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# An Anisotropic Mumford-Shah Model

David Vicente

March 16, 2015

## Abstract

We introduce an anisotropic Mumford-Shah functional in dimension  $n = 2, 3$ . The classical Hausdorff measure with dimension  $n - 1$  is replaced by an anisotropic surface measure depending on a riemannian metric  $\mathbf{M}$ . This term contains the geometric a priori on the image we want to get. We give a relaxation of this energy in the set of *SBV* functions and we prove that with this formulation, the minimizing problem admits solution under suitable conditions. We also prove that a relaxed solution provides in fact a regular solution of the initial problem. Methods are given to construct the metric  $\mathbf{M}$ .

## Introduction

This work is a contribution to the problem of detection of thin structures, namely tubes, in a digital image with dimension  $n = 2$  or  $n = 3$ . In a previous work [1], we have introduced an energy in the binary context. In this paper we remove this assumption and generalize our previous results. To solve this problem, we modify the so-called Mumford-Shah model [2] by introducing a geometric a priori which favors tubes. The domain of the image is an open and bounded set  $\Omega \subset \mathbb{R}^n$ ,  $g : \Omega \rightarrow \mathbb{R}$  is a given image with gray level normalized in  $[0; 1]$  and the Mumford-Shah energy associated to this image is defined as

$$E(u, K) = \int_{\Omega \setminus K} (u - g)^2 + \int_{\Omega \setminus K} |\nabla u|^2 + \mathcal{H}^{n-1}(K),$$

where  $K$  a compact subset of  $\Omega$  and  $u \in W^{1,2}(\Omega \setminus K)$ . To introduce a geometric a priori, we set  $\mathbf{M}$  a riemannian metric associated to  $g$ . Formally,  $\mathbf{M}$  is a function defined on  $\Omega$  which takes its values in the set  $\mathcal{S}_n^+(\mathbb{R})$  of symmetric positive definite matrices. To favor the detection of a set  $T$  which has a tubular geometry, at each point of the tube, the unit ball of the metric must be elongated in direction of the tube.

Then, the energy associated to this metric is defined as

$$E_{\mathbf{M}}(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_K \langle \mathbf{M}\nu, \nu \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \quad (0.1)$$

where  $K$  is a compact hypersurface with class  $\mathcal{C}^1$ ,  $\nu : K \rightarrow \mathbb{S}^{n-1}$  an unitary and normal vector to  $K$ , and  $u \in W^{1,2}(\Omega \setminus K)$ .

In section 1, we will introduce a relaxed formulation of this problem and prove that it admits a solution. In section 2, we will show a regularity result and prove that it provides a solution of the initial problem. In

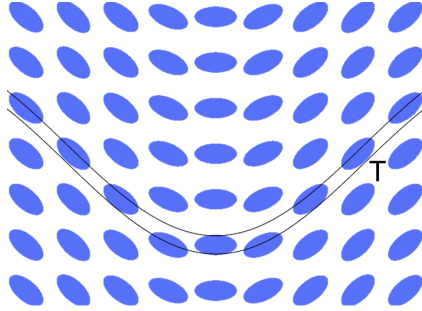


Figure 0.1: Unit Balls for  $\mathbf{M}$  adapted to a tube  $T$

section 3, we will present various techniques to construct the metric  $\mathbf{M}$ . In a forthcoming work, we will introduce an approximation of this problem with  $\Gamma$ -convergence and numerical experimentations.

## 1 Relaxed problem

In order to prove that the minimization of  $E_{\mathbf{M}}$  is a well-posed problem, we will introduce in this section a relaxed formulation and prove that this new formulation admits a solution.

### 1.1 Functional framework

A function  $u \in L^1(\Omega)$  is said with *bounded variation*, denoted  $u \in BV(\Omega)$ , if its derivative, in the sense of the distribution, is a Radon measure. We are interested by the property for this space to allow function with *jump* discontinuities. To this aim we introduce

$$\begin{cases} B_r^+(x, \nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle > 0\}, \\ B_r^-(x, \nu) = \{y \in B_r(x) : \langle y - x, \nu \rangle < 0\}, \end{cases}$$

for the two half balls contained in the ball  $B_r(x)$  determined by  $\nu \in \mathbb{S}^{n-1}$ .

**Definition 1.1.** Let  $u \in L^1(\Omega)$  and  $x \in \Omega$ . We say that  $x$  is an *approximate jump point* of  $u$  if there exist  $a, b \in \mathbb{R}$  and  $\nu \in \mathbb{S}^{n-1}$  such that  $a \neq b$  and

$$\lim_{r \rightarrow 0^+} \oint_{B_r^+(x, \nu)} |u(y) - a| dy = 0, \quad \lim_{r \rightarrow 0^+} \oint_{B_r^-(x, \nu)} |u(y) - b| dy = 0.$$

The set of approximate jump points is denoted by  $J_u$ .

The triplet  $(a, b, \nu)$ , uniquely determined up to a permutation of  $(a, b)$  and a change of sign of  $\nu$ , is denoted by  $(u^+(x), u^-(x), \nu_u(x))$ .

The set  $J_u$  inherits the following structure theorem (see [3], theorem 10.3.4.).

**Theorem 1.1.** Let  $u$  be a given function in  $BV(\Omega)$ . Then,  $J_u$  is countably  $(n-1)$ -rectifiable. That is, there exists a countable family  $(K_i)_i$  of compact hypersurface with class  $\mathcal{C}^1$  such that  $J_u = N \cup (\bigcup_i K_i)$ , where  $\mathcal{H}^{n-1}(N) = 0$ .

We say that  $u \in BV(\Omega)$  is a *special function with bounded variation* and we write  $u \in SBV(\Omega)$ , if the Cantor part of its derivative is zero, we obtain (see [4]):

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where  $\nabla u$  is the density of  $Du$  with respect to the Lebesgue measure  $\mathcal{L}^n$ ,  $\nu_u$  the normal of the jump set  $J_u$  and  $\mathcal{H}^{n-1} \llcorner J_u$  the restriction of the Hausdorff measure to the jump set. For  $u \in SBV(\Omega)$ , replacing  $K$  by  $J_u$  in 0.1, we define its relaxed energy by

$$\tilde{E}_{\mathbf{M}}(u) = \int_{\Omega} (u - g)^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \quad (1.1)$$

where  $\nabla u$ ,  $J_u$  and  $\nu_u$  are defined in the sense of  $SBV(\Omega)$ . We denote by  $(\mathcal{P}_r)$  the relaxed problem

$$(\tilde{\mathcal{P}}_{\mathbf{M}}) : \quad \min\{\tilde{E}_{\mathbf{M}}(u) : u \in SBV(\Omega)\}.$$

We introduce the following constraints on  $\mathbf{M}$ :

i) ellipticity:

$$(H_1) : \quad \exists \lambda > 0, \exists \Lambda > 0, \forall (x, v) \in \Omega \times \mathbb{R}^n, \quad \lambda |v|^2 \leq \langle \mathbf{M}(x)v, v \rangle \leq \Lambda |v|^2,$$

ii) Hölder-regularity:

$$(H_2) : \quad \exists \alpha > 0, \exists C > 0, \forall (x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq C|x - y|^\alpha.$$

## 1.2 Existence result

In this section, we will prove the following theorem.

**Theorem 1.2.** *Let  $\tilde{E}_{\mathbf{M}}$  as defined in (1.1) and  $\mathbf{M}$  a metric which satisfies  $(H_1)$  and  $(H_2)$ . Then, the problem  $(\tilde{\mathcal{P}}_{\mathbf{M}})$  admits at least one solution.*

To prove this result, we will use the *direct method* of calculus of variation. The key tools are the theorem 4.8. (compactness) and theorem 4.7. (lower semi-continuity) of [4] in the context of a constant and homogeneous metric (that is  $\mathbf{M} \equiv \text{Id}$ ). Our result is a generalization with  $\mathbf{M}$  not necessary the identity (anisotropy) and depending on  $x \in \Omega$  (non homogeneity).

In the two following results we assume that the hypothesis of theorem 1.2 are satisfied.

**Proposition 1.1** (Compactness). *Let  $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$  be such that  $(\tilde{E}_{\mathbf{M}}(u_k))_k$  is bounded. Then, there exists a sequence weakly\* convergent to  $u \in SBV(\Omega)$ .*

*Proof.* According to ellipticity condition  $(H_1)$ , we have

$$\int_{\Omega} |\nabla u_k|^2 + \mathcal{H}^{n-1}(J_{u_k}) \leq \max\left\{1; \lambda^{-\frac{1}{2}}\right\} \tilde{E}_{\mathbf{M}}(u_k). \quad (1.2)$$

In [4] (theorem 4.8.), it is proved that the boundedness of the left hand side of (1.2) ensures the existence of a subsequence  $(u_k)_k$  weakly\* converging in  $BV(\Omega)$  to  $u \in SBV(\Omega)$ . □

**Proposition 1.2** (Lower semicontinuity). *Let  $(u_k)_k \subset SBV(\Omega)$  be a weakly\* convergent sequence to  $u \in SBV(\Omega)$ . Then, we have*

$$\tilde{E}_{\mathbf{M}}(u) \leq \liminf_{k \rightarrow \infty} \tilde{E}_{\mathbf{M}}(u_k).$$

*Proof.* As  $(u_k)_k$  weakly\* converges to  $u$  then it converges in  $L^1(\Omega)$ , it gives

$$\lim_{k \rightarrow \infty} \int_{\Omega} (u_k - g)^2 dx = \int_{\Omega} (u - g)^2 dx. \quad (1.3)$$

We may assume that  $(\tilde{E}_{\mathbf{M}}(u_k))_k$  is bounded, otherwise the result is ensured. So, according to inequality (1.2),  $\int_{\Omega} |\nabla u_k|^2 dx + \mathcal{H}^{n-1}(J_{u_k})$  is bounded with respect to  $k$ . In [4] (theorem 4.7.), it implies the two following limits:

$$\int_{\Omega} |\nabla u|^2 dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 dx, \quad (1.4)$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k \rightarrow \infty} \mathcal{H}^{n-1}(J_{u_k}). \quad (1.5)$$

According to (1.3), (1.4) and (1.5), it suffices to prove that

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}. \quad (1.6)$$

This result is proved in [4] (theorem 5.2.) for an homogeneous and fixed media, that is, if the metric  $\mathbf{M}$  does not depend on  $x \in \Omega$ . In order to overpass these constraint, we introduce a piecewise constant approximation. Then, we apply the result of [4] in each piece of the approximation.

Let  $\eta > 0$  be arbitrary small and  $\mathcal{A}$  be a finite partition of  $\Omega$ , such that, for any  $A \in \mathcal{A}$ ,  $\text{diam}(A) < \eta$ . For each set  $A \in \mathcal{A}$ , we fix one point  $x_A \in A$ . We denote  $\mathbf{M}^A$  the metric such as its restriction on  $A$  is equal to  $\mathbf{M}(x_A)$ . Moreover, for each vector of the canonical basis  $e_i \in \mathbb{S}^{n-1}$ , we denote

$$\begin{cases} \Pi_t^i = \{x \in \Omega : \langle x, e_i \rangle = t\}, \\ N_t^i = \{t \in \mathbb{R} : \mathcal{H}^{n-1}(J_u \cap \Pi_t^i) > 0\} \cup \{t \in \mathbb{R} : \exists k \in \mathbb{N}, \mathcal{H}^{n-1}(J_{u_k} \cap \Pi_t^i) > 0\}. \end{cases}$$

As  $\mathcal{H}^{n-1}(J_u) < \infty$  (1.5) and  $\mathcal{H}^{n-1}(J_{u_k}) < \infty$ , then  $N_t^i$  is at most countable. For any fixed  $\eta > 0$ , we may choose a partition  $\mathcal{A}$  of  $\Omega$  such that any  $A \in \mathcal{A}$  has its boundary  $\partial A$  supported by  $n$  planes  $(\Pi_{t_i}^i)_{i=1 \dots n}$  and which satisfies

$$\forall A \in \mathcal{A}, \forall k \in \mathbb{N}, \quad \mathcal{H}^{n-1}(J_u \cap \partial A) = 0, \quad \mathcal{H}^{n-1}(J_{u_k} \cap \partial A) = 0. \quad (1.7)$$

We will estimate the limits of each of the following integrals.

$$\begin{aligned} \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} &= \int_{J_{u_k}} \left( \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1} \\ &\quad + \int_{J_{u_k}} \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}. \end{aligned} \quad (1.8)$$

*Claim 1: The sequence*

$$\int_{J_{u_k}} \left( \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1}$$

*converges to 0 uniformly with respect to  $k \in \mathbb{N}$  when  $\eta$  converges to  $0^+$ .*

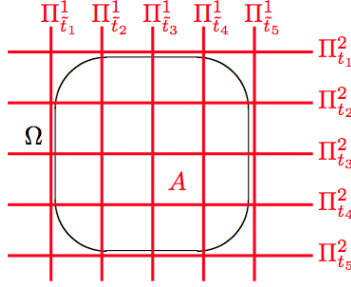


Figure 1.1: Construction of  $\mathcal{A}$

We set  $A \in \mathcal{A}$ ,  $x \in A$  and estimate

$$\begin{aligned} \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} &= \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}}, \\ &= \frac{\langle (\mathbf{M}(x) - \mathbf{M}(x_A))\nu_{u_k}, \nu_{u_k} \rangle}{\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} + \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}}}. \end{aligned} \quad (1.9)$$

According to ellipticity condition  $(H_1)$ , it yields

$$\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} + \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \geq 2\lambda^{\frac{1}{2}}, \quad \text{for } \mathcal{H}^{n-1} \llcorner J_{u_k} \text{-a.e. } x \in A. \quad (1.10)$$

According to regularity assumption  $(H_2)$ , there exist constants  $C > 0$  and  $\alpha > 0$  such that

$$\forall x \in A, \quad \|\mathbf{M}(x) - \mathbf{M}(x_A)\| \leq C|x - x_A|^\alpha \leq C\eta^\alpha. \quad (1.11)$$

So, (1.9), (1.10) and (1.11) give

$$\left| \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A(x)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \right| \leq \frac{C\eta^\alpha}{2\lambda^{\frac{1}{2}}}, \quad \text{for } \mathcal{H}^{n-1} \llcorner J_{u_k} \text{-a.e. } x \in A.$$

As  $\mathcal{A}$  is a partition of  $\Omega$ , we have

$$\left| \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^A\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_{u_k})}{2\lambda^{\frac{1}{2}}}.$$

As ellipticity condition gives  $\mathcal{H}^{n-1}(J_{u_k}) \leq \lambda^{-\frac{1}{2}} \tilde{E}_{\mathbf{M}}(u_k)$ , then  $(\mathcal{H}^{n-1}(J_{u_k}))_k$  is a bounded sequence and it concludes the proof of *Claim 1*.

*Claim 2: We have the following result*

$$\int_{J_u} \langle \mathbf{M}^A\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^A\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

We denote by  $\overset{\circ}{A}$  the interior of the set  $A$ . According to [4], theorem 5.2., we have

$$\int_{J_u \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

According to (1.7), the energy on the boundaries is null, it gives

$$\int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

So,

$$\begin{aligned} \sum_{A \in \mathcal{A}} \int_{J_u \cap A} \langle \mathbf{M}(x_A) \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} &\leq \sum_{A \in \mathcal{A}} \liminf_{k \rightarrow \infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A) \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \\ &\leq \liminf_{k \rightarrow \infty} \sum_{A \in \mathcal{A}} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A) \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}. \end{aligned}$$

As  $\mathcal{A}$  is a partition of  $\Omega$ , we conclude the proof of *Claim 2*

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Let  $\delta > 0$  be an arbitrary small number. According to *Claim 1*, there exists  $\eta > 0$  and a partition  $\mathcal{A}$  defined above which satisfies

$$\limsup_{k \rightarrow \infty} \left| \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \right| \leq \delta. \quad (1.12)$$

According to decomposition (1.8), (1.12) and *Claim 2*, we have

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \delta + \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

As for *Claim 2*, we have

$$\left| \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_u)}{2\lambda^{\frac{1}{2}}}.$$

We may conclude

$$\int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \leq \liminf_{k \rightarrow \infty} \int_{J_{u_k}} \langle \mathbf{M} \nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

□

We have the following chain rule for  $SBV(\Omega)$ .

**Theorem 1.3.** *Let  $u \in SBV(\Omega)$  and let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a Lipschitz function. Then,  $v = \varphi \circ u$  belongs to  $SBV(\Omega)$  and*

$$Dv = \varphi'(u) \nabla u \mathcal{L}^n + (\varphi(u^+) - \varphi(u^-)) \nu_u \mathcal{H}^{n-1} \llcorner J_u. \quad (1.13)$$

This result is a straightforward consequence of theorem 3.99 in [4]. It is the key tool for the following proof of theorem 1.2.

*Proof.* We denote by  $(u_k)_k \subset SBV(\Omega)$  a minimizing sequence for  $\tilde{E}_{\mathbf{M}}$ . As we assumed that  $g \in L^\infty(\Omega)$ , we may introduce

$$\forall t \in \mathbb{R}, \quad \varphi(t) = \begin{cases} -\|g\|_{L^\infty(\Omega)} & \text{if } t \leq -\|g\|_{L^\infty(\Omega)}, \\ t & \text{if } |t| \leq \|g\|_{L^\infty(\Omega)}, \\ \|g\|_{L^\infty(\Omega)} & \text{if } t \geq \|g\|_{L^\infty(\Omega)}. \end{cases}$$

We denote  $v_k = \varphi \circ u_k$ . As the function  $\varphi$  is 1-Lipshitz, we may apply theorem 1.3, then  $v_k \in SBV(\Omega) \cap L^\infty(\Omega)$ . According the decomposition (1.13), we have

$$\forall k, \quad \tilde{E}_{\mathbf{M}}(v_k) \leq \tilde{E}_{\mathbf{M}}(u_k),$$

so  $(v_k)_k$  is a minimizing sequence for  $\tilde{E}_{\mathbf{M}}$ . According to theorem 1.1, there exist  $v \in SBV(\Omega)$  and a subsequence, still denoted  $(v_k)_k$  weakly\* convergent to  $v$ . According to theorem 1.2, we have  $\tilde{E}_{\mathbf{M}}(v) \leq \liminf \tilde{E}_{\mathbf{M}}(v_k)$ . So,  $v$  is a minimizer of  $\tilde{E}_{\mathbf{M}}$ .

□

## 2 Regularity result

An important question is whether an *SBV* minimizer of the relaxed problem  $(\tilde{\mathcal{P}}_{\mathbf{M}})$  is a "classical" one i.e. its jump set is closed and so the function is locally smooth in the complement of the jump set. A positive answer was given by De Giorgi, Carriero and Leaci in [5] for the Mumford-Shah functional. We generalize this result and prove that a minimizer of the relaxed problem  $(\tilde{\mathcal{P}}_{\mathbf{M}})$ , whose existence is proved in section 2, provides a minimizer of the general problem

$$(\mathcal{P}_{\mathbf{M}}) : \min\{E_{\mathbf{M}}(u, K) : K \subset \Omega \text{ is a compact } \mathcal{C}^1 \text{ hypersurface, } u \in W^{1,2}(\Omega \setminus K)\}.$$

We give the definition of *local almost-quasi minimizer of a free discontinuity problem* and a regularity result for its jump set which is proved in [6].

**Definition 2.1.** *We say that  $w \in SBV(U)$  is an almost-quasi minimizer of a free discontinuity problem, if there exists  $\Lambda \geq 1$ ,  $\alpha > 0$  and  $c_{\alpha} \geq 0$  such that*

$$\begin{aligned} v \in SBV(U), \quad x \in U, \quad \overline{B_r}(x) \subset U, \quad [w \neq v] \subset B_r(x) \quad \Rightarrow \\ \int_{B_r(x)} |\nabla w|^2 dx + \mathcal{H}^{n-1}(J_w \cap \overline{B_r}(x)) \leq \int_{B_r(x)} |\nabla v|^2 dx + \Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B_r}(x)) + c_{\alpha} r^{n-1+\alpha}. \end{aligned} \quad (2.1)$$

**Theorem 2.1.** *Let  $u$  be an almost-quasi minimizer of a free discontinuity problem, then  $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$ .*

We use this key tool to prove the following.

**Theorem 2.2.** *Let  $u$  be a minimizer of  $(\tilde{\mathcal{P}}_{\mathbf{M}})$ , then  $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$ .*

*Proof.* Let  $u \in SBV(\Omega)$  a minimizer of  $(\tilde{\mathcal{P}}_{\mathbf{M}})$ . For  $\beta > 0$ , we denote

$$\forall x \in \beta\Omega, \quad u_{\beta}(x) = u\left(\frac{x}{\beta}\right), \quad g_{\beta}(x) = g\left(\frac{x}{\beta}\right).$$

As

$$\mathcal{H}^{n-1}(\overline{J_{u_{\beta}}} \setminus J_{u_{\beta}}) = 0 \Rightarrow \mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$$

then, according to theorem 2.1, it suffices to prove the following assertion.

*Claim : There exists  $\beta > 0$  such that  $u_{\beta} \in SBV(\beta\Omega)$  is an almost-quasi minimizer of a free discontinuity problem*

With the same argument as in the proof of theorem 1.2, we have  $u \in SBV(\Omega) \cap L^{\infty}(\Omega)$ . As  $u$  is a minimizer of  $(\tilde{\mathcal{P}}_{\mathbf{M}})$ , then  $u_{\beta}$  is a minimizer of the rescaled problem

$$(\tilde{\mathcal{P}}_{\mathbf{M}}^{\beta}) : \min\{E_{\mathbf{M}}^{\beta}(v), v \in SBV(\beta\Omega)\},$$

where

$$\tilde{E}_{\mathbf{M}}^{\beta}(v) = \beta^2 \int_{\beta\Omega} (v - g_{\beta})^2 dx + \int_{\beta\Omega} |\nabla v|^2 dx + \beta \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

We set  $v \in SBV(\beta\Omega)$ ,  $x \in \beta\Omega$ ,  $\overline{B_r}(x) \subset \beta\Omega$ ,  $[u_{\beta} \neq v] \subset B_r(x)$  and  $\tilde{v} = \varphi \circ v$ , where  $\varphi$  is introduced in the proof of theorem 1.2. As  $u_{\beta}$  is a minimizer of  $(\tilde{\mathcal{P}}_{\mathbf{M}}^{\beta})$  then  $E_{\mathbf{M}}^{\beta}(u_{\beta}) \leq E_{\mathbf{M}}^{\beta}(\tilde{v})$  and it implies

$$\begin{aligned} & \int_{B_r(x)} |\nabla u_{\beta}|^2 dx + \beta \int_{J_{u_{\beta}} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_{\beta}}, \nu_{u_{\beta}} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \\ & \leq \int_{B_r(x)} |\nabla \tilde{v}|^2 dx + \beta \int_{J_{\tilde{v}} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + \int_{B_r(x)} (\tilde{v} - g_{\beta})^2 dx. \end{aligned}$$



Then

$$\begin{aligned} & \int_{B_r(x)} |\nabla u_\beta|^2 dx + \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \\ & \leq \int_{B_r(x)} |\nabla v|^2 dx + \beta \int_{J_v \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_v, \nu_v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + 4\|g\|_{L^\infty(\Omega)}^2 \omega_n r^n, \end{aligned}$$

where  $\omega_n = \mathcal{L}^n(B_1(x))$ . Now, we set  $\beta = \lambda^{-1}$  where  $\lambda$  is the ellipticity coefficient introduced in section 1.1. The left hand side of inequality  $(H_1)$  gives

$$\begin{aligned} \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) &= \beta \lambda \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)), \\ &\leq \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta} \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}, \end{aligned}$$

so

$$\begin{aligned} & \int_{B_r(x)} |\nabla u_\beta|^2 dx + \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) \\ & \leq \int_{B_r(x)} |\nabla v|^2 dx + \beta \int_{J_v \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_v, \nu_v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + 4\|g\|_{L^\infty(\Omega)}^2 \omega_n r^n. \end{aligned}$$

The right hand side of  $(H_1)$  gives

$$\begin{aligned} & \int_{B_r(x)} |\nabla u_\beta|^2 dx + \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) \\ & \leq \int_{B_r(x)} |\nabla v|^2 dx + \beta \lambda \mathcal{H}^{n-1}(J_v \cap \overline{B_r}(x)) + 4\|g\|_{L^\infty(\Omega)}^2 \omega_n r^n. \end{aligned}$$

So, we may conclude that  $u_\beta$  satisfies the definition of an almost quasi-minimizer of a free discontinuity problem and the *Claim* is proved.  $\square$

We deduce from the previous theorem that a minimizer of the relaxed problem provides a minimizer of the general problem. Moreover, we have

**Proposition 2.1.** *Let  $u \in SBV(\Omega)$  be a minimizer of  $(\mathcal{P}_M)$ , then  $u \in \mathcal{C}^1(\Omega \setminus \overline{J}_u)$ .*

*Proof.* Let  $\overline{B_r}(x) \subset \Omega \setminus \overline{J}_u$ ; then  $u \in W^{1,2}(B_r(x))$  and it is a minimizer of the functional

$$\int_{B_r(x)} (v - g)^2 dx + \int_{B_r(x)} |\nabla v|^2 dx$$

among the functions  $v$  in  $u + W_0^{1,2}(B_r(x))$  and then classical regularity results give  $u \in \mathcal{C}^1(B_r(x))$ .  $\square$

### 3 Construction of M

In the previous sections, we admit the existence of a riemannian metric  $\mathbf{M}$  adapted to the problem of detection of tubes. Moreover, our results are true if ellipticity  $(H_1)$  and Holder-regularity  $(H_2)$  are satisfied. So, we propose two possible definitions of such a metric which may be used in practice.

### 3.1 In dimension 2

We present a definition adapted to dimension 2. For that, we search for an unitary vector field  $\mathbf{c} : \Omega \rightarrow \mathbb{S}^{n-1}$  such that it follows the direction of the tubes.

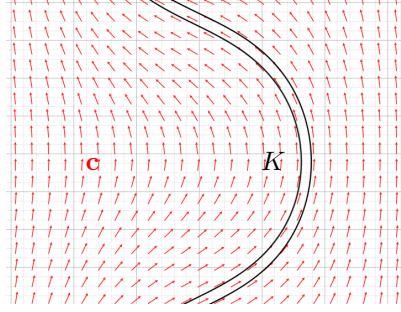


Figure 3.1: Vector field  $\mathbf{c}$  along a tube  $K$

We introduce the following functional

$$F(\mathbf{c}) = \int_{\Omega} \langle Dg, \mathbf{c} \rangle^2 + \int_{\Omega} |D\mathbf{c}|^r$$

and the following minimization problem

$$(\mathcal{P}_c) : \min\{F(\mathbf{c}) : \mathbf{c}(x) \in \mathbb{S}^{n-1} \text{ a.e. } x \in \Omega, \mathbf{c} \in W^{1,r}(\Omega)\}.$$

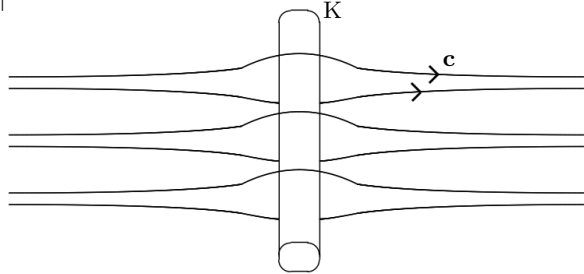
If we set  $r > n$  then, Sobolev embedding theorem ensures that  $\mathbf{c}$  is  $\alpha$ -Holder regular with  $\alpha > 0$ . It is easy to prove that a solution  $\mathbf{c}_0$  of  $(\mathcal{P}_c)$  exists and we set

$$\mathbf{M} = \text{Id} + \mu^t \mathbf{c}\mathbf{c},$$

where  $\mu > 0$  corresponds to the elongation of the unit ball of  $\mathbf{M}$  along the direction  $\mathbf{c}(x)$ .

### 3.2 In dimension 3

In dimension 3, the previous approach is not adapted. In fact, a vector field can avoid *laterally* a tube without penalizing the regularization term  $\int_{\Omega} |$



To overpass this problem, we introduce the second order derivative of  $\mathbf{H}$  of  $g$ . We introduce the following minimization problem

$$F(\mathbf{M}) = \int_{\Omega} \|\mathbf{M} - \mathbf{H}\|^2 + \int_{\Omega} |D\mathbf{M}|^r$$

and the following minimization problem

$$(\mathcal{P}_H) : \min\{F(\mathbf{M}) : \mathbf{M} \text{ satisfies } (H_1), (H_2), \mathbf{M} \in W^{1,r}(\Omega)\}.$$

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## References

- [1] D. Vicente. Anisotropic bimodal energy for segmentation of thin tubes and its approximation with gamma-convergence. *Advances in Calculus of Variation*, 2015.
- [2] D. Mumford and J. Shah. Optimal approximations by piecewise smooth functions and associated variational problems. *Communications on Pure and Applied Mathematics*, XLVII(5):577–685, 1989.
- [3] H. Attouch, G. Buttazzo, and G. Michaille. *Variational Analysis in Sobolev and BV spaces*. SIAM, 2005.
- [4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Science Publications, 2000.
- [5] E. De Giorgi, M. Carriero, and A. Leaci. Existence theorem for a minimum problem with free discontinuity set. *Archive for Rational Mechanics and Analysis*, 108:195–218, 1989.
- [6] D. Bucur and S. Luckhaus. Monocity formula and regularity for general free discontinuity problems. *Archive for Rational Mechanics and Analysis*, 2013.