quad (1.3)$$

We may assume that $(\tilde{E}_{\mathbf{M}}(u_k))_k$ is bounded, otherwise the result is ensured. So, according to inequality (1.2), $\int_{\Omega} |\nabla u_k|^2 dx + \mathcal{H}^{n-1}(J_{u_k})$ is bounded with respect to k . With [4] (Theorem 4.7.), it implies that

$$\int_{\Omega} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 dx, \quad (1.4)$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k \rightarrow \infty} \mathcal{H}^{n-1}(J_{u_k}). \quad (1.5)$$

According to (1.3), (1.4) and (1.5), it is sufficient to prove that

$$\int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}. \quad (1.6)$$

This result is proved in [4] (Theorem 5.2.) for an homogeneous and fixed media (if the metric \mathbf{M} does not depend on $x \in \Omega$). In order to overpass this constraint, we introduce a piecewise constant approximation. Then, we apply the result of [4] for each piece of the approximation.

Let $\eta > 0$ be arbitrary small and \mathcal{A} be a finite partition of Ω , such that, for any $A \in \mathcal{A}$, $\text{diam}(A) < \eta$. For each set $A \in \mathcal{A}$, we fix one point $x_A \in A$. We denote \mathbf{M}^A the metric such as its restriction on A is equal to $\mathbf{M}(x_A)$. Moreover, for any vector of the canonical basis $e_i \in \mathbb{S}^{n-1}$, we denote

$$\begin{cases} \Pi_t^i = \{x \in \Omega : \langle x, e_i \rangle = t\}, \\ N_t^i = \{t \in \mathbb{R} : \mathcal{H}^{n-1}(J_u \cap \Pi_t^i) > 0\} \cup \{t \in \mathbb{R} : \exists k \in \mathbb{N}, \mathcal{H}^{n-1}(J_{u_k} \cap \Pi_t^i) > 0\}. \end{cases}$$

As $\mathcal{H}^{n-1}(J_u) < \infty$ (1.5) and $\mathcal{H}^{n-1}(J_{u_k}) < \infty$, then N_t^i is at most countable. So, for any fixed $\eta > 0$, there exists a finite partition \mathcal{A} of Ω such that any $A \in \mathcal{A}$ satisfies

$$\begin{cases} \partial A \subset \bigcup_{i,j} \Pi_{t_{i,j}}^i, \\ \forall (x, y) \in A^2, |x - y| \leq \eta, \\ \mathcal{H}^{n-1}(J_u \cap \partial A) = 0, \\ \forall k \in \mathbb{N}, \mathcal{H}^{n-1}(J_{u_k} \cap \partial A) = 0. \end{cases} \quad (1.7)$$

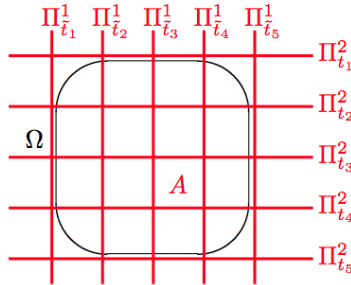


Figure 1.1: Construction of \mathcal{A}

We will estimate the limits of the following integral.

$$\begin{aligned} \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} &= \int_{J_{u_k}} \left(\langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1} \\ &\quad + \int_{J_{u_k}} \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}. \end{aligned} \quad (1.8)$$

Claim 1: The sequence

$$\int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1}$$

converges to 0 uniformly with respect to $k \in \mathbb{N}$ when η converges to 0^+ .

Let be $A \in \mathcal{A}$, $x \in A$ and estimate

$$\begin{aligned} \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} &= \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}}, \\ &= \frac{\langle (\mathbf{M}(x) - \mathbf{M}(x_A))\nu_{u_k}, \nu_{u_k} \rangle}{\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} + \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}}}. \end{aligned} \quad (1.9)$$

According to ellipticity condition (H_1) , it yields

$$\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} + \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \geq 2\lambda^{\frac{1}{2}}, \quad \text{for } \mathcal{H}^{n-1} \llcorner J_{u_k} - \text{a.e. } x \in A. \quad (1.10)$$

According to regularity assumption (H_2) , there exist constants $C > 0$ and $\alpha > 0$ such that

$$\forall x \in A, \quad \|\mathbf{M}(x) - \mathbf{M}(x_A)\| \leq C|x - x_A|^\alpha \leq C\eta^\alpha. \quad (1.11)$$

So, (1.9), (1.10) and (1.11) give

$$\left| \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right| \leq \frac{C\eta^\alpha}{2\lambda^{\frac{1}{2}}}, \quad \text{for } \mathcal{H}^{n-1} \llcorner J_{u_k} - \text{a.e. } x \in A.$$

As \mathcal{A} is a partition of Ω , we have

$$\left| \int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1} \right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_{u_k})}{2\lambda^{\frac{1}{2}}}.$$

As ellipticity condition gives $\mathcal{H}^{n-1}(J_{u_k}) \leq \lambda^{-\frac{1}{2}} \tilde{E}_{\mathbf{M}}(u_k)$, then $(\mathcal{H}^{n-1}(J_{u_k}))_k$ is a bounded sequence and it concludes the proof of *Claim 1*.

Claim 2: We have the following result

$$\int_{J_u} \langle \mathbf{M}^A\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^A\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

We denote by $\overset{\circ}{A}$ the interior of the set A . According to [4], Theorem 5.2., we have

$$\int_{J_u \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

According to (1.7), the energy on the boundaries is null. It gives

$$\int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

So,

$$\begin{aligned} \sum_{A \in \mathcal{A}} \int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} &\leq \sum_{A \in \mathcal{A}} \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \\ &\leq \liminf_{k \rightarrow \infty} \sum_{A \in \mathcal{A}} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}. \end{aligned}$$

As \mathcal{A} is a partition of Ω , this concludes the proof of *Claim 2*:

$$\int_{J_u} \langle \mathbf{M}^A\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^A\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Now, let $\delta > 0$ be an arbitrary small number. According to *Claim 1*, there exists $\eta > 0$ and a partition \mathcal{A} defined as above which satisfies

$$\limsup_{k \rightarrow \infty} \left| \int_{J_{u_k}} \left(\langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1} \right| \leq \delta. \quad (1.12)$$

According to decomposition (1.8), (1.12) and *Claim 2*, we have

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \delta + \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

As for *Claim 2*, we have

$$\left| \int_{J_u} \left(\langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}} \nu_u, \nu_u \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1} \right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_u)}{2\lambda^{\frac{1}{2}}}.$$

We may conclude

$$\int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

□

Recall that we have the following chain rule for $SBV(\Omega)$.

Theorem 1.3. *Let $u \in SBV(\Omega)$ and let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. Then, $v = \varphi \circ u$ belongs to $SBV(\Omega)$ and*

$$Dv = \varphi'(u) \nabla u \mathcal{L}^n + (\varphi(u^+) - \varphi(u^-)) \nu_u \mathcal{H}^{n-1} \llcorner J_u. \quad (1.13)$$

This result is a straightforward consequence of Theorem 3.99 in [4]. It is the key tool for the proof of Theorem 1.2 that follows.

Proof. We denote by $(u_k)_k \subset SBV(\Omega)$ a minimizing sequence for $\tilde{E}_{\mathbf{M}}$. As we assumed that $g \in L^\infty(\Omega)$, we may introduce

$$\forall t \in \mathbb{R}, \quad \varphi(t) = \begin{cases} -\|g\|_{L^\infty(\Omega)} & \text{if } t \leq -\|g\|_{L^\infty(\Omega)}, \\ t & \text{if } |t| \leq \|g\|_{L^\infty(\Omega)}, \\ \|g\|_{L^\infty(\Omega)} & \text{if } t \geq \|g\|_{L^\infty(\Omega)}. \end{cases}$$

We denote $v_k = \varphi \circ u_k$. As the function φ is 1-Lipshitz, we may apply Theorem 1.3, then $v_k \in SBV(\Omega) \cap L^\infty(\Omega)$. According to the decomposition (1.13), we have

$$\forall k, \quad \tilde{E}_{\mathbf{M}}(v_k) \leq \tilde{E}_{\mathbf{M}}(u_k),$$

so $(v_k)_k$ is a minimizing sequence for $\tilde{E}_{\mathbf{M}}$. According to Theorem 1.1, there exists $v \in SBV(\Omega)$ and a subsequence, still denoted $(v_k)_k$ weakly* convergent to v . With Theorem 1.2, we have $\tilde{E}_{\mathbf{M}}(v) \leq \liminf \tilde{E}_{\mathbf{M}}(v_k)$. So, v is a minimizer of $\tilde{E}_{\mathbf{M}}$.

□

2 Regularity result

An important question is to check if a SBV minimizer of the relaxed problem ($\tilde{\mathcal{P}}_{\mathbf{M}}$) is a "classical" one i.e. its jump set is closed and so the function is locally smooth in the complement of the jump set. A positive answer was given by De Giorgi, Carriero and Leaci in [5] for the Mumford-Shah functional. We generalize this result and prove that a

minimizer of the relaxed problem $(\tilde{\mathcal{P}}_{\mathbf{M}})$, whose existence is proved in section 2, provides a minimizer of the original problem

$$(\mathcal{P}_{\mathbf{M}}) : \min\{E_{\mathbf{M}}(u, K) : K \subset \Omega \text{ is a compact } \mathcal{C}^1 \text{ hypersurface, } u \in W^{1,2}(\Omega \setminus K)\}.$$

We give the definition of a *local almost-quasi minimizer of a free discontinuity problem* and a regularity result for its jump set which is proved in [6].

Definition 2.1. *We say that $w \in SBV(U)$ is an almost-quasi minimizer of a free discontinuity problem, if there exists $\Lambda \geq 1$, $\alpha > 0$ and $c_\alpha \geq 0$ such that*

$$\begin{aligned} v \in SBV(U), \quad x \in U, \quad \overline{B_r}(x) \subset U, \quad [w \neq v] \subset B_r(x) \quad \Rightarrow \\ \int_{B_r(x)} |\nabla w|^2 dx + \mathcal{H}^{n-1}(J_w \cap \overline{B_r}(x)) \leq \int_{B_r(x)} |\nabla v|^2 dx + \Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B_r}(x)) + c_\alpha r^{n-1+\alpha}. \end{aligned} \quad (2.1)$$

Theorem 2.1. *Let u be an almost-quasi minimizer of a free discontinuity problem, then $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$.*

We use this key tool to prove the following.

Theorem 2.2. *Let u be a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}})$, then $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$.*

Proof. Let $u \in SBV(\Omega)$ be a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}})$. For $\beta > 0$, we denote

$$\forall x \in \beta\Omega, \quad u_\beta(x) = u\left(\frac{x}{\beta}\right), \quad g_\beta(x) = g\left(\frac{x}{\beta}\right).$$

As

$$\mathcal{H}^{n-1}(\overline{J_{u_\beta}} \setminus J_{u_\beta}) = 0 \Rightarrow \mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$$

then, according to Theorem 2.1, it suffices to prove the following assertion.

Claim : *There exists $\beta > 0$ such that $u_\beta \in SBV(\beta\Omega)$ is an almost-quasi minimizer of a free discontinuity problem*

With the same argument as in the proof of Theorem 1.2, we have $u \in SBV(\Omega) \cap L^\infty(\Omega)$. As u is a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}})$, then u_β is a minimizer of the rescaled problem

$$(\tilde{\mathcal{P}}_{\mathbf{M}}^\beta) : \min\{E_{\mathbf{M}}^\beta(v), v \in SBV(\beta\Omega)\},$$

where

$$\tilde{E}_{\mathbf{M}}^\beta(v) = \beta^2 \int_{\beta\Omega} (v - g_\beta)^2 dx + \int_{\beta\Omega} |\nabla v|^2 dx + \beta \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Let be $v \in SBV(\beta\Omega)$, $x \in \beta\Omega$, $\overline{B_r}(x) \subset \beta\Omega$, $[u_\beta \neq v] \subset B_r(x)$ and $\tilde{v} = \varphi \circ v$, where φ is introduced in the proof of Theorem 1.2. As u_β is a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}}^\beta)$ then $E_{\mathbf{M}}^\beta(u_\beta) \leq E_{\mathbf{M}}^\beta(\tilde{v})$ and it implies

$$\begin{aligned} & \int_{B_r(x)} |\nabla u_\beta|^2 dx + \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \\ & \leq \int_{B_r(x)} |\nabla \tilde{v}|^2 dx + \beta \int_{J_{\tilde{v}} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + \int_{B_r(x)} (\tilde{v} - g_\beta)^2 dx. \end{aligned}$$

Then

$$\begin{aligned} & \int_{B_r(x)} |\nabla u_\beta|^2 dx + \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \\ & \leq \int_{B_r(x)} |\nabla v|^2 dx + \beta \int_{J_v \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_v, \nu_v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + 4\|g\|_{L^\infty(\Omega)}^2 \omega_n r^n, \end{aligned}$$

where $\omega_n = \mathcal{L}^n(B_1(x))$. Now, we set $\beta = \lambda^{-1}$ where λ is the ellipticity coefficient introduced in section 1.1. The left hand side of inequality (H_1) gives

$$\begin{aligned} \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) &= \beta \lambda \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)), \\ &\leq \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M} \nu_{u_\beta}, \nu_{u_\beta} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \end{aligned}$$

so

$$\begin{aligned} &\int_{B_r(x)} |\nabla u_\beta|^2 dx + \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) \\ &\leq \int_{B_r(x)} |\nabla v|^2 dx + \beta \int_{J_v \cap \overline{B_r}(x)} \langle \mathbf{M} \nu_v, \nu_v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + 4 \|g\|_{L^\infty(\Omega)}^2 \omega_n r^n. \end{aligned}$$

The right hand side of (H_1) gives

$$\begin{aligned} &\int_{B_r(x)} |\nabla u_\beta|^2 dx + \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) \\ &\leq \int_{B_r(x)} |\nabla v|^2 dx + \beta \Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B_r}(x)) + 4 \|g\|_{L^\infty(\Omega)}^2 \omega_n r^n. \end{aligned}$$

So, we may conclude that u_β satisfies the definition of an almost quasi-minimizer of a free discontinuity problem and the *Claim* is proved. \square

We deduce from the previous Theorem that a minimizer of the relaxed problem provides a minimizer of the general problem. Moreover, we have

Proposition 2.1. *Let $u \in SBV(\Omega)$ be a minimizer of (\mathcal{P}_M) , then $u \in \mathcal{C}^1(\Omega \setminus \overline{J}_u)$.*

Proof. Let $\overline{B_r}(x) \subset \Omega \setminus \overline{J}_u$; then $u \in W^{1,2}(B_r(x))$ and it is a minimizer of the functional

$$\int_{B_r(x)} (v - g)^2 dx + \int_{B_r(x)} |\nabla v|^2 dx$$

among the functions v in $u + W_0^{1,2}(B_r(x))$ and then classical regularity results give $u \in \mathcal{C}^1(B_r(x))$. \square

3 Construction of M

In the previous sections, we assumed the existence of a riemannian metric \mathbf{M} adapted to the problem of detection of tubes. Moreover, our results are true if ellipticity (H_1) and Holder-regularity (H_2) are satisfied. So, we propose two possible definitions of such a metric which may be used in practice.

3.1 2D Case

We give a definition adapted to dimension 2. For that, we search for an unitary vector field $\mathbf{c} : \Omega \rightarrow \mathbb{S}^{n-1}$ following the direction of the tubes.

We introduce the following functional

$$F(\mathbf{c}) = \int_{\Omega} \langle Dg, \mathbf{c} \rangle^2 dx + \int_{\Omega} |D\mathbf{c}|^r dx$$

and the following minimization problem

$$(\mathcal{P}_c) : \min\{F(\mathbf{c}) : \mathbf{c}(x) \in \mathbb{S}^{n-1} \text{ a.e. } x \in \Omega, \mathbf{c} \in W^{1,r}(\Omega)\}.$$

If we set $r > n$ then, Sobolev embedding Theorem ensures that \mathbf{c} is α -Holder regular with $\alpha > 0$. It is easy to prove that a solution \mathbf{c}_0 of (\mathcal{P}_c) exists and we set

$$\mathbf{M} = \text{Id} + \mu^t \mathbf{c}_0 \mathbf{c}_0,$$

where $\mu > 0$ corresponds to the elongation of the unit ball of \mathbf{M} along the direction $\mathbf{c}(x)$.

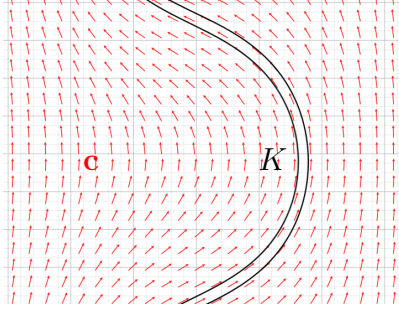
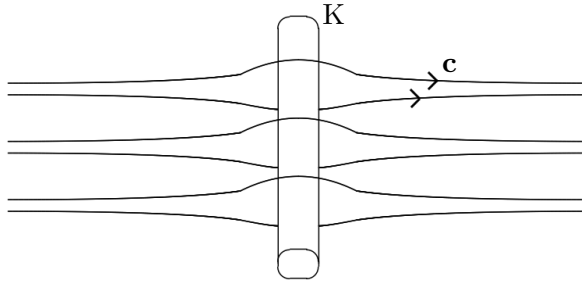


Figure 3.1: Vector field \mathbf{c} along a tube K

3.2 3D Case

In dimension 3, the previous approach is not adapted. In fact, a vector field can avoid *laterally* a tube without penalizing the regularization term $\int_{\Omega} |D\mathbf{c}|^r$.



To overpass this problem, we introduce the second order derivative of \mathbf{H} of g . We introduce the following minimization problem

$$F(\mathbf{M}) = \int_{\Omega} \|\mathbf{M} - \mathbf{H}\|^2 dx + \int_{\Omega} |D\mathbf{M}|^r dx$$

and the following minimization problem

$$(\mathcal{P}_H) : \min\{F(\mathbf{M}) : \mathbf{M} \text{ satisfies } (H_1), (H_2), \mathbf{M} \in W^{1,r}(\Omega)\}.$$

Conclusion

We have introduced a new model and we have proved that the associated minimizing problem is well posed. In a forthcoming work, we will introduce an approximation of this problem with Γ -convergence. It allows us to solve the minimizing problem with PDE technics.

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