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Abstract. Let H be a Krull monoid with class group G such that every class contains a prime divisor. Then every nonunit $a \in H$ can be written as a finite product of irreducible elements. If $a = u_1 \cdot \ldots \cdot u_k$, with irreducibles $u_1, \ldots u_k \in H$, then k is called the length of the factorization and the set $\mathsf{L}(a)$ of all possible k is called the set of lengths of a. It is well-known that the system $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ depends only on the class group G. In the present paper we study the inverse question asking whether or not the system $\mathcal{L}(H)$ is characteristic for the class group. Consider a further Krull monoid H' with class group G' such that every class contains a prime divisor and suppose that $\mathcal{L}(H) = \mathcal{L}(H')$. We show that, if one of the groups G and G' is finite and has rank at most two, then G and G' are isomorphic (apart from two well-known pairings).

1. Introduction

Let H be a cancelative semigroup with unit element. If an element $a \in H$ can be written as a product of k irreducible elements, say $a = u_1 \cdot \ldots \cdot u_k$, then k is called the length of the factorization. The set $\mathsf{L}(a)$ of all possible factorization lengths is the set of lengths of a, and $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ is called the system of sets of lengths of H. Clearly, if H is factorial, then $|\mathsf{L}(a)| = 1$ for each $a \in H$. Suppose there is some $a \in H$ with $|\mathsf{L}(a)| > 1$, say $k, l \in \mathsf{L}(a)$ with $k < l$. Then, for every $m \in \mathbb{N}$, we observe that $\mathsf{L}(a^m) \supset \{km + \nu(l-k) \mid \nu \in [0, m]\}$ which shows that sets of lengths can become arbitrarily large. Under mild conditions on the ideal theory of H every nonunit of H has a factorization into irreducibles and all sets of lengths are finite.

Sets of lengths (together with parameters controlling their structure) are the most investigated invariants in factorization theory, in settings ranging from numerical monoids, noetherian domains, monoids of modules to maximal orders in central simple algebras. The focus of the present paper is on Krull monoids with finite class group such that every class contains a prime divisor. Rings of integers in algebraic number fields are such Krull monoids, and classical philosophy in algebraic number theory (dating back to the 19th century) states that the class group determines the arithmetic. This idea has been formalized and justified. In the 1970s Narkiewicz posed the inverse question whether or not arithmetical phenomena (in other words, phenomena describing the non-uniqueness of factorizations) characterize the class group ([36, Problem 32; page 469]). Very quickly first affirmative answers were given by Halter-Koch, Kaczorowski, and Rush ([32, 28, 39]). Indeed, it is not too difficult to show that the system of sets of factorizations determines the class group ([18, Sections 7.1 and 7.2]).

All these answers are not really satisfactory because the given characterizations are based on rather abstract arithmetical properties which are designed to do the characterization and which play only a little role in other parts of factorization theory. Since on the other hand sets of lengths are of central interest in factorization theory it has been natural to ask whether their structure is rich enough to do characterizations.

2010 *Mathematics Subject Classification.* 11B30, 11R27, 13A05, 13F05, 20M13.

Key words and phrases. Krull monoids, maximal orders, seminormal orders; class groups, arithmetical characterizations, sets of lengths, zero-sum sequences, Davenport constant.

This work was supported by the Austrian Science Fund FWF, Project Number P26036-N26, by the Austrian-French Amadée Program FR03/2012, and by the ANR Project Caesar, Project Number ANR-12-BS01-0011.

Let H be a commutative Krull monoid with finite class group G and suppose that every class contains a prime divisor. It is classical that H is factorial if and only if $|G| = 1$, and by a result due to Carlitz in 1960 we know that all sets of lengths are singletons (i.e., $|L| = 1$ for all $L \in \mathcal{L}(H)$) if and only if $|G| \le 2$. Let us suppose now that $|G| \ge 3$. Then the monoid $\mathcal{B}(G)$ of zero-sum sequences over G is again a Krull monoid with class group isomorphic to G, every class contains a prime divisors, and the systems of sets of lengths of H and that of $\mathcal{B}(G)$ coincide. Thus $\mathcal{L}(H) = \mathcal{L}\big(B(G)\big)$, and it is usual to set $\mathcal{L}(G) := \mathcal{L}\big(\mathcal{B}(G)\big)$. The Characterization Problem can be formulated as follows ([18, Section 7.3], [22, page 42], [43]).

Given two finite abelian groups G and G' such that $\mathcal{L}(G) = \mathcal{L}(G')$. Does it follow that $G \cong G'$?

The system of sets of lengths $\mathcal{L}(G)$ for finite abelian groups is studied with methods from Additive Combinatorics (it has been written down explicitly only for a handful small groups, see Proposition 4.2). Zero-sum theoretical invariants, such as the Davenport constant, play a central role. Recall that, although the precise value of the Davenport constant is well-known for p-groups and for groups of rank at most two since the 1960s (see Proposition 2.3), the precise value is unknown in general (even for groups of the form $G = C_n^3$). Thus it is not surprising that all answers to the Characterization Problem so far are restricted to very special groups including cyclic groups, elementary 2-groups, and groups of the form $C_n \oplus C_n$. Apart from two well-known pairings (see Proposition 4.2) the answer is always positive.

The goal of the present paper is to settle the Characterization Problem for groups of rank at most two. Here is our main result.

Theorem 1.1. *Let G be an abelian group such that $\mathcal{L}(G) = \mathcal{L}(C_{n_1} \oplus C_{n_2})$ where $n_1, n_2 \in \mathbb{N}$ with $n_1 \,|\, n_2$ and $n_1 + n_2 > 4$. Then $G \cong C_{n_1} \oplus C_{n_2}$.*

Theorem 1.1 does not only apply to Krull monoids with class group G but also to certain maximal orders in central simple algebras and to certain seminormal orders in algebraic number fields. This will be outlined in Section 2 (see Proposition 2.1 and 2.2). The proof of Theorem 1.1 is based substantially on

- Prior work on this problem (as summarized in Propositions 6.1 and 6.2), in particular on the recent paper [7].
- The structure theorem for sets of lengths (see Proposition 3.2) and an associated inverse result ([18, Proposition 9.4.9]; see the start of the proof of Proposition 6.5).
- The characterization of minimal zero-sum sequences of maximal length over groups of rank two (Lemma 5.2) which is crucial also for the above mentioned paper [7].

The difficulty of the Characterization Problem stems from the fact that most sets of lengths over any finite abelian group are arithmetical progressions with difference 1 (see Proposition 3.2.4, or [18, Theorem 9.4.11] for a density result of this flavor). Moreover, G and G' are finite abelian groups with $G \subset G'$, then clearly $\mathcal{L}(G) \subset \mathcal{L}(G')$. Thus in order to characterize a group G, we first have to find distinctive sets of lengths for G (i.e., sets of lengths which do occur in $\mathcal{L}(G)$, but in no other or only in a small number of further groups), and second we will have to show that certain sets are not sets of lengths in $\mathcal{L}(G)$. These distinctive sets of lengths for rank two groups are identified in Proposition 6.5 which is the core of our whole approach, and Proposition 5.1 provides sets which do not occur as sets of lengths for rank two groups. In order to pull this through we proceed as follows. After gathering some background material in Section 2, we summarize key results on the structure of sets of lengths in Propositions 3.2. 3.3, and 3.4. Furthermore, we provide some explicit constructions which will turn out to be crucial (Propositions 3.4 – 3.8). In Section 4 we characterize groups whose sets of lengths have a very special structure (e.g., arithmetical progressions) which allows us to settle the Characterization Problem for small groups (see Theorem 4.1). After that we are well-prepared for the main parts given in Sections 5 and 6.

2. The arithmetic of Krull monoids: Background

In this section we gather the required tools from the algebraic and arithmetic theory of Krull monoids. Our notation and terminology are consistent with the monographs [18, 22, 27].

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let $A, B \subset \mathbb{Z}$ be subsets of the integers. We denote by $A + B = \{a + b \mid a \in A, b \in B\}$ their *sumset*, and by $\Delta(A)$ the *set of* (*successive*) *distances* of A (that is, $d \in \Delta(A)$ if and only if $d = b - a$ with $a, b \in A$ distinct and $[a, b] \cap A = \{a, b\}$). For $k \in \mathbb{N}$, we denote by $k \cdot A = \{ka \mid a \in A\}$ the dilation of A by k. If $A \subset \mathbb{N}$, then

$$\rho(A) = \sup\Big\{\frac{m}{n} \mid m, n \in A\Big\} = \frac{\sup A}{\min A} \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$$

is the *elasticity* of A, and we set $\rho(\{0\}) = 1$.

Monoids and Factorizations. By a *monoid*, we always mean a commutative semigroup with identity which satisfies the cancelation law (that is, if a, b, c are elements of the monoid with $ab = ac$, then $b = c$ follows). The multiplicative semigroup of non-zero elements of an integral domain is a monoid. Let H be a monoid. We denote by $H^\times$ the set of invertible elements of H, by $\mathcal{A}(H)$ the set of atoms (irreducible elements) of H, and by $H_{\rm red} = H/H^\times = \{aH^\times \mid a \in H\}$ the associated reduced monoid of H.

A monoid F is free abelian, with basis $P \subset F$, and we write $F = \mathcal{F}(P)$ if every $a \in F$ has a unique representation of the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)}, \quad \text{where} \quad \mathsf{v}_p(a) \in \mathbb{N}_0 \ \text{ with } \ \mathsf{v}_p(a) = 0 \ \text{ for almost all } \ p \in P\,,$$

and we call

$$|a|_F = |a| = \sum_{p \in P} \mathsf{v}_p(a) \quad \text{the } \textit{length} \text{ of } a\,.$$

The monoid $\mathsf{Z}(H) = \mathcal{F}\big(\mathcal{A}(H_{\rm red})\big)$ is called the *factorization monoid* of H, and the unique homomorphism

$$\pi\colon \mathsf{Z}(H) \to H_{\rm red} \quad \text{satisfying} \quad \pi(u) = u \quad \text{for each} \quad u \in \mathcal{A}(H_{\rm red})$$

is the *factorization homomorphism* of H. For $a \in H$,

$$\mathsf{Z}_H(a) = \mathsf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H) \quad \text{is the } \textit{set of factorizations} \text{ of } a\,, \quad \text{and}$$
$$\mathsf{L}_H(a) = \mathsf{L}(a) = \big\{|z| \ \big| \ z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0 \quad \text{is the } \textit{set of lengths} \text{ of } a\,.$$

Thus H is factorial if and only if $H_{\rm red}$ is free abelian (equivalently, $|\mathsf{Z}(a)| = 1$ for all $a \in H$). The monoid H is called atomic if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$ (equivalently, every nonunit can be written as a finite product of irreducible elements). From now on we suppose that H is atomic. Note that, $\mathsf{L}(a) = \{0\}$ if and only if $a \in H^\times$, and $\mathsf{L}(a) = \{1\}$ if and only if $a \in \mathcal{A}(H)$. We denote by

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\} \qquad \text{the } \textit{system of sets of lengths} \text{ of } H\,, \quad \text{and by}$$
$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \ \subset \mathbb{N} \qquad \text{the } \textit{set of distances} \text{ of } H\,.$$

For $k \in \mathbb{N}$, we set $\rho_k(H) = k$ if $H = H^\times$, and

$$\rho_k(H) = \sup\{\sup L \mid L \in \mathcal{L}(H), k \in L\} \in \mathbb{N} \cup \{\infty\}, \quad \text{if} \quad H \ne H^\times\,.$$

Then

$$\rho(H) = \sup\{\rho(L) \mid L \in \mathcal{L}(H)\} = \lim_{k \to \infty} \frac{\rho_k(H)}{k} \in \mathbb{R}_{\ge 1} \cup \{\infty\}$$

is the *elasticity* of H. The monoid H is said to be

- *half-factorial* if $\Delta(H) = \emptyset$ (equivalently, $\rho(H) = 1$). If H is not half-factorial, then $\min \Delta(H) = \gcd \Delta(H)$.

- *decomposable* if there exist submonoids H_1, H_2 with $H_i \not\subset H^\times$ for $i \in [1,2]$ such that $H = H_1 \times H_2$ (and H is called *indecomposable* else).

For a free abelian monoid $\mathcal{F}(P)$, we introduce a distance function $\mathsf{d}\colon \mathcal{F}(P) \times \mathcal{F}(P) \to \mathbb{N}_0$, by setting

$$\mathsf{d}(a,b) = \max\Big\{\Big|\frac{a}{\gcd(a,b)}\Big|, \Big|\frac{b}{\gcd(a,b)}\Big|\Big\} \in \mathbb{N}_0 \quad \text{for} \quad a,b \in \mathcal{F}(P)\,,$$

and we note that $\mathsf{d}(a,b) = 0$ if and only if $a = b$. For a subset $\Omega \subset \mathcal{F}(P)$, we define the *catenary degree* $\mathsf{c}(\Omega)$ as the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: for each $a, b \in \Omega$, there are elements $a_0, \ldots a_k \in \Omega$ such that $a = a_0, a_k = b$, and $\mathsf{d}(a_{i-1}, a_i) \le N$ for all $i \in [1,k]$. Note that $\mathsf{c}(\Omega) = 0$ if and only if $|\Omega| \le 1$. For an element $a \in H$, we call $\mathsf{c}_H(a) = \mathsf{c}(a) := \mathsf{c}(\mathsf{Z}_H(a))$ the catenary degree of a, and

$$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

is the *catenary degree* of H. The monoid H is factorial if and only if $\mathsf{c}(H) = 0$, and if H is not factorial, then $2 + \sup \Delta(H) \le \mathsf{c}(H)$.

Krull monoids and transfer homomorphisms. A monoid homomorphism $\varphi\colon H \to F$ is said to be a divisor homomorphism if $\varphi(a) \,|\, \varphi(b)$ in F implies that $a \,|\, b$ in H for all $a, b \in H$. A monoid H is said to be a *Krull monoid* if one of the following equivalent properties is satisfied ([18, Theorem 2.4.8] or [30]):

(a) H is completely integrally closed and satisfies the ascending chain condition on divisorial ideals.
(b) H has a divisor homomorphism into a free abelian monoid.
(c) H has a divisor theory: this is a divisor homomorphism $\varphi\colon H \to F = \mathcal{F}(P)$ into a free abelian monoid such that for each $p \in P$ there is a finite set $E \subset H$ with $p = \gcd\big(\varphi(E)\big)$.

Suppose that H is a Krull monoid. Then a divisor theory $\varphi\colon H \to F = \mathcal{F}(P)$ is essentially unique. The class group $\mathcal{C}(H) = \mathsf{q}(F)/\mathsf{q}\big(\varphi(H)\big)$ depends only on H and it is isomorphic to the v-class group $\mathcal{C}_v(H)$. For $a \in \mathsf{q}(F)$, we denote by $[a] = a\mathsf{q}\big(\varphi(H)\big) \in \mathcal{C}(H)$ the class containing a. Thus every class $g \in \mathcal{C}(H)$ is considered as a subset of $\mathsf{q}(F)$ and $P \cap g$ is the set of prime divisors lying in g. We use additive notation for the class group.

An integral domain R is a Krull domain if and only if its multiplicative monoid $R \setminus \{0\}$ is a Krull monoid (this generalizes to Marot rings: indeed, a Marot ring is a Krull ring if and only if the monoid of regular elements is a Krull monoid, [21]). Property (a) shows that a noetherian domain is Krull if and only if it is integrally closed. Rings of integers, holomorphy rings in algebraic function fields, and regular congruence monoids in these domains are Krull monoids with finite class group such that every class contains a prime divisor ([18, Section 2.11]). Monoid domains and power series domains that are Krull are discussed in [26, 34, 35]. For monoids of modules which are Krull we refer the reader to [6, 3, 11].

Much of the arithmetic of a Krull monoid can be studied in an associated monoid of zero-sum sequences. This is a Krull monoid again which can be studied with methods from Additive Combinatorics. To introduce the necessary concepts, let G be an additively written abelian group, $G_0 \subset G$ a subset, and let $\mathcal{F}(G_0)$ be the free abelian monoid with basis G_0. In Combinatorial Number Theory, the elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0. If $S = g_1 \cdot \ldots \cdot g_l \in \mathcal{F}(G_0)$, where $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G_0$, then $\sigma(S) = g_1 + \ldots + g_l$ is called the sum of S, and the monoid

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \subset \mathcal{F}(G_0)$$

is called the *monoid of zero-sum sequences* over G_0. Since the embedding $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, Property (b) shows that $\mathcal{B}(G_0)$ is a Krull monoid. The monoid $\mathcal{B}(G)$ is factorial if and only if $|G| \le 2$. If $|G| \ne 2$, then $\mathcal{B}(G)$ is a Krull monoid with class group isomorphic to G and every class contains precisely one prime divisor. This is well-known and will also follow from a more general result given in Proposition 2.5. For every arithmetical invariant $*(H)$ defined for a monoid H, it is usual to write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$ (although this is an abuse of language, there will be no danger of confusion). In particular, we set $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ and $\mathcal{L}(G_0) = \mathcal{L}(\mathcal{B}(G_0))$. Similarly, arithmetical properties of $\mathcal{B}(G_0)$ are attributed to G_0. Thus, G_0 is said to be

- *(in)decomposable* if $\mathcal{B}(G_0)$ is (in)decomposable,
- *(non-) half-factorial* if $\mathcal{B}(G_0)$ is (non-)half-factorial.

A monoid homomorphism $\theta\colon H \to B$ is called a *transfer homomorphism* if if it has the following properties:

(T 1) $B = \theta(H)B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

(T 2) If $u \in H$, $b, c \in B$ and $\theta(u) = bc$, then there exist $v, w \in H$ such that $u = vw$, $\theta(v) \simeq b$ and $\theta(w) \simeq c$.

Transfer homomorphisms preserve sets of lengths and further arithmetical properties. We formulate the relevant results in the settings we need.

Proposition 2.1. *Let H be a Krull monoid with divisor theory $\varphi\colon H \to \mathcal{F}(P)$, class group G, and suppose that that each class contains a prime divisor. Let $\widetilde{\boldsymbol{\beta}}\colon \mathcal{F}(P) \to \mathcal{F}(G)$ be the homomorphism defined by $\boldsymbol{\beta}(p) = [p] \in G$ for each $p \in P$. Then the homomorphism $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \circ \varphi\colon H \to \mathcal{B}(G)$ is a transfer homomorphism. In particular, we have*

1. $\mathsf{L}_H(a) = \mathsf{L}_{\mathcal{B}G)}\big(\boldsymbol{\beta}(a)\big)$ *for each $a \in H$ and $\mathcal{L}(H) = \mathcal{L}(G)$.*
2. *If $|G| \ge 3$, then $\mathsf{c}(H) = \mathsf{c}\big(\mathcal{B}(G)\big)$ (i.e., the catenary degrees of H and of $\mathcal{B}(G)$ coincide).*

Proof. See [18, Section 3.4]. □

There are recent deep results showing that there are non-Krull monoids which allow transfer homomorphisms to monoids of zero-sum sequences.

Proposition 2.2.

1. *Let $\mathcal{O}$ be a holomorphy ring in a global field K, A a central simple algebra over K, and H a classical maximal $\mathcal{O}$-order of A such that every stably free left R-ideal is free. Then $\mathcal{L}(H) = \mathcal{L}(G)$, where G is a ray class group of $\mathcal{O}$ and hence finite abelian.*
2. *Let H be a seminormal order in a holomorphy ring of a global field with principal order $\widehat{H}$ such that the natural map $\mathfrak{X}(\widehat{H}) \to \mathfrak{X}(H)$ is bijective and there is an isomorphism $\overline{\vartheta}\colon \mathcal{C}_v(H) \to \mathcal{C}_v(\widehat{H})$ between the v-class groups. Then $\mathcal{L}(H) = \mathcal{L}(G)$, where $G = \mathcal{C}_v(H)$ is finite abelian.*

Proof. 1. See [48, Theorem 1.1], and [5] for related results of this flavor. Note that H is a non-commutative Dedekind prime ring.

2. See [19, Theorem 5.8] for a more general result in the setting of weakly Krull monoids. Note, if $H \neq \widehat{H}$, then H is not a Krull domain. □

Thus, beyond the Krull monoids occurring in Proposition 2.1, there are classes of objects H, where sets of lengths depend only on an abelian group G and where $\mathcal{L}(H) = \mathcal{L}(G)$ holds. Hence all characterization results, such as Theorem 1.1, applies to them. To provide an example where the opposite phenomenon holds, consider the class of numerical monoids. There we can find (infinitely many) non-isomorphic numerical monoids H and H' with $\mathcal{L}(H) = \mathcal{L}(H')$ ([1]).

Zero-Sum Theory. Let G be an additive abelian group, $G_0 \subset G$ a subset, and $G_0^\bullet = G_0 \setminus \{0\}$. Then $[G_0] \subset G$ denotes the subsemigroup and $\langle G_0 \rangle \subset G$ the subgroup generated by G_0. For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0)\,,$$

we set $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ for any homomorphism $\varphi\colon G \to G'$, and in particular, we have $-S = (-g_1) \cdot \ldots \cdot (-g_l)$. We call

$$\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \text{ the } \textit{support} \text{ of } S\,, \quad \mathsf{v}_g(S) \text{ the } \textit{multiplicity} \text{ of } g \text{ in } S\,,$$

$$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ the } \textit{length} \text{ of } S\,, \quad \mathsf{k}(S) = \sum_{g \in G} \frac{1}{\mathrm{ord}(g)} \in \mathbb{Q} \text{ the } \textit{cross number} \text{ of } S\,,$$

$$\sigma(S) = \sum_{i=1}^{l} g_i \text{ the } \textit{sum} \text{ of } S\,, \text{ and } \Sigma(S) = \Big\{ \sum_{i \in I} g_i \mid \emptyset \neq I \subset [1,l] \Big\} \text{ the } \textit{set of subsequence sums} \text{ of } S\,.$$

The sequence S is said to be

- *zero-sum free* if $0 \notin \Sigma(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a nontrivial zero-sum sequence and every proper subsequence is zero-sum free.

Clearly, if S is zero-sum free, then $(-\sigma(S))S$ is a minimal zero-sum sequence, and the minimal zero-sum sequences over G_0 are precisely the atoms of the monoid $\mathcal{B}(G_0)$. Their study is basic for all arithmetical investigations of Krull monoids. The three invariants,

- the (small) *Davenport constant* $\mathsf{d}(G_0) = \sup\big\{|S| \;\big|\; S \in \mathcal{F}(G_0) \text{ is zero-sum free}\big\} \in \mathbb{N}_0 \cup \{\infty\}$,
- the (large) *Davenport constant* $\mathsf{D}(G_0) = \sup\big\{|U| \;\big|\; U \in \mathcal{A}(G_0)\big\} \in \mathbb{N}_0 \cup \{\infty\}$, and
- the *cross number* $\mathsf{K}(G_0) = \sup\big\{\mathsf{k}(U) \;\big|\; U \in \mathcal{A}(G_0)\big\} \in \mathbb{Q} \cup \{\infty\}$

are classical tools describing minimal zero-sum sequences (all three of them are finite for finite subsets G_0). For $n \in \mathbb{N}$, let C_n denote a cyclic group with n elements. Suppose that G is finite. A tuple $(e_i)_{i \in I}$ is called a *basis* of G if all elements are nonzero and $G = \oplus_{i \in I} \langle e_i \rangle$. For $p \in \mathbb{P}$, let $\mathsf{r}_p(G)$ denote the p-rank of G, $\mathsf{r}(G) = \max\{\mathsf{r}_p(G) \mid p \in \mathbb{P}\}$ denote the *rank* of G, and let $\mathsf{r}^*(G) = \sum_{p \in \mathbb{P}} \mathsf{r}_p(G)$ be the *total rank* of G. If $|G| > 1$, then

$$G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}\,, \quad \text{and we set} \quad \mathsf{d}^*(G) = \sum_{i=1}^{r} (n_i - 1)\,,$$

where $r, n_1, \ldots, n_r \in \mathbb{N}$ with $1 < n_1 \mid \ldots \mid n_r$, $r = \mathsf{r}(G)$, and $n_r = \exp(G)$ is the exponent of G. Similarly, we have

$$G \cong C_{q_1} \oplus \ldots \oplus C_{q_s}\,, \quad \text{and we set} \quad \mathsf{K}^*(G) = \frac{1}{\exp(G)} + \sum_{i=1}^{s} \frac{q_i - 1}{qi}\,,$$

where $s = \mathsf{r}^*(G)$ and $q_1, \ldots, q_s$ are prime powers. If $|G| = 1$, then $\mathsf{r}(G) = \mathsf{r}^*(G) = 0$, $\exp(G) = 1$, and $\mathsf{d}^*(G) = 0$. We will use the following well-known results without further mention.

Proposition 2.3. *Let G be a finite abelian group.*

1. $1 + \mathsf{d}^*(G) \le 1 + \mathsf{d}(G) = \mathsf{D}(G) \le |G|$. *If G is a p-group or $\mathsf{r}(G) \le 2$, then $\mathsf{d}(G) = \mathsf{d}^*(G)$.*
2. $\mathsf{K}^*(G) \le \mathsf{K}(G) \le \frac{1}{2} + \log|G|$, *and the left inequality is an equality if G is a p-group or $\mathsf{r}^*(G) \le 2$.*

Proof. See [18, Chapter 5]. □

There are more groups G with $\mathsf{d}^*(G) = \mathsf{d}(G)$ but we do not have equality in general. On the other hand there is known no group G with $\mathsf{K}^*(G) < \mathsf{K}(G)$. We refer to [8, 20, 47, 46, 10, 31, 33] for recent progress. We will make substantial use of the following result, which highlights the role of the Davenport constant for the arithmetic of Krull monoids.

Proposition 2.4. *Let H be a Krull monoid with finite class group G where $|G| \ge 3$ and every class contains a prime divisor. Then $\mathsf{c}(H) \in [3, \mathsf{D}(G)]$, and we have*

1. $\mathsf c(H) = \mathsf D(G)$ *if and only if* G *is either cyclic or an elementary* 2*-group.*
2. $\mathsf c(H) = \mathsf D(G) - 1$ *if and only if* G *is isomorphic either to* $C_2^{r-1} \oplus C_4$ *for some* $r \ge 2$ *or to* $C_2 \oplus C_{2n}$ *for some* $n \ge 2$.

Proof. See [18, Theorem 6.4.7] and [24, Theorem 1.1]. □

We gather some simple facts on sets of lengths which will be used without further mention. Let $A \in \mathcal B(G_0)$ and $d = \min\{|U| \mid U \in \mathcal A(G_0)\}$. If $A = BC$ with $B, C \in \mathcal B(G_0)$, then

$$\mathsf L(B) + \mathsf L(C) \subset \mathsf L(A)\,.$$

If $A = U_1 \cdot \ldots \cdot U_k = V_1 \cdot \ldots \cdot V_l$ with $U_1, \ldots, U_k, V_1, \ldots, V_l \in \mathcal A(G_0)$ and $k < l$, then

$$ld \le \sum_{\nu=1}^{l} |V_\nu| = |A| = \sum_{\nu=1}^{k} |U_\nu| \le k\mathsf D(G_0)\,,$$

and hence

$$\frac{|A|}{\mathsf D(G_0)} \le \min \mathsf L(A) \le \max \mathsf L(A) \le \frac{|A|}{d}\,.$$

For sequences over cyclic groups the g-norm plays a similar role as the length does for sequences over arbitrary groups. Let $g \in G$ with $\mathrm{ord}(g) = n \ge 2$. For a sequence $S = (n_1 g) \cdot \ldots \cdot (n_l g) \in \mathcal F(\langle g \rangle)$, where $l \in \mathbb N_0$ and $n_1, \ldots, n_l \in [1, n]$, we define

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n}\,.$$

Note that $\sigma(S) = 0$ implies that $n_1 + \ldots + n_l \equiv 0 \mod n$ whence $\|S\|_g \in \mathbb N_0$. Thus, $\|\cdot\|_g \colon \mathcal B(\langle g \rangle) \to \mathbb N_0$ is a homomorphism, and $\|S\|_g = 0$ if and only if $S = 1$. If $S \in \mathcal A(G_0)$, then $\|S\|_g \in [1, n-1]$, and if $\|S\|_g = 1$, then $S \in \mathcal A(G_0)$. Arguing as above we obtain that

$$\frac{\|A\|_g}{n-1} \le \min \mathsf L(A) \le \max \mathsf L(A) \le \|A\|_g\,.$$

We will need the concept of relative block monoids (as introduced by F. Halter-Koch in [29], and recently studied by N. Baeth et al. in [4]). Let G be an abelian group. For a subgroup $K \subset G$ let

$$\mathcal B_K(G) = \{S \in \mathcal F(G) \mid \sigma(S) \in K\} \subset \mathcal F(G)\,,$$

and let $\mathsf D_K(G)$ denote the smallest $l \in \mathbb N \cup \{\infty\}$ with the following property:

- Every sequence $S \in \mathcal F(G)$ of length $|S| \ge l$ has a subsequence T with $\sigma(T) \in K$.

Clearly, $\mathcal B_K(G) \subset \mathcal F(G)$ is a submonoid with

$$\mathcal B(G) = \mathcal B_{\{0\}}(G) \subset \mathcal B_K(G) \subset \mathcal B_G(G) = \mathcal F(G)$$

and $\mathsf D_{\{0\}}(G) = \mathsf D(G)$. The following result is well-known ([4, Theorem 2.2]). Since there seems to be no proof in the literature and we make substantial use of it, we provide the simple arguments.

Proposition 2.5. *Let* G *be an abelian group and* $K \subset G$ *a subgroup.*

1. $\mathcal B_K(G)$ *is a Krull monoid. If* $|G| = 2$ *and* $K = \{0\}$*, then* $\mathcal B_K(G) = \mathcal B(G)$ *is factorial. In all other cases the embedding* $\mathcal B_K(G) \hookrightarrow \mathcal F(G)$ *is a divisor theory with class group isomorphic to* G/K *and every class contains precisely* $|K|$ *prime divisors.*
2. *The monoid homomorphism* $\theta \colon \mathcal B_K(G) \to \mathcal B(G/K)$*, defined by* $\theta(g_1 \cdot \ldots \cdot g_l) = (g_1 + K) \cdot \ldots \cdot (g_l + K)$ *is a transfer homomorphism. If* $|G/K| \ge 3$*, then* $\mathsf c\big(\mathcal B_K(G)\big) = \mathsf c\big(\mathcal B(G/K)\big)$.
3. $\mathsf D_K(G) = \sup\{|U| \mid U$ *is an atom of* $\mathcal B_K(G)\} = \mathsf D(G/K)$.

Proof. 1. If $|G| = 1$, then $\mathcal{B}(G) = \mathcal{F}(G)$ is factorial, the class group is trivial, and there is precisely one prime divisor. If $|G| = |K| = 2$, then $\mathcal{B}_K(G) = \mathcal{F}(G)$ is factorial, the class group is trivial, and there are precisely two prime divisors. If $|G| = 2$ and $|K| = 1$, then $\mathcal{B}_K(G) = \mathcal{B}(G)$ is factorial, and hence a Krull monoid with trivial class group. Suppose that $|G| \ge 3$. Clearly, the embedding $\mathcal{B}_K(G) \hookrightarrow \mathcal{F}(G)$ is a divisor homomorphism. To verify that it is a divisor theory, let $g \in G$ be given. If $\mathrm{ord}(g) = n \ge 3$, then $g = \gcd\big(g^n, g(-g)\big)$. If $\mathrm{ord}(g) = 2$, then there is an element $h \in G \setminus \{0, g\}$ and $g = \gcd\big(g^2, gh(g-h)\big)$. If $\mathrm{ord}(g) = \infty$, then $g = \gcd\big((-g)g, g(2g)(-3g)\big)$.

The map $\varphi\colon \mathcal{F}(G) \to G/K$, defined by $\varphi(S) = \sigma(S) + K$ for every $S \in \mathcal{F}(G)$, is a monoid epimorphism. If $S, S' \in \mathcal{F}(G)$, then $\sigma(S) + K = \sigma(S') + K$ if and only if $S' \in [S] = S\mathsf{q}\big(\mathcal{B}_K(G)\big)$. Thus φ induces a group isomorphism $\Phi\colon \mathsf{q}(\mathcal{F}(G))/\mathsf{q}(\mathcal{B}_K(G)) \to G/K$, defined by $\Phi([S]) = \sigma(S) + K$, and we have $[S] \cap G = \sigma(S) + K$. Thus the class $[S]$ contains precisely $|K|$ prime divisors.

2. If $|G| \le 2$, then θ is the identity map. If $|G| \ge 3$, then this follows from 1. and from Proposition 2.1.

3. Since a sequence $S \in \mathcal{F}(G)$ is an atom of $\mathcal{B}_K(G)$ if and only if $S \ne 1$, $\sigma(S) \in K$ and $\sigma(T) \notin K$ for all proper subsequences T of S, it follows that $\mathsf{D}_K(G) = \sup\{|U| \mid U \text{ is an atom of } \mathcal{B}_K(G)\}$. Since θ is a transfer homomorphism, we get $\theta\big(\mathcal{A}(\mathcal{B}_K(G))\big) = \mathcal{A}(G/K)$ and $\theta^{-1}\big(\mathcal{A}(G/K)\big) = \mathcal{A}(\mathcal{B}_K(G))$. Therefore $|U| = |\theta(U)|$ for all $U \in \mathcal{B}_K(G)$, and it follows that

$$\sup\{|U| \mid U \in \mathcal{A}(\mathcal{B}_K(G))\} = \sup\{|V| \mid V \in \mathcal{A}(G/K)\} = \mathsf{D}(G/K)\,. \qquad \square$$

3. Structural results on $\mathcal{L}(G)$ and first basic constructions

Let G be an abelian group. Recall that all sets of lengths $L \in \mathcal{L}(G)$ are finite (indeed, $\max \mathsf{L}(A) \le |A|$ for all $A \in \mathcal{B}(G)$) and that G is half-factorial (i.e., $|L| = 1$ for each $L \in \mathcal{L}(G)$) if and only if $|G| \le 2$. If G is infinite, then every finite subset $L \subset \mathbb{N}_{\ge 2}$ is contained in $\mathcal{L}(G)$ ([18, Theorem 7.4.1]). If G is finite with $|G| > 2$, then sets of lengths have a well-studied structure which is the basis for all characterizations of class groups. First we repeat the results needed in the sequel and then we start with some basic constructions which will be used in all forthcoming sections.

Definition 3.1. Let $d \in \mathbb{N}$, $l, M \in \mathbb{N}_0$ and $\{0, d\} \subset \mathcal{D} \subset [0, d]$. A subset $L \subset \mathbb{Z}$ is called an

- *arithmetical multiprogression* (AMP for short) with *difference* d, *period* $\mathcal{D}$ and *length* l, if L is an interval of $\min L + \mathcal{D} + d\mathbb{Z}$ (this means that L is finite nonempty and $L = (\min L + \mathcal{D} + d\mathbb{Z}) \cap [\min L, \max L]$), and l is maximal such that $\min L + ld \in L$.
- *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal{D}$, *length* l and *bound* M, if
$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z}$$
where L^* is an AMP with difference d (whence $L^* \ne \emptyset$), period $\mathcal{D}$ and length l such that $\min L^* = 0$, $L' \subset [-M, -1]$, $L'' \subset \max L^* + [1, M]$ and $y \in \mathbb{Z}$.
- *almost arithmetical progression* (AAP for short) with *difference* d, *bound* M and *length* l, if it is an AAMP with difference d, period $\{0, d\}$, bound M and length l.

The subset $\Delta^*(G)$ of $\Delta(G)$, defined as

$$\Delta^*(G) = \{\min \Delta(G_0) \mid G_0 \subset G \text{ with } \Delta(G_0) \ne \emptyset\} \subset \Delta(G)\,,$$

plays a crucial role throughout this paper.

Proposition 3.2 (Structural results on $\mathcal{L}(G)$)**.**
Let G be a finite abelian group with $|G| \ge 3$.

1. *There exists some* $M \in \mathbb{N}_0$ *such that every set of lengths* $L \in \mathcal{L}(G)$ *is an* AAMP *with some difference* $d \in \Delta^*(G)$ *and bound* M.
2. *For every* $M \in \mathbb{N}_0$ *and every finite nonempty set* $\Delta^* \subset \mathbb{N}$, *there is a finite abelian group* G^* *such that the following holds*: *for every* AAMP L *with difference* $d \in \Delta^*$ *and bound* M *there is some* $y_L \in \mathbb{N}$ *such that*
$$y + L \in \mathcal{L}(G^*) \quad \text{for all} \quad y \ge y_L\,.$$
3. *Let* $G_0 \subset G$ *be a subset. Then there exist a bound* $M \in \mathbb{N}_0$ *and some* $A^* \in \mathcal{B}(G_0)$ *such that for all* $A \in A^*\mathcal{B}(G_0)$ *the set of lengths* $\mathsf{L}(A)$ *is an* AAP *with difference* $\min \Delta(G_0)$ *and bound* M.
4. *If* $A \in \mathcal{B}(G)$ *such that* $\mathrm{supp}(A) \cup \{0\}$ *is a subgroup of* G, *then* $\mathsf{L}(A)$ *is an arithmetical progression with difference* 1.

Proof. The first statement gives the Structure Theorem for Sets of Lengths ([18, Theorem 4.4.11]), which is sharp by the second statement proved in [44]. The third and the fourth statements show that sets of lengths are extremely smooth provided that the associated zero-sum sequence contains all elements of its support sufficiently often ([18, Theorems 4.3.6 and 7.6.8]). □

Proposition 3.3 (Structural results on $\Delta(G)$ and on $\Delta^*(G)$)**.**
Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ *where* $r, n_1, \ldots, n_r \in \mathbb{N}$ *with* $r = \mathsf{r}(G)$, $1 < n_1 \,|\, \ldots \,|\, n_r$, *and* $|G| \ge 3$.

1. $\Delta(G)$ *is an interval with*
$$\big[1, \max\{\exp(G) - 2,\, k - 1\}\big] \subset \Delta(G) \subset \big[1, \mathsf{D}(G) - 2\big] \quad \text{where} \quad k = \sum_{i=1}^{\mathsf{r}(G)} \Big\lfloor \frac{n_i}{2} \Big\rfloor\,.$$
2. $1 \in \Delta^*(G) \subset \Delta(G)$, $[1, \mathsf{r}(G) - 1] \subset \Delta^*(G)$, *and* $\max \Delta^*(G) = \max\{\exp(G) - 2, \mathsf{r}(G) - 1\}$.
3. *If* G *is cyclic of order* $|G| = n \ge 4$, *then* $\max\big(\Delta^*(G) \setminus \{n-2\}\big) = \lfloor \frac{n}{2} \rfloor - 1$.

Proof. The statement on $\max \Delta^*(G)$ follows from [25]. For all remaining statements see [18, Section 6.8]. A more detailed analysis of $\Delta^*(G)$ in case of cyclic groups can be found in [37]. □

Proposition 3.4 (Results on $\rho_k(G)$ and on $\rho(G)$)**.**
Let G *be a finite abelian group with* $|G| \ge 3$, *and let* $k \in \mathbb{N}$.

1. $\rho_{2k}(G) = k\mathsf{D}(G)$.
2. $1 + k\mathsf{D}(G) \le \rho_{2k+1}(G) \le k\mathsf{D}(G) + \mathsf{D}(G)/2$. *If* G *is cyclic, then equality holds on the left side.*
3. $\rho(G) = \mathsf{D}(G)/2$.

Proof. See [18, Chapter 6.3], [15, Theorem 5.3.1], and [17] for recent progress. □

In the next propositions we provide examples of sets of lengths over cyclic groups, over groups of rank two, and over groups of the form $C_2^{r-1} \oplus C_n$ with $r, n \in \mathbb{N}_{\ge 2}$. All examples will have difference $d = \max \Delta^*(G)$ and period $\mathcal{D}$ with $\{0, d\} \subset \mathcal{D} \subset [0, d]$ and $|\mathcal{D}| = 3$, and we write them down in a form used in Definition 3.1 in order to highlight their periods. It will be crucial for our approach (see Proposition 6.5) that the sets given in Proposition 3.5.2 do not occur over cyclic groups (Proposition 3.6). It is well-known that sets of lengths over cyclic groups and over elementary 2-groups have many features in common, and this carries over to rank two groups and groups of the form $C_2^{r-1} \oplus C_n$ (see Propositions 3.5.2, 3.7.2, and 6.5). Let G be an abelian group and $L \in \mathcal{L}(G)$. Then there is a $B \in \mathcal{B}(G)$ such that $L = \mathsf{L}(B)$ and hence $m + L = \mathsf{L}(0^m B) \in \mathcal{L}(G)$ for all $m \in \mathbb{N}_0$. Therefore the interesting sets of lengths $L \in \mathcal{L}(G)$ are those which do not stem from such a shift. These are those sets $L \in \mathcal{L}(G)$ with $-m + L \notin \mathcal{L}(G)$ for every $m \in \mathbb{N}$.

Proposition 3.5. *Let* $G = C_{n_1} \oplus C_{n_2}$ *where* $n_1, n_2 \in \mathbb{N}$ *with* $2 < n_1 \,|\, n_2$, *and let* $d \in [3, n_1]$.

1. *For each $k \in \mathbb{N}$ we have*
$$(2k+2) + \{0, d-2, n_2-2\} + \{\nu(n_2-2) \mid \nu \in [0,k-1]\} \cup \{(kn+2)+(d-2)\} =$$
$$(2k+2) + \{0, d-2\} + \{\nu(n_2-2) \mid \nu \in [0,k]\} \in \mathcal{L}(G)\,.$$

2. *For each $k \in \mathbb{N}$ we have*
$$\Big((2k+3) + \{0, n_1-2, n_2-2\} + \{\nu(n_2-2) \mid \nu \in [0,k]\}\Big) \cup \Big\{(kn_2+3)+(n_1-2)+(n_2-2)\Big\} \in \mathcal{L}(G)\,.$$

Proof. Let (e_1, e_2) be a basis of G with $\mathrm{ord}(e_i) = n_i$ for $i \in [1,2]$, and let $k \in \mathbb{N}$. For $i \in [1,2]$, we set $U_i = e_i^{n_i}$ and $V_i = (-e_i)e_i$. Then
$$(-U_i)^k U_i^k = (-U_i)^{k-\nu} U_i^{k-\nu} V_i^{\nu n_i} \quad \text{for all} \quad \nu \in [0,k]\,,$$
and hence
$$\mathsf{L}\big((-U_i)^k U_i^k\big) = 2k + \{\nu(n_i-2) \mid \nu \in [0,k]\}\,.$$

1. We set $h = (d-1)e_1$, $W_1 = (-e_1)^{d-1}h$, and $W_2 = e_1^{n_1-(d-1)}h$. Then $\mathsf{Z}(U_1W_1) = \{U_1W_1, V_1^{d-1}W_2\}$ and $\mathsf{L}(U_1W_1) = \{2,d\}$. Therefore
$$\begin{aligned}\mathsf{L}\big((-U_2)^k U_2^k U_1W_1\big) &= \mathsf{L}\big((-U_2)^k U_2^k\big) + \mathsf{L}\big(U_1W_1\big)\\ &= \{2k + \nu(n_2-2) \mid \nu \in [0,k]\} + \{2,d\}\\ &= (2k+2) + \{\nu(n_2-2) \mid \nu \in [0,k]\} + \{0, d-2\}\,.\end{aligned}$$

2. We define
$$W_1 = e_1^{n_1-1}e_2^{n_2-1}(e_1+e_2),\ W_2 = (-e_1)e_2^{n_2-1}(e_1+e_2),\ W_3 = e_1^{n_1-1}(-e_2)(e_1+e_2),\ W_4 = (-e_1)(-e_2)(e_1+e_2)\,,$$
and
$$B_k = W_1(-U_1)(-U_2)U_2^k(-U_2)^k\,.$$
Then any factorization of B_k is divisible by precisely one of $W_1, \ldots, W_4$, and we obtain that
$$\begin{aligned}B_k &= W_1(-U_1)(-U_2)U_2^k(-U_2)^k = W_2V_1^{n_1-1}(-U_2)U_2^k(-U_2)^k\\ &= W_3(-U_1)V_2^{n_2-1}U_2^k(-U_2)^k = W_4V_1^{n_1-1}V_2^{n_2-1}U_2^k(-U_2)^k\,.\end{aligned}$$
Thus it follows that
$$\begin{aligned}\mathsf{L}(B_k) &= \{3, n_1+1, n_2+1, n_1+n_2-1\} + \mathsf{L}\big(U_2^k(-U_2)^k\big)\\ &= (2k+3) + \{\nu(n_2-2) \mid \nu \in [0,k]\} \cup\\ &\quad (2k+3) + (n_1-2) + \{\nu(n_2-2) \mid \nu \in [0,k]\} \cup\\ &\quad (2k+3) + (n_2-2) + \{\nu(n_2-2) \mid \nu \in [0,k]\} \cup\\ &\quad (2k+3) + (n_1-2) + (n_2-2) + \{\nu(n_2-2) \mid \nu \in [0,k]\}\,.\end{aligned}$$
Thus $\max \mathsf{L}(B_k) = (kn_2+3) + (n_1-2) + (n_2-2)$ and
$$\mathsf{L}(B_k) = \Big((2k+3) + \{0, n_1-2, n_2-2\} + \{\nu(n_2-2) \mid \nu \in [0,k]\}\Big) \cup \{\max \mathsf{L}(B_k)\}\,. \qquad \square$$

Proposition 3.6. *Let G be a cyclic group of order $|G| = n \ge 4$, and let $d \in [3, n-1]$.*

1. *For each $k \in \mathbb{N}_0$, we have*
$$(2k+2) + \{0, d-2\} + \{\nu(n-2) \mid \nu \in [0,k]\} \in \mathcal{L}(G)\,.$$

2. *For each $k \in \mathbb{N}_0$, we set*
$$L_k = \Big((2k+3) + \{0, d-2, n-2\} + \{\nu(n-2) \mid \nu \in [0,k]\}\Big) \cup \Big\{(kn+3)+(d-2)+(n-2)\Big\}\,.$$
Then for each $k \in \mathbb{N}_0$ and each $m \in \mathbb{N}_0$, we have $-m + L_k \notin \mathcal{L}(G)$.

Proof. Let $k \in \mathbb{N}_0$.

1. Let $g \in G$ with $\mathrm{ord}(g) = n$, $U = g^n$, $V = (-g)g$, $W_1 = \big((d-1)g\big)(-g)^{d-1}$, $W_2 = \big((d-1)g\big)g^{n-(d-1)}$, and

$$B_k = \big((-U)U\big)^k U W_1 \,.$$

Then $\mathsf{Z}(UW_1) = \{UW_1, W_2V^{d-1}\}$ and $\mathsf{L}(UW_1) = \{2, d\}$. Since every factorization of B_k is divisible either by W_1 or by W_2, it follows that

$$\begin{aligned} \mathsf{L}(B_k) &= \mathsf{L}\big((-U)^k U^k\big) + \mathsf{L}\big(UW_1\big) \\ &= \{2k + \nu(n-2) \mid \nu \in [0,k]\} + \{2, d\} \\ &= (2k+2) + \{\nu(n-2) \mid \nu \in [0,k]\} + \{0, d-2\} \,. \end{aligned}$$

2. Note that $\max L_k = (kn+3) + (d-2) + (n-2) = (k+1)n + (d-1)$. Assume to the contrary that there is a $B_k \in \mathcal{B}(G)$ such that $\mathsf{L}(B_k) = L_k$. Then $\min \mathsf{L}(B_k) = 2k+3$ and, by Proposition 3.4,

$$(k+1)n + (d-1) = \max \mathsf{L}(B_k) \le \rho_{2k+3}(G) = (k+1)n + 1 \,,$$

a contradiction. If $m \in \mathbb{N}_0$ and $B_{m,k} \in \mathcal{B}(G)$ such that $\mathsf{L}(B_{m,k}) = -m + L_k$, then $\mathsf{L}(0^m B_{m,k}) = L_k \in \mathcal{L}(G)$. Thus $-m + L_k \notin \mathcal{L}(G)$ for any $m \in \mathbb{N}_0$. □

Proposition 3.7. *Let $G = C_2^{r-1} \oplus C_n$ where $r, n \in \mathbb{N}_{\ge 2}$ and n is even.*

1. *For each $k \in \mathbb{N}_0$ we have*

$$L_k = (2k+2) + \{0, n-2, n+r-3\} + \{\nu(n-2) \mid \nu \in [0,k]\} \in \mathcal{L}(G) \,.$$

   If $r \ge 2$, then $L_k \notin \mathcal{L}(C_n)$.

2. *For each $k \in \mathbb{N}_0$, we have*

$$\Big((2k+3) + \{0, r-1, n-2\} + \{\nu(n-2) \mid \nu \in [0,k]\}\Big) \cup \Big\{(kn+3) + (r-1) + (n-2)\Big\} \in \mathcal{L}(G) \,.$$

Proof. Let $(e_1, \ldots, e_{r-1}, e_r)$ be a basis of G with $\mathrm{ord}(e_1) = \ldots = \mathrm{ord}(e_{r-1}) = 2$ and $\mathrm{ord}(e_r) = n$. We set $e_0 = e_1 + \ldots + e_{r-1}$, $U_i = e_i^{\mathrm{ord}(e_i)}$ for each $i \in [1,r]$, $U_0 = (e_0 + e_r)(e_0 - e_r)$, $V_r = (-e_r)e_r$,

$$V = e_1 \cdot \ldots \cdot e_{r-1}(e_0 + e_r)(-e_r) \,, \quad \text{and} \quad W = e_1 \cdot \ldots \cdot e_{r-1}(e_0 + e_r)e_r^{n-1} \,.$$

Let $k \in \mathbb{N}_0$.

1. Obviously, $\mathsf{L}\big((-W)W\big) = \{2, n, n+r-1\}$ and

$$\begin{aligned} \mathsf{L}\big((-W)W(-U_r)^k U_r^k\big) &= \mathsf{L}\big((-W)W\big) + \mathsf{L}\big((-U_r)^k U_r^k\big) \\ &= \{2, n, n+r-1\} + \{2k + \nu(n-2) \mid \nu \in [0,k]\} \\ &= (2k+2) + \{0, n-2, n+r-3\} + \{\nu(n-2) \mid \nu \in [0,k]\} \end{aligned}$$

Since $\min L_k = 2k+2$, $\max L_k = (k+1)n + r - 1$, and $\rho_{2k+2}(C_n) = (k+1)n$ by Proposition 3.4, $r \ge 2$ implies that $L_k \notin \mathcal{L}(C_n)$.

2. Let L_k denote the set in the statement. We define

$$B_k = U_0 U_1 \cdot \ldots \cdot U_{r-1}(-U_r)^{k+1} U_r^{k+1}$$

and assert that $\mathsf{L}(B_k) = L_k$. Let z be a factorization of B_k. We distinguish two cases.

CASE 1: $U_1 \mid z$.

Then $U_0 U_1 \cdot \ldots U_{r-1} \mid z$ which implies that $z = U_0 U_1 \cdot \ldots \cdot U_{r-1}\big((-U_r)U_r\big)^{k+1-\nu} V_r^{\nu n}$ for some $\nu \in [0, k+1]$ and hence $|z| \in r + (2k+2) + \{\nu(n-2) \mid \nu \in [0, k+1]\}$.

CASE 2: $U_1 \nmid z$.

Then either $V \,|\, z$ or $W \,|\, z$. If $V \,|\, z$, then $z = (-V)VV_r^{n-1}\big((-U_r)U_r\big)^{k-\nu}V_r^{\nu n}$ for some $\nu \in [0,k]$ and hence $|z| \in (n+1)+2k+\{\nu(n-2) \mid \nu \in [0,k]\}$. If $W \,|\, z$, then $z = (-W)WV_r\big((-U_r)U_r\big)^{k-\nu}V_r^{\nu n}$ for some $\nu \in [0,k]$ and hence $|z| \in 3+2k+\{\nu(n-2) \mid \nu \in [0,k]\}$.

Putting all together the assertion follows. □

Proposition 3.8. *Let G be a finite abelian group, $g \in G$ with $\mathrm{ord}(g) = n \ge 5$, and $B \in \mathcal{B}(G)$ such that $\big((-g)g\big)^{2n} \,|\, B$. Suppose $\mathsf{L}(B)$ is an* AAMP *with period $\{0, d, n-2\}$ for some $d \in [1, n-3] \setminus \{(n-2)/2\}$.*

1. *If $S \in \mathcal{A}\big(\mathcal{B}_{\langle g\rangle}(G)\big)$ with $S \,|\, B$, then $\sigma(S) \in \{0, g, -g, (d+1)g, -(d+1)g\}$.*
2. *If $S_1, S_2 \in \mathcal{A}\big(\mathcal{B}_{\langle g\rangle}(G)\big)$ with $S_1S_2 \,|\, B$, then $\sigma(S_i) \in \{0, g, -g\}$ for at least one $i \in [1,2]$.*

Proof. By definition, there is a $y \in \mathbb{Z}$ such that

$$\mathsf{L}(B) \subset y + \{0, d, n-2\} + (n-2)\mathbb{Z}\,.$$

We set $U = g^n$ and $V = (-g)g$.

1. Let $S \in \mathcal{A}\big(\mathcal{B}_{\langle g\rangle}(G)\big)$ with $S \,|\, B$ and set $\sigma(S) = kg$ with $k \in [0, n-1]$. If $k \in \{0, 1, n-1\}$, then we are done. Suppose that $k \in [2, n-2]$. Since S is an atom in $\mathcal{B}_{\langle g\rangle}(G)$, it follows that $W_1 = S(-g)^k \in \mathcal{A}(G)$ and $W_1' = Sg^{n-k} \in \mathcal{A}(G)$. We consider a factorization $z \in \mathsf{Z}(B)$ with $UW_1 \,|\, z$, say $z = UW_1y$. Then $z' = W_1'V^ky$ is a factorization of B of length $|z'| = |z| + k - 1$. Since $\mathsf{L}(B)$ is an AAMP with period $\{0, d, n-2\}$ for some $d \in [1, n-3] \setminus \{(n-2)/2\}$ it follows that $k - 1 \in \{d, n-2-d\}$.

2. Let $S_1, S_2 \in \mathcal{A}\big(\mathcal{B}_{\langle e\rangle}(G)\big)$ with $S_1S_2 \,|\, B$, and assume to the contrary $\sigma(S_i) = k_ie$ with $k_i \in [2, n-2]$ for each $i \in [1,2]$. As in 1. it follows that

$$W_1 = S_1(-g)^{k_1},\ W_1' = S_1g^{n-k_1},\ W_2 = S_2(-g)^{k_2}, \quad \text{and} \quad W_2' = S_2g^{n-k_2}$$

are in $\mathcal{A}(G)$. We consider a factorization $z \in \mathsf{Z}(B)$ with $UW_1UW_2 \,|\, z$, say $z = UW_1UW_2y$. Then $z_1 = W_1'V^{k_1-1}UW_2y \in \mathsf{Z}(B)$ with $|z_1| = |z| + k_1 - 1$ and hence $k_1 - 1 \in \{d, n-2-d\}$. Similarly, $z_2 = UW_1W_2'V^{k_2-1}y \in \mathsf{Z}(B)$, hence $k_2 - 1 \in \{d, n-2-d\}$, and furthermore it follows that $k_1 = k_2$. Now $z_3 = W_1'V^{k_1-1}W_2'V^{k_2-1}y \in \mathsf{Z}(B)$ is a factorization of length $|z_3| = |z|+k_1+k_2-2$. Thus, if $k_1-1 = d$, then $2d \in \{n-2, n-2+d\}$, a contradiction, and if $k_1-1 = n-2-d$, then $2(n-2-d) \in \{n-2, n-2+(n-2-d)\}$, a contradiction. □

4. Characterizations of extremal cases

Let G be a finite abelian group. By the Structure Theorem for Sets of Lengths (Proposition 3.2.1), all sets of lengths are AAMPs (with difference in $\Delta^*(G)$ and some universal bound). By definition, the concept of an AAMP comprises arithmetical progressions, AAPs, and AMPs. The goal of this section is to characterize those groups where all sets of lengths are not only AAMPs, but have one of these more special forms. As a consequence we establish characterizations of all involved class groups. Since $\mathcal{L}(C_1) = \mathcal{L}(C_2)$ and $\mathcal{L}(C_3) = \mathcal{L}(C_2 \oplus C_2)$ (see Proposition 4.2 below), small groups require special attention in the study of the Characterization Problem. All results of this section are gathered in the following Theorem 4.1.

Theorem 4.1. *Let G be a finite abelian group.*

1. *The following statements are equivalent:*
   (a) *All sets of lengths in $\mathcal{L}(G)$ are arithmetical progressions with difference in $\Delta^*(G)$.*
   (b) *All sets of lengths in $\mathcal{L}(G)$ are arithmetical progressions.*
   (c) *G is cyclic of order $|G| \le 4$ or isomorphic to a subgroup of C_2^3 or isomorphic to a subgroup of C_3^2.*
2. *The following statements are equivalent:*

   (a) *There is a constant $M \in \mathbb{N}$ such that all sets of lengths in $\mathcal{L}(G)$ are* AAP*s with bound M.*
   (b) *G is isomorphic to a subgroup of C_3^3 or isomorphic to a subgroup of C_4^3.*
3. *The following statements are equivalent*:
   (a) *All sets of lengths in $\mathcal{L}(G)$ are* AMP*s with difference in $\Delta^*(G)$.*
   (b) *G is cyclic with $|G| \le 5$ or isomorphic to a subgroup of C_2^3 or isomorphic to a subgroup of C_3^2.*
4. *Suppose that $\mathsf{D}(G) \ge 4$ and that $\mathcal{L}(G)$ satisfies the property in 1., 2., or 3. If G' is a finite abelian group such that $\mathcal{L}(G) = \mathcal{L}(G')$, then $G \cong G'$.*

We proceed in a series of lemmas. The proof of Theorem 4.1 will be given at the end of this section.

Proposition 4.2.

1. $\mathcal{L}(C_1) = \mathcal{L}(C_2) = \big\{\{m\} \mid m \in \mathbb{N}_0\big\}$.
2. $\mathcal{L}(C_3) = \mathcal{L}(C_2 \oplus C_2) = \big\{y + 2k + [0,k] \bigm| y,\, k \in \mathbb{N}_0\big\}$.
3. $\mathcal{L}(C_4) = \big\{y + k + 1 + [0,k] \bigm| y,\, k \in \mathbb{N}_0\big\} \cup \big\{y + 2k + 2 \cdot [0,k] \bigm| y,\, k \in \mathbb{N}_0\big\}$.
4. $\mathcal{L}(C_2^3) = \big\{y + (k+1) + [0,k] \bigm| y \in \mathbb{N}_0,\ k \in [0,2]\big\}$
   $\cup \big\{y + k + [0,k] \bigm| y \in \mathbb{N}_0,\ k \ge 3\big\} \cup \big\{y + 2k + 2 \cdot [0,k] \bigm| y,\, k \in \mathbb{N}_0\big\}$.
5. $\mathcal{L}(C_3^2) = \{[2k,l] \mid k \in \mathbb{N}_0, l \in [2k, 5k]\}$
   $\cup \{[2k+1,l] \mid k \in \mathbb{N}, l \in [2k+1, 5k+2]\} \cup \{\{1\}\}$.

Proof. 1. This is straightforward and well-known. A proof of 2.,3., and 4. can be found in [18, Theorem 7.3.2]. For 5. we refer to [23, Proposition 3.12]. □

Lemma 4.3. *Let G be a cyclic group of order $|G| = n \ge 7$, $g \in G$ with $\mathrm{ord}(g) = n$, $k \in \mathbb{N}$, and*

$$A_k = \begin{cases} g^{nk}(-g)^{nk}(2g)^n & \text{if } n \text{ is even,} \\ g^{nk}(-g)^{nk}\big((2g)^{(n-1)/2}g\big)^2 & \text{if } n \text{ is odd.} \end{cases}$$

Then there is a bound $M \in \mathbb{N}$ such that, for all $k \ge n-1$, the sets $\mathsf{L}(A_k)$ are AAPs *with difference* 1 *and bound M, but they are not arithmetical progressions with difference* 1.

Proof. We set $G_0 = \{g, -g, 2g\}$, $U_1 = (-g)g$, $U_2 = (-g)^2(2g)$ and, if n is odd, then $V_1 = (2g)^{(n+1)/2}(-g)$. Furthermore, for $j \in [0, n/2]$, we define $W_j = (2g)^j g^{n-2j}$. Then, together with $-W_0 = (-g)^n$, these are all minimal zero-sum sequences which divide A_k for $k \in \mathbb{N}$. Note that

$$\|-W_0\|_g = n-1,\ \|U_2\|_g = \|V_1\|_g = 2, \quad \text{and} \quad \|U_1\|_g = \|W_j\|_g = 1 \quad \text{for all } j \in [0, n/2]\,.$$

It is sufficient to prove the following two assertions.

A1. There is a bound $M \in \mathbb{N}_0$ such that $\mathsf{L}(A_k)$ is an AAP with difference 1 and bound M for all $k \ge n-1$.

A2. For each $k \in \mathbb{N}$, $\mathsf{L}(A_k)$ is not an arithmetical progression with difference 1.

Proof of **A1**. By Proposition 3.2.1 there is a bound $M' \in \mathbb{N}_0$ such that, for each $k \in \mathbb{N}$, $\mathsf{L}(A_k)$ is an AAMP with difference $d_k \in \Delta^*(G) \subset [1, n-2]$ and bound M'. Suppose that $k \ge n-1$. Then $(W_0U_2)^{n-1}$ divides A_k. Since $W_0U_2 = W_1U_1^2$, it follows that

$$(W_0U_2)^{n-1} = (W_0U_2)^{n-1-\nu}(W_1U_1^2)^\nu \quad \text{for all} \quad \nu \in [0, n-1]$$

and hence $\mathsf{L}\big((W_0U_2)^{n-1}\big) \supset [2n-2, 3n-3]$. Thus $\mathsf{L}(A_k)$ contains an arithmetical progression of difference 1 and length $n-1$. Therefore there is a bound $M \in \mathbb{N}_0$ such that $\mathsf{L}(A_k)$ is an AAP with difference 1 and bound M for all $k \ge n-1$.

Proof of **A2**. Let $k \in \mathbb{N}$. Observe that

$$A_k = \begin{cases} W_0^k(-W_0)^k W_{n/2}^2 & \text{if } n \text{ is even,} \\ W_0^k(-W_0)^k\big(W_{(n-1)/2}\big)^2 & \text{if } n \text{ is odd,} \end{cases}$$

and it can be seen that $\min \mathsf L(A_k) = 2k+2$. We assert that $2k+3 \notin \mathsf L(A_k)$. If n is even, then

$$W_0 W_{n/2} = W_j W_{n/2-j} \quad \text{for each} \quad j \in [0, n/2]\,,$$

and similarly, for odd n we have

$$W_0 W_{(n-1)/2} = W_j W_{(n-1)/2-j} \quad \text{for each} \quad j \in [0, (n-1)/2]\,.$$

In both cases, all factorizations of A_k of length $2k+2$ contain only atoms with g-norm 1 and with g-norm $n-1$. Let z' be any factorization of A_k containing only atoms with g-norm 1 and with g-norm $n-1$. Then $|z'| - |z|$ is a multiple of $n-2$ whence if $|z'| > |z|$, then $|z'| - |z| \ge n-2 > 1$.

Next we consider a factorization z' of A_k containing at least one atom with g-norm 2, say z' has r atoms with g-norm $n-1$, $s \ge 1$ atoms with g-norm 2, and t atoms with g-norm 1. Then $k > r$,

$$\|A_k\|_g = k(n-1) + (k+2) = r(n-1) + 2s + t\,,$$

and we study

$$\begin{aligned} |z'| - |z| &= r + s + t - (2k+2) \\ &= r + s + k(n-1) + (k+2) - r(n-1) - 2s - (2k+2) \\ &= (k-r)(n-2) - s\,. \end{aligned}$$

Note that $s \le \mathsf v_{2g}(A_k) \le n$. Thus, if $k - r \ge 2$, then

$$(k-r)(n-2) - s \ge 2n - 4 - s \ge n - 4 > 1\,.$$

Suppose that $k - r = 1$. Then we cancel $(-W_0)^{k-1}$, and consider a relation where $-W_0$ occurs precisely once. Suppose that all s atoms of g-norm 2 are equal to U_2. Since $\mathsf v_{-g}(U_2) = 2$, it follows that $s \le \mathsf v_{-g}(-W_0)/2 = n/2$ whence

$$(k-r)(n-2) - s \ge n - 2 - n/2 = n/2 - 2 > 1\,.$$

Suppose that V_1 occurs among the s atoms with g-norm 2. Then n is odd, V_1 occurs precisely once, and

$$s - 1 \le \mathsf v_{2g}(A_k) - \frac{n+1}{2} = (n-1) - \frac{n+1}{2} = \frac{n-3}{2}\,,$$

whence

$$(k-r)(n-2) - s \ge (n-2) - \frac{n-1}{2} = \frac{n+1}{2} - 2 > 1\,. \qquad \square$$

Lemma 4.4. *Let G be a cyclic group of order $|G| = 6$, $g \in G$ with* $\mathrm{ord}(g) = 6$ *and, for each $k \in \mathbb{N}$, $A_k = g^{6k}(-g)^{6k}(4g)(-g)^4(3g)g^3$. Then there is a bound $M \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, the sets $\mathsf L(A_k)$ are* AAPs *with difference* 1 *and bound M, but they are not arithmetical progressions with difference* 1.

Proof. We set $U = g^6$, $W_1 = (4g)(-g)^4$, and $W_2 = (3g)g^3$. Then, for each $k \in \mathbb{N}$, we have $A_k = U^k(-U)^k W_1 W_2$. By Proposition 3.3, we obtain that $\Delta^*(G) = \{1, 2, 4\}$. By Proposition 3.2.1, there is a bound $M' \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, $\mathsf L(A_k)$ is an AAMP with difference $d_k \in \Delta^*(G)$ and bound M'. We show that $2k+4, 2k+5, 2k+6, 2k+7 \in \mathsf L(A_k)$ which implies that there is a bound $M \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, $\mathsf L(A_k)$ is an AAP with difference 1 and bound M.

Let $k \in \mathbb{N}$. We set $V = (-g)g$, $W_3 = (4g)(3g)(-g)$, $W_4 = (4g)g^2$, and obtain that

$$\begin{aligned} A_k &= U^k(-U)^k W_1 W_2 = U^k(-U)^k W_3 V^3 \\ &= U^{k-1}(-U)^k W_4 W_2 V^4 = U^{k-1}(-U)^{k-1} W_1 W_2 V^6 = U^{k-1}(-U)^{k-1} W_4 (-W_2) V^7\,, \end{aligned}$$

and hence $\{2k+2, 2k+4, 2k+5, 2k+6, 2k+7\} \subset \mathsf{L}(A_k)$. Furthermore, $\min \mathsf{L}(A_k) = 2k+2$, and $z = U^k(-U)^k W_1 W_2$ is the only factorization of A_k of length $2k+2$. From this we see that there is no factorization of length $2k+3$, and hence $\mathsf{L}(A_k)$ is not an arithmetical progression with difference 1. □

Lemma 4.5. *Let G be a cyclic group of order $|G| = 5$. Then every $L \in \mathcal{L}(G)$ has one of the following forms:*

- *L is an arithmetical progression with difference 1.*
- *L is an arithmetical progression with difference 3.*
- *L is an* AMP *with period $\{0, 2, 3\}$ or with period $\{0, 1, 3\}$.*

Proof. By Proposition 3.3 we obtain that $\Delta^*(G) = \{1, 3\}$. Let $A' \in \mathcal{B}(G)$. If $A' = 0^m A$ with $m \in \mathbb{N}_0$ and $A \in \mathcal{B}(G^\bullet)$, then $\mathsf{L}(A') = m + \mathsf{L}(A)$. Thus it is sufficient to prove the assertion for $\mathsf{L}(A)$. If $|\operatorname{supp}(A)| = 1$, then $|\mathsf{L}(A)| = 1$. If $|\operatorname{supp}(A)| = 4$, then $\mathsf{L}(A)$ is an arithmetical progression with difference 1 by Proposition 3.2.4. Suppose that $|\operatorname{supp}(A)| = 2$. Then there is a $g \in G^\bullet$ such that $\operatorname{supp}(A) = \{g, 2g\}$ or $\operatorname{supp}(A) = \{g, 4g\}$. If $\operatorname{supp}(A) = \{g, 2g\}$, then $\mathsf{L}(A)$ is an arithmetical progression with difference 1 (this can be checked directly by arguing with the g-norm). If $\operatorname{supp}(A) = \{g, 4g\}$, then $\mathsf{L}(A)$ is an arithmetical progression with difference 3.

Thus it remains to consider the case $|\operatorname{supp}(A)| = 3$. We set $G_0 = \operatorname{supp}(A)$. Then there is an element $g \in G_0$ such that $-g \in G_0$. Thus either $G_0 = \{g, 2g, -g\}$ or $G_0 = \{g, 3g, -g\}$. Since $\{g, 3g, -g\} = \{-g, 2(-g), -(-g)\}$, we may suppose without restriction that $G_0 = \{g, 2g, -g\}$.

If $\Delta(\mathsf{L}(A)) \subset \{1\}$, then $\mathsf{L}(A)$ is an arithmetical progression with difference 1. If $3 \in \Delta(\mathsf{L}(A))$, then $\Delta(\mathsf{L}(A)) = \{3\}$ by [9, Theorem 3.2], which means that $\mathsf{L}(A)$ is an arithmetical progression with difference 3. Thus it remains to consider the case where $2 \in \Delta(\mathsf{L}(A)) \subset [1, 2]$. We show that $\mathsf{L}(A)$ is an AMP with period $\{0, 2, 3\}$ or with period $\{0, 1, 3\}$. Since $2 \in \Delta(\mathsf{L}(A))$, there exist $k \in \mathbb{N}$, $A_1, \ldots, A_k, B_1, \ldots, B_{k+2} \in \mathcal{A}(G_0)$ such that

$$A = A_1 \cdot \ldots \cdot A_k = B_1 \cdot \ldots \cdot B_{k+2}, \quad \text{and} \quad k+1 \notin \mathsf{L}(A)\,.$$

For convenience we list the elements of $\mathcal{A}(G_0)$, and we order them by their lengths:

- $g^5,\ (-g)^5,\ (2g)^5$,
- $g^3(2g),\ (2g)^3(-g)$,
- $g(2g)^2,\ (2g)(-g)^2$,
- $g(-g)$.

Clearly, $\{\|S\|_g \mid S \in \mathcal{A}(G_0)\} = \{1, 2, 4\}$, and $(-g)^5$ is the only atom having g-norm 4. We distinguish two cases.

CASE 1: $(-g)^5 \notin \{A_1, \ldots, A_k\}$.

Then $\{A_1, \ldots, A_k\}$ must contain atoms with g-norm 2. These are the atoms $(2g)^5, (2g)(-g)^2, (2g)^3(-g)$. If g^5 or $g^3(2g)$ occurs in $\{A_1, \ldots, A_k\}$, then $k+1 \in \mathsf{L}(A)$, a contradiction. Thus none of the elements $(-g)^5, g^5$, and $g^3(2g)$ lies in $\{A_1, \ldots, A_k\}$, and hence

$$\{A_1, \ldots, A_k\} \subset \{(2g)^5, (2g)^3(-g), g(2g)^2, (2g)(-g)^2, g(-g)\}\,.$$

Now we set $h = 2g$ and obtain that

$$\{A_1, \ldots, A_k\} \subset \{(2g)^5, (2g)^3(-g), g(2g)^2, (2g)(-g)^2, g(-g)\} = \{h^5, h^3(2h), h^2(3h), h(2h)^2, (2h)(3h)\}\,.$$

Since the h-norm of all these elements equals 1, it follows that $\max \mathsf{L}(A) = k$, a contradiction.

CASE 2: $(-g)^5 \in \{A_1, \ldots, A_k\}$.

If $(2g)^5$, or $g(2g)^2$, or $(2g)^3(-g)$ occurs in $\{A_1, \ldots, A_k\}$, then $k+1 \in \mathsf{L}(A)$, a contradiction. Since $\Delta(\{-g, g\}) = \{3\}$, it follows that

$$\Omega = \{A_1, \ldots, A_k\} \cap \{g^3(2g), (2g)(-g)^2\} \neq \emptyset\,.$$

Since $\Big(g^3(2g)\Big)\Big((2g)(-g)^2\Big) = \Big((-g)g\Big)^2\Big(g(2g)^2\Big)$ and $k+1 \notin \mathsf L(A)$, it follows that $|\Omega| = 1$. We distinguish two cases.

CASE 2.1: $\{A_1, \ldots, A_k\} \subset \{g^5, (-g)^5, g(-g), (2g)(-g)^2\}$.

We set $h = -g$, and observe that

$$\{A_1, \ldots, A_k\} \subset \{g^5, (-g)^5, g(-g), (2g)(-g)^2\} = \{h^5, (-h)^5, h(-h), h^2(3h)\}\,.$$

Since $(-h)^5$ is the only element with h-norm greater than 1, it follows that $(-h)^5 \in \{A_1, \ldots, A_k\}$. Since $\Delta(\{h, -h\}) = \{3\}$, it follows that $h^2(3h) \in \{A_1, \ldots, A_k\}$. Since $\Big((-h)^5\Big)\Big(h^2(3h)\Big) = \Big(h(-h)\Big)^2\Big((3h)(-h)^3\Big)$, we obtain that $k+1 \in \mathsf L(A)$, a contradiction.

CASE 2.2: $\{A_1, \ldots, A_k\} \subset \{g^5, (-g)^5, g(-g), g^3(2g)\}$.

Since $\Big(g^3(2g)\Big)^2\Big((-g)^5\Big) = \Big(g^5\Big)\Big(g(-g)\Big)\Big((2g)(-g)^2\Big)^2$ and $k+1 \notin \mathsf L(A)$, it follows that

$$|\{i \in [1,k] \mid A_i = g^3(2g)\}| = 1\,,$$

and hence $\mathsf v_{2g}(A) = 1$. Thus every factorization z of A has the form

$$z = \big((2g)g^3\big)z_1 \quad \text{or} \quad z = \big((2g)(-g)^2\big)z_2\,,$$

where z_1, z_2 are factorizations of elements $B_1, B_2 \in \mathcal B(\{-g, g\})$. Since $\mathsf L(B_1)$ and $\mathsf L(B_2)$ are arithmetical progressions of difference 3, $\mathsf L(A)$ is a union of two shifted arithmetical progression of difference 3. We set

$$A = \big(g^5\big)^{m_1}\big((-g)^5\big)\big((-g)g\big)^{m_3}\big((2g)g^3\big)\,,$$

where $m_1 \in \mathbb N_0$, $m_2 \in \mathbb N$, and $m_3 \in [0,4]$. Suppose that $m_1 \ge 1$. Note that

$$A' = \big(g^5\big)\big((-g)^5\big)\big((2g)g^3\big) = \big((-g)g\big)^3\big((2g)(-g)^2\big)\big(g^5\big) = \big((-g)g\big)^5\big((2g)g^3\big)\,,$$

and hence $\mathsf L(A') = \{3, 5, 6\}$. We set $A = A'A''$ with $A'' \in \mathcal B(\{g, -g\})$. The above argument on the structure of the factorizations of A implies that $\mathsf L(A)$ is the sumset of $\mathsf L(A')$ and $\mathsf L(A'')$ whence

$$\mathsf L(A) = \mathsf L(A') + \mathsf L(A'') = 3 + \{0, 2, 3\} + \mathsf L(A'')\,.$$

Since $\mathsf L(A'')$ is an arithmetical progression with difference 3, $\mathsf L(A)$ is an AMP with period $\{0, 2, 3\}$. Suppose that $m_1 = 0$. If $m_3 \in [2, 4]$, then $\mathsf L(A) = \{m_2 + m_3, m_2 + m_3 + 1, m_2 + m_3 + 3\}$ is an AMP with period $\{0, 1, 3\}$. If $m_3 = 1$, then $\mathsf L(A) = \{m_2 + 2, m_2 + 4\}$. If $m_3 = 0$, then $\mathsf L(A) = \{m_2 + 1, m_2 + 3\}$. □

Lemma 4.6. *Let $G = C_{n_1} \oplus C_{n_2}$ where $n_1, n_2 \in \mathbb N$ with $4 \le n_1 \,|\, n_2$, (e_1, e_2) be a basis of G with $\mathrm{ord}(e_i) = n_i$ for $i \in [1,2]$, and set $W = e_1^{n_1-1}e_2^{n_2-1}(e_1 + e_2)$. Then there is a bound $M \in \mathbb N$ such that, for all sufficiently large k, the sets $\mathsf L\big(W^k(-W)^k\big)$ are* AAPs *with difference* 1 *and bound M, but they are not arithmetical progressions with difference* 1.

Proof. We set $e_0 = e_1 + e_2$, $G_0 = \{e_\nu, -e_\nu \mid \nu \in [0,2]\}$, $U_\nu = e_\nu^{\mathrm{ord}(e_\nu)}$ and $V_\nu = (-e_\nu)e_\nu$ for $\nu \in [0,2]$. For $k \in \mathbb N$ we set $A_k = W^k(-W)^k$ and $L_k = \mathsf L(A_k)$. Since $\gcd \Delta(G_0) \,|\, \gcd(\{n_1 - 2, n_2 - 2, |W| - 2 = n_1 + n_2 - 3\}) = 1$, it follows that $\min \Delta(G_0) = 1$. Thus, by Proposition 3.2.3, there are $M, k_0 \in \mathbb N$ such that for all $k \ge k_0$, the set L_k is an AAP with difference 1 and bound M.

Let $k \in \mathbb N$. We assert that $1 + \min L_k \notin L_k$. This implies that L_k is not an arithmetical progression with difference 1. Since $|W| = |-W| = \mathsf D(G)$, it follows that $\min L_k = 2k$, and clearly $W^k(-W)^k$ is the only factorization of A_k having length $2k$. If $S = (-e_1)e_2^{n_2-1}(e_1 + e_2)$, then $W(-W) = S(-S)V_1^{n_1-2}$, $2k + n_1 - 2 \in L_k$, and this is the second shortest factorization length of A_k. □

Lemma 4.7. *Let $G = C_2^4$, (e_1, e_2, e_3, e_4) be a basis of G, $e_0 = e_1 + \ldots + e_4$, $U_4 = e_0 \cdot \ldots \cdot e_4$, $U_3 = e_1e_2e_3(e_1 + e_2 + e_3)$, and $U_2 = e_1e_2(e_1 + e_2)$.*

1. *There is a bound $M \in \mathbb{N}$ such that, for all sufficiently large k, the sets $\mathsf{L}\big((U_3U_4)^{2k}\big)$ are* AAPs *with difference* 1 *and bound M, but they are not arithmetical progressions with difference* 1.
2. *For each $k \in \mathbb{N}$, we have*

$$\mathsf{L}(U_4^{2k}U_2) = (2k+1) + \{0,1,3\} + 3 \cdot [0, k-1]\,.$$

Proof. 1. We set $G_0 = \mathrm{supp}(U_3U_4)$, $A_k = U_3^{2k}U_4^{2k}$ and $L_k = \mathsf{L}(A_k)$ for each $k \in \mathbb{N}$. Since $\gcd \Delta(G_0) \mid \gcd\{|U_3|-2 = 2, |U_4|-2 = 3\}$, it follows that $\min \Delta(G_0) = 1$. Thus, by Proposition 3.2.3, there are $M, k_0 \in \mathbb{N}$ such that for all $k \ge k_0$, the set L_k is an AAP with difference 1 and bound M.

Let $k \in \mathbb{N}$. Then $\min L_k = 4k$, and we assert that $1 + 4k \notin L_k$. For $\nu \in [0,4]$, we set $V_\nu = e_\nu^2$ and $V_5 = (e_1+e_2+e_3)^2$. Since $\mathsf{Z}(U_3^2) = \{U_3^2, V_1V_2V_3V_5\}$, $\mathsf{Z}(U_4^2) = \{U_4^2, V_1V_2V_3V_4V_0\}$, and $\mathsf{Z}(U_3U_4) = \{U_3U_4, V_1V_2V_3W\}$ where $W = (e_1+e_2+e_3)e_0e_4$, it follows that $\min(L_k \setminus \{4k\}) = 4k+2$.

2. Setting $W = (e_1+e_2)e_3e_4e_0$ we infer that $U_4^2U_2 = U_4(e_1^2)(e_2^2)W = U_2(e_0^2)\cdot\ldots\cdot(e_4^2)$ and hence $\mathsf{L}(U_4^2U_2) = \{3,4,6\}$. Thus for each $k \in \mathbb{N}$ we obtain that

$$\begin{aligned}
\mathsf{L}(U_4^{2k}U_2) &= \big(\{1\} + \mathsf{L}(U_4^{2k})\big) \cup \big(\mathsf{L}(U_4^{2k-2}) + \mathsf{L}(U_4^2U_2)\big) \\
&= \big(2k+1+3\cdot[0,k]\big) \cup \big(2k-2+3\cdot[0,k-1] + \{3,4,6\}\big) \\
&= \big(2k+1+3\cdot[0,k]\big) \cup \big(2k+2+3\cdot[0,k-1]\big) \cup \big(2k+4+3\cdot[0,k-1]\big) \\
&= (2k+1) + \{0,1,3\} + 3\cdot[0,k-1]\,. \quad \square
\end{aligned}$$

Lemma 4.8. *Let $G = C_3^r$ with $r \in [3,4]$, $(e_1,\ldots,e_r)$ a basis of G, $e_0 = e_1+\ldots+e_r$, and $U = (e_1\cdot\ldots\cdot e_r)^2e_0$.*

1. *If $r = 3$, then there is a bound $M \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$, the sets $\mathsf{L}\big(U^{6k+1}(-U)\big)$ are* AAPs *with difference* 1 *and bound M, but they are not arithmetical progressions with difference* 1.
2. *If $r = 4$ and $V_1 = e_1^2e_2^2(e_1+e_2)$, then for each $k \in \mathbb{N}$ we have*

$$\mathsf{L}(U^{3k}V_1) = (3k+1) + \{0,1,3\} + 3\cdot[0,2k-1]\,.$$

Proof. 1. Let $r = 3$ and $k \in \mathbb{N}$. We set $A_k = U^{6k+1}(-U)$ and $L_k = \mathsf{L}(A_k)$. For $\nu \in [0,3]$, we set $U_\nu = e_\nu^3$, $V_\nu = (-e_\nu)e_\nu$, and we define $X = e_0^2e_1e_2e_3$.

First, consider $\mathsf{L}(U^{6k})$. We observe that $\mathsf{Z}(U^2) = \{U^2, U_1U_2U_3X\}$ and $\mathsf{Z}(U^3) = \{U^3, UU_1U_2U_3X, U_0U_1^2U_2^2U_3^2\}$. Furthermore, $\min \mathsf{L}(U^{6k}) = 6k$, $\max \mathsf{L}(U^{6k}) = 14k$, $\Delta(\{e_0,\ldots,e_3\} = \{2\}$, and hence

$$\mathsf{L}(U^{6k}) = 6k + 2\cdot[0,4k]\,.$$

Next, consider $\mathsf{L}\big((-U)U\big)$. For subsets $I, J \subset [1,3]$ with $[1,3] = I \uplus J$, we set

$$W_I = e_0 \prod_{i\in I} e_i^2 \prod_{j\in J}(-e_j)\,.$$

Since

$$\mathsf{Z}\big(U(-U)\big) = \Big\{V_0V_1^2V_2^2V_3^2\Big\} \uplus \Big\{W_I(-W_I)\prod_{j\in J} V_j \mid I, J \subset [1,3] \text{ with } [1,3] = I \uplus J\Big\}\,,$$

it follows that

$$\mathsf{L}\big((-U)U\big) = \Big\{7\Big\} \uplus \Big\{2+|J| \mid I, J \subset [1,3] \text{ with } [1,3] = I \uplus J\Big\} = \{2,3,4,5,7\}\,.$$

This implies that

$$[6k+2, 14k+5] \cup \{14k+7\} = \mathsf{L}\big((-U)U\big) + \mathsf{L}(U^{6k}) \subset \mathsf{L}(A_k) \subset [6k+2, 14k+7]\,,$$

and we claim that $[6k+2, 14k+5] \cup \{14k+7\} = \mathsf{L}(A_k)$. Then the assertion of the lemma follows.

To prove this, we consider the unique factorization $z \in \mathsf{Z}(A_k)$ of length $|z| = 14k+7$ which has the form

$$z = \big(U_0U_1^2U_2^2U_3^2\big)^{2k}\big(V_0V_1^2V_2^2V_3^2\big)\,.$$

Assume to the contrary that there is a factorization $z' \in \mathsf Z(A_k)$ of length $|z'| = 14k+6$. If $V_0 \,|\, z'$, then $V_0V_1^2V_2^2V_3^2 \,|\, z'$ and $z' = V_0V_1^2V_2^2V_3^2x$ with $x \in \mathsf Z(U^{6k})$, whence $|x| \in \mathsf L(U^{6k})$ and $|z'| \in 7 + \mathsf L(U^{6k})$, a contradiction. Suppose that $V_0 \nmid z'$. Then there are $I, J \subset [1,3]$ with $[1,3] = I \uplus J$ such that $W_I(-W_I)\prod_{j\in J} V_j \,|\, z'$ and hence $z' = W_I(-W_I)\big(\prod_{j\in J} V_j\big)x$ with $x \in \mathsf Z(U^{6k})$. Thus $|z'| \in [2,5] + \mathsf L(U^{6k})$, a contradiction.

2. Let $r = 4$ and $k \in \mathbbm N$. We have $\mathsf L(U^2) = \{2,5\}$ and $\mathsf L(U^{3k}) = 3k + 3\cdot[0,2k]$. We define

$$V_2 = (e_1+e_2)e_1e_2e_3^2e_4^2e_0\,,\ V_3 = (e_1+e_2)e_3e_4e_0^2, \quad \text{and} \quad W = e_1\cdot\ldots\cdot e_4e_0^2\,,$$

and observe that

$$U^3V_1 = U^2V_2(e_1^3)(e_2^3) = UV_3(e_1^3)^2(e_2^3)^2(e_3^3)(e_4^3)$$

whence $\mathsf L(U^3V_1) = \{4,5,7,8\}$. Clearly, each factorization of $U^{3k}V_1$ contains exactly one of the atoms V_1, V_2, V_3, and it contains it exactly once. Therefore we obtain that

$$\begin{aligned}
\mathsf L(U^{3k}V_1) &= \big(\{1\} + \mathsf L(U^{3k})\big) \cup \big(\mathsf L(U^3V_1) + \mathsf L(U^{3k-3})\big) \\
&= \big((3k+1) + 3\cdot[0,2k]\big) \cup \big(\{4,5,7,8\} + (3k-3) + 3\cdot[0,2k-2]\big) \\
&= \big((3k+1) + 3\cdot[0,2k]\big) \cup \big((3k+1) + \{0,1,3,4\} + 3\cdot[0,2k-2]\big) \\
&= (3k+1) + \{0,1,3\} + 3\cdot[0,2k-1]\,. \quad \square
\end{aligned}$$

Proof of Theorem 4.1. 1. (c) $\Rightarrow$ (a) Proposition 4.2 shows that, for all groups mentioned, all sets of lengths are arithmetical progressions. Proposition 3.3 shows that all differences lie in $\Delta^*(G)$.

(a) $\Rightarrow$ (b) Obvious.

(b) $\Rightarrow$ (c) Suppose that $\exp(G) = n$, and that G is not isomorphic to any of the groups listed in (c). We have to show that there is an $L \in \mathcal L(G)$ which is not an arithmetical progression. We distinguish four cases.

CASE 1: $n \ge 5$.

Then Proposition 3.6.1 provides examples of sets of lengths which are not arithmetical progressions.

CASE 2: $n = 4$.

Since G is not cyclic, it has a subgroup isomorphic to $C_2 \oplus C_4$. Then [18, Theorem 6.6.5] shows that $\{2,4,5\} \in \mathcal L(C_2 \oplus C_4\} \subset \mathcal L(G)$.

CASE 3: $n = 3$.

Then G is isomorphic to C_3^r with $r \ge 3$, and Lemma 4.8.1 provides examples of sets of lengths which are not arithmetical progressions.

CASE 4: $n = 2$.

Then G is isomorphic to C_2^r with $r \ge 4$, and Lemma 4.7.1 provides examples of sets of lengths which are not arithmetical progressions.

2. (b) $\Rightarrow$ (a) Suppose that G is a subgroup of C_4^3 or a subgroup of C_3^3. Then Proposition 3.3.2 implies that $\Delta^*(G) \subset \{1,2\}$, and hence Proposition 3.2.1 implies the assertion.

(a) $\Rightarrow$ (b) Suppose that (b) does not hold. Then G has a subgroup isomorphic to a cyclic group of order $n \ge 5$, or isomorphic to C_2^4, or isomorphic to C_3^4. We show that in none of these cases (a) holds.

If G has a subgroup isomorphic to C_n for some $n \ge 5$, then Proposition 3.6.1 shows that (a) does not hold. If G has a subgroup isomorphic to C_2^4, then Lemma 4.7.2 shows that (a) does not hold. If G has a subgroup isomorphic to C_3^4, then Lemma 4.8.2 shows that (a) does not hold.

3. Suppose that G is cyclic. If $|G| \le 4$, then all sets of lengths are arithmetical progressions with difference in $\Delta^*(G)$ by 1. and hence they are AMPs with difference in $\Delta^*(G)$. If $|G| \ge 5$, then the assertion follows from the Lemmas 4.3, 4.4, and 4.5.

Suppose that G has rank $r \ge 2$ and $\exp(G) \in [2,5]$, say $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \,|\, \ldots \,|\, n_r$. If $n_1 \ge 4$, then Lemma 4.6 shows that there are sets of lengths which are not AMPs with difference in $\Delta^*(G)$. Thus it suffices to consider the cases where G is isomorphic to one of the following groups: $C_2^r, C_2^{r-1} \oplus C_4, C_3^r$.

If $G = C_2^{r-1} \oplus C_4$, then $\mathcal{L}(G)$ contains (arbitrarily long) AAPs with difference 2 which are not arithmetical progressions and hence no AMPs ([15, Example 3.2.1]).

Suppose that $G = C_2^r$. If $r \le 3$, then the assertion follows from 1. If $r \ge 4$, then the assertion follows from Lemma 4.7.1.

Suppose that $G = C_3^r$. If $r \le 2$, then the assertion follows from 1. If $r \ge 3$, then the assertion follows from Lemma 4.8.1.

4. Let G' be a finite abelian group such that $\mathcal{L}(G) = \mathcal{L}(G')$. Then, by Proposition 3.4, $\mathsf{D}(G) = \rho_2(G) = \rho_2(G') = \mathsf{D}(G')$, and $\mathcal{L}(G)$ satisfies one of the properties 1., 2., or 3. if and only if the same is true for $\mathcal{L}(G')$. We distinguish three cases.

CASE 1: $\mathcal{L}(G)$ satisfies the property in 1.

By 1., G is cyclic of order $|G| \le 4$ or isomorphic to a subgroup of C_2^3 or isomorphic to a subgroup of C_3^2, and the same is true for G'. Since $\mathsf{D}(G) \ge 4$, the assertion follows from Proposition 4.2.

CASE 2: $\mathcal{L}(G)$ satisfies the property in 2.

By CASE 1, we may suppose that $\mathcal{L}(G)$ and $\mathcal{L}(G')$ do not satisfy the property in 1. Then by 2., G and G', are isomorphic to one of the following groups: C_3^3, $C_2 \oplus C_4$, $C_2^2 \oplus C_4$, $C_2 \oplus C_4^2$, C_4^2, or C_4^3. Since C_3^3 and C_4^2 are the only non-isomorphic groups having the same Davenport constant, it remains to show that $\mathcal{L}(C_3^3) \ne \mathcal{L}(C_4^2)$. Since $\max \Delta(C_4^2) = 3$ (by [24, Lemma 3.3]) and $\max \Delta(C_3^3) = 2$ (by [16, Proposition 5.5]), the assertion follows.

CASE 3: $\mathcal{L}(G)$ satisfies the property in 3.

By CASE 1, we may suppose that G and G' do not satisfy the property in 1. But then 3. implies that G and G' are both cyclic of order five. □

5. A set of lengths not contained in $\mathcal{L}(C_{n_1} \oplus C_{n_2})$

The aim of this section is to prove the following proposition.

Proposition 5.1. *Let $G = C_{n_1} \oplus C_{n_2}$ where $n_1, n_2 \in \mathbb{N}$ with $n_1 \,|\, n_2$ and $6 \le n_1 < n_2$. Then $\{2, n_2, n_1 + n_2 - 2\} \notin \mathcal{L}(G)$.*

Let G be a finite abelian group. Sets of lengths $L \in \mathcal{L}(G)$ with $\{2, \mathsf{D}(G)\} \subset L$ have been studied frequently in the literature (e.g., [18, Chapter 6.6], [7]). Let $G = C_{n_1} \oplus C_{n_2}$ where $n_1, n_2 \in \mathbb{N}$ with $n_1 \,|\, n_2$. We show that $\{2, n_2, n_1 + n_2 - 2 = \mathsf{D}(G) - 1\} \notin \mathcal{L}(G)$ under the parameter restrictions that $6 \le n_1 < n_2$ because this is precisely what is needed for our results in Section 6. If $6 \le n_1 < n_2$ does not hold, then $\{2, n_2, n_1 + n_2 - 2\}$ may or may not be a set of lengths (e.g., if $2 = n_1 \le n_2$, then $\{2, n_2\} \in \mathcal{L}(G)$). By Proposition 3.7, $\{2, n_2, n_1 + n_2 - 2\} \in \mathcal{L}(C_2^{n_1 - 2} \oplus C_{n_2})$, and we will use Proposition 5.1 to show that $\mathcal{L}(C_{n_1} \oplus C_{n_2}) \ne \mathcal{L}(C_2^{n_1 - 2} \oplus C_{n_2})$.

The proof of Proposition 5.1 is based on the characterization of all minimal zero-sum sequences of maximal length over groups of rank two. This characterization is due to Gao, Grynkiewicz, Reiher, and the present authors ([12, 13, 45, 38]). We use the formulation given in [7, Theorem 3.1].

Lemma 5.2. *Let $G = C_{n_1} \oplus C_{n_2}$ where $n_1, n_2 \in \mathbb{N}$ with $1 < n_1 \,|\, n_2$. A sequence U over G of length $\mathsf{D}(G) = n_1 + n_2 - 1$ is a minimal zero-sum sequence if and only if it has one of the following two forms:*

- $$U = e_j^{\mathrm{ord}(e_j)-1} \prod_{\nu=1}^{\mathrm{ord}(e_i)} (x_\nu e_j + e_i)$$

  where
  (a) $\{i,j\} = \{1,2\}$ *and* (e_1, e_2) *is a basis of* G *with* $\mathrm{ord}(e_1) = n_1$ *and* $\mathrm{ord}(e_2) = n_2$,
  (b) $x_1, \ldots, x_{\mathrm{ord}(e_i)} \in [0, \mathrm{ord}(e_j) - 1]$ *and* $x_1 + \ldots + x_{\mathrm{ord}(e_i)} \equiv 1 \mod \mathrm{ord}(e_j)$.
  In this case, we say that U *is of type I with respect to basis* (e_i, e_j).

- $$U = (e_1 + ye_2)^{sn_1-1} e_2^{n_2-sn_1+\epsilon} \prod_{\nu=1}^{n_1-\epsilon} (-x_\nu e_1 + (-x_\nu y + 1)e_2),$$

  where
  (a) (e_1, e_2) *is a basis of* G *with* $\mathrm{ord}(e_1) = n_1$ *and* $\mathrm{ord}(e_2) = n_2$,
  (b) $y \in [0, n_2 - 1]$, $\epsilon \in [1, n_1 - 1]$, *and* $s \in [1, n_2/n_1 - 1]$,
  (c) $x_1, \ldots, x_{n_1-\epsilon} \in [1, n_1 - 1]$ *with* $x_1 + \ldots + x_{n_1-\epsilon} = n_1 - 1$,
  (d) $n_1 y e_2 \neq 0$, *and*
  (e) *either* $s = 1$ *or* $n_1 y e_2 = n_1 e_2$.
  In this case, we say that U *is of type II with respect to basis* (e_1, e_2).

We continue with a simple corollary, and provide two lemmas before we start the actual proof of Proposition 5.1.

Corollary 5.3. *Let* $G = C_{n_1} \oplus C_{n_2}$ *where* $n_1, n_2 \in \mathbb{N}$ *with* $n_1 \,|\, n_2$ *and* $6 \le n_1 < n_2$, *and let* $U \in \mathcal{A}(G)$ *with* $|U| = \mathsf{D}(G) = n_1 + n_2 - 1$.

1. *If* $\mathsf{h}(U) = n_2 - 1$, *then* U *is of type I with respect to a basis* (e_1, e_2) *with* $\mathrm{ord}(e_1) = n_1$ *and* $\mathrm{ord}(e_2) = n_2$, *that is*
$$U = e_2^{\mathrm{ord}(e_2)-1} \prod_{\nu=1}^{\mathrm{ord}(e_1)} (x_\nu e_2 + e_1)$$
where $x_1, \ldots, x_{n_1} \in [0, n_2 - 1]$ *with* $x_1 + \ldots + x_{n_1} \equiv 1 \mod n_2$.
2. *If* $\mathsf{h}(U) = n_2 - 2$, *then*
$$U = (e_1 + ye_2)^{n_1-1} e_2^{n_2-2} \big(-xe_1 + (-xy+1)e_2 \big) \big(-(n_1 - 1 - x)e_1 + (-(n_1 - 1 - x)y + 1)e_2 \big),$$
where (e_1, e_2) *is a basis with* $\mathrm{ord}(e_1) = n_1$, $\mathrm{ord}(e_2) = n_2$, $y \in [0, n_2 - 1]$, *and* $x \in [1, (n_1 - 1)/2]$.
3. *If* $\mathsf{h}(U) = n_2 - 3$, *then*
$$U = (e_1 + ye_2)^{n_1-1} e_2^{n_2-3} \prod_{\nu=1}^{3} (-x_\nu e_1 + (-x_\nu y + 1)e_2),$$
where (e_1, e_2) *is a basis with* $\mathrm{ord}(e_1) = n_1$, $\mathrm{ord}(e_2) = n_2$, $y \in [0, n_2 - 1]$, *and* $x_1, x_2, x_3 \in [1, n_1 - 1]$ *with* $x_1 + x_2 + x_3 \equiv n_1 - 1 \mod n_1$ *(if* $y \neq 0$, *then* $x_1 + x_2 + x_3 = n_1 - 1$*).*
4. *There is at most one element* $g \in G$ *with* $\mathsf{v}_g(U) \ge n_2 - 3$. *In particular, if* $\mathsf{h}(U) \ge n_2 - 3$, *then there is precisely one element* $g \in G$ *with* $\mathsf{v}_g(U) = \mathsf{h}(U)$.

Proof. We use all notation as in Lemma 5.2.

1. If U is of type II with respect to basis (e_1, e_2), then $s = 1$, $\epsilon = n_1 - 1$, and
$$U = (e_1 + ye_2)^{n_1-1} e_2^{n_2-1} \big(e_1 + ((-n_1 + 1)y + 1)e_2 \big),$$
which shows that U is also of type I with respect to basis (e_1, e_2). If U is of type I with respect to the basis (e_2, e_1) then $\mathsf{h}(U) = n_2 - 1$ implies that U is also of type I with respect to the basis (e_1, e_2).

2. Suppose that U is of type I with respect to basis (f_2, f_1). Then U has the form

$$U = f_1^{n_1-1}(x_1f_1 + f_2)^{n_2-2}(x_2f_1 + f_2)(x_3f_1 + f_2)\,.$$

Thus U has the asserted form with $y = 0$, $e_1 = f_1$, and with $e_2 = x_1f_1 + f_2$. In this case we only have two summands the congruence condition modulo n_2, and hence we obtain an equality in the integers. Suppose that U is of type II with respect to basis (e_1, e_2). Then $s = 1$, $\epsilon = n_1 - 2$, and thus the assertion follows.

3. Suppose that U is of type I with respect to basis (f_1, f_2). Then U has the form

$$U = f_1^{n_1-1}(x_1f_1 + f_2)^{n_2-3}(x_2f_1 + f_2)(x_3f_1 + f_2)(x_4f_1 + f_2)\,.$$

Thus U has the asserted form with $y = 0$, $e_1 = f_1$, and with $e_2 = x_1f_1 + f_2$.

Suppose that U is of type II with respect to basis (e_1, e_2). Then $s = 1$, $\epsilon = n_1 - 3$, and thus the assertion follows.

4. Assume to the contrary that there are two distinct elements $g_1, g_2 \in G$ with $\mathsf{v}_{g_1}(U) \ge n_2 - 3$ and $\mathsf{v}_{g_2}(U) \ge n_2 - 3$. Then

$$(n_2 - 3) + (n_2 - 3) \le \mathsf{v}_{g_1}(U) + \mathsf{v}_{g_2}(U) \le |U| = n_1 + n_2 - 1\,,$$

which implies that $n_2 \le n_1 + 5$. Hence $2n_1 \le n_2 \le n_1 + 5$ and $n_1 \le 5$, a contradiction. □

The argument used in the proof of Lemma 5.4 occurs frequently in [16].

Lemma 5.4. *Let G be a finite abelian group and let $S \in \mathcal{F}(G)$ be a zero-sum free sequence. If $\prod_{g\in\mathrm{supp}(S)}(1 + \mathsf{v}_g(S)) > |G|$, then there is an $A \in \mathcal{A}(G)$ with $|A| \ge 3$ such that $(-A)A \mid (-S)S$.*

Proof. We observe that

$$|\{T \in \mathcal{F}(G) \mid T \text{ is a subsequence of } S\}| = \prod_{g\in\mathrm{supp}(S)} (1 + \mathsf{v}_g(S))\,.$$

Thus, if $\prod_{g\in\mathrm{supp}(S)}(1 + \mathsf{v}_g(S)) > |G|$, then there exist distinct sequences $T_1', T_2' \in \mathcal{F}(G)$ such that $T_1' \mid S$, $T_2' \mid S$, and $\sigma(T_1') = \sigma(T_2')$. We set $T_1' = TT_1$ and $T_2' = TT_2$ where $T = \gcd(T_1', T_2')$ and $T_1, T_2 \in \mathcal{F}(G)$. Then $\sigma(T_1) = \sigma(T_2)$ and $(-T_1)T_2$ is a zero-sum subsequence of $(-S)S$. Let $A \in \mathcal{A}(G)$ with $A \mid (-T_1)T_2$. Assume to the contrary that $|A| = 2$. Then $A = (-g)g$ for some $g \in G$. Since S is zero-sum free, we infer (after renumbering if necessary) that $(-g) \mid (-T_1)$ and $g \mid T_2$, a contradiction to $\gcd(T_1, T_2) = 1$. Therefore we obtain that $|A| \ge 3$, which implies that $|\gcd(A, (-g)g)| \le 1$ for each $g \in G$, and thus $(-A)A \mid (-S)S$. □

The next lemma is a minor modification of [16, Lemma 5.3].

Lemma 5.5. *Let $t \in \mathbbm{N}$ and $\alpha, \alpha_1, \ldots, \alpha_t, \alpha_1', \ldots, \alpha_t' \in \mathbbm{R}$ with $\alpha_1 \ge \ldots \ge \alpha_t \ge 0$, $\alpha_1' \ge \ldots \ge \alpha_t' \ge 0$, $\alpha_i' \le \alpha_i$ for each $i \in [1, t]$, and $\sum_{i=1}^t \alpha_i \ge \alpha \ge \sum_{i=1}^t \alpha_i'$. Then*

$$\prod_{\nu=1}^{t}(1 + x_\nu) \qquad \textit{is minimal}$$

over all $(x_1, \ldots, x_t) \in \mathbbm{R}^t$ with $\alpha_i' \le x_i \le \alpha_i$ for each $i \in [1, t]$ and $\sum_{i=1}^t x_i = \alpha$, if

$$x_i = \alpha_i \text{ for each } i \in [1, s] \quad \text{and} \quad x_i = \alpha_i' \quad \text{for each } i \in [s + 2, t]$$

where $s \in [0, t]$ is maximal with $\sum_{i=1}^s \alpha_i \le \alpha$.

Proof. Since continuous functions attain minima on compact sets, the above function has a minimum at some point $(m_1, \ldots, m_t) \in \mathbbm{R}^t$. Suppose there are $i, j \in [1, t]$ such that $i < j$ and $m_i < m_j$. Then $\alpha_j' \le \alpha_i' \le m_i < m_j \le \alpha_j \le \alpha_i$, and thus we can exchange m_i and m_j. Therefore, after renumbering if necessary, we may suppose that $m_1 \ge \cdots \ge m_t$. Since for $x \ge y \ge 0$ and $\delta > 0$ we have

$$(1 + x + \delta)(1 + y - \delta) = (1 + x)(1 + y) - \delta(x - y) - \delta^2 < (1 + x)(1 + y)$$

it follows that all but at most one of the m_i is equal to α_i or α_i'. It remains to show that there is an $s \in [1,t]$ such that $m_i = \alpha_i$ for $i \in [1,s]$ and $m_i = \alpha_i'$ for each $i \in [s+2,t]$. Assume to the contrary that this is not the case. Then there are $i,j \in [1,t]$ with $i < j$ such that $m_i < \alpha_i$ and $\alpha_j' < m_j$. Using again the just mentioned inequality and that $m_i \ge m_j$, we obtain a contradiction to the minimum being attained at $(m_1, \ldots, m_t)$. □

Proof of Proposition 5.1. Assume to the contrary that there is an $A \in \mathcal{B}(G)$ such that $\mathsf{L}(A) = \{2, n_2, n_1 + n_2 - 2\}$. Then there are $U, V \in \mathcal{A}(G)$ with $|U| \ge |V|$ such that $A = UV$. We set

$$U = SU' \quad \text{and} \quad V = (-S)V' \,,$$

where $U', V' \in \mathcal{F}(G)$, and $S = \gcd(U, -V)$. Since

$$2(n_1 + n_2 - 2) \le |A| = |U| + |V| \le 2\mathsf{D}(G) = 2(n_1 + n_2 - 1) \,,$$

there are the following three cases:

(I) $|A| = 2(n_1 + n_2 - 2)$. Then, a factorization of A of length $n_1 + n_2 - 2$ must contain only minimal zero-sum sequences of length 2 and thus $U' = V' = 1$.

(II) $|A| = 2(n_1 + n_2 - 2) + 1$. Then, a factorization of A of length $n_1 + n_2 - 2$ must contain one minimal zero-sum of length 3 and otherwise only minimal zero-sum sequences of length 2, thus $U' = g_1 g_2$ and $V' = (-g_1 - g_2)$ for some $g_1, g_2 \in S$.

(III) $|A| = 2(n_1 + n_2 - 1)$. Then a factorization of A of length $n_1 + n_2 - 2$ must contain either one minimal zero-sum subsequence of length 4 and otherwise minimal zero-sum sequences of length 2, or two minimal zero-sum sequences of length 3 and otherwise only minimal zero-sum sequences of length 2. Thus, there are the following two subcases.
- $U' = g_1 g_2$, $V' = h_1 h_2$ where $g_1, g_2, h_1, h_2 \in G$ such that $g_1 g_2 h_1 h_2 \in \mathcal{A}(G)$.
- $U' = g_1 g_2 (-h_1 - h_2)$ and $V' = h_1 h_2 (-g_1 - g_2)$ where $g_1, g_2, h_1, h_2 \in G$.

We start with the following two assertions.

A1. Let $W \in \mathcal{A}(G)$ with $|W| < |U|$ and $W \,|\, (-S)S$. Then $|W| \in \{2, n_1\}$.

A2. Let $W_1, W_2 \in \mathcal{A}(G)$ such that $W_1(-W_1)W_2(-W_2) \,|\, S(-S)$. Then $\{|W_1|, |W_2|\} \ne \{n_1\}$.

Proof of **A1.** Suppose $|W| > 2$. Then $(-W)W \,|\, (-S)S$ and we set $(-S)S = (-W)WT(-T)$ with $T \in \mathcal{F}(G)$ and obtain that

$$UV = (-W)WT(-T)(U'V') \,.$$

Let z be a factorization of $U'V'$. Then $|z| \in [0,2]$. If $T = 1$, then UV has a factorization of length $2 + |z| \in \{2, n_2, n_1 + n_2 - 2\}$ which implies $|z| = 0$ and hence $|W| = |U|$, a contradiction. Thus $T \ne 1$. Since $T(-T)$ has a factorization of length $|T| = |S| - |W|$, the above decomposition gives rise to a factorization of UV of length t where

$$3 \le t = 2 + |T| + |z| = 2 + |S| - |W| + |z| \in \{n_2, n_1 + n_2 - 2\} \,.$$

We distinguish four cases.

Suppose that $U' = V' = 1$. Then $z = 1$, $|z| = 0$, and $|S| = |U| = n_1 + n_2 - 2$. Thus $t = n_1 + n_2 - |W| \in \{n_2, n_1 + n_2 - 2\}$, and the assertion follows.

Suppose that $U' = g_1 g_2$ and $V' = (-g_1 - g_2)$ for some $g_1, g_2 \in S$. Then $|z| = 1$ and $|S| = n_1 + n_2 - 3$. Thus $t = n_1 + n_2 - |W| \in \{n_2, n_1 + n_2 - 2\}$, and the assertion follows.

Suppose that $U' = g_1 g_2$, $V' = h_1 h_2$ where $g_1, g_2, h_1, h_2 \in G$ such that $g_1 g_2 h_1 h_2 \in \mathcal{A}(G)$. Then $|z| = 1$ and $|S| = n_1 + n_2 - 3$. Thus $t = n_1 + n_2 - |W| \in \{n_2, n_1 + n_2 - 2\}$, and the assertion follows.

Suppose that $U' = g_1 g_2 (-h_1 - h_2)$ and $V' = h_1 h_2 (-g_1 - g_2)$ where $g_1, g_2, h_1, h_2 \in G$. Then $|z| = 2$ and $|S| = n_1 + n_2 - 4$. Thus $t = n_1 + n_2 - |W| \in \{n_2, n_1 + n_2 - 2\}$, and the assertion follows. □(**A1**)

Proof of **A2**. Assume to the contrary that $|W_1| = |W_2| = n_1$. Then there are $W_5, \ldots, W_k \in \mathcal{A}(G)$ such that

$$UV = W_1(-W_1)W_2(-W_2)W_5 \cdot \ldots \cdot W_k \,,$$

where $k \in \mathsf{L}(UV) = \{2, n_2, n_1 + n_2 - 2\}$ and hence $k = n_2$. Since

$$|W_5 \cdot \ldots \cdot W_k| = |UV| - 4n_1 \le 2(n_1 + n_2 - 1) - 4n_1 = 2(n_2 - n_1 - 1) \,,$$

it follows that

$$k - 4 \le \max \mathsf{L}(W_5 \cdot \ldots \cdot W_k) \le |W_5 \cdot \ldots \cdot W_k|/2 \le n_2 - n_1 - 1 < n_2 - 4 \,,$$

a contradiction. □(**A2**)

We distinguish two cases depending on the size of $\mathsf{h}(S)$.

CASE 1: $\mathsf{h}(S) \ge n_2/2$.

We set $S = g^v S'$ where $g \in G$, $v = \mathsf{h}(S)$, and $S' \in \mathcal{F}(G)$. Then

$$U_1 = (-g)^{n_2 - v} S'U' \in \mathcal{B}(G), \ V_1 = g^{n_2 - v}(-S')V' \in \mathcal{B}(G) \,.$$

Clearly, we have

$$(U')^{-1}U_1 = (-g)^{n_2 - v} S' = -\big((V')^{-1}V_1\big) \,.$$

We will often use that if some $W \in \mathcal{A}(G)$ divides $(U')^{-1}U_1$, then $(-W)$ divides $(V')^{-1}V_1$ and hence $(-W)W \,|\, (-S)S$.

Now we choose a factorization $x_1 \in \mathsf{Z}(U_1)$ and a factorization $y_1 \in \mathsf{Z}(V_1)$. Note that $|x_1| \le n_2 - v$ and $|y_1| \le n_2 - v$ as each minimal zero-sum sequence in x_1 and y_1 contains $(-g)$ and g, respectively. Then

$$UV = U_1V_1\big((-g)g\big)^{2v - n_2}$$

has a factorization of length t where

$$2 + (2v - n_2) \le t = |x_1| + |y_1| + (2v - n_2) \le 2(n_2 - v) + (2v - n_2) = n_2 \,.$$

Assume to the contrary that $t = 2$. Then $v = n_2/2$ and both, $U = g^{n_2/2}S'U'$ and $U' = (-g)^{n_2/2}S'U'$, are minimal zero-sum sequences, a contradiction, as $SU' \notin \mathcal{A}_{\langle g \rangle}(G)$ as its length is greater than n_1. Thus $t = n_2$, $|x_1| = |y_1| = n_2 - v$, and hence $\mathsf{L}(U_1) = \mathsf{L}(V_1) = \{n_2 - v\}$. If $W \in \mathcal{A}(G)$ with $|W| = 2$ and $W \,|\, U_1$, then $W = (-g)g$. Similarly, if $W' \in \mathcal{A}(G)$ with $|W'| = 2$ and $W' \,|\, V_1$, then $W = (-g)g$. By definition of v, not both U_1 and V_1 are divisible by an atom of length 2. Now we distinguish four cases depending on the form of U' and V', which we determined above.

CASE 1.1: $U' = V' = 1$.

Then $V_1 = -U_1$, say $V_1 = W_1 \cdot \ldots \cdot W_{n_2 - v}$. Since none of the W_i has length 2, it follows that $|W_1| = \ldots = |W_{n_2 - v}| = n_1$ and hence

$$2(n_2 + n_1 - 2) = 2|V| = 2(2v - n_2) + 2|V_1| = 2(2v - n_2) + 2n_1(n_2 - v) \,,$$

which implies that $v = n_2 - 1$. Consequently $|S| = n_1 - 1$ and $S \in \mathcal{A}_{\langle g \rangle}(G)$. This implies that (use an elementary direct argument or [15, Theorem 5.1.8]),

$$S = (2e_1 + a_1e_2) \prod_{\nu=2}^{n_1 - 1} (e_1 + a_\nu e_2) \,,$$

where (e_1, e_2) is a basis of G. Let $r \in [0, n_2 - 1]$ such that $r \equiv -a_1 + a_2 + a_3 \mod n_2$. Then

$$W_1 = (2e_1 + a_1e_2)(-e_1 - a_2e_2)(-e_1 - a_3e_2)e_2^r \quad \text{and} \quad W_2 = (2e_1 + a_1e_2)(-e_1 - a_2e_2)(-e_1 - a_3e_2)(-e_2)^{n_2 - r}$$

are minimal zero-sum sequences dividing $(-V)V$. Since $|W_1| = 3 + r$, $|W_2| = 3 + n_2 - r$, and $|W_1W_2| = n_2 + 6 > 2n_1$, at least one of them does not have length n_1, a contradiction.

CASE 1.2: $U' = g_1g_2$ and $V' = (-g_1 - g_2)$ for some $g_1, g_2 \in S$.

We set

$$x_1 = X_1 \cdot \ldots \cdot X_{n_2-v} \quad \text{and} \quad y_1 = Y_1 \cdot \ldots \cdot Y_{n_2-v}$$

where all $X_i, Y_j \in \mathcal{A}(G)$, $g_1g_2 \,|\, X_1X_2$ (or even $g_1g_2 \,|\, X_1$), and $(-g_1-g_2) \,|\, Y_1$. We distinguish three subcases.

CASE 1.2.1: $v = n_2 - 1$.

By Corollary 5.3, with all notations as introduced there, we get

$$U = e_2^{n_2-1} \prod_{\nu=1}^{n_1} (e_1 + x_\nu e_2)\,.$$

Thus $g = e_2$ and $U' \mid \prod_{\nu=1}^{n_1}(e_1 + x_\nu e_2)$, whence after renumbering if necessary we have $g_i = x_ie_1 + e_2$ for each $i \in [1,2]$. Therefore we have

$$V = \big(-2e_1 - (x_1+x_2)e_2\big)(-e_2)^{n_2-1} \prod_{\nu=3}^{n_1}(-e_1 - x_\nu e_2) = (-g_1-g_2)(-S)\,.$$

Assume to the contrary that there are $i, j \in [3, n_1]$ distinct with $x_i \neq x_j$. If $q \in [1, n_2-1]$ with $q \equiv -(x_i - x_j) \mod n_2$, then

$$W_1' = (e_1 + x_ie_2)(-e_1 - x_je_2)e_2^q \quad \text{and} \quad W_2' = (e_1 + x_ie_2)(-e_1 - x_je_2)(-e_2)^{n_2-q}$$

are atoms dividing $(-S)S$, both have length greater than two but not both have length n_1, a contradiction to **A1**. Therefore we have $x_3 = \ldots = x_{n_1}$. Since $x_1 + \ldots + x_{n_1} \equiv 1 \mod n_2$, it follows that $x_1 \neq x_3$ or $x_2 \neq x_3$, say $x_2 \neq x_3$. Therefore there is an $r \in [1, n_2-1]$ with $r \equiv x_2 - x_3$ such that

$$W_1 = \big(-(x_1+x_2)e_2 - 2e_1\big)(x_1e_2 + e_1)(x_3e_2 + e_1)e_2^r \in \mathcal{A}(G)$$

and

$$W_2 = (x_2e_2 + e_1)(-x_3e_2 - e_1)(-e_2)^r \in \mathcal{A}(G)\,.$$

Thus it follows that

$$UV = W_1W_2 \prod_{\nu=4}^{n_1} \big((x_\nu e_2 + e_1)(-x_\nu e_2 - e_1)\big)\big((-e_2)e_2\big)^{n_2-1-r}$$

has a factorization of length

$$2 + (n_1 - 3) + (n_2 - 1 - r) = n_1 + n_2 - 2 - r \in \{2, n_2, n_1+n_2-2\}\,,$$

and hence $r = n_1 - 2$. Now we define

$$W_1' = \big(-(x_1+x_2)e_2 - 2e_1\big)(x_1e_2 + e_1)(x_3e_2 + e_1)(-e_2)^{n_2-r}$$

and

$$W_2' = (x_2e_2 + e_1)(-x_3e_2 - e_1)e_2^{n_2-r} \in \mathcal{A}(G)\,.$$

Thus it follows that

$$UV = W_1'W_2' \prod_{\nu=4}^{n_1} \big((x_\nu e_2 + e_1)(-x_\nu e_2 - e_1)\big)\big((-e_2)e_2\big)^{r-1}$$

has a factorization of length

$$2 + (n_1 - 3) + (r - 1) = n_1 - 2 + r = 2n_1 - 4 \notin \{2, n_2, n_1+n_2-2\}\,,$$

a contradiction.

CASE 1.2.2: $v = n_2 - 2$.

Since $V' \mid Y_1$ it follows that $Y_2 \,|\, (-S)S$ and we thus may assume that $X_2 = -Y_2$. Furthermore, $|Y_2|$ cannot have length 2, since $-g \nmid S'$. Hence **A1** gives that $|Y_2|$ has length n_1. It follows that $|Y_1| = 2$ and

$|X_1| = 3$. This implies that $-g_1 - g_2 = -g$ whence $\mathsf v_{-g}(V) = n_2 - 1$ and $\mathsf v_g(U) = n_2 - 2$. By Corollary 5.3, with all notations as introduced there, we obtain that

$$U = (e_1 + ye_2)^{n_1-1}e_2^{n_2-2}\big(- xe_1 + (-xy+1)e_2\big)\ \big(- (n_1 - 1 - x)e_1 + (-(n_1 - 1 - x)y + 1)e_2\big)\,.$$

Since $g_1 + g_2 = g = e_2$, it follows that $x = 1$ and that

$$V = (-(e_1 + ye_2))^{n_1-2}(-e_2)^{n_2-1}((n_1 - 2)e_1 + ((n_1 - 2)y + 1)e_2)\,.$$

Then $W = (e_1 + ye_2)^2((n_1 - 2)e_1 + ((n_1 - 2)y + 1)e_2)(-e_2)^r$, where $r \in [0, n_2 - 1]$ such that $r \equiv n_1y + 1 \mod n_2$, is a minimal zero-sum sequence. Since $r \equiv 1 \mod n_1$, it follows that $r \in [1, n_2 - n_1 + 1]$. Thus, $W \,|\, (-S)S$, hence $|W| = n_1$, and thus $r = n_1 - 3$. We consider $W' = (e_1 + ye_2)^2((n_1 - 2)e_1 + ((n_1 - 2)y + 1)e_2)e_2^{n_2-(n_1-3)}$. Again, $W' \,|\, (-S)S$. Yet $|W'| = 3 + (n_2 - (n_1 - 3)) = n_2 - n_1 + 6$, a contradiction.

CASE 1.2.3: $v \le n_2 - 3$.

Then $Y_2Y_3 \,|\, (V')^{-1}V_1$, and since $|Y_2| \ne 2 \ne |Y_3|$, we infer that $|Y_2| = |Y_3| = n_1$. Thus $(-Y_2)(-Y_3) \,|\, (U')^{-1}U_1$ and $Y_2(-Y_2)Y_3(-Y_3) \,|\, S(-S)$, a contradiction to **A2**.

CASE 1.3: $U' = g_1g_2$, $V' = h_1h_2$ where $g_1, g_2, h_1, h_2 \in G$ such that $g_1g_2h_1h_2 \in \mathcal A(G)$.

We set

$$x_1 = X_1 \cdot \ldots \cdot X_{n_2-v} \quad \text{and} \quad y_1 = Y_1 \cdot \ldots \cdot Y_{n_2-v}$$

where all $X_i, Y_j \in \mathcal A(G)$, $g_1g_2 \,|\, X_1X_2$ (or even $g_1g_2 \,|\, X_1$), and $h_1h_2 \,|\, Y_1Y_2$ (or even $h_1h_2 \,|\, Y_1$). We distinguish four cases.

CASE 1.3.1: $v = n_2 - 1$.

By Corollary 5.3, we have

$$U = e_2^{n_2-1}\prod_{\nu=1}^{n_1}(e_1 + x_\nu e_2)$$

$$-V = e_2^{n_2-1}\prod_{\nu=1}^{n_1}(e_1' + x_\nu' e_2)\,,$$

where (e_1, e_2) and (e_1', e_2) are both bases with $\mathrm{ord}(e_2) = n_2$ and $x_i, x_i' \in [0, n_2 - 1]$ for each $i \in [1, n_1]$. Since $|S| = |U| - 2$, it follows that, after renumbering if necessary, $\prod_{\nu=3}^{n_1}(e_1 + x_\nu e_2) \,|\, (-V)$ and hence, after a further renumbering if necessary, $e_1 + x_ie_2 = e_1' + x_i'e_2$ for each $i \in [3, n_1]$. Thus, if we write $-V$ with respect to the basis (e_1, e_2), it still has the above structure. Therefore we may assume that $e_1 = e_1'$ and $x_i = x_i'$ for each $i \in [3, n_1]$. Therefore

$$g_i = e_1 + x_ie_2 \quad \text{and} \quad h_i = -e_1 - x_i'e_2 \quad \text{for each } i \in [1, 2]\,.$$

Since $g_1 + g_2 = -h_1 - h_2$, it follows that $x_1 + x_2 \equiv -x_1' - x_2' \mod n_2$ and hence $x_1 - x_1' \equiv x_2' - x_2 \mod n_2$. Let $r \in [0, n_2 - 1$ such that $r \equiv x_1 - x_1' \mod n_2$. Then

$$W_1 = g_1h_1(-e_2)^r\,,\ W_1' = g_1h_1e_2^{n_2-r}\,,\ W_2 = g_2h_2e_2^r\,, \quad \text{and} \quad W_2' = g_2h_2(-e_2)^{n_2-r}$$

are minimal zero-sum sequences which give rise to the factorizations

$$\begin{aligned} UV &= W_1W_2\big((-e_2)e_2\big)^{n_2-r-1}\prod_{\nu=3}^{n_1}\big((e_1 + x_\nu e_2)(-e_1 - x_\nu e_2)\big) \\ &= W_1'W_2'\big((-e_2)e_2\big)^{r-1}\prod_{\nu=3}^{n_1}\big((e_1 + x_\nu e_2)(-e_1 - x_\nu e_2)\big)\,. \end{aligned}$$

These factorizations have length $2 + (n_2 - r - 1) + (n_1 - 2) = n_1 + n_2 - 2 - (r - 1)$ and $2 + (r - 1) + (n_1 - 2) = n_1 + r - 1$. Since not both of them can be in $\{2, n_2, n_1 + n_2 - 2\}$, we have arrived at a contradiction.

CASE 1.3.2: $v = n_2 - 2$.

Assume to the contrary that $\mathsf h(U) = \mathsf h(V) = n_2 - 1$. Since, by Corollary 5.3.3, the elements $g', g'' \in G$ with $\mathsf v_{g'}(U) = n_2 - 1$ and $\mathsf v_{g''}(V) = n_2 - 1$ are uniquely determined, it follows that $g = g' = -g''$ and

hence $v = \mathsf h(S) = n_2 - 1$, a contradiction. Thus, after exchanging U and V if necessary, we may assume that $\mathsf h(U) = n_2 - 2$ and it remains to consider the two cases $\mathsf h(V) = n_2 - 1$ and $\mathsf h(V) = n_2 - 2$.

CASE 1.3.2.1: $\mathsf h(U) = n_2 - 2$ and $\mathsf h(V) = n_2 - 1$.

By Corollary 5.3, we infer that

$$U = (e_1 + ye_2)^{n_1-1}e_2^{n_2-2}\big(-xe_1 + (-xy+1)e_2\big)\,\big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big)\,,$$

$$-V = e_2^{n_2-1}\prod_{\nu=1}^{n_1}(e_1' + x_\nu e_2)\,,$$

where (e_1, e_2) and (e_1', e_2) are bases and all parameters are as in Corollary 5.3. Since $|S| = |U|-2$, it follows that $(e_1 + ye_2)^{n_1-3} \,|\, (-V)$ and hence, after renumbering if necessary, $e_1' + x_1e_2 = \ldots = e_1' + x_{n_1-3}e_2 = e_1 + ye_2$. Thus, if we write $-V$ with respect to the basis (e_1, e_2), it still has the above structure. Therefore we may assume that $e_1 = e_1'$ and $y = x_1 = \ldots = x_{n_1-3}$. Thus we obtain that

$$-V = e_2^{n_2-1}(e_1 + ye_2)^{n_1-3}\prod_{\nu=n_1-2}^{n_1}(e_1 + x_\nu e_2)\,.$$

Note that $e_2 \in \{-h_1, -h_2\}$, $e_2 \notin \{g_1, g_2\}$, say $-h_1 = e_2$ and $-h_2 = e_1 + x_{n_1}e_2$, and $g_1 + g_2 = -(h_1+h_2) = e_1 + (x_{n_1}+1)e_2$. This condition on the sum shows that

$$\{g_1, g_2\} = \{-xe_1 + (-xy+1)e_2, -(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\}\,.$$

This implies that $(e_1 + ye_2)^{n_1-1} \,|\, (-V)$ and hence, after renumbering if necessary,

$$-V = e_2^{n_2-1}(e_1 + ye_2)^{n_1-1}(e_1 + x_{n_1}e_2)\,.$$

Since $g_1 + g_2 = -(n_1-1)e_1 - (y(n_1-1)-2)e_2$, it follows that the sequence

$$W_1 = g_1g_2(-e_1 - ye_2)(-e_2)^r\,,$$

where $r \in [0, n_2-1]$ and $r \equiv -yn_1 + 2 \mod n_2$, is a minimal zero-sum sequence. Since $r \equiv 2 \mod n_1$, we infer that $r \in [2, n_2-2]$ and that $W_1 \,|\, UV$. Since $g_1 + g_2 = -(h_1+h_2)$, we obtain that

$$W_2 = h_1h_2(e_1 + ye_2)e_2^r \in \mathcal B(G)\ ,\ \mathsf L(W_2) = \{2\} \quad \text{and} \quad W_2 \,|\, UV\,.$$

Therefore it follows that

$$UV = W_1W_2\big((-e_2)e_2\big)^{n_2-2-r}\big((e_1 + ye_2)(-e_1 - ye_2)\big)^{n_1-2}\,,$$

and hence $n_1 + n_2 - 1 - r \in \mathsf L(UV) = \{2, n_2, n_1+n_2-2\}$, a contradiction, since $r \equiv 2 \mod n_1$.

CASE 1.3.2.2: $\mathsf h(U) = \mathsf h(V) = n_2 - 2$.

By Corollary 5.3, we infer that

$$U = (e_1 + ye_2)^{n_1-1}e_2^{n_2-2}U''$$

$$-V = (e_1' + y'e_2)^{n_1-1}e_2^{n_2-2}(-V'')$$

where (e_1, e_2) and (e_1', e_2) are bases, $U'', V'' \in \mathcal F(G)$ with $|U''| = |V''| = 2$, and $y, y' \in [0, n_2-1]$. Since $|S| = |U| - 2$, it follows that $(e_1' + y'e_2)^{n_1-3} \,|\, U$ and hence $e_1' + y'e_2 = e_1 + ye_2$. Thus, if we write $-V$ with respect to the basis (e_1, e_2), it still has the above structure. Therefore we may assume that $e_1 = e_1'$ and $y = y'$. Therefore it follows that

$$U = (e_1 + ye_2)^{n_1-1}e_2^{n_2-2}g_1g_2 \quad \text{and} \quad V = (-e_1 - ye_2)^{n_1-1}(-e_2)^{n_2-2}h_1h_2\,,$$

and hence $g_1 + g_2 = -(n_1-1)e_1 - (y(n_1-1)-2)e_2$. Thus

$$W_1 = g_1g_2(-e_1 - ye_2)(-e_2)^r\,,$$

where $r \in [0, n_2-1]$ and $r \equiv -yn_1 + 2 \mod n_2$, is a minimal zero-sum sequence. Since $r \equiv 2 \mod n_1$, we infer that $r \in [2, n_2-2]$ and that $W_1 \,|\, UV$. Since $g_1 + g_2 = -(h_1+h_2)$, we obtain that

$$W_2 = h_1h_2(e_1 + ye_2)e_2^r \in \mathcal A(G) \quad \text{and} \quad W_2 \,|\, UV\,.$$

Therefore it follows that

$$UV = W_1W_2\big((-e_2)e_2\big)^{n_2-2-r}\big((e_1+ye_2)(-e_1-ye_2)\big)^{n_1-2}\,,$$

and hence $n_1+n_2-2-r \in \mathsf L(UV) = \{2, n_2, n_1+n_2-2\}$, a contradiction, since $r \equiv 2 \mod n_1$.

CASE 1.3.3: $v = n_2-3$.

Then $X_3 \,|\, (U')^{-1}U_1$ and hence $|X_3| = n_1$. Since

$$|X_1X_2X_3| = |U_1| = |U| - v + (n_2 - v) = n_1 + 5\,,$$

it follows that $|X_1X_2| = 5$, and hence X_1 or X_2 has length two. Similarly, we obtain that Y_1 or Y_2 has length two, a contradiction to the earlier mentioned fact that not both, U_1 and V_1 are divisible by an atom of length two.

CASE 1.3.4: $v \le n_2-4$.

Then $Y_3Y_4 \,|\, (V')^{-1}V_1$, and since $|Y_3| \ne 2 \ne |Y_4|$, we infer that $|Y_3| = |Y_4| = n_1$. Thus $(-Y_3)(-Y_4) \,|\, (U')^{-1}U_1$ and $Y_3(-Y_3)Y_4(-Y_4) \,|\, S(-S)$, a contradiction to **A2**.

CASE 1.4: $U' = g_1g_2(-h_1-h_2)$ and $V' = h_1h_2(-g_1-g_2)$ where $g_1, g_2, h_1, h_2 \in G$.

We set

$$x_1 = X_1 \cdot \ldots \cdot X_{n_2-v} \quad \text{and} \quad y_1 = Y_1 \cdot \ldots \cdot Y_{n_2-v}$$

where all $X_i, Y_j \in \mathcal A(G)$, $g_1g_2(-h_1-h_2) \,|\, X_1X_2X_3$ (or even $g_1g_2(-h_1-h_2) \,|\, X_1X_2$ or $g_1g_2(-h_1-h_2) \,|\, X_1$) , and $h_1h_2(-g_1-g_2) \,|\, Y_1Y_2Y_3$ (or even $h_1h_2(-g_1-g_2) \,|\, Y_1Y_2$ or $h_1h_2(-g_1-g_2) \,|\, Y_1$).

CASE 1.4.1: $v = n_2-1$.

By Corollary 5.3 we have

$$U = e_2^{n_2-1}\prod_{\nu=1}^{n_1}(e_1+x_\nu e_2)$$

$$-V = e_2^{n_2-1}\prod_{\nu=1}^{n_1}(e_1'+x_\nu' e_2)\,,$$

where (e_1, e_2) and (e_1', e_2) are both bases with $\mathrm{ord}(e_2) = n_2$ and $x_\nu, x_\nu' \in [0, n_2-1]$ for each $\nu \in [1, n_1]$. Since $|S| = |U| - 3$, it follows that, after renumbering if necessary, $\prod_{\nu=4}^{n_1}(e_1+x_\nu e_2) \,|\, (-V)$ and hence, after a further renumbering if necessary, $e_1 + x_\nu e_2 = e_1' + x_\nu' e_2$ for each $\nu \in [4, n_1]$. Thus, if we write $-V$ with respect to the basis (e_1, e_2), it still has the above structure. Therefore we may assume that $e_1 = e_1'$ and $x_\nu = x_\nu'$ for each $\nu \in [4, n_1]$. Furthermore, we obtain that

$$g_1 = e_1 + x_1e_2,\ \ g_2 = e_1 + x_2e_2,\ \ \ -h_1 - h_2 = e_1 + x_3e_2$$

$$h_1 = -e_1 - x_1'e_2,\ \ h_2 = -e_1 - x_2'e_2, \quad \text{and} \quad -g_1 - g_2 = -e_1 - x_3'e_2\,,$$

a contradiction, since $\mathrm{ord}(e_1) = n_1 > 3$

CASE 1.4.2: $v = n_2-2$.

Arguing as at the beginning of CASE 1.3.2 we may assume $\mathsf h(U) = n_2-2$ and it is sufficient to consider the two subcases $\mathsf h(V) = n_2-1$ and $\mathsf h(V) = n_2-2$.

CASE 1.4.2.1: $\mathsf h(U) = n_2-2$ and $\mathsf h(V) = n_2-1$.

By Corollary 5.3, we infer that

$$U = (e_1+ye_2)^{n_1-1}e_2^{n_2-2}\big(-xe_1 + (-xy+1)e_2\big)\,\big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big)\,,$$

$$-V = e_2^{n_2-1}\prod_{\nu=1}^{n_1}(e_1'+x_\nu e_2)\,,$$

where (e_1, e_2) and (e_1', e_2) are bases and all parameters are as in Corollary 5.3. Since $|S| = |U|-3$, it follows that $(e_1+ye_2)^{n_1-4} \,|\, (-V)$ and hence, after renumbering if necessary, $e_1' + x_1e_2 = \ldots = e_1' + x_{n_1-4}e_2 =$

e_1+ye_2. Thus, if we write $-V$ with respect to the basis (e_1, e_2), it still has the above structure. Therefore we may assume that $e_1 = e_1'$ and $y = x_5 = \ldots = x_{n_1}$. Thus we obtain that

$$-V = e_2^{n_2-1}(e_1 + ye_2)^{n_1-4} \prod_{\nu=1}^{4} (e_1 + x_\nu e_2)\,.$$

Since

$$\gcd\Big(\big(-xe_1 + (-xy+1)e_2\big)\ \big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big),\ -V\Big) = 1\,,$$

it follows that

$$g_1g_2(-h_1-h_2) = (e_1+ye_2)\big(-xe_1 + (-xy+1)e_2\big)\ \big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big)\,.$$

Thus $\mathsf{v}_{e_1+ye_2}(S) = n_1 - 2$ and hence, after renumbering if necessary,

$$-V = e_2^{n_2-1}(e_1+ye_2)^{n_1-2}(e_1+x_1e_2)(e_1+x_2e_2)\,.$$

We observe that

$$\big(-xe_1 + (-xy+1)e_2\big) + \big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big) = e_1 + (-(n_1-1)y+2)e_2\,,$$
$$(-e_1 - x_1e_2) + (-e_1 - x_2e_2) = (n_1-2)(e_1+ye_2) + (n_2-1)e_2 = -2e_1 + ((n_1-2)y-1)e_2\,.$$

Consequently, there are $r, r' \in [0, n_2-1]$ such that

$$W_1 = \big(-xe_1 + (-xy+1)e_2\big)\big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big)(-e_1-ye_2)(-e_2)^r \in \mathcal{A}(G)\,, \text{and}$$
$$W_2 = (-e_1-x_1e_2)(-e_1-x_2e_2)(e_1+ye_2)^2e_2^{r'} \in \mathcal{A}(G)\,,$$

(note that $y \notin \{-x_1, -x_2\}$), and clearly we have

$$r \equiv 2 - n_1y \mod n_2 \quad \text{and} \quad r' \equiv 1 - n_1y \mod n_2\,.$$

This implies that $r \equiv 2 \mod n_1$ and $r' = r-1$. Therefore, we obtain that

$$UV = W_1W_2\big((e_1+ye_2)(-e_1-ye_2)\big)^{n_1-3}\big((-e_2)e_2\big)^{n_2-r-1}\,,$$

and thus $n_1+n_2-2-r \in \mathsf{L}(UV) = \{2, n_2, n_1+n_2-2\}$, a contradiction, since $r \equiv 2 \mod n_1$.

CASE 1.4.2.2: $\mathsf{h}(U) = \mathsf{h}(V) = n_2 - 2$.

By Corollary 5.3, we infer that

$$U = (e_1+ye_2)^{n_1-1}e_2^{n_2-2}U''$$
$$-V = (e_1'+y'e_2)^{n_1-1}e_2^{n_2-2}(-V'')$$

where (e_1, e_2) and (e_1', e_2) are bases, $U'', V'' \in \mathcal{F}(G)$ with $|U''| = |V''| = 2$, and $y, y' \in [0, n_2-1]$. Since $|S| = |U| - 3$, it follows that $(e_1'+y'e_2)^{n_1-4} \,|\, U$. If $n_1 > 6$, it follows that $e_1'+y'e_2 = e_1+ye_2$. If $n_1 = 6$, so n_1 even, then

$$U'' = \big(-xe_1 + (-xy+1)e_2\big)\ \big(-(n_1-1-x)e_1 + (-(n_1-1-x)y+1)e_2\big)$$

is not a square whence $(e_1'+y'e_2)^2 \neq U''$ and it follows again that $e_1'+y'e_2 = e_1+ye_2$. Thus, if we write $-V$ with respect to the basis (e_1, e_2), it still has the above structure. Therefore we may assume that $e_1 = e_1'$ and $y = y'$. Therefore it follows that

$$U = (e_1+ye_2)^{n_1-1}e_2^{n_2-2}U'' \quad \text{and} \quad V = (-e_1-ye_2)^{n_1-1}(-e_2)^{n_2-2}V''\,,$$

a contradiction to $|S| = |U| - 3$.

CASE 1.4.3: $v = n_2 - 3$.

Note that

$$|X_1X_2X_3| = |U_1| = |U| - v + (n_2 - v) = n_1 + n_2 - 1 + n_2 - (2n_2 - 6) = n_1 + 5\,.$$

Suppose that U' divides a product of two of the X_1, X_2, X_3, say $U' \,|\, X_1X_2$. Then $X_3 \,|\, (U')^{-1}U_1$ and hence $|X_3| = n_1$. Thus $|X_1X_2| = 5$ and either X_1 or X_2 has length two. Since X_3 divides $(U')^{-1}U_1$, it follows that $-X_3$ divides $V_1(V')^{-1}$. After considering a new factorization of V_1 if necessary we may suppose without restriction that $Y_3 = -X_3$. Arguing as above we infer that Y_1 or Y_2 has length two, a contradiction to the earlier mentioned fact that not both, U_1 and V_1 are divisible by an atom of length two.

Thus from now on we may assume that for every $X \in \mathcal{A}(G)$ dividing U_1 we have $|\gcd(X, U')| = 1$, and similarly for every $Y \in \mathcal{A}(G)$ dividing V_1 we have $|\gcd(Y, V')| = 1$.

Arguing as at the beginning of CASE 1.3.2 we obtain that $\mathsf h(U) = n_2 - 3$ or $\mathsf h(V) = n_2 - 3$, say $\mathsf h(U) = n_2 - 3$. By Corollary 5.3.3, we infer that

$$U = (e_1 + ye_2)^{n_1-1}e_2^{n_2-3}\prod_{\nu=1}^{3}(-x_ie_1 + (-x_\nu y + 1)e_2)\,,$$

with all parameters as described there. Since $|S| = |U| - 3$, it follows that $(e_1 + ye_2)^{n_1-4} \,|\, (-V)$ and thus

$$V = (-e_1 - ye_2)^{n_1-4}(-e_2)^{n_2-3}V'' \quad \text{where} \quad V'' \in \mathcal{F}(G) \text{ with } |V''| = 6\,.$$

Since

$$U_1 = (-e_2)^3(e_1 + ye_2)^{n_1-1}\prod_{\nu=1}^{3}(-x_\nu e_1 + (-x_\nu y + 1)e_2) = X_1X_2X_3\,,$$

it follows that $(-e_2) \,|\, X_\nu$ for each $\nu \in [1, 3]$. Since $(e_1 + ye_2)^{n_1-1}(-e_2)^3$ is zero-sum free, each of the X_ν is divisible by at least one of the elements from $\prod_{\nu=1}^{3}(-x_\nu e_1 + (-x_\nu y + 1)e_2)$. Thus, after renumbering if necessary, it follows that for each $\nu \in [1, 3]$

$$X_\nu = (-e_2)(-x_\nu e_1 + (-x_\nu y + 1)e_2)(e_1 + ye_2)^{x_\nu}\,.$$

This implies that $x_1 + x_2 + x_3 = n_1 - 1$. Since $|\gcd(X_\nu, U')| = 1$ for each $\nu \in [1, 3]$, it follows that $U' = \prod_{\nu=1}^{3}(-x_\nu e_1 + (-x_\nu y + 1)e_2)$, and hence

$$V = (-e_1 - ye_2)^{n_1-1}(-e_2)^{n_2-3}V'\,.$$

Since

$$V_1 = e_2^3(-e_1 - ye_2)^{n_1-1}V' = Y_1Y_2Y_3\,,$$

it follows that $e_2 \,|\, Y_\nu$ for each $\nu \in [1, 3]$. Since $(-e_1 - ye_2)^{n_1-1}e_2^3$ is zero-sum free, each of the Y_ν is divisible by at least one of the elements from V'. Setting $h_3 = -g_1 - g_2$ and renumbering if necessary, it follows that for each $\nu \in [1, 3]$

$$Y_\nu = e_2h_\nu(-e_1 - ye_2)^{y_\nu}\,,$$

where $y_1, y_2, y_3 \in \mathbb{N}_0$ with $y_1 + y_2 + y_3 = n_1 - 1$. For each $\nu \in [1, 3]$ it follows that $h_\nu = y_\nu e_1 + (yy_\nu - 1)e_2$. Therefore we obtain that

$$\begin{aligned} 0 = g_1 + g_2 + h_3 &= \Big(-x_1e_1 + (-x_1y + 1)e_2\Big) + \Big(-x_2e_1 + (-x_2y + 1)e_2\Big) + \Big(y_3e_1 + (yy_3 - 1)e_2\Big) \\ &= \Big(-x_1 - x_2 + y_3\Big)e_1 + \Big((-x_1 - x_2 + y_3)y + 1\Big)e_2\,, \end{aligned}$$

a contradiction, as not both $-x_1 - x_2 + y_3$ and $(-x_1 - x_2 + y_3)y + 1$ can be 0 modulo n_1.

CASE 1.4.4: $v = n_2 - 4$.

Then $X_4 \,|\, (U')^{-1}U_1$ and hence $|X_4| = n_1$. Since

$$|X_1X_2X_3X_4| = |U_1| = |U| - v + (n_2 - v) = n_1 + 7\,,$$

it follows that $|X_1X_2X_3| = 7$, and hence X_1, X_2, or X_3 has length two. Similarly, we obtain that Y_1, Y_2, or Y_3 has length two, a contradiction to the earlier mentioned fact that not both, U_1 and V_1 are divisible by an atom of length two.

CASE 1.4.5: $v \le n_2 - 5$.

Then $Y_4Y_5 \,|\, (V')^{-1}V_1$, and since $|Y_4| \neq 2 \neq |Y_5|$, we infer that $|Y_4| = |Y_5| = n_1$. Thus $(-Y_4)(-Y_5) \,|\, (U')^{-1}U_1$ and $Y_4(-Y_4)Y_5(-Y_5) \,|\, S(-S)$, a contradiction to **A2**.

CASE 2: $\mathsf h(S) < n_2/2$.

We distinguish two subcases, depending on the parity of n_2.

CASE 2.1: n_2 is odd.

Since n_2 is odd, we have $3n_1 \le n_2$ and $n_1 \ge 7$. We write $S = \prod_{\nu=1}^k a_\nu^{\alpha_\nu}$ with $a_1, \ldots, a_k \in G$ pairwise distinct and $\alpha_1 \ge \ldots \ge \alpha_k \ge 1$. Since $|S| = \sum_{\nu=1}^k \alpha_\nu \ge n_2 + n_1 - 4$ and $\alpha_1 \le (n_2-1)/2$, it follows that $k \ge 3$.

We define $T_1 = S(a_1a_2a_3)^{-1}$ and set $T_1 = \prod_{i=1}^l b_i^{\beta_i}$ with $b_1, \ldots, b_l \in G$ pairwise distinct and $\beta_1 \ge \ldots \ge \beta_l \ge 1$. Since $\beta_1 \le \alpha_1 \le (n_2-1)/2$, $\sum_{i=1}^k \beta_i = |S| - 3 \ge n_2 + n_1 - 7$, and $n_2 - 1 < n_1 + n_2 - 7$, it follows that $l \ge 3$. Applying Lemma 5.5 (with parameters $t = l$, $\alpha = |T_1|$, $\alpha_1' = \ldots = \alpha_t' = 0$ and $\alpha_1 = \ldots = \alpha_t = (n_2-1)/2$; note that we have $s + 1 = 3$) we infer that

$$\begin{aligned}\prod_{\nu=1}^l (1+\beta_\nu) &\ge \left(1 + \frac{n_2-1}{2}\right)^2 \big(1 + (|T_1| - (n_2-1)\big) \\ &\ge \left(1 + \frac{n_2-1}{2}\right)^2 (1+1) = \frac{n_2^2 + 2n_2 + 1}{2} > n_1n_2\,.\end{aligned}$$

Thus Lemma 5.4 implies that there is a $W_1 \in \mathcal A(G)$ with $|W_1| \ge 3$ such that $(-W_1)W_1 \,|\, (-T_1)T_1$. Since $|W_1| < |U|$, **A1** implies that $W_1 = n_1$. We write $W_1 = W_1'(-W_1'')$ with $W_1'W_1'' \,|\, T_1$.

We define $T_2 = S(W_1'W_1'')^{-1}$ and note that $(-S)S = W_1(-W_1)T_2(-T_2)$. Furthermore,

$$|T_2| = |S| - n_1 \ge n_2 - 4 \quad \text{and} \quad |\operatorname{supp}(T_2)| \ge 3\,.$$

We set $T_2 = \prod_{\nu=1}^m c_\nu^{\gamma_\nu}$ with $c_1, \ldots, c_m \in G$ pairwise distinct and $\gamma_1 \ge \ldots \ge \gamma_m \ge 1$. Applying Lemma 5.5 (with parameters $t = m$, $\alpha = |T_2|$, $\alpha_1' = \ldots = \alpha_3' = 1$, $\alpha_4' = \ldots = \alpha_t' = 0$ and $\alpha_1 = \ldots = \alpha_t = (n_2-1)/2$; note that we have $s + 1 = 3$) we infer that

$$\begin{aligned}\prod_{\nu=1}^m (1+\gamma_\nu) &\ge \left(1 + \frac{n_2-1}{2}\right)\left(1 + |T_2| - \left(\frac{n_2-1}{2} + 1\right)\right)(1+1) \\ &= \frac{n_2+1}{2}\frac{n_2-7}{2}2 = \frac{1}{2}(n_2^2 - 6n_2 - 7) > n_1n_2\,.\end{aligned}$$

Thus Lemma 5.4 implies that there is a $W_2 \in \mathcal A(G)$ with $|W_2| \ge 3$ such that $(-W_2)W_2 \,|\, (-T_2)T_2$. Since $|W_2| < |U|$, **A1** implies that $W_2 = n_1$. Therefore we obtain that $W_1(-W_1)W_2(-W_2) \,|\, (-S)S$, a contradiction to **A2**.

CASE 2.2: n_2 is even.

We distinguish three cases; the first one is that $|U| = |V| = n_1 + n_2 - 2$ and the two others deal with the case $|U| = n_1 + n_2 - 2$, further distinguishing based on the structural description recalled in Lemma 5.2.

CASE 2.2.1: $|U| = |V| = n_1 + n_2 - 2$.

First we handle the case $n_2 > 12$. The special case $n_2 = 12$ will follow by the same strategy but the details will be different.

We write $S = \prod_{\nu=1}^k a_\nu^{\alpha_\nu}$ with $\alpha_1 \ge \ldots \ge \alpha_k$. Since $\sum_{\nu=1}^k \alpha_\nu = n_2 + n_1 - 2$ and $\alpha_1 \le (n_2-2)/2$, it follows that $k \ge 3$.

We define $T_1 = \prod_{\nu=1}^l b_\nu^{\beta_\nu}$ with $\beta_1 \ge \ldots \ge \beta_l$ to be a subsequence of S of length $n_2 - 2$ such that $\beta_2 \le n_2/2 - 3$ and such that $T_1^{-1}S$ contains at least 4 distinct elements or 3 elements with multiplicity at least 2. Applying Lemma 5.5 (with parameters $t = l$, $\alpha = |T_1| = n_2 - 2$, $\alpha_1' = \ldots = \alpha_t' = 0$, and

$\alpha_1 = (n_2-2)/2$, $\alpha_2 = \ldots = \alpha_t = (n_2-6)/2$; note that we have $s+1 = 3$) we infer that

$$\prod_{\nu=1}^{l}(1+\beta_\nu) \ge \left(1+\frac{n_2-2}{2}\right)\left(1+\frac{n_2-6}{2}\right)(1+2) = 3\left(\frac{n_2^2}{4}-n_2\right) > n_1n_2\,,$$

where the last inequality holds because $n_2 > 12$. Thus Lemma 5.4 implies that there is a $W_1 \in \mathcal{A}(G)$ with $|W_1| \ge 3$ such that $(-W_1)W_1 \,|\, (-T_1)T_1$. Since $|W_1| < |U|$, **A2** implies that $|W_1| = n_1$. We write $W_1 = W_1'(-W_1'')$ with $W_1'W_1'' \,|\, T_1$.

We define $T_2 = S(W_1'W_1'')^{-1} = \prod_{\nu=1}^{m} c_\nu^{\gamma_\nu}$ with $\gamma_1 \ge \ldots \ge \gamma_m$. We note that $T_1^{-1}S \,|\, T_2$, $(-S)S = W_1(-W_1)T_2(-T_2)$, and $|T_2| = n_2 - 2$. By construction of $T_1^{-1}S$, we obtain that either ($\gamma_3 \ge 2$) or ($\gamma_3 \ge 1$ and $\gamma_4 \ge 1$). Applying Lemma 5.5 (with parameters $t = m$, $\alpha = |T_2|$, $\alpha_1 = \ldots = \alpha_t = (n_2-2)/2$, and either ($\alpha_1' = \ldots = \alpha_3' = 2$, $\alpha_4' = \ldots = \alpha_t' = 0$) or ($\alpha_1' = \ldots = \alpha_4' = 1$, $\alpha_5' = \ldots = \alpha_t' = 0$) we infer that either

$$\prod_{\nu=1}^{m}(1+\gamma_\nu) \ge \left(1+\frac{n_2}{2}-1\right)\left(1+(|T_2|-\left(\frac{n_2}{2}-1\right)-2)\right)(1+2) = \frac{n_2}{2}\left(\frac{n_2}{2}-2\right)3 > n_1n_2\,,$$

or

$$\prod_{\nu=1}^{m}(1+\gamma_\nu) \ge \left(1+\frac{n_2}{2}-1\right)\left(1+|T_2|-\left(\frac{n_2}{2}-1\right)-2\right)\,)(1+1)(1+1) = \frac{n_2}{2}\left(\frac{n_2}{2}-2\right)4 > n_1n_2\,.$$

Thus Lemma 5.4 implies that there is a $W_2 \in \mathcal{A}(G)$ with $|W_2| \ge 3$ such that $(-W_2)W_2 \,|\, (-T_2)T_2$. Since $|W_2| < |U|$, **A1** implies that $|W_2| = n_1$. Therefore we obtain that $W_1(-W_1)W_2(-W_2) \,|\, (-S)S$, a contradiction to **A2**.

Now suppose that $n_2 = 12$. Then $n_1 = 6$ and $|S| = 16$. Again we set $S = \prod_{\nu=1}^{k} a_\nu^{\alpha_\nu}$ with $\alpha_1 \ge \ldots \ge \alpha_k$. Since $\mathsf{h}(S) \le 5$, we infer that $k \ge 4$. We define $T_1 = S(a_1a_2a_3a_4)^{-1}$ and set $T_1 = \prod_{\nu=1}^{l} b_\nu^{\beta_\nu}$ with $\beta_1 \ge \ldots \ge \beta_l$. Observe that $\beta_1 \le 4$. Applying Lemma 5.5 (with parameters $t = l$, $\alpha = |T_1| = 12$, $\alpha_1' = \ldots = \alpha_t' = 0$, and $\alpha_1 = \ldots = \alpha_t = 4$) we infer that

$$\prod_{\nu=1}^{l}(1+\beta_\nu) \ge (1+4)^3 > n_1n_2\,.$$

Thus Lemma 5.4 implies that there is a $W_1 \in \mathcal{A}(G)$ with $|W_1| \ge 3$ such that $(-W_1)W_1 \,|\, (-T_1)T_1$. Since $|W_1| < |U|$, **A1** implies that $|W_1| = n_1 = 6$. We write $W_1 = W_1'(-W_1'')$ with $W_1'W_1'' \,|\, T_1$.

We define $T_2 = S(W_1'W_1'')^{-1} = \prod_{\nu=1}^{m} c_\nu^{\gamma_\nu}$ with $\gamma_1 \ge \ldots \ge \gamma_m$. We note that $|T_2| = n_2 - 2 = 10$ and $m \ge 4$. Applying Lemma 5.5 (with parameters $t = m$, $\alpha = |T_2| = 10$, $\alpha_1' = \ldots = \alpha_4' = 1$, $\alpha_5' = \ldots = \alpha_t' = 0$, and $\alpha_1 = \ldots = \alpha_t = 5$) we infer that

$$\prod_{\nu=1}^{m}(1+\gamma_\nu) \ge (1+5)(1+3)(1+1)(1+1) > n_1n_2\,,$$

and we obtain a contradiction as above.

CASE 2.2.2: U is of type I, as given in Lemma 5.2.

Then

$$\frac{n_2}{2} - 1 \ge \mathsf{h}(S) \ge \mathsf{h}(U) - 3 \ge \mathrm{ord}(e_j) - 4\,,$$

which implies that $\mathrm{ord}(e_j) = n_1$ so $j = 1$. We assert that

$$\mathsf{v}_{e_1}(UV) + \mathsf{v}_{-e_1}(UV) \ge n_1 + 1\,.$$

If this holds, then Lemma 5.2 in [16] implies that $\mathsf{L}(UV) \cap [3, n_1] \ne \emptyset$, a contradiction. Since $\mathsf{v}_{-e_1}(V) \ge \mathsf{v}_{-e_1}(-S) \ge \mathsf{v}_{e_1}(U) - 3$, we obtain that

$$\mathsf{v}_{e_1}(UV) + \mathsf{v}_{-e_1}(UV) \ge (n_1-1) + (n_1-1) - 3 = 2n_1 - 5 \ge n_1 + 1\,.$$

CASE 2.2.3: U is of type II, as given in Lemma 5.2.

We observe that

$$\frac{n_2}{2} - 1 \ge \mathsf h(S) \ge \mathsf h(U) - 3 = \max\{sn_1 - 1, n_2 - sn_1 + \epsilon\} - 3\,.$$

This implies that $s = \frac{n_2}{2n_1}$, hence $\mathsf v_{e_2}(U) = \frac{n_2}{2} + \epsilon$. Thus $\epsilon \in [1,2]$ and $e_2^{\epsilon+1} \,|\, U'$.

Assume to the contrary that $U' = g_1g_2(-h_1 - h_2) = e_2^3$. Then $V' = (-2e_2)h_1h_2$, $|V| = \mathsf D(G)$, and

$$\mathsf v_{-e_2}(V) \ge \mathsf v_{-e_2}(-S) = \mathsf v_{e_2}(S) = \mathsf v_{e_2}(U) - 3 \ge \frac{n_2}{2} + \epsilon - 3\,.$$

Thus $\mathsf v_{-e_2}(V) \ge 1$, which implies that $V^* = (-2e_2)^{-1}(-e_2)^2V \in \mathcal A(G)$, but $|V^*| = |V| + 1 = \mathsf D(G) + 1$, a contradiction.

Since $\epsilon = 2$ implies that $U' = e_2^3$, we obtain that $\epsilon = 1$, $\mathsf v_{e_2}(U') = 2$, $\mathsf v_{-e_2}(V') = 0$, and

$$\mathsf v_{-e_2}(V) = \mathsf v_{-e_2}(-S) = \mathsf v_{e_2}(S) = \mathsf v_{e_2}(U) - 2 = \frac{n_2}{2} - 1\,.$$

We consider

$$U_1 = e_2^{-\mathsf v_{e_2}(U)}(-e_2)^{\mathsf v_{-e_2}(V)}U \quad \text{and} \quad V_1 = (-e_2)^{-\mathsf v_{-e_2}(V)}e_2^{\mathsf v_{e_2}(U)}V\,.$$

Neither U_1 nor V_1 is divisible by an atom of length 2, and since $|V_1| = |V| + 2 > \mathsf D(G)$, $V_1 \notin \mathcal A(G)$. Therefore we obtain that

$$2 < \max \mathsf L(U_1) + \max \mathsf L(V_1) \le \frac{|U_1|}{3} + \frac{|V_1|}{3} = \frac{|UV|}{3} \le \frac{2n_1 + 2n_2 - 2}{3} < n_2\,,$$

a contradiction. □

6. Characterization of the system $\mathcal L(C_{n_1} \oplus C_{n_2})$

In this section we finally provide the proof of Theorem 1.1. We start with two propositions which gather various special cases which have been settled before. The first groups, for which the Characterization Problem has been solved, are cyclic groups and elementary 2-groups ([14]) for which we now have a variety of proofs. We use the characterization of groups $C_n \oplus C_n$ ([42]) and [7]. The core of this section is Proposition 6.5, whose proof covers almost the whole section.

Proposition 6.1. *Let G be an abelian group such that $\mathcal L(G) = \mathcal L(C_{n_1} \oplus C_{n_2})$ where $n_1, n_2 \in \mathbb N$ with $n_1 \,|\, n_2$ and $n_1 + n_2 > 4$. Then G is finite, and we have*

1. *$\mathsf d(G) = \mathsf d(C_{n_1} \oplus C_{n_2}) = n_1 + n_2 - 2$ and $\exp(G) = n_2$.*
2. *If $n_1 = n_2$, then $G \cong C_{n_1} \oplus C_{n_2}$.*

Proof. 1. The finiteness of G and the equality of the Davenport constants follows from [18, Proposition 7.3.1]. The statement on the exponents follows from [46, Proposition 5.2] or from [7, Proposition 5.4].

2. This follows from [42, Theorem 4.1]. □

Proposition 6.2. *Let $n_1, n_2 \in \mathbb N$ with $n_1 \,|\, n_2$ and $n_1 + n_2 > 4$, and let $G = H \oplus C_{n_2}$ where $H \subset G$ is a subgroup with $\exp(H) \,|\, n_2$. Suppose that $\mathcal L(G) = \mathcal L(C_{n_1} \oplus C_{n_2})$.*

1. *$\mathsf d(H) \le n_1 - 1$, and $\mathsf d^*(H) = n_1 - 1$ implies that $\mathsf d(G) = \mathsf d^*(G)$.*
2. *If $\mathsf d(G) = \mathsf d^*(G)$, then $G \cong C_{n_1} \oplus C_{n_2}$.*
3. *If $n_1 \in [1,5]$, then $G \cong C_{n_1} \oplus C_{n_2}$.*

Proof. 1. Proposition 6.1 implies that

$$n_1 + n_2 - 2 = \mathsf{d}(G) \ge \mathsf{d}(H) + (n_2 - 1) \quad \text{and hence} \quad \mathsf{d}(H) \le n_1 - 1\,.$$

If $\mathsf{d}^*(H) = n_1 - 1$, then

$$n_1 + n_2 - 2 = \mathsf{d}(G) \ge \mathsf{d}^*(G) = \mathsf{d}^*(H) + (n_2 - 1) = n_1 + n_2 - 2\,.$$

2. This follows from [7, Theorem 5.6].

3. By 2., it is sufficient to show that $\mathsf{d}(G) = \mathsf{d}^*(G)$. Suppose that H is cyclic. Then $\mathsf{r}(G) \le 2$ and Proposition 2.3.1 implies that $\mathsf{d}(G) = \mathsf{d}^*(G)$. Suppose that H is noncyclic. Then $2 \le \mathsf{r}(H) \le \mathsf{d}(H) \le n_1 - 1$, and hence $n_1 \in [3, 5]$.

Suppose that $n_1 = 3$. Then $\mathsf{d}(H) = 2$ and $H \cong C_2 \oplus C_2$. Thus $\mathsf{d}^*(H) = 2 = n_1 - 1$, and the assertion follows from 1.

Suppose that $n_1 = 4$. Then $\mathsf{d}(H) \in [2, 3]$ and H is isomorphic to $C_2 \oplus C_2$ or to C_2^3. If $H \cong C_2^3$, then $\mathsf{d}^*(H) = n_1 - 1$, and the assertion follows from 1. Suppose that $H \cong C_2 \oplus C_2$ and set $n_2 = 2m$. If m is even, then $\mathsf{d}(G) = \mathsf{d}^*(G)$ by [15, Corollary 4.2.13]. If m is odd, then $\mathsf{d}(G) = \mathsf{d}^*(G)$ by [2] (in [46, Theorem 3.13] even the structure of all minimal zero-sum sequences of length $\mathsf{D}(G)$ has been determined).

Suppose that $n_1 = 5$. Then $\mathsf{d}(H) \in [2, 4]$ and H is isomorphic to one of the following groups: $C_2^2, C_2^3, C_2^4, C_2 \oplus C_4, C_3 \oplus C_3$. If H is isomorphic to one of the groups in $\{C_2^4, C_2 \oplus C_4, C_3 \oplus C_3\}$, then $\mathsf{d}^*(H) = n_1 - 1$. If $H \cong C_2 \oplus C_2$, then $\mathsf{d}(G) = \mathsf{d}^*(G)$ as outlined above. Suppose that $H \cong C_2^3$. Then $G = C_2^3 \oplus C_{n_2}$ and we set $n_2 = 2m$. If m is even, then again [15, Corollary 4.2.13] implies that $\mathsf{d}(G) = \mathsf{d}^*(G)$. If m is odd, then this follows from [2]. □

We need the following characterization of decomposable subsets.

Lemma 6.3. *Let G be a finite abelian group and $G_0 \subset G$ a subset.*

1. *The following statements are equivalent:*
   (a) *G_0 is decomposable.*
   (b) *There are nonempty subsets $G_1, G_2 \subset G_0$ such that $G_0 = G_1 \uplus G_2$ and $\mathcal{B}(G_0) = \mathcal{B}(G_1) \times \mathcal{B}(G_2)$.*
   (c) *There are nonempty subsets $G_1, G_2 \subset G_0$ such that $G_0 = G_1 \uplus G_2$ and $\mathcal{A}(G_0) = \mathcal{A}(G_1) \uplus \mathcal{A}(G_2)$.*
   (d) *There are nonempty subsets $G_1, G_2 \subset G_0$ such that $\langle G_0 \rangle = \langle G_1 \rangle \oplus \langle G_2 \rangle$.*
2. *There exist uniquely determined $t \in \mathbb{N}$ and (up to order) uniquely determined nonempty indecomposable sets $G_1, \ldots, G_t \subset G_0$ such that*

$$G_0 = \biguplus_{\nu=1}^{t} G_\nu \quad \text{and} \quad \langle G_0 \rangle = \bigoplus_{\nu=1}^{t} \langle G_\nu \rangle\,.$$

Proof. 1. See [40, Lemma 3.7] and [3, Lemma 3.2].
2. See [40, Proposition 3.10]. □

We need the invariant

$$\mathsf{m}(G) = \max\{\min \Delta(G_0) \mid G_0 \subset G \text{ is a non-half-factorial subset with } \mathsf{k}(A) \ge 1 \text{ for all } A \in \mathcal{A}(G_0)\}\,.$$

Lemma 6.4. *Let G be a finite abelian group, $G_0 \subset G$ a subset with $\min \Delta(G_0) = \max \Delta^*(G)$, and let $G_0 = \bigcup_{\nu=1}^{t} G_\nu$ be the decomposition into indecomposable components. If $\exp(G) > \mathsf{m}(G) + 2$, then each component G_ν is either half-factorial or equal to $\{-g_\nu, g_\nu\}$ for some $g_\nu \in G$ with $\operatorname{ord}(g_\nu) = \exp(G)$, and there exists at least one non-half-factorial component.*

Proof. See [41, Corollary 5.2]. □

Proposition 6.5. *Let $n_1, n_2 \in \mathbb{N}$ with $n_1 \mid n_2$ and $6 \le n_1 < n_2$, and let G be a finite abelian group with $\exp(G) = n_2$ and $\mathsf d(G) = n_1 + n_2 - 2$. Suppose that, for all $k \in \mathbb{N}$, the sets*

$$L_k = \Big\{(kn_2+3)+(n_1-2)+(n_2-2)\Big\} \cup \Big((2k+3)+\{0, n_1-2, n_2-2\}+\{\nu(n_2-2) \mid \nu \in [0,k]\}\Big) \in \mathcal{L}(G)\,.$$

Then G is isomorphic to one of the following groups

$$C_{n_1} \oplus C_{n_2}\,,\ C_2^s \oplus C_{n_2} \text{ with } s \in \{n_1-2, n_1-1\}\,, C_2^{n_1-4} \oplus C_4 \oplus C_{n_2}\,,\ C_2 \oplus C_{n_1-1} \oplus C_{n_2} \text{ with } 2 \mid (n_1-1) \mid n_2\,.$$

Proof. We set $G = H \oplus C_{n_2}$ where $H \subset G$ is a subgroup with $\exp(H) \mid n_2$. If H is cyclic, then $\mathsf d(G) = |H| + n_2 - 2$ whence $|H| = n_1$ and $G \cong C_{n_1} \oplus C_{n_2}$. From now on we suppose that H is non-cyclic. Since $\mathsf d(H) + n_2 - 1 \le \mathsf d(G) = n_1 + n_2 - 2$, it follows that $\mathsf d(H) \le n_1 - 1$, and hence $\exp(H) \le \mathsf D(H) \le n_1$. Since $\exp(H) = n_1$ would imply that H is cyclic of order n_1, it follows that $\exp(H) \le n_1 - 1$. We have $\mathsf r(H) \le \mathsf d(H) \le n_1 - 1$. If $\mathsf r(H) = n_1 - 1$, then $H \cong C_2^{n_1-1}$ and hence $G \cong C_2^{n_1-1} \oplus C_{n_2}$. Thus from now on we suppose that $\mathsf r(H) \in [2, n_1 - 2]$.

We start with the following two assertions.

A1. $\exp(G) > \mathsf m(G) + 2$.

A2. Let $G_0 \subset G$ with $\min \Delta(G_0) = n_2 - 2$. Then $G_0 = \{g, -g\} \cup G_1$ where $\mathrm{ord}(g) = n_2$, $G_1 \subset G$ is half-factorial, and $\langle G_1 \rangle \cap \langle g \rangle = \{0\}$.

Proof of **A1**. Assume to the contrary that $n_2 \le \mathsf m(G) + 2$, and hence $\mathsf m(G) \ge n_2 - 2$. By [42, Proposition 3.6], we have

$$\mathsf m(G) \le \max\{\mathsf r^*(G) - 1, \mathsf K(G) - 1\}\,.$$

We have $\mathsf r^*(G) \le \log_2 |G|$, $\mathsf K(G) \le \frac12 + \log |G| \le \frac12 + \log_2 |G|$ by Proposition 2.3, and hence

$$\mathsf m(G) \le -\frac12 + \log_2 |G|\,.$$

If $H = C_{m_1} \oplus \ldots \oplus C_{m_s}$, where $s, m_1, \ldots, m_s \in \mathbb{N}$ with $s = \mathsf r(H) \ge 2$ and $1 < m_1 \mid \ldots \mid m_s \mid n_2$, then

$$\log_2 |H| = \sum_{i=1}^s \log_2 m_i \le \sum_{i=1}^s (m_i - 1) = \mathsf d^*(H) \le \mathsf d(H)\,.$$

Therefore we obtain that

$$\begin{aligned} n_2 - 2 \le \mathsf m(G) &\le -\frac12 + \log_2 |G| = -\frac12 + \log_2 n_2 + \log_2 |H| \le -\frac12 + \log_2 n_2 + \mathsf d(H) \\ &\le -\frac32 + \log_2 n_2 + n_1 \le -\frac32 + \log_2 n_2 + \frac{n_2}{2} \end{aligned}$$

and hence

$$\frac{n_2}{2} \le \log_2 n_2 + \frac12\,,$$

a contradiction to $n_2 \ge 7$. □(Proof of **A1**)

Proof of **A2**. By Lemma 6.3, G_0 has a decomposition into indecomposable subsets, say $G_0 = \cup_{\nu=1}^t G_\nu$. Proposition 3.3.2 implies that $\max \Delta^*(G) = n_2 - 2 = \min \Delta(G_0)$. By **A1** and Lemma 6.4, the sets G_ν have the following structure: there is an $s \in [1, t]$ such that $G_\nu = \{-g_\nu, g_\nu\}$ with $\mathrm{ord}(g_\nu) = n_2$ for each $\nu \in [1, s]$, and $G_{s+1}, \ldots, G_t$ are half-factorial. Since, by Lemma 6.3.2,

$$\langle G_0 \rangle = \bigoplus_{\nu=1}^t \langle G_\nu \rangle \subset G = H \oplus C_{n_2} \quad \text{and} \quad \exp(H) < n_2\,,$$

it follows that $s = 1$. □(Proof of **A2**)

By assumption, for every $k \in \mathbb{N}$ the sets $L_k =$

$$\Big\{(kn_2+3)+(n_1-2)+(n_2-2)\Big\} \cup \Big((2k+3)+\{0, n_1-2, n_2-2\}+\{\nu(n_2-2) \mid \nu \in [0,k]\}\Big) \in \mathcal{L}(G)\,.$$

Clearly, these sets are AAMPs with difference n_2-2 and period $\{0, n_1-2, n_2-2\}$ and, for all sufficiently large $k \in \mathbb{N}$, L_k is not an AAMP with some difference d which is not a multiple of n_2-2 ([18, Theorem 4.2.7]). Let $k \in \mathbb{N}$ be sufficiently large. In the course of the proof we will meet certain bounds and will assume that k exceeds all of them.

We choose $B_k \in \mathcal{B}(G)$ such that $\mathsf{L}(B_k) = L_k$. By [18, Proposition 9.4.9], there exist an $M_1 \in \mathbb{N}$ (not depending on k) such that $B_k = V_k S_k$, where V_k and S_k are zero-sum sequences with the following properties:

$$\min \Delta\big(\operatorname{supp}(V_k)\big) = n_2 - 2 \quad \text{and} \quad |S_k| \le M_1\,,$$

(indeed in the terminology of Proposition 9.4.9, we have $V_k \in V\,[\![V]\!]$ and $S_k \in \mathcal{B}(G)[\mathcal{U}, V]$ for a given full almost generating set $\mathcal{U}$; but we do not need these additional properties). By **A2**, we obtain that

$$\operatorname{supp}(V_k) = \{-g_k, g_k\} \cup A_k, \quad \text{where } \operatorname{ord}(g_k) = n_2 \text{ and } A_k \subset G \text{ is half-factorial with } \langle g_k\rangle \cap \langle A_k\rangle = \{0\}\,.$$

Since for each two elements $g, g' \in G$ with $\operatorname{ord}(g) = \operatorname{ord}(g') = n_2$, there is a group automorphism $\varphi\colon G \to G$ with $\varphi(g) = g'$, and since $\mathsf{L}(B) = \mathsf{L}\big(\varphi(B)\big)$ for all $B \in \mathcal{B}(G)$, we may assume without restriction that there is a $g \in G$ such that $g = g_k$. Applying a further automorphism if necessary we may suppose that $G = H \oplus \langle g\rangle$.

We continue with the assertion

A3. There exist a constant $M_2 \in \mathbb{N}$ (not depending on k), $C_k \in \mathcal{B}(\operatorname{supp}(V_k))$ and $D_k \in \mathcal{B}(G^\bullet)$ with the following properties:
- $B_k = C_k D_k$,
- $|D_k| \le M_2$,
- For any factorization $z = W_1 \cdot \ldots \cdot W_\gamma \in \mathsf{Z}(B_k)$ with $W_1, \ldots, W_\gamma \in \mathcal{A}(G)$ there are I, J such that $[1, \gamma] = I \uplus J$, $\prod_{i\in I} W_i = C_k$ and $\prod_{j\in J} W_j = D_k$.

Proof of **A3**. Let $z = X_1 \cdot \ldots \cdot X_\alpha Y_1 \cdot \ldots \cdot Y_\beta$ be a factorization of B_k, where $X_1, \ldots, X_\alpha, Y_1, \ldots, Y_\beta$ are atoms, and $Y_1, \ldots, Y_\beta$ are precisely those atoms which contain some element from S_k. Then $\beta \le |S_k| \le M_1$ and $X_1 \cdot \ldots \cdot X_\alpha$ divides V_k (in $\mathcal{B}(G)$). For any element $a \in \operatorname{supp}(V_k)$ let $m_a(z) \in \mathbb{N}_0$ be maximal such that $a^{\operatorname{ord}(a)m_a(z)}$ divides $X_1 \cdot \ldots \cdot X_\alpha$. Since $\beta \le M_1$, there is a constant $M_3(z) \in \mathbb{N}$ (not depending on k) such that $\mathsf{v}_a(B_k) - \operatorname{ord}(a)m_a(z) \le M_3(z)$. Now we define, for each $a \in \operatorname{supp}(V_k)$,

$$m_a = \min\{m_a(z) \mid z \in \mathsf{Z}(B_k)\}\,,$$

$$C_k = \prod_{a\in\operatorname{supp}(V_k)} a^{\operatorname{ord}(a)m_a} \quad \text{and} \quad D_k = C_k^{-1} B_k\,.$$

Since there is a constant $M_3 \in \mathbb{N}$ (not depending on k) such that $\mathsf{v}_a(B_k) - \operatorname{ord}(a)m_a \le M_3$ for all $a \in \operatorname{supp}(V_k)$, there is a constant $M_2 \in \mathbb{N}$ (not depending on k) such that

$$|D_k| = |B_k| - |C_k| \le M_2\,. \qquad \square(\text{Proof of } \mathbf{A3})$$

Since $C_k \in \mathcal{B}(\operatorname{supp}(V_k))$, $\mathsf{L}(C_k)$ is an arithmetical progression with difference $n_2 - 2$, and by **A3** we have

$$\mathsf{L}(B_k) = \mathsf{L}(C_k) + \mathsf{L}(D_k) = \bigcup_{m\in\mathsf{L}(D_k)} \big(m + \mathsf{L}(C_k)\big)\,.$$

Assume to the contrary that $\mathsf{L}(D_k) = \{m\}$. Then $-m + L_k = -m + \mathsf{L}(B_k) = \mathsf{L}(C_k) \in \mathcal{L}(C_{n_2})$, a contradiction to Proposition 3.6.2. This implies that $|\mathsf{L}(D_k)| > 1$. Since $\operatorname{supp}(C_k) \subset \operatorname{supp}(V_k) \subset \{-g, g\} \cup A_k$, where A_k is half-factorial and $\langle g\rangle \cap \langle A_k\rangle = \{0\}$, it follows that $C_k = C_k' C_k''$, with $C_k' \in \mathcal{B}(\{g, -g\})$, $C_k'' \in \mathcal{B}(A_k)$, $\mathsf{L}(C_k) = \mathsf{L}(C_k') + \mathsf{L}(C_k'')$ and $|\mathsf{L}(C_k'')| = 1$. Thus, if $\mathsf{L}(C_k'') = \{m_k\}$, then

$$\mathsf{L}(C_k) = \mathsf{L}(C_k' 0^{m_k}) \quad \text{and} \quad \mathsf{L}(B_k) = \mathsf{L}(C_k D_k) = \mathsf{L}(C_k' 0^{m_k} D_k)\,.$$

Therefore, after changing notation if necessary, we suppose from now on that

$$B_k = C_k D_k,\ \mathsf{L}(B_k) = \mathsf{L}(C_k) + \mathsf{L}(D_k),\ \text{where } \operatorname{supp}(C_k) \subset \{0, g, -g\} \text{ and } D_k \in \mathcal{B}(G) \text{ with } |D_k| \le M_2\,.$$

We continue with the assertion

A4.
- Let $T \in \mathcal{F}(G)$ with $T \,|\, D_k$. If $T \in \mathcal{A}(\mathcal{B}_{\langle g\rangle}(G))$, then $\sigma(T) \in \{0, g, -g, (n_1-1)g, -(n_1-1)g\}$.
- If $z = T_1 \cdot \ldots \cdot T_\gamma \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k)$ with $T_1, \ldots, T_\gamma \in \mathcal{A}(\mathcal{B}_{\langle g\rangle}(G))$, then at most one of the elements $\sigma(T_1), \ldots, \sigma(T_\gamma)$ does not lie in $\{0, g, -g\}$.

Proof of **A4**. This follows from Proposition 3.8. □(Proof of **A4**)

We shall use the following notation. If $z = T_1 \cdot \ldots \cdot T_\gamma$ is as above, then we set

$$\sigma(z) = \sigma(T_1) \cdot \ldots \cdot \sigma(T_\gamma) \in \mathcal{F}(\langle g\rangle)\,,$$

and we continue with the assertion

A5.

$$\mathsf{L}_{\mathcal{B}(G)}(B_k) \;=\; \bigcup_{z \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k)} \mathsf{L}_{\mathcal{B}(\langle g\rangle)}\big(C_k\sigma(z)\big)\,,$$

and the union on the right hand side consists of at least two distinct sets which are not contained in each other.

Proof of **A5**. Assume to the contrary that all sets of lengths on the right hand side are contained in one fixed set $L_1 = L_{\mathcal{B}(\langle g\rangle)}\big(C_k\sigma(z^*)\big)$ with $z^* \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k)$. Then $\mathsf{L}(B_k) \in \mathcal{L}(C_{n_2})$, a contradiction to Proposition 3.6.

To show that the set on the left side is in the union on the right side, we choose a factorization $z^* = W_1 \cdot \ldots \cdot W_\gamma \in \mathsf{Z}_{\mathcal{B}(G)}(B_k)$, where $W_1, \ldots, W_\gamma \in \mathcal{A}(G)$. For each $\nu \in [1, \gamma]$, we set $W_\nu = X_\nu Y_\nu$ where $X_\nu, Y_\nu \in \mathcal{F}(G)$ such that

$$C_k = X_1 \cdot \ldots \cdot X_\gamma \quad \text{and} \quad D_k = Y_1 \cdot \ldots \cdot Y_\gamma\,.$$

For each $\nu \in [1, \gamma]$, we have $\sigma(W_\nu) = 0 \in G$, hence $\sigma(Y_\nu) = -\sigma(X_\nu) \in \langle g\rangle$, $Y_\nu \in \mathcal{B}_{\langle g\rangle}(G)$, and we choose a factorization $z_\nu \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(Y_\nu)$. Then

$$z = z_1 \cdot \ldots \cdot z_\gamma \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k)\,.$$

Then, for each $\nu \in [1, \gamma]$, $W'_\nu = X_\nu\sigma(z_\nu) \in \mathcal{A}(\langle g\rangle)$ and $W'_1 \cdot \ldots \cdot W'_\gamma = C_k\sigma(z) \in \mathcal{F}(\langle g\rangle)$. Therefore $z' = W'_1 \cdot \ldots \cdot W'_\gamma \in \mathsf{Z}_{\mathcal{B}(\langle g\rangle)}\big(C_k\sigma(z)\big)$ and

$$|z^*| = \gamma = |z'| \in \mathsf{L}_{\mathcal{B}(\langle g\rangle)}\big(C_k\sigma(z)\big)\,.$$

Conversely, let $z = S_1 \cdot \ldots \cdot S_\beta \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k)$ and $z' = W'_1 \cdot \ldots \cdot W'_\gamma \in \mathsf{Z}_{\mathcal{B}(\langle g\rangle)}\big(C_k\sigma(z)\big)$ be given, where $S_1, \ldots, S_\beta \in \mathcal{A}(\mathcal{B}_{\langle g\rangle}(G))$, $W'_1, \ldots, W'_\gamma \in \mathcal{A}(\langle g\rangle)$, and we write

$$\sigma(z) = s_1 \cdot \ldots \cdot s_\beta, \text{ where } s_1 = \sigma(S_1), \ldots, s_\beta = \sigma(S_\beta)\,.$$

Note that $s_1, \ldots, s_\beta$ satisfy the properties given in **A4**. We continue with the following

Assertion: We can find a renumbering such that

$$W'_\nu = s_\nu T_\nu \quad \text{with} \quad T_\nu \in \mathcal{F}(\{-g, g\}) \quad \text{for all } \nu \in [1, \beta]\,.$$

Proof of the Assertion. We proceed in three steps.

First, we may assume without restriction that $s_1 = \ldots = s_\delta = 0$ and $0 \notin \{s_{\delta+1}, \ldots, s_\beta\}$. Then at least δ of the $W'_1, \ldots, W'_\gamma$ are equal to 0. After renumbering if necessary, we may suppose that $W'_1 = \ldots = W'_\delta = 0$, and we set $T_1 = \ldots = T_\delta = 1 \in \mathcal{F}(\{-g, g\})$.

Second, suppose there is a $\nu \in [\delta{+}1, \beta]$ such that $s_\nu \in \{(n_1{-}1)g, n_2{-}(n_1{-}1)g\}$, say $\nu = \delta{+}1$. Then $s_{\delta+1}$ divides (in $\mathcal{F}(G)$) one element of $\{W'_{\delta+1}, \ldots, W'_\gamma\}$, say $W'_{\delta+1}$. Then we set $T_{\delta+1} = s_{\delta+1}^{-1}W'_{\delta+1} \in \mathcal{F}(\{-g, g\})$.

To handle the last step, we observe that, by **A4**, all remaining s_ν lie in $\{-g, g\}$. Since $\beta \le |D_k| \le M_2$ and the multiplicities of g and of $-g$ in C_k are growing with k, and k is sufficiently large, for each $\nu \le \beta$ the product $\prod_{\lambda=\nu}^{\gamma} W'_\lambda$ is divisible by g and by $-g$. Thus we can pick a suitable W'_ν and the assertion follows. □

Now we define

$$W''_\nu = \begin{cases} S_\nu T_\nu & \text{for each } \nu \in [1,\beta], \\ W'_\nu & \text{for each } \nu \in [\beta+1,\gamma]\,. \end{cases}$$

Then, by construction, we have

$$B_k = W''_1 \cdot \ldots \cdot W''_\gamma\,.$$

Let $\nu \in [1,\beta]$. Since $W'_\nu \in \mathcal{A}(\langle g\rangle)$, it follows that $T'_\nu \in \mathcal{F}(\langle g\rangle)$ is zero-sum free. Since $S_\nu \in \mathcal{A}(\mathcal{B}_{\langle g\rangle}(G))$, it follows that $W''_\nu \in \mathcal{A}(G)$. Thus $W''_1, \ldots, W''_\beta \in \mathcal{A}(G)$, and we have constructed a factorization of B_k of length $\gamma = |z'|$. □(Proof of **A5**)

A6. Let

$$z = T_1 \cdot \ldots \cdot T_\gamma \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k) \quad \text{and} \quad z' = T'_1 \cdot \ldots \cdot T'_{\gamma'} \in \mathsf{Z}_{\mathcal{B}_{\langle g\rangle}(G)}(D_k)\,,$$

where $\gamma, \gamma' \in \mathbb{N}$, $T_1, \ldots, T_\gamma, T'_1, \ldots, T'_{\gamma'} \in \mathcal{A}(\mathcal{B}_{\langle g\rangle}(G))$, and $z \ne z'$. Furthermore, let

$$F = C_k\sigma(T_1)\cdot\ldots\cdot\sigma(T_\gamma) \in \mathcal{B}(\langle g\rangle) \quad \text{and} \quad F' = C_k\sigma(T'_1)\cdot\ldots\cdot\sigma(T'_{\gamma'}) \in \mathcal{B}(\langle g\rangle)\,,$$

and define

$$F = SF_1 \text{ and } F' = SF_2, \quad \text{where} \quad S, F_1, F_2 \in \mathcal{F}(\langle g\rangle) \text{ and } S = \gcd_{\mathcal{F}(\langle g\rangle)}(F, F')\,.$$

Then one of the following statements holds:

(i) $\mathsf{d}(z, z') \ge n_1 - 1$.

(ii) $\{F_1, F_2\} = \{\big((-g)g\big)^v, 0^v\}$ with $v \in \mathbb{N}$.

Proof of **A6**. Note that $\gcd(F_1, F_2) = 1$, $\sigma(F_1) = \sigma(F_2) = -\sigma(S)$, $C_k \,|\, S$, and

$$(*) \qquad \mathsf{d}_{\mathsf{Z}(\mathcal{B}_{\langle g\rangle}(G))}(z, z') \ge \mathsf{d}_{\mathcal{F}(\langle g\rangle)}(F, F') = \mathsf{d}_{\mathcal{F}(\langle g\rangle)}(F_1, F_2) = \max\{|F_1|, |F_2|\}\,.$$

Since $|F| = |C_k| + |z|$, $|F_1| + |S| = |C_k| + |z|$, $|F_2| + |S| = |C_k| + |z'|$, we obtain that $|F_2| - |F_1| = |z'| - |z|$ and

$$(**) \qquad \mathsf{d}(z, z') \ge \big||z| - |z'|\big| + 2 = \big||F_1| - |F_2|\big| + 2\,.$$

Using $(*)$ and $(**)$ we observe that $\max\{|F_1|, |F_2|\} \ge n_1 - 1$ as well as $\big||F_1| - |F_2|\big| \ge n_1 - 3$ implies (i). To simplify the discussion we suppose that $\max\{|F_1|, |F_2|\} \le n_1 - 1$ (of course we could also assume that $\max\{|F_1|, |F_2|\} \le n_1 - 2$; the slightly weaker assumption allows us to give a more complete description of (F_1, F_2) without additional efforts). Based on the structural description of $\sigma(z)$ and $\sigma(z')$ given in **A4** we distinguish four cases.

CASE 1: $\sigma(z)\sigma(z') \in \mathcal{F}(\{0, g, -g\})$.

We set

$$S = \big(g^{n_2}\big)^{k_1}\big((-g)^{n_2}\big)^{k_2}\big((-g)g\big)^{k_3}\big(\delta g\big)^{k_4}0^{k_5}\,,$$

where $\delta \in \{-1, 1\}$, $k_1, \ldots, k_5 \in \mathbb{N}_0$ and $k_3 < n_2$. We distinguish two cases.

CASE 1.1: $F_1 = 1$ or $F_2 = 1$, say $F_2 = 1$.

Then $\sigma(S) = 0$, $\sigma(F_1) = 0$, and $k_4 = 0$. We have $F_1 = 0^{\mathsf{v}_0(F_1)}\big((-g)g\big)^{\mathsf{v}_g(F_1)}$ and $|F_1| > 0$. Then

$$\min \mathsf{L}(SF_2) = \min \mathsf{L}(S) = k_1+k_2+k_3+k_5 \quad \text{and} \quad \min \mathsf{L}(SF_1) = k_1+k_2+k_3+k_5+\mathsf{v}_0(F_1)+\mathsf{v}_g(F_1)-\epsilon(n_2-2)$$

where

$$\epsilon = \begin{cases} 0 & k_3 + \mathsf{v}_g(F_1) < n_2 \\ 1 & \text{otherwise} \end{cases}\,.$$

Thus $\mathsf{v}_0(F_1)+\mathsf{v}_g(F_1)$ is congruent to $\min \mathsf{L}(SF_1) - \min \mathsf{L}(SF_2)$ modulo n_2-2 and hence congruent either to 0 or to n_1-2 or to $(n_2-2)-(n_1-2) = n_2-n_1$ modulo n_2-2. Since $0 < |F_1| = \mathsf{v}_0(F_1)+2\mathsf{v}_g(F_1) \le n_1-1$, it follows that $(\mathsf{v}_0(F_1), \mathsf{v}_g(F_1)) \in \{(n_1-2, 0), \{n_1-3, 1)\}$, hence $|F_1| - |F_2| = |F_1| \ge n_1 - 2$, and thus (i) holds.

CASE 1.2: $F_1 \ne 1$ and $F_2 \ne 1$.

By symmetry we may suppose that $0 \nmid F_1$. Then $g \,|\, F_1$ or $(-g) \,|\, F_1$, and by symmetry we may suppose that $g \,|\, F_1$. We distinguish two cases.

CASE 1.2.1: $(-g) \,|\, F_1$.

Then $F_2 = 0^{\mathsf v_0(F_2)}$, and hence $\sigma(S) = 0 = \sigma(F_1)$. This implies $k_4 = 0$ and $F_1 = \big((-g)g\big)^{\mathsf v_g(F_1)}$. Then

$$\min \mathsf L(SF_2) = k_1 + k_2 + k_3 + \mathsf v_0(F_2) + k_5 \quad \text{and} \quad \min \mathsf L(SF_1) = k_1 + k_2 + k_3 + \mathsf v_g(F_1) - \epsilon(n_2 - 2) + k_5$$

where $\epsilon \in \{0, 1\}$. Thus $\mathsf v_0(F_2) - \mathsf v_g(F_1)$ is congruent to $\min \mathsf L(SF_1) - \min \mathsf L(SF_2)$ modulo $n_2 - 2$ and hence congruent either to 0 or to $n_1 - 2$ or to $n_2 - n_1$ modulo $n_2 - 2$. This implies that either

$$\mathsf v_0(F_2) = \mathsf v_g(F_1) \quad \text{or} \quad \mathsf v_0(F_2) = \mathsf v_g(F_1) + n_1 - 2 \quad \text{or} \quad \mathsf v_g(F_1) = \mathsf v_0(F_2) + n_1 - 2\,.$$

If $\mathsf v_0(F_2) = \mathsf v_g(F_1) + n_1 - 2$, then $\mathsf v_g(F_1) \ge 1$ implies that $|F_2| \ge \mathsf v_0(F_2) \ge n_1 - 1$, and hence (i) holds.

If $\mathsf v_g(F_1) = \mathsf v_0(F_2) + n_1 - 2$, then $\mathsf v_0(F_2) \ge 1$ implies that $\mathsf v_g(F_1) \ge n_1 - 1$ whence $|F_1| = 2\mathsf v_g(F_1) \ge 2(n_1 - 1) > n_1$, a contradiction.

If $\mathsf v_0(F_2) = \mathsf v_g(F_1)$, then (ii) holds.

CASE 1.2.2: $(-g) \nmid F_1$.

Then $F_1 = g^{\mathsf v_g(F_1)}$ and $F_2 = (-g)^{\mathsf v_{-g}(F_2)} 0^{\mathsf v_0(F_2)}$. Note that $\mathsf v_g(F_1) + \mathsf v_{-g}(F_2) > 0$, $\mathsf v_g(F_1), \mathsf v_{-g}(F_2) \in [0, n_1 - 1]$, and $n_2 \ge 2n_1$. However, $\sigma(F_1) = \sigma(F_2)$ implies that $\mathsf v_g(F_1) + \mathsf v_{-g}(F_2) \equiv 0 \mod n_2$, a contradiction.

CASE 2: $\sigma(z)\sigma(z') \in \big((n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$ or $\sigma(z)\sigma(z') \in \big(-(n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$.

After applying the group automorphism which sends each $h \in G$ onto its negative if necessary, we may suppose that $\sigma(z)\sigma(z') \in \big((n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$. After exchanging z and z' if necessary we may suppose that $\sigma(z) \in \mathcal F(\{0, -g, g\})$ and $\sigma(z') \in \big((n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$.

We set

$$S = \big(g^{n_2}\big)^{k_1} \big((-g)^{n_2}\big)^{k_2} \big((-g)g\big)^{k_3} \big(\delta g\big)^{k_4} 0^{k_5}\,,$$

where $\delta \in \{-1, 1\}$, $k_1, \ldots, k_5 \in \mathbb N_0$ and $k_3 < n_2$. If $\sigma(F_1) = 0$, then $\sigma(F_2) = 0$ and hence $|F_2| \ge n_1$, a contradiction. Thus it follows that $\sigma(F_1) \ne 0$, and hence there are the following three cases.

CASE 2.1: $g \,|\, F_1$ and $(-g) \,|\, F_1$.

Then $F_2 = \big((n_1 - 1)g\big)0^{\mathsf v_0(F_2)}$ and hence $\sigma(F_2) = (n_1 - 1)g = \sigma(F_1) = \big(\mathsf v_g(F_1) - \mathsf v_{-g}(F_1)\big)g$, a contradiction to $|F_1| \le n_1 - 1$.

CASE 2.2: $g \,|\, F_1$ and $(-g) \nmid F_1$.

Then $F_1 = g^{\mathsf v_g(F_1)} 0^{\mathsf v_0(F_1)}$, $F_2 = \big((n_1 - 1)g\big)(-g)^{n_1 - 1 - \mathsf v_g(F_1)} 0^{\mathsf v_0(F_2)}$, and we can write SF_1 and SF_2 as follows:

$$\begin{aligned} SF_1 &= \big(g^{n_2}\big)^{l_1} \big((-g)^{n_2}\big)^{l_2} \big((-g)g\big)^{l_3} 0^{l_4 + \mathsf v_0(F_1)} \\ SF_2 &= \big(g^{n_2}\big)^{l_1} \big((-g)^{n_2}\big)^{l_2} \big((-g)g\big)^{l_3 - \mathsf v_g(F_1)} 0^{l_4 + \mathsf v_0(F_2)} \Big(\big((n_1 - 1)g\big)(-g)^{n_1 - 1}\Big) \end{aligned}$$

where $l_1, \ldots, l_4 \in \mathbb N_0$, $l_4 = \mathsf v_0(S)$, and $l_3 \ge \mathsf v_g(F_1)$ (the last inequality holds because k is large enough). Therefore

$$m_1 = l_1 + l_2 + l_3 + l_4 + \mathsf v_0(F_1) \in L_k$$

and

$$m_2 = l_1 + l_2 + l_3 - \mathsf v_g(F_1) + l_4 + \mathsf v_0(F_2) + 1 \in L_k$$

which implies that $m_1 - m_2 = \mathsf v_0(F_1) + \mathsf v_g(F_1) - \mathsf v_0(F_2) - 1$ is congruent to either 0 or to $n_1 - 2$ or to $n_2 - n_1$ modulo $n_2 - 2$. We distinguish three cases.

CASE 2.2.1: $\mathsf v_0(F_1) + \mathsf v_g(F_1) \equiv \mathsf v_0(F_2) + 1 \mod n_2 - 2$.

Since $|F_1| < n_1$ and $|F_2| < n_1$, it follows that $\mathsf v_0(F_1) + \mathsf v_g(F_1) = \mathsf v_0(F_2) + 1$. Since

$$\begin{aligned} |F_2| &= \mathsf v_0(F_2) + 1 + \big(n_1 - 1 - \mathsf v_g(F_1)\big) \\ &= \mathsf v_0(F_1) + \mathsf v_g(F_1) + \big(n_1 - 1 - \mathsf v_g(F_1)\big) = n_1 - 1 + \mathsf v_0(F_1)\,, \end{aligned}$$

it follows that $|F_2| \ge n_1 - 1$ and hence (i) holds.

CASE 2.2.2: $\mathsf v_0(F_1)+\mathsf v_g(F_1)\equiv \mathsf v_0(F_2)+n_1-1 \mod n_2-2$.

Similarly, we obtain that $\mathsf v_0(F_1)+\mathsf v_g(F_1)=\mathsf v_0(F_2)+n_1-1$. Thus $|F_1|\ge n_1-1$ and hence (i) holds.

CASE 2.2.3: $\mathsf v_0(F_1)+\mathsf v_g(F_1)\equiv n_2-n_1+\mathsf v_0(F_2)+1 \mod n_2-2$.

We obtain that $\mathsf v_0(F_1)+\mathsf v_g(F_1)=-(n_1-3)+\mathsf v_0(F_2)$ which implies that $\mathsf v_0(F_2)>n_1-3$. Therefore

$$\begin{aligned}|F_2| &= \mathsf v_0(F_2)+1+n_1-1-\mathsf v_g(F_1)=n_1+\mathsf v_0(F_1)+(n_1-3)-\mathsf v_0(F_2)+\mathsf v_0(F_2)\\ &= 2n_1-3+\mathsf v_0(F_1)\ge n_1\,,\end{aligned}$$

a contradiction.

CASE 2.3: $g\nmid F_1$ and $(-g)\,|\,F_1$.

Then

$$F_1=(-g)^{\mathsf v_{-g}(F_1)}0^{\mathsf v_0(F_1)} \quad \text{and} \quad F_2=\big((n_1-1)g\big)g^{n_2-(n_1-1)-\mathsf v_{-g}(F_1)}0^{\mathsf v_0(F_2)}\,.$$

We can write SF_1 and SF_2 as

$$\begin{aligned}SF_1 &= \big(g^{n_2}\big)^{l_1}\big((-g)^{n_2}\big)^{l_2}\big((-g)g\big)^{l_3}0^{l_4+\mathsf v_0(F_1)} \quad \text{and}\\ SF_2 &= \big(g^{n_2}\big)^{l_1}\big((-g)^{n_2}\big)^{l_2}\big((-g)g\big)^{l_3-\mathsf v_{-g}(F_1)}0^{l_4+\mathsf v_0(F_2)}\Big(g^{n_2-(n_1-1)}\big((n_1-1)g\big)\Big)\end{aligned}$$

where $l_1,\dots,l_4\in\mathbb N_0$, $l_4=\mathsf v_0(S)$, and $l_3\ge \mathsf v_{-g}(F_1)$ (the last inequality holds because k is large enough). Therefore

$$m_1=l_1+l_2+l_3+l_4+\mathsf v_0(F_1)\in L_k$$

and

$$m_2=l_1+l_2+l_3-\mathsf v_{-g}(F_1)+l_4+\mathsf v_0(F_2)+1\in L_k\,.$$

which implies that $m_1-m_2=\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)-\mathsf v_0(F_2)-1$ is congruent to either 0 or to n_1-2 or to n_2-n_1 modulo n_2-2. We distinguish three cases.

CASE 2.3.1: $\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)\equiv \mathsf v_0(F_2)+1 \mod n_2-2$.

We obtain that $\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)=\mathsf v_0(F_2)+1$ and hence

$$|F_2|=1+\mathsf v_0(F_2)+n_2-(n_1-1)-\mathsf v_{-g}(F_1)=\mathsf v_0(F_1)+n_2-(n_1-1)\ge n_1\,,$$

a contradiction.

CASE 2.3.2: $\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)\equiv \mathsf v_0(F_2)+n_1-1 \mod n_2-2$.

We obtain that $\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)=\mathsf v_0(F_2)+n_1-1$. Therefore $|F_1|\ge n_1-1$ and hence (i) holds.

CASE 2.3.3: $\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)\equiv n_2-n_1+\mathsf v_0(F_2)+1 \mod n_2-2$.

We obtain that $\mathsf v_0(F_1)+\mathsf v_{-g}(F_1)=\mathsf v_0(F_2)-n_1+3$ and therefore

$$\begin{aligned}|F_2| &= \mathsf v_0(F_2)+1+n_2-(n_1-1)-\mathsf v_{-g}(F_1)=\mathsf v_0(F_1)+n_1-3+1+n_2-(n_1-1)\\ &= \mathsf v_0(F_1)-1+n_2\ge n_1\,,\end{aligned}$$

a contradiction.

CASE 3: $\sigma(z)\sigma(z')\in\big((n_1-1)g\big)^2\mathcal F(\{0,-g,g\})$ or $\sigma(z)\sigma(z')\in\big(-(n_1-1)g\big)^2\mathcal F(\{0,-g,g\})$.

After applying the group automorphism which sends each $h\in G$ onto its negative if necessary, we may suppose that $\sigma(z)\sigma(z')\in\big((n_1-1)g\big)^2\mathcal F(\{0,-g,g\})$, whence $\sigma(z)\in\big((n_1-1)g\big)\mathcal F(\{0,-g,g\})$ and $\sigma(z')\in\big((n_1-1)g\big)\mathcal F(\{0,-g,g\})$.

We set

$$S=\big((n_1-1)g\big)\big(g^{n_2}\big)^{k_1}\big((-g)^{n_2}\big)^{k_2}\big((-g)g\big)^{k_3}\big(\delta g\big)^{k_4}0^{k_5}\,,$$

where $\delta\in\{-1,1\}$, $k_1,\dots,k_5\in\mathbb N_0$ and $k_3<n_2$. We distinguish two cases.

CASE 3.1: $F_1=1$ or $F_2=1$, say $F_2=1$.

Since $F_2=1$, it follows that $\mathsf L(S)\subset L_k$. Let $l_1\in\mathsf L(F_1)$. Then $l_1+\mathsf L(S)\subset L_k$ and hence l_1 is congruent either to 0 or to n_1-2 or to n_2-n_1 modulo n_2-2. Since $l_1>0$, it follows that $|F_1|-|F_2|=|F_1|\ge n_1-2$, and hence (i) holds.

CASE 3.2: $F_1\ne 1$ and $F_2\ne 1$.

We have $0 \nmid F_1$ or $0 \nmid F_2$, say $0 \nmid F_1$. Then $g \,|\, F_1$ or $(-g) \,|\, F_1$. We distinguish three cases.

CASE 3.2.1: $g \,|\, F_1$ and $(-g) \nmid F_1$.

Then $F_1 = g^{\mathsf v_g(F_1)}$ and $F_2 = (-g)^{\mathsf v_{-g}(F_2)}0^{\mathsf v_0(F_2)}$. Since $\sigma(F_1) = \sigma(F_2)$, it follows that $\mathsf v_g(F_1)+\mathsf v_{-g}(F_2) \equiv 0 \mod n_2$, and hence

$$\max\{\mathsf v_g(F_1), \mathsf v_{-g}(F_2)\} \ge \frac{n_2}{2} \ge n_1\,,$$

a contradiction.

CASE 3.2.2: $g \,|\, F_1$ and $(-g) \,|\, F_1$.

Then $(-g)g \,|\, F_1$ whence $F_2 = 0^{\mathsf v_0(F_2)}$. This implies that $0 = \sigma(F_2) = \sigma(F_1)$ and thus $F_1 = \big((-g)g\big)^{\mathsf v_g(F_1)}$. As above it follows that $\mathsf v_g(F_1) - \mathsf v_0(F_2)$ is congruent either to 0 or to $n_1 - 2$ or to $n_2 - n_1$ modulo $n_2 - 2$.

If $\mathsf v_g(F_1) = \mathsf v_0(F_2)$, then (ii) holds.

If $\mathsf v_g(F_1) = \mathsf v_0(F_2) + n_1 - 2$, then $|F_1| = 2\mathsf v_g(F_1) \ge 2n_1 - 4 \ge n_1$, a contradiction.

Suppose that $\mathsf v_g(F_1) - \mathsf v_0(F_2) \equiv n_2 - n_1 \mod n_2 - 2$. Then $|F_1| < n_1$ implies that $\mathsf v_g(F_1) - \mathsf v_0(F_2) = -n_1 + 2$. Since $\mathsf v_g(F_1) \ge 1$, it follows that $|F_2| \ge \mathsf v_0(F_2) \ge n_1 - 1$, and hence (i) holds.

CASE 3.2.3: $g \nmid F_1$ and $(-g) \,|\, F_1$.

Then $F_1 = (-g)^{\mathsf v_{-g}(F_1)}$ and $F_2 = g^{\mathsf v_g(F_2)}0^{\mathsf v_0(F_2)}$. Since $\sigma(F_1) = \sigma(F_2)$, it follows that $\mathsf v_{-g}(F_1)+\mathsf v_g(F_2) \equiv 0 \mod n_2$, and hence

$$\max\{\mathsf v_{-g}(F_1), \mathsf v_g(F_2)\} \ge \frac{n_2}{2} \ge n_1\,,$$

a contradiction.

CASE 4: $\sigma(z)\sigma(z') \in \big((n_1 - 1)g\big)\big(-(n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$.

After exchanging z and z' if necessary we may suppose that $\sigma(z) \in \big((n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$ and $\sigma(z') \in \big(-(n_1 - 1)g\big)\mathcal F(\{0, -g, g\})$. We set

$$SF_1 = \big(g^{n_2}\big)^{l_1}\big((-g)^{n_2}\big)^{l_2}\big((g(-g)\big)^{l_3}\big((n_1 - 1)g(-g)^{n_1-1}\big)0^{l_4+\mathsf v_0(F_1)}$$

and

$$SF_2 = \big(g^{n_2}\big)^{l_1'}\big((-g)^{n_2}\big)^{l_2'}\big((g(-g)\big)^{l_3'}\Big(\big(-(n_1 - 1)g\big)g^{n_1-1}\Big)0^{l_4+\mathsf v_0(F_2)}$$

where $l_1, l_1', \ldots, l_3, l_3', l_4 \in \mathbb N_0$.

Since

$$F_1 = \big((n_1 - 1)g\big)g^{\mathsf v_g(F_1)}(-g)^{\mathsf v_{-g}(F_1)}0^{\mathsf v_0(F_1)} \quad \text{and} \quad F_2 = \big(-(n_1 - 1)g\big)g^{\mathsf v_g(F_2)}(-g)^{\mathsf v_{-g}(F_2)}0^{\mathsf v_0(F_2)}\,,$$

it follows that

$$\big(n_1 - 1 + \mathsf v_g(F_1) - \mathsf v_{-g}(F_1)\big)g = \sigma(F_1) = \sigma(F_2) = \big(-n_1 + 1 + \mathsf v_g(F_2) - \mathsf v_{-g}(F_2)\big)g$$

and hence

$$2n_1 - 2 \equiv \big(\mathsf v_g(F_2) - \mathsf v_g(F_1)\big) + \big(\mathsf v_{-g}(F_1) - \mathsf v_{-g}(F_2)\big) \mod n_2\,.$$

We distinguish four cases.

CASE 4.1: $g \,|\, F_1$ and $(-g) \,|\, F_1$.

Then $\mathsf v_g(F_2) = 0 = \mathsf v_{-g}(F_2)$ and hence

$$2n_1 - 2 \equiv -\mathsf v_g(F_1) + \mathsf v_{-g}(F_1) \mod n_2\,.$$

If $n_2 \ge 3n_1$, then $|F_1| \ge n_1$, a contradiction. Thus $n_2 = 2n_1$, $\mathsf v_{-g}(F_1) + 2 \equiv \mathsf v_g(F_1) \mod n_2$, and so $\mathsf v_{-g}(F_1) + 2 = \mathsf v_g(F_1)$. Therefore we obtain that

$$SF_2 = \big(g^{n_2}\big)^{l_1}\big((-g)^{n_2}\big)^{l_2}\big((g(-g)\big)^{l_3-\mathsf v_g(F_1)-(n_1-1)}(-g)^{n_2}\Big(\big(-(n_1 - 1)g\big)g^{n_1-1}\Big)0^{l_4+\mathsf v_0(F_2)}$$

Therefore

$$m_1 = l_1 + l_2 + l_3 + l_4 + 1 + \mathsf v_0(F_1) \in L_k$$

and

$$m_2 = l_1 + l_2 + l_3 + l_4 - (\mathsf v_g(F_1) + n_1 - 1) + 2 + \mathsf v_0(F_2) \in L_k$$

which implies that $m_1 - m_2 = \mathsf v_0(F_1) - \mathsf v_0(F_2) + \mathsf v_g(F_1) + n_1 - 2$ is congruent to either 0 or to $n_1 - 2$ or to $n_2 - n_1$ modulo $n_2 - 2$. We distinguish three cases.

CASE 4.1.1: $\mathsf v_0(F_1) + \mathsf v_g(F_1) + n_1 \equiv \mathsf v_0(F_2) + 2 \mod n_2 - 2$.

The left and the right hand side cannot be equal, since $\mathsf v_g(F_1) \ge 2$ would imply that $|F_2| \ge \mathsf v_0(F_2) \ge n_1$. Therefore we have

$$\mathsf v_0(F_1) + \mathsf v_g(F_1) + n_1 = \mathsf v_0(F_2) + n_2$$

and thus $|F_1| \ge \mathsf v_0(F_1) + \mathsf v_g(F_1) \ge n_2 - n_1 = n_1$, a contradiction.

CASE 4.1.2: $\mathsf v_0(F_1) + \mathsf v_g(F_1) + n_1 \equiv \mathsf v_0(F_2) + n_1 \mod n_2 - 2$.

This implies that $\mathsf v_0(F_1) + \mathsf v_g(F_1) = \mathsf v_0(F_2)$ whence $\mathsf v_0(F_2) \ge \mathsf v_g(F_1) \ge 2$, $\mathsf v_0(F_1) = 0$, and $\mathsf v_g(F_1) = \mathsf v_0(F_2)$. Therefore we obtain $F_1 = \big((n_1-1)g\big)g^{\mathsf v_0(F_2)}(-g)^{\mathsf v_0(F_2)-2}$ and $F_2 = \big(-(n_1-1)g\big)0^{\mathsf v_0(F_2)}$. Now consider a factorization z_1 of SF_1 which is divisible by the atom $X = \big((n_1-1)g\big)g^{n_1+1}$ and by $\big(g(-g)\big)^{\mathsf v_0(F_2)-2}$. It gives rise to a factorization

$$z_2 = z_1X^{-1}\big(g(-g)\big)^{-(\mathsf v_0(F_2)-2)}\Big(\big(-(n_1-1)g\big)g^{n_1-1}\Big)0^{\mathsf v_0(F_2)} \in \mathsf Z(SF_2)$$

of length $|z_2| = |z_1| - (1 + \mathsf v_0(F_2) - 2) + 1 + \mathsf v_0(F_2) = |z_1| + 2$. Since $n_1 \ge 5$ and $\min \Delta(L_k) = \min\{n_1 - 2, n_2 - n_1\} \ge 3$, L_k cannot contain the lengths $|z_1|$ and $|z_1| + 2 = |z_2|$, a contradiction.

CASE 4.1.3: $\mathsf v_0(F_1) + \mathsf v_g(F_1) \equiv \mathsf v_0(F_2) + 2 \mod n_2 - 2$.

This implies that $\mathsf v_0(F_1) + \mathsf v_g(F_1) = \mathsf v_0(F_2) + 2$. Since $\mathsf v_g(F_1) \ge 3$, it follows that $\mathsf v_0(F_2) > 0$ and hence $\mathsf v_0(F_1) = 0$. Therefore we obtain $F_1 = \big((n_1-1)g\big)g^{\mathsf v_0(F_2)+2}(-g)^{\mathsf v_0(F_2)}$ and $F_2 = \big(-(n_1-1)g\big)0^{\mathsf v_0(F_2)}$. Now consider a factorization z_2 of SF_2 which is divisible by the atom $X = \big(-(n_1-1)g\big)(-g)^{n_1+1}$. It gives rise to a factorization

$$z_1 = z_2X^{-1}0^{-\mathsf v_0(F_2)}\big((n_1-1)g(-g)^{n_1-1}\big)\big((-g)g\big)^{\mathsf v_0(F_2)+2}$$

of length $|z_1| = |z_2| + 2$, a contradiction.

CASE 4.2: $g \,|\, F_1$ and $(-g) \nmid F_1$.

Then $\mathsf v_g(F_2) = 0 = \mathsf v_{-g}(F_1)$, hence

$$n_2 - 2n_1 + 2 \equiv \mathsf v_g(F_1) + \mathsf v_{-g}(F_2) \mod n_2$$

and thus $n_2 - 2n_1 + 2 = \mathsf v_g(F_1) + \mathsf v_{-g}(F_2)$. Furthermore, we obtain that

$$\max\{\mathsf v_g(F_1), \mathsf v_{-g}(F_2)\} \ge \frac{n_2 - 2n_1 + 2}{2},$$

and hence $n_2 \in \{2n_1, 3n_2\}$. We obtain that

$$SF_2 = \big(g^{n_2}\big)^{l_1}\big((-g)^{n_2}\big)^{l_2}\big(\big(g(-g)\big)^{l_3-\mathsf v_g(F_1)-(n_1-1)}(-g)^{n_2}\Big(\big(-(n_1-1)g\big)g^{n_1-1}\Big)0^{l_4+\mathsf v_0(F_2)}$$

Therefore

$$m_1 = l_1 + l_2 + l_3 + l_4 + 1 + \mathsf v_0(F_1) \in L_k$$

and

$$m_2 = l_1 + l_2 + l_3 + l_4 - (\mathsf v_g(F_1) + n_1 - 1) + 2 + \mathsf v_0(F_2) \in L_k$$

which implies that $m_1 - m_2 = \mathsf v_0(F_1) - \mathsf v_0(F_2) + \mathsf v_g(F_1) + n_1 - 2$ is congruent to either 0 or to $n_1 - 2$ or to $n_2 - n_1$ modulo $n_2 - 2$. We distinguish three cases.

CASE 4.2.1: $\mathsf v_0(F_1) + \mathsf v_g(F_1) + n_1 \equiv \mathsf v_0(F_2) + 2 \mod n_2 - 2$.

The left and the right hand side cannot be equal, because otherwise we would have $|F_2| \ge \mathsf v_0(F_2) + 1 \ge n_1$. Therefore we have

$$\mathsf v_0(F_1) + \mathsf v_g(F_1) + n_1 = \mathsf v_0(F_2) + n_2$$

and thus $|F_1| \ge \mathsf v_0(F_1) + \mathsf v_g(F_1) \ge n_2 - n_1 \ge n_1$, a contradiction.

CASE 4.2.2: $\mathsf v_0(F_1) + \mathsf v_g(F_1) + n_1 \equiv \mathsf v_0(F_2) + n_1 \mod n_2 - 2$.

This implies that $\mathsf v_0(F_1)+\mathsf v_g(F_1)=\mathsf v_0(F_2)$ whence $\mathsf v_0(F_2)\ge \mathsf v_g(F_1)\ge 1$, $\mathsf v_0(F_1)=0$, and $\mathsf v_g(F_1)=\mathsf v_0(F_2)$. Therefore we obtain $F_1=\big((n_1-1)g\big)g^{\mathsf v_0(F_2)}$ and $F_2=\big(-(n_1-1)g\big)(-g)^{n_2-2n_1+2-\mathsf v_0(F_2)}0^{\mathsf v_0(F_2)}$ and hence $|F_2|=1+n_2-2n_1+2$ which implies that $n_2=2n_1$ and $|F_2|=3$. Thus $\mathsf v_0(F_2)\in\{1,2\}$.

Suppose that $\mathsf v_0(F_2)=1$. Then $\mathsf v_g(F_1)=1$, $F_1=\big((n_1-1)g\big)g$, and $F_2=\big(-(n_1-1)g\big)(-g)0$. Consider a factorization z_1 of SF_1 divisible by $X=\big((n_1-1)g\big)g^{n_1+1}$. This gives rise to a factorization

$$z_2=z_1X^{-1}0\big((-g)g\big)\Big(\big(-(n_1-1)g\big)g^{n_1-1}\Big)$$

of length $|z_2|=|z_1|+2$, a contradiction.

Suppose that $\mathsf v_0(F_2)=2$. Then $\mathsf v_g(F_1)=2$, $F_1=\big((n_1-1)g\big)g^2$, and $F_2=\big(-(n_1-1)g\big)0^2$. Consider a factorization z_1 of SF_1 divisible by $X=\big((n_1-1)g\big)g^{n_1+1}$. This gives rise to a factorization

$$z_2=z_1X^{-1}0^2\Big(\big(-(n_1-1)g\big)g^{n_1-1}\Big)$$

of length $|z_2|=|z_1|+2$, a contradiction.

CASE 4.2.3: $\mathsf v_0(F_1)+\mathsf v_g(F_1)\equiv \mathsf v_0(F_2)+n_2-2n_1+2 \mod n_2-2$.

Suppose that $n_2=3n_1$. Then $\mathsf v_0(F_1)+\mathsf v_g(F_1)\equiv \mathsf v_0(F_2)+n_1+2 \mod n_2-2$, and equality cannot hold because $|F_1|\ge \mathsf v_0(F_1)+\mathsf v_g(F_1)$. This implies that $(n_2-2)+\mathsf v_0(F_1)+\mathsf v_g(F_1)=\mathsf v_0(F_2)+n_1+2$ and hence $2n_1-4+\mathsf v_0(F_1)+\mathsf v_g(F_1)=\mathsf v_0(F_2)$, a contradiction to $\mathsf v_0(F_2)\le |F_2|\le n_1-1$.

This implies that $n_2=2n_1$ and $\mathsf v_0(F_1)+\mathsf v_g(F_1)=\mathsf v_0(F_2)+2$. Since $2=\mathsf v_g(F_1)+\mathsf v_{-g}(F_2)$, we infer that $\mathsf v_g(F_1)\in[1,2]$.

Suppose $\mathsf v_g(F_1)=2$. Then $\mathsf v_{-g}(F_2)=0$ and $\mathsf v_0(F_1)=\mathsf v_0(F_2)=0$, and we have $F_1=\big((n_1-1)g\big)g^2$ and $F_2=\Big(-(n_1-1)g\Big)$. Consider a factorization z_2 of SF_2 containing the atom $X=\big(-(n_1-1)g\big)(-g)^{n_1+1}$. This gives rise to a factorization

$$z_1=z_2X^{-1}\Big(\big((n_1-1)g\big)(-g)^{n_1-1}\Big)\big((-g)g\big)^2$$

of length $|z_1|=|z_2|+2$, a contradiction.

Suppose $\mathsf v_g(F_1)=1$. Then $\mathsf v_{-g}(F_2)=1$, $\mathsf v_0(F_1)=1$, $\mathsf v_0(F_2)=0$, and we have $F_1=\big((n_1-1)g\big)g0$ and $F_2=\Big(-(n_1-1)g\Big)(-g)$. Consider a factorization z_2 of SF_2 containing the atom $X=\big(-(n_1-1)g\big)(-g)^{n_1+1}$. This gives rise to a factorization

$$z_1=z_2X^{-1}\Big(\big((n_1-1)g\big)(-g)^{n_1-1}\Big)\big((-g)g\big)0$$

of length $|z_1|=|z_2|+2$, a contradiction.

CASE 4.3: $g\nmid F_1$ and $(-g)\mid F_1$.

Then $\mathsf v_g(F_1)=0$ and $\mathsf v_{-g}(F_2)=0$ and hence

$$2n_1-2\equiv \mathsf v_g(F_2)+\mathsf v_{-g}(F_1) \mod n_2\,.$$

This implies that $\mathsf v_g(F_2)=\mathsf v_{-g}(F_1)=n_1-1$ and hence $|F_1|\ge n_1$ and $|F_2|\ge n_1$, a contradiction.

CASE 4.4: $g\nmid F_1$ and $(-g)\nmid F_1$.

Then $\mathsf v_g(F_1)=0=\mathsf v_{-g}(F_1)$ and hence

$$2n_1-2\equiv \mathsf v_g(F_2)-\mathsf v_{-g}(F_2) \mod n_2\,.$$

If $n_2\ge 3n_1$, then $|F_2|\ge n_1$, a contradiction. Thus $n_2=2n_1$ and hence $\mathsf v_g(F_2)=\mathsf v_{-g}(F_2)-2$. Therefore we obtain that

$$SF_2=\big(g^{n_2}\big)^{l_1}\big((-g)^{n_2}\big)^{l_2}\big(g(-g)\big)^{l_3+\mathsf v_g(F_2)-(n_1-1)}(-g)^{n_2}\big(((-n_1+1)g)g^{n_1-1}\big)0^{l_4+\mathsf v_0(F_2)}$$

Therefore

$$m_1=l_1+l_2+l_3+l_4+1+\mathsf v_0(F_1)\in L_k$$

and

$$m_2 = l_1 + l_2 + l_3 + l_4 + \mathsf v_g(F_2) - (n_1 - 1) + 2 + \mathsf v_0(F_2) \in L_k$$

which implies that $m_1 - m_2 = \mathsf v_0(F_1) - \mathsf v_0(F_2) - \mathsf v_g(F_2) + (n_1 - 1) - 1$ is congruent to either 0 or to $n_1 - 2$ or to $n_2 - n_1$ modulo $n_2 - 2$. We distinguish three cases.

CASE 4.4.1: $\mathsf v_0(F_2) + \mathsf v_g(F_2) \equiv \mathsf v_0(F_1) + n_1 - 2 \mod n_2 - 2$.

This implies that $\mathsf v_0(F_2) + \mathsf v_g(F_2) = \mathsf v_0(F_1) + n_1 - 2$, and hence $|F_2| \ge \mathsf v_{-g}(F_2) \ge \mathsf v_g(F_2) + 2 \ge n_1$, a contradiction.

CASE 4.4.2: $\mathsf v_0(F_2) + \mathsf v_g(F_2) + (n_1 - 2) \equiv \mathsf v_0(F_1) + n_1 - 2 \mod n_2 - 2$.

This implies that $\mathsf v_0(F_2) + \mathsf v_g(F_2) = \mathsf v_0(F_1)$ and hence $\mathsf v_0(F_2) = 0$. Therefore we obtain that $F_1 = \big((n_1 - 1)g\big)0^{\mathsf v_0(F_1)}$ and $F_2 = \big(- (n_1 - 1)g\big)g^{\mathsf v_0(F_1)}(-g)^{\mathsf v_0(F_1)+2}$. Now consider a factorization z_1 of SF_1 containing the atom $X = \big((n_1 - 1)g\big)g^{n_1+1}$. This gives rise to a factorization

$$z_2 = z_1 X^{-1}\Big(\big(- (n_1 - 1)g\big)g^{n_1-1}\Big)\big((-g)g\big)^{\mathsf v_0(F_1)+2}0^{-\mathsf v_0(F_1)}$$

of length $|z_2| = |z_1| + 2$, a contradiction.

CASE 4.4.3: $\mathsf v_0(F_2) + \mathsf v_g(F_2) + (n_2 - n_1) \equiv \mathsf v_0(F_1) + n_1 - 2 \mod n_2 - 2$.

Since $n_2 = 2n_1$, the congruence simplifies to $\mathsf v_0(F_2) + \mathsf v_g(F_2) + 2 \equiv \mathsf v_0(F_1) \mod n_2 - 2$ which implies that $\mathsf v_0(F_2) + \mathsf v_g(F_2) + 2 = \mathsf v_0(F_1)$. Thus $\mathsf v_0(F_2) = 0$, $F_1 = \big((n_1 - 1)g\big)0^{\mathsf v_0(F_1)}$, and $F_2 = \big(- (n_1 - 1)g\big)g^{\mathsf v_0(F_1)-2}(-g)^{\mathsf v_0(F_1)}$. Now consider a factorization z_1 of SF_1 containing the atom $X = \big((n_1 - 1)g\big)(-g)^{n_1-1}$. This gives rise to a factorization

$$z_2 = z_1 X^{-1}\Big(\big(- (n_1 - 1)g\big)(-g)^{n_1+1}\Big)\big((-g)g\big)^{\mathsf v_0(F_1)-2}0^{-\mathsf v_0(F_1)}$$

of length $|z_2| = |z_1| - 2$, a contradiction. □(Proof of **A6**)

We state the final assertion

A7. $n_1 - 1 \le \mathsf c_{\mathcal B_{\langle g\rangle}(G)}(D_k)$.

Proof of **A7**. By **A5**, we have

$$\mathsf L_{\mathcal B(G)}(B_k) \;=\; \bigcup_{z \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)} \mathsf L_{\mathcal B(\langle g\rangle)}\big(C_k\sigma(z)\big)\,,$$

and the union on the right hand side consists of at least two distinct sets which are not contained in each other. Assume to the contrary that $\mathsf c_{\mathcal B_{\langle g\rangle}(G)}(D_k) \le n_1 - 2$ and choose a factorization $z_0 \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)$.

We assert that for each $z \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)$ there exists an $l(z) \in \mathbb Z$ such that $\sigma(z) = \sigma(z_0)0^{-l(z)}\big((-g)g\big)^{l(z)}$. Let $z \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)$ be given, and let $z_0, \ldots, z_k = z$ be an $(n_1 - 2)$-chain of factorizations concatenating z_0 and z. Since $\mathsf d(z_{i-1}, z_i) < n_1 - 1$, it follows that the pair (z_{i-1}, z_i) is of type (ii) in **A6** for each $i \in [1, k]$. Therefore $\sigma(z_i) = \sigma(z_{i-1})0^{-l_i}\big((-g)g\big)^{l_i}$ for some $l_i \in \mathbb Z$ and each $i \in [1, k]$, and hence the assertion follows with $l(z) = l_1 + \ldots + l_k$.

We choose a factorization $z^* \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)$ such that

$$l(z^*) = \max\{l(z) \mid z \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)\}\,,$$

and assert that

$$\mathsf L_{\mathcal B(\langle g\rangle)}(C_k\sigma(z)) \subset \mathsf L_{\mathcal B(\langle g\rangle)}(C_k\sigma(z^*)) \quad \text{for each} \quad z \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)\,.$$

Let $z \in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)$ be given. Then $\sigma(z^*) = \sigma(z)0^{-(l(z^*)-l(z))}\big((-g)g\big)^{l(z^*)-l(z)}$. If

$$y \in \mathsf Z_{\mathcal B(\langle g\rangle)}(C_k\sigma(z))\,, \quad \text{then} \quad y0^{-(l(z^*)-l(z))}\big((-g)g\big)^{l(z^*)-l(z)} \in \mathsf Z_{\mathcal B(\langle g\rangle)}(C_k\sigma(z^*))$$

is a factorization of length $|y|$, and hence $\mathsf L_{\mathcal B(\langle g\rangle)}(C_k\sigma(z)) \subset \mathsf L_{\mathcal B(\langle g\rangle)}(C_k\sigma(z^*))$.

Therefore we obtain that

$$\mathsf L_{\mathcal B(G)}(B_k) \;=\; \bigcup_{z\in \mathsf Z_{\mathcal B_{\langle g\rangle}(G)}(D_k)} \mathsf L_{\mathcal B(\langle g\rangle)}\big(C_k\sigma(z)\big) \;=\; \mathsf L(C_k\sigma(z^*))\,,$$

a contradiction to the fact that this union consists of least two distinct sets which are not contained in each other. □(Proof of **A7**)

Using **A7** and Proposition 2.5.2, we infer that

$$n_1-1 \le \mathsf c_{\mathcal B_{\langle g\rangle}(G)}(D_k) \le \mathsf c(\mathcal B_{\langle g\rangle}(G)) = \mathsf c(\mathcal B(G/\langle g\rangle)) \le \mathsf D(G/\langle g\rangle) = \mathsf D(H) \le n_1\,.$$

We distinguish two cases.

CASE 1: $\mathsf c(\mathcal B(G/\langle g\rangle)) = n_1$.

Then $\mathsf D(G/\langle g\rangle) = n_1$, Proposition 2.4.1 implies that $G/\langle g\rangle$ is either cyclic of order n_1 or an elementary 2-group of rank n_1-1. Since $H \cong G/\langle g\rangle$, it follows that $G \cong C_{n_1}\oplus C_{n_2}$ or $G \cong C_2^{n_1-1}\oplus C_{n_2}$.

CASE 2: $\mathsf c(\mathcal B(G/\langle g\rangle)) = n_1-1$.

We distinguish two cases.

CASE 2.1: $\mathsf D(G/\langle g\rangle) = n_1$.

Then Proposition 2.4.2 implies that $G/\langle g\rangle$ is isomorphic either to $C_2\oplus C_{n_1-1}$, where n_1-1 is even, or to $C_2^{n_1-4}\oplus C_4$. Since $H\cong G/\langle g\rangle$, it follows that $G\cong C_2\oplus C_{n_1-1}\oplus C_{n_2}$ or $G\cong C_2^{n_1-4}\oplus C_4\oplus C_{n_2}$.

CASE 2.2: $\mathsf D(G/\langle g\rangle) = n_1-1$.

Then $\mathsf c(\mathcal B(G/\langle g\rangle)) = \mathsf D(G/\langle g\rangle) = n_1-1$, and (again by Proposition 2.4.1) $G/\langle g\rangle$ is cyclic of order n_1-1 or an elementary 2-group of rank n_1-2. If $G/\langle g\rangle$ is cyclic, then G has rank two and $\mathsf d(G) = \mathsf d(G/\langle g\rangle)+\mathsf d(\langle g\rangle) = n_1-2+n_2-1 < n_1+n_2-2 = \mathsf d(G)$, a contradiction. Thus $G/\langle g\rangle$ is an elementary 2-group and $G\cong C_2^{n_1-2}\oplus C_{n_2}$. □

Finally we are able to prove the main result of this paper.

Proof of Theorem 1.1. Let G be an abelian group such that $\mathcal L(G) = \mathcal L(C_{n_1}\oplus C_{n_2})$ where $n_1,n_2\in\mathbbm N$ with $n_1\,|\,n_2$ and $n_1+n_2>4$.

Proposition 6.1 implies that G is finite with $\exp(G)=n_2$ and $\mathsf d(G) = \mathsf d(C_{n_1}\oplus C_{n_2}) = n_1+n_2-2$. If $n_1=n_2$, then $G\cong C_{n_1}\oplus C_{n_2}$ by Proposition 6.1.2. Thus we may suppose that $n_1<n_2$, and we set $G = H\oplus C_{n_2}$ where $H\subset G$ is a subgroup with $\exp(H)\,|\,n_2$. If $n_1\in[1,5]$, then the assertion follows from Proposition 6.2.3, and hence we suppose that $n_1\ge 6$. Since $\mathcal L(G)=\mathcal L(C_{n_1}\oplus C_{n_2})$, Proposition 3.5 implies that, for each $k\in\mathbbm N$, the sets

$$L_k = \Big\{(kn_2+3)+(n_1-2)+(n_2-2)\Big\} \cup \Big((2k+3)+\{0,n_1-2,n_2-2\}+\{\nu(n_2-2)\mid \nu\in[0,k]\}\Big)$$

are in $\mathcal L(G)$. Therefore Proposition 6.5 implies that G is isomorphic to one of the following groups

$$C_{n_1}\oplus C_{n_2}\,,\; C_2^s\oplus C_{n_2} \text{ with } s\in\{n_1-2,n_1-1\}\,, C_2^{n_1-4}\oplus C_4\oplus C_{n_2}\,,\; C_2\oplus C_{n_1-1}\oplus C_{n_2} \;\text{ with } 2\,|\,(n_1-1)\,|\,n_2\,.$$

Since

$$\mathsf d^*(C_2^{n_1-4}\oplus C_4\oplus C_{n_2}) = n_1+n_2-2 = \mathsf d(G) \quad\text{and}\quad \mathsf d^*(C_2\oplus C_{n_1-1}\oplus C_{n_2}) = n_1+n_2-2 = \mathsf d(G)\,,$$

Proposition 6.2.2 implies that G cannot be isomorphic to any of these two groups. Proposition 3.7 (with $k=0$, $n=n_2$, and $r=n_1-1$) implies that

$$\{2,n_2,n_1+n_2-2\}\in\mathcal L(C_2^{n_1-2}\oplus C_{n_2})\subset\mathcal L(C_2^{n_1-1}\oplus C_{n_2})\,.$$

However, Proposition 5.1 shows that $\{2,n_2,n_1+n_2-2\}\notin\mathcal L(C_{n_1}\oplus C_{n_2}) = \mathcal L(G)$. Therefore it follows that $G\cong C_{n_1}\oplus C_{n_2}$. □

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