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# Spectral minimal partitions for a family of tori

Corentin Léna \*

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## Abstract

We study partitions of the two-dimensional flat torus  $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$  into  $k$  domains, with  $b$  a parameter in  $(0, 1]$  and  $k$  an integer. We look for partitions that minimize an energy defined from the first eigenvalue of the Dirichlet Laplacian on the domains. We are in particular interested in the way these minimal partitions change when  $b$  is varied. We recall previous results on transition values by Helffer and Hoffmann-Ostenhof (2014), present a slight improvement when  $k$  is odd, and state some conjectures. We support these conjectures by looking for candidates to be minimal partitions using an optimization algorithm adapted from Bourdin, Bucur, and Oudet (2009). Guided by these numerical results, we construct  $k$ -partitions that are tilings of the torus by hexagons. We compute their energy and thus obtain an improved upper bound of the minimal energy.

**MSC classification.** Primary 49Q10; Secondary 35J05, 65K10, 65N06, 65N25.

**Keywords.** Minimal partitions, shape optimization, Laplacian, eigenvalues, eigenfunctions, finite difference method, projected gradient algorithm.

## 1 Introduction

Optimal partition problems are a field of shape optimization that has been the object of recent interest, see e.g. [4, 5, 6, 13]. It can be connected to the study of nodal sets for the Laplacian (see e.g. [13]) or of steady states for competition-diffusion systems (see e.g. [7]). This paper studies minimal partitions of the two-dimensional flat torus  $T(1, b) = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$ , with  $b \in (0, 1]$ . To be more precise, for a given integer  $k$ , we look for  $k$  open, connected, and mutually disjoint subsets  $D_1, \dots, D_k$  of  $T(1, b)$  such that the quantity  $\max\{\lambda_1(D_i); 1 \leq i \leq k\}$ , which we call the *energy*, is minimal. Here  $\lambda_1(D_i)$  is the lowest eigenvalue of the Dirichlet Laplacian on  $D_i$ . This problem was studied by Bernard Helffer, Thomas Hoffmann-Ostenhof, and Susanna Terracini in [13], where existence and regularity were proved for a large class of domains. Similar shape optimization problems were studied in [5] and [6]. In the specific setting that we consider here, B. Helffer and T. Hoffmann-Ostenhof showed, in [12], that when  $b$  is small enough, a minimal  $k$ -partition is obtained by dividing the torus in  $k$  equal 'vertical' strips.

One of the motivations of this work is to get information on the minimal partitions when  $b$  increases. In particular, we are interested in the value of  $b$  for which the 'vertical' partition mentioned above is no longer minimal. This problem was studied in [12], where the transition value was found for  $k$  even, and a lower bound on the transition value was obtained for  $k$  odd. We show that in this latter case, the lower bound in [12] is not optimal, although we are unable to give an explicit lower bound. However, we present a conjecture on the transition value, supported by some partial results.

To check this conjecture, and to look for minimal partitions for larger  $b$ 's (with  $b \in (0, 1]$ ), we need a numerical method to find minimal partitions. We have adapted the optimization algorithm used by Blaise Bourdin, Dorin Bucur and Édouard Oudet in [2]. Our numerical computations support the conjectures concerning the transition values, improve known upper bounds of the minimal energy, and produce some quite striking candidates to be minimal partitions.

Guided by the numerical results, we have constructed families of  $k$ -partitions, for  $k \in \{3, 4, 5\}$ . These partitions are tilings of the torus by isometric polygons (in general hexagons) that satisfy a

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geometrical constraint called the 'equal angle meeting property'. For some values of  $b$ , we obtain in this way partitions that closely resemble the best candidates produced by the optimization algorithm. For other values of  $b$ , such a tiling does not seem to be optimal. In any case, however, the first eigenvalue of the Dirichlet Laplacian of the fundamental domain of the tiling gives an upper bound of the minimal energy, which often improves the previously known ones.

The paper is organized in the following way. In Section 2, we recall some relevant facts concerning minimal partitions, including the results of [12] for flat tori. In section 3, we present some results on the transition values, using ideas introduced in [14, 12]. When  $b$  is equal to the conjectured transition value, we also present some candidates to be minimal partitions which are obtained from the nodal domains of eigenfunctions of the Laplacian on a covering of the torus. In Section 4, we describe the numerical method that we have used. The numerical results are presented in Section 5. For a fixed  $k$  ( $k \in \{3, 4, 5\}$ ), we compute candidates to be a minimal  $k$ -partition of  $T(1, b)$ , for a discrete set of values of  $b$  in  $(0, 1]$ . These computations suggest the existence of hexagonal tilings of tori under consideration. In Section 6, we compute the energy of these tilings (which are constructed in Appendix A), and compare it to the formerly obtained bounds. This allows us to improve the bounds of the minimal energy deduced from the general theorems of Section 2. In some cases, we are even able to formulate a conjecture on the value of the minimal energy. We conclude with a summary of the results and some suggestions for future research.

## 2 Minimal partitions

### 2.1 Definitions

In the following,  $M$  is a compact two-dimensional Riemannian manifold without boundary. For any open set  $\omega \subset M$ , the sequence  $(\lambda_k(\omega))_{k \geq 1}$  denotes the eigenvalues of the Laplace-Beltrami operator on  $\omega$  with Dirichlet boundary conditions. They are arranged in increasing order and counted with multiplicity. The following definitions and results were introduced in [13] for the case of planar domains, and their extension to a two-dimensional manifold was discussed in [14].

**Definition 2.1.** *For any integer  $k \geq 1$ , a  $k$ -partition is a finite family  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  whose  $k$  elements are open, connected, and mutually disjoint subsets of  $M$ . We define its energy by*

$$\Lambda_k(\mathcal{D}) = \max_{1 \leq i \leq k} \lambda_1(D_i).$$

In the rest of the paper, we call the  $D_i$ 's *domains* of the  $k$ -partition and, for any given  $k$ , denote by  $\mathfrak{P}_k$  the set of all  $k$ -partitions. Furthermore, we simply use the term *partition* if we do not want to specify the number of domains.

**Definition 2.2.** *For any integer  $k \geq 1$ , we set  $\mathfrak{L}_k(M) = \inf_{\mathcal{D} \in \mathfrak{P}_k} \Lambda_k(\mathcal{D})$ . A partition  $\mathcal{D} \in \mathfrak{P}_k$  such that  $\Lambda_k(\mathcal{D}) = \mathfrak{L}_k(M)$  is called a minimal  $k$ -partition.*

The following existence result is proved in the paper [8].

**Theorem 2.3.** *For any integer  $k \geq 1$ , there exists a minimal  $k$ -partition of  $M$ .*

This paper in fact proves a much stronger result, namely the existence of a regular minimal partition in the sense of Definition 2.5.

### 2.2 Properties of minimal partitions

Following [13], let us now define what we mean by regular partition.

**Definition 2.4.** *A  $k$ -partition  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  is called strong if  $\bigcup_{i=1}^k \overline{D_i} = M$ . Its boundary is then defined as  $N(\mathcal{D}) = \bigcup_{i=1}^k \partial D_i$ .*

**Definition 2.5.** *A partition  $\mathcal{D}$  is called regular if it is strong and its boundary  $N(\mathcal{D})$  satisfies the following properties.*

- (i) The set  $N(\mathcal{D})$  is locally a  $C^{1,1^-} = \cap_{\alpha < 1} C^{1,\alpha}$  curve, except in the neighborhood of a finite set  $\{x_i; 1 \leq i \leq \ell\}$ . The elements of this set are called singular points.
- (ii) For each  $1 \leq i \leq \ell$ , there is an integer  $\nu_i \geq 3$  such that, in a neighborhood of  $x_i$ , the set  $N(\mathcal{D})$  is the union of  $\nu_i$  half curves of class  $C^{1,+} = \cup_{\alpha > 0} C^{1,\alpha}$ , which meet at  $x_i$ .
- (iii) At each singular point, the half curves meet with equal angles.

The third property in Definition 2.5 is called the 'equal angle meeting property' in [14]. As proved in [13], we have the following regularity result:

**Theorem 2.6.** *For any  $k \geq 1$ , minimal  $k$ -partitions are strong and regular (up to zero capacity sets).*

**Definition 2.7.** *A  $k$ -partition  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  is called equispectral if  $\lambda_1(D_1) = \dots = \lambda_1(D_k)$ .*

**Proposition 2.8.** *Minimal partitions are equispectral.*

Let us note that this last property holds only for the energy in Definition 2.1. It is in general false for partitions that minimize the energy in Definition 4.1 (see [11] for a discussion of this point).

We give additional definitions that help us to describe the topology of a partition.

**Definition 2.9.** *Let  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  be a strong partition. Two domains  $D_i$  and  $D_j$  are said to be neighbors if  $\text{Int}(\overline{D_i \cup D_j})$  is connected. A strong partition is said to be bipartite if one can color its domains, using only two colors, in such a way that two neighbors have different colors.*

The paper [13] shows that there exists a connection between the minimal partitions of  $M$  and the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on  $M$ . Since we will use this connection in the rest of this work, we recall some relevant definitions and results. With an eigenfunction  $u$  of the Laplace-Beltrami operator on  $M$ , we associate the nodal set  $N(u) = \{x \in M; u(x) = 0\}$ , and we call nodal domains the connected components of  $M \setminus N(u)$ . We denote by  $\mu(u)$  the number of nodal domains and the nodal domains themselves by  $D_i$ . The following result was proved by Courant (cf. [9]).

**Theorem 2.10.**

*If  $k \geq 1$  and  $u$  is an eigenfunction associated with  $\lambda_k(M)$ ,  $\mu(u) \leq k$ .*

The family  $(D_i)_{1 \leq i \leq \mu(u)}$  is therefore a partition in the sense of Definition 2.1. According to classical results on the nodal set, it is a regular partition, that we will call the nodal partition associated with  $u$ . Following [13], we introduce a new definition.

**Definition 2.11.** *Let  $k \geq 1$ . An eigenfunction  $u$  of the Laplace-Beltrami operator, associated with the eigenvalue  $\lambda$ , is said to be Courant-sharp if  $\mu(u) = \min\{\ell; \lambda_\ell(M) = \lambda\}$ .*

To give an upper-bound of  $\mathfrak{L}_k(M)$ , one can use  $k$ -partitions obtained from eigenfunctions. To make this explicit, we introduce a new spectral quantity.

**Definition 2.12.** *For  $k \geq 1$ , we denote by  $L_k(M)$  the smallest eigenvalue of the Dirichlet Laplacian that has an eigenfunction with  $k$  nodal domains. If there is no such eigenvalue, we set  $L_k(M) = +\infty$ .*

We can now state the results of [13] that link minimal and nodal partitions.

**Theorem 2.13.** *A minimal partition is nodal if, and only if, it is bipartite. Furthermore, for any integer  $k \geq 1$ ,  $\lambda_k(M) \leq \mathfrak{L}_k(M) \leq L_k(M)$ , and, if  $\mathfrak{L}_k(M) = L_k(M)$  or  $\lambda_k(M) = \mathfrak{L}_k(M)$ , then  $\lambda_k(M) = \mathfrak{L}_k(M) = L_k(M)$ . In this latter case, any minimal  $k$ -partition is nodal.*

Let us point out a few consequences of Theorem 2.13.

**Corollary 2.14.** *A nodal partition associated with a Courant-sharp eigenfunction is minimal. Minimal 2-partitions are the nodal partitions of eigenfunctions associated with  $\lambda_2(M)$ .*

### 2.3 Results on tori

In order to apply Theorem 2.13 we need the eigenvalues of the Laplace-Beltrami operator on the torus, which we recall here for future reference. As we will see in Section 3, the tori  $T(2, 2b)$  and  $T(2, b)$  play a part in our study of  $T(1, b)$ . It will therefore be useful to have statements dealing with the more general torus

$$T(a, b) = (\mathbb{R}/a\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z}) ,$$

with  $0 < b \leq a$ . The Laplace-Beltrami operator on  $T(a, b)$  is of course unitarily equivalent to the Laplacian on the rectangle  $(0, a) \times (0, b)$  with periodic boundary conditions. We will therefore often describe  $T(a, b)$  as the rectangle  $[0, a] \times [0, b]$  with the opposite sides identified. The eigenvalues are then easy to compute using separation of variables and the periodic boundary condition.

**Proposition 2.15.** *The eigenvalues of the Laplace-Beltrami operator on  $T(a, b)$  are  $\lambda_{m,n}(a, b) = 4\pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$ , with  $m$  and  $n$  in  $\mathbb{N}$ .*

The following notation will also be useful.

**Definition 2.16.** *For  $0 < b \leq a$  and  $k \in \mathbb{N}^*$ , we denote by  $\mathcal{D}_k(a, b)$  the  $k$ -partition of  $T(a, b)$  with domains*

$$D_i = \left( \frac{i-1}{k}a, \frac{i}{k}a \right) \times (0, b) , \quad \text{for } i = 1, \dots, k .$$

We have  $\Lambda_k(\mathcal{D}_k(a, b)) = k^2\pi^2/a^2$ , and any partition obtained from  $\mathcal{D}_k(a, b)$  by a translation has the same energy. Let us note that if  $k$  is even,  $\mathcal{D}_k(a, b)$  is the nodal partition associated with the eigenfunction  $(x, y) \mapsto \sin(k\pi x/a)$  of the Laplace-Beltrami operator on  $T(a, b)$ .

Let us now go back to the study of  $T(1, b)$  with  $b \in (0, 1]$ . We want to know for which values of  $b$  the partition  $\mathcal{D}_k(1, b)$  is minimal. We have to distinguish between an even and an odd  $k$ . If  $k$  is even, we can answer the question by applying Theorem 2.13.

**Proposition 2.17.** *Let  $k$  be an even integer,  $k = 2\ell$ . If  $b \leq 1/\ell$ ,  $\mathcal{D}_k(1, b)$  is a minimal  $k$ -partition of  $T(1, b)$ .*

*Furthermore, if  $b < 1/\ell$ ,  $\mathcal{D}_k(1, b)$  is, up to a translation, the only minimal  $k$ -partition of  $T(a, b)$ .*

*Finally, if  $b > 1/\ell$ , the partition  $\mathcal{D}_k(1, b)$  is not minimal.*

If  $k = 2\ell$  and  $b = 1/\ell$ ,  $\mathcal{D}_k(1, b)$  is not the only minimal partition of  $T(1, b)$  up to a translation, as we will see in Section 3.

The answer is less clear when  $k$  is odd. However, a more complicated analysis, performed by B. Helffer and T. Hoffmann-Ostenhof in [12], gives a range of values of  $b$  for which  $\mathcal{D}_k(1, b)$  is minimal.

**Proposition 2.18.** *Let  $k > 1$  be an odd integer. If  $b < 1/k$ , then  $\mathcal{D}_k(1, b)$  is a minimal  $k$ -partition of  $T(1, b)$ .*

Let us conclude this section by a monotonicity result.

**Proposition 2.19.** *Let  $k \geq 1$ . The function  $b \mapsto \mathfrak{L}_k(T(1, b))$ , defined on  $(0, 1]$ , is non-increasing.*

*Proof.* Let us pick  $b$  and  $b'$  in  $(0, 1]$ , with  $b' < b$ . We define a mapping  $A$  from  $T(1, b')$  to  $T(1, b)$  by

$$\begin{aligned} A : T(1, b') &\rightarrow T(1, b) \\ (x, y) &\mapsto \left( x, \frac{b}{b'}y \right) . \end{aligned}$$

If  $\omega$  is an open set in  $T(1, b')$ ,  $A(\omega)$  is an open set in  $T(1, b)$  and one can see, using the change of variable defined by  $A$ , that  $\lambda_1(A(\omega)) \leq \lambda_1(\omega)$ . As a consequence, a minimal partition of  $T(1, b')$  is transformed by  $A$  into a partition of  $T(1, b)$  whose energy is lower than  $\mathfrak{L}_k(T(1, b'))$ . This implies  $\mathfrak{L}_k(T(1, b)) \leq \mathfrak{L}_k(T(1, b'))$ .  $\square$

### 3 Transitions

#### 3.1 Definition

**Definition 3.1.** Let  $k \geq 1$ . We define the transition value  $b_k$  by

$$b_k = \sup\{b > 0 ; \forall b' \in (0, b), \mathcal{D}_k(1, b') \text{ is a minimal } k\text{-partition of } T(1, b')\}.$$

Due to the monotonicity of  $b \mapsto \mathfrak{L}_k(T(1, b))$ , we can give the following alternative definition of  $b_k$ .

**Remark 3.2.** For  $k \geq 1$ ,  $b_k = \sup\{b > 0 ; \mathcal{D}_k(1, b) \text{ is a minimal } k\text{-partition of } T(1, b)\}$ .

Propositions 2.17 and Proposition 2.18 can be summarized in the following statement.

**Proposition 3.3.**

If  $k$  is even,  $b_k = 2/k$ . If  $k$  is odd,  $b_k \geq 1/k$ .

We are interested in finding  $b_k$  for  $k$  odd, and also in understanding the transition from  $\mathcal{D}_k(1, b)$  to another type of minimal partition when  $b$  becomes greater than  $b_k$ .

#### 3.2 Further results for an odd $k$

We will now show that the method used in [12] can be refined to obtain a little more information on  $b_k$  when  $k$  is odd.

**Theorem 3.4.** If  $k$  is odd,  $b_k > 1/k$ .

Following [12], we consider the torus

$$T(2, 2b) = (\mathbb{R}/2\mathbb{Z}) \times (\mathbb{R}/2b\mathbb{Z}).$$

There is a natural projection map  $\Pi_4 : (x, y) \mapsto (x \bmod 1, y \bmod b)$  from  $T(2, 2b)$  to  $T(1, b)$ . Since every point of  $T(1, b)$  has four antecedents by  $\Pi_4$ ,  $T(2, 2b)$  can be seen as a four-sheeted covering of  $T(1, b)$ . The pullback  $\Pi_4^{-1}(D)$  of a connected open set  $D$  in  $T(1, b)$  is an open set in  $T(2, 2b)$  that has at most four connected components. With any  $k$ -partition  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$ , we associate the partition  $\Pi_4^{-1}(\mathcal{D})$  whose domains are the all the connected components of all the sets  $\Pi_4^{-1}(D_i)$ , for  $i \in \{1, \dots, k\}$ . We call it the partition *lifted from*  $\mathcal{D}$ . It is a regular  $\ell$ -partition, with  $k \leq \ell \leq 4k$ .

Let us note that  $\Pi_4^{-1}(\mathcal{D}_k(1, b)) = \mathcal{D}_{2k}(2, 2b)$ . In particular, while  $\mathcal{D}_k(1, b)$  is not nodal, the partition  $\Pi_4^{-1}(\mathcal{D}_k(1, b))$  is a nodal  $2k$ -partition of  $T(2, 2b)$ .

Let us now define the following mapping on  $T(2, 2b)$ :

$$\begin{aligned} \sigma : T(2, 2b) &\rightarrow T(2, 2b) \\ (x, y) &\mapsto (x + 1 \bmod 2, y). \end{aligned}$$

We have  $\sigma((x, y)) \neq (x, y)$  and  $\Pi_4(\sigma((x, y))) = \Pi_4((x, y))$  for all  $(x, y) \in T(2, 2b)$ . We define the corresponding unitary operator  $\Sigma$  on  $L^2(T(2, 2b))$  by  $\Sigma u = u \circ \sigma$ .

**Definition 3.5.** We say that  $u \in L^2(T(2, 2b))$  is symmetric (resp. antisymmetric) if  $\Sigma u = u$  (resp.  $\Sigma u = -u$ ). We write  $\mathcal{S}_\sigma$  (resp.  $\mathcal{A}_\sigma$ ) for the space of symmetric (resp. antisymmetric) functions.

**Remark 3.6.** Since  $\Sigma$  is unitary and  $\Sigma^2 = I$ , we have the orthogonal decomposition

$$L^2(T(2, 2b)) = \mathcal{S}_\sigma \oplus \mathcal{A}_\sigma.$$

Let us note that for any smooth function  $u$  on  $T(2, 2b)$ ,  $\Sigma(\Delta u) = \Delta(\Sigma u)$ . We write  $H_\sigma^a$  for the Friedrichs's extension of the differential operator  $-\Delta$  acting on  $C^\infty(T(2, 2b)) \cap \mathcal{A}_\sigma$ . The operator  $H_\sigma^a$  is self-adjoint with domain  $H^2(T(2, 2b)) \cap \mathcal{A}_\sigma$ . It has compact resolvent and we write  $(\lambda_k^{\sigma, a})_{k \geq 1}$  for the sequence of its eigenvalues, arranged in non-decreasing order and counted with multiplicities.

Following [11], we will focus on partitions of a specific type

**Definition 3.7.** We say that a  $k$ -partition  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  of  $T(2, 2b)$  is pair symmetric if, for any  $i \in \{1, \dots, k\}$ ,  $\sigma(D_i) = D_j$  with  $j \neq i$ .

**Remark 3.8.** According to the definition, a pair symmetric partition has an even number of domains. Let us also note that the nodal domains of an antisymmetric eigenfunction form an pair symmetric partition.

**Proposition 3.9.** Let  $\mathcal{D}$  be an pair symmetric equispectral  $2k$ -partition of  $T(2, 2b)$ . We have

$$\lambda_k^{\sigma, a} \leq \Lambda_{2k}(\mathcal{D}).$$

The proof is an application of the minmax characterization of eigenvalues, with test function taken in the space  $\mathcal{A}_\sigma$ , as allowed by the geometrical condition in Definition 3.7.

Let us now consider an auxiliary optimization problem. For  $b \in (0, 1]$ , we consider the infinite strip

$$S_b = \mathbb{R} \times \left( -\frac{b}{2}, \frac{b}{2} \right).$$

We now look at the problem of minimizing the first eigenvalue of the Laplacian under two constraints: an upper bound on the area, and inclusion in  $S_b$ :

$$J(b, A) = \inf_{\Omega \subset S_b, |\Omega| \leq A} \lambda_1(\Omega). \quad (3.1)$$

It follows from the definition that  $(b, A) \mapsto J(b, A)$  is non-increasing with respect to both  $b$  and  $A$ . According to Faber-Krahn inequality, the disk is a minimizer for sufficiently large  $b$ . For a small  $b$ , the following lower bound, deduced from domain monotonicity, gives more information (see [12]):

$$J(b, A) \geq \frac{\pi^2}{b^2}.$$

Finally, it can be shown, using a concentration-compactness result for shapes proved by D. Bucur (see [3]), that there exists a quasi-open minimizer  $\Omega^*$  for Problem (3.1), which satisfies  $|\Omega^*| = A$ .

Let us now define, for any integer  $k$  (in practice we are only interested in the case where  $k$  is odd), the following quantity:

$$b_k^{FK} = \sup \{ b \in (0, 1] : J(b, b) > k^2 \pi^2 \}.$$

**Lemma 3.10.** For any integer  $k$ ,

$$\frac{1}{k} < b_k^{FK} < \frac{1}{\sqrt{k^2 - 1}}.$$

Let us outline the proof of Lemma 3.10. It can be shown, by an elementary argument using monotonicity, that the function  $b \mapsto J(b, b)$  is non-increasing and continuous. Furthermore, in the case  $b = 1/k$ , we have seen that there exists a quasi-open set  $\Omega^* \subset S_b$  such that  $\lambda_1(\Omega^*) = J(1/k, 1/k)$  and  $|\Omega^*| = 1/k$ . This condition on the area implies in particular that  $J(1/k, 1/k) = \lambda_1(\Omega^*) > k^2 \pi^2$ . We conclude that  $b_k^{FK} > 1/k$ . To obtain the lower bound let us consider the rectangle

$$\mathcal{R}(1, b) = \left( -\frac{1}{2}, \frac{1}{2} \right) \times \left( -\frac{b}{2}, \frac{b}{2} \right).$$

We have  $\mathcal{R}(1, b) \subset S_b$  and  $|\mathcal{R}(1, b)| = b$ . If  $b = 1/\sqrt{k^2 - 1}$ , we have  $\lambda_1(\mathcal{R}(1, b)) = k^2 \pi^2$ . Furthermore, we know that  $\mathcal{R}(1, b)$  is not minimal, since the normal derivative of a first eigenfunction is not constant on the free boundary. We conclude that  $b_k^{FK} < 1/\sqrt{k^2 - 1}$ .

We can now prove the following improvement of Proposition 2.18.

**Proposition 3.11.** If  $k$  is odd and  $b < b_k^{FK}$ , then  $\mathcal{D}_k(1, b)$  is a minimal partition of  $T(1, b)$ .

The proof follows closely the method used in [12].

**Lemma 3.12.** If  $D \subset T(1, b)$  is homeomorphic to a disk,  $\lambda_1(D) \geq J(b, b)$ .

*Proof.* Let us consider the following covering of the two dimensional torus  $T(1, b)$  by the plane  $\mathbb{R}^2$ :

$$\begin{aligned} \Pi_\infty : \quad \mathbb{R}^2 &\rightarrow T(1, b) \\ (x, y) &\mapsto (x \bmod 1, y \bmod b). \end{aligned}$$

Let  $D_0$  be one of the connected components of  $\Pi_\infty^{-1}(D)$ . It is homeomorphic to a disk, and  $|D_0| = |D| \leq b$ . Furthermore, since  $T(1, b)$  is of width  $b$ , the total length of the set  $\{y; (x_0, y) \in D_0\}$  is smaller than  $b$  for any  $x_0 \in \mathbb{R}$ . Let us call  $D_0^S$  the Steiner symmetrization of  $D_0$  with respect to the axis  $\{(x, y); y = 0\}$ . It has the same area as  $D_0$ , and it is contained in  $S_b$  according to the geometrical property mentioned above. Since Steiner symmetrization does not increase the first eigenvalue, we obtain

$$J(b, b) \leq \lambda_1(D_0^S) \leq \lambda_1(D_0) = \lambda_1(D). \quad \square$$

Let us now fix an odd integer  $k$ . Let us assume that  $b < b_k^{FK}$ , and let us consider a minimal  $k$ -partition  $\mathcal{D}$  of  $T(1, b)$ . If  $\mathcal{D}$  had at least one domain homeomorphic to a disk, we would have, according to Lemma 3.12,

$$\Lambda_k(\mathcal{D}) \geq J(b, b) > k^2\pi^2.$$

On the other hand,  $\Lambda_k(\mathcal{D}(1, b)) = k^2\pi^2$ , contradicting the minimality of  $\mathcal{D}$ . We conclude that  $\mathcal{D}$  has no domain homeomorphic to a disk.

As shown in [12], this implies that  $N(\mathcal{D})$  contains no singular points, and consists in  $k$  simple curves in the same homotopy class. This implies in turn that the lifted partition  $\Pi_4^{-1}(\mathcal{D})$  is a  $2k$ -partition of  $T(2, 2b)$ , more precisely that  $\Pi_4^{-1}(D_i)$  has two distinct connected components for each  $i \in \{1, \dots, k\}$ , which are exchanged by the map  $\sigma$ . Consequently,  $\Pi_4^{-1}(\mathcal{D})$  is a pair symmetric partition, in the sense of Definition 3.7. According to Proposition 3.9, we obtain

$$\lambda_k^{\sigma, a} \leq \Lambda_{2k}(\Pi_4^{-1}(\mathcal{D})) = \Lambda_k(\mathcal{D}).$$

A direct computation, using the fact that  $b < b_k^{FK} < 1/\sqrt{k^2 - 1}$ , shows that  $\lambda_k^{\sigma, a} = k^2\pi^2$ . Therefore  $\mathcal{D}_k(1, b)$  is a minimal partition of  $T(1, b)$ .

Let us make some additional remarks. Instead of a the fourfold covering  $\Pi_4 : T(2, 2b) \rightarrow T(1, b)$ , we can define the twofold covering  $\Pi_2 : T(2, b) \rightarrow T(1, b)$ , equipped with the map  $\sigma : (x, y) \mapsto (x+1 \bmod 2, y)$ . In same manner as before, we can define for this covering lifted partitions, symmetric and antisymmetric functions, antisymmetric eigenvalues, and pair symmetric partitions. In the rest of this section, these terms will be understood with respect to the covering  $\Pi_2 : T(2, b) \rightarrow T(1, b)$ . Proposition 3.9 also holds in that case. We then have the following conditional result.

**Proposition 3.13.** *If  $b < 2/\sqrt{k^2 - 1}$ , and if  $\mathcal{D}$  is a  $k$ -partition of  $T(1, b)$  such that  $\Pi_2^{-1}(\mathcal{D})$  is a  $2k$ -partition of  $T(1, b)$ , then*

$$\lambda_k^{\sigma, a} \leq \Lambda_{2k}(\Pi_2^{-1}(\mathcal{D})) = \Lambda_k(\mathcal{D}).$$

*Proof.* Since we have assumed that  $\Pi_2^{-1}(\mathcal{D})$  is a  $2k$ -partition, the pullback  $\Pi_2^{-1}(D_i)$  of a domain of  $\mathcal{D}$  has two connected components, and the map  $\sigma$  exchanges them. This implies that  $\Pi_2^{-1}(\mathcal{D})$  is a pair symmetric partition, and the result follows from Proposition 3.9.  $\square$

A direct computation shows that if  $k$  is odd and if  $b < 2/\sqrt{k^2 - 1}$ ,  $\lambda_k^{\sigma, a} = k^2\pi^2$ . If we were able to prove that a minimal  $k$ -partition of  $T(1, b)$  can be lifted to a  $2k$ -partition of  $T(2, b)$  when  $b < 2/\sqrt{k^2 - 1}$ , we would obtain  $b_k \geq 2/\sqrt{k^2 - 1}$ . However, this is not obvious, even assuming that the boundary set of the partition does not contain singular points (see [12]).

We have on the other hand the following result:

**Proposition 3.14.** *If  $\mathcal{D}$  is a nodal  $2k$ -partition associated with an antisymmetric eigenvalue, and if  $\mathcal{D}$  has minimal energy among pair symmetric partitions, then  $\Lambda_{2k}(\mathcal{D}) = \lambda_k^{\sigma, a}$ .*

The proof is outlined in [14, Prop. 6.3] in the case of a double covering of the sphere. It consists in reproducing the arguments in the proof of [13, Th. 1.17], while preserving the antisymmetry. Proposition 3.14 has the following consequence.

**Corollary 3.15.** *If  $b > 2/\sqrt{k^2 - 1}$ , the partition  $\mathcal{D}_k(1, b)$  is not minimal.*

*Proof.* We know that  $\mathcal{D}_{2k}(2, b)$  is the partition lifted from  $\mathcal{D}_k(1, b)$ , and is also the nodal partition of  $T(2, b)$  associated with the antisymmetric eigenfunction  $(x, y) \mapsto \sin(k\pi x)$ . Let us assume by contradiction that  $\mathcal{D}_k(1, b)$  is minimal. This would imply that  $\mathcal{D}_{2k}(2, b)$  is minimal among pair symmetric  $2k$ -partitions, and thus, according to Proposition 3.14, that  $k^2\pi^2 = \Lambda_{2k}(\mathcal{D}_{2k}(2, b)) = \lambda_k^{\sigma, a}$ . A direct computation shows that the condition  $b > 2/\sqrt{k^2 - 1}$  implies  $\lambda_k^{\sigma, a} > k^2\pi^2$ . We have reached a contradiction.  $\square$



Corollary 3.15 implies that  $b_k \leq 2/\sqrt{k^2-1}$  for all odd integers  $k$ . The partial result stated in Proposition 3.13, and the numerical results presented in Section 5, suggest the following conjecture.

**Conjecture 3.16.** *For any odd integer  $k \geq 3$ ,*

$$b_k = \frac{2}{\sqrt{k^2-1}}.$$

### 3.3 Minimal partitions at the transition value

According to Proposition 2.17,  $b_4 = 1/2$ . More precisely, the minimal 4-partitions of  $T(1, 1/2)$  are simply the nodal partitions, associated with the eigenvalue  $16\pi^2$ , which have 4 domains. Since the eigenvalue  $16\pi^2$  has multiplicity 4, we obtain in this way minimal partitions which are not merely a translation of  $\mathcal{D}_4(1, 1/2)$ . Figure 1 shows an example whose boundary contains singular points. We

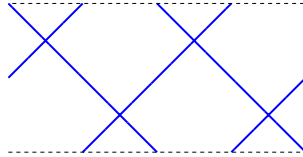


Figure 1: A nodal 4-partition of  $T(1, 1/2)$  (associated with  $\sin(4\pi x) + \sin(4\pi y)$ .)

conjecture that this partition is a starting point for the appearance of non-nodal 4-partitions of  $T(1, b)$  when  $b = 1/2 + \varepsilon$ , with  $0 < \varepsilon \ll 1$ . More precisely, we conjecture that each singular point of order four splits into two singular points of order three (see Figures 7(d), 7(e), and 7(f) in Subsection 5.2 for numerical simulations). This deformation mechanism was already suggested by the numerical simulation in [1, 7.], where the authors consider rectangles rather than tori, and where the singular point appears on the boundary of the rectangle.

In the case of an odd  $k$ , we are not able to give explicit examples of minimal  $k$ -partitions which are not translations of  $\mathcal{D}_k(1, b)$ . We can however construct candidates that are minimal providing Conjecture 3.16 is true. For instance, for  $k = 3$ , Conjecture 3.16 implies  $\mathfrak{L}_3(T(1, 1/\sqrt{2})) = 9\pi^2$ , which means that any 3-partition with energy  $9\pi^2$  is minimal. We now look for antisymmetric eigenfunctions on  $T(2, 1/\sqrt{2})$ , associated with the eigenvalue  $9\pi^2$ , which have 6 nodal domains. After projecting the corresponding nodal partition on  $T(1, 1/\sqrt{2})$ , we obtain a 3-partition with energy  $9\pi^2$ . Figure 2 shows an example, in which the boundary contains singular points.

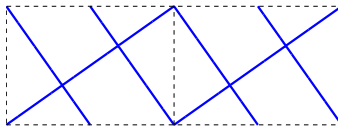


Figure 2: A nodal partition of  $T(1, 1/\sqrt{2})$  (associated with  $\cos(3\pi x) - \cos(\pi x) \cos(2\sqrt{2}\pi y) + \sin(\pi x) \sin(2\sqrt{2}\pi y)$ ).

In the same way, we can obtain 5-partitions of the torus  $T(1, 1/\sqrt{6})$  from nodal 10-partitions of the torus  $T(2, 1/\sqrt{6})$ . Figure 3 shows a nodal 10-partition associated with the eigenvalue  $25\pi^2$ . It gives, after projection, a 5-partition of  $T(1, 1/\sqrt{6})$ , which is minimal if Conjecture 3.16 is true. For  $k = 3$  and  $k = 5$ , the partitions obtained numerically for  $b = 2/\sqrt{k^2-1} + \varepsilon$  seem close to these examples, with each singular point of order 4 splitted into a pair of singular points of order 3 (see Figures 5(d) and 9(c)).

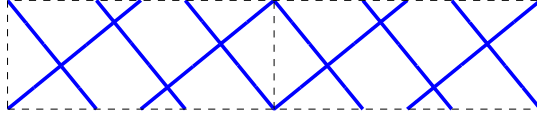


Figure 3: A nodal partition of  $T(1, 1/\sqrt{6})$  (associated with  $\cos(5\pi x) + \sin(\pi x) \sin(2\pi\sqrt{6}y) - \cos(\pi x) \cos(2\pi\sqrt{6}y)$ ).

## 4 Numerical method

### 4.1 Relaxed formulation

Our numerical investigations are based on the method introduced by B. Bourdin, D. Bucur, and É. Oudet in [2].

Let us start with relaxing the original minimization problem. In this subsection, we still denote by  $M$  a two-dimensional Riemannian manifold without boundary.

The first step of the relaxation consists in modifying the energy. We define a family of energies associated with a partition.

**Definition 4.1.** For any  $1 \leq p < \infty$  and  $k \in \mathbb{N}^*$ , if  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  is a  $k$ -partition, we define

$$\Lambda_{k,p}(\mathcal{D}) = \left( \frac{1}{k} \sum_{i=1}^k \lambda_1(D_i)^p \right)^{\frac{1}{p}}.$$

Then, we define  $\mathfrak{L}_{k,p}(M) = \inf\{\Lambda_{k,p}(\mathcal{D}); \mathcal{D} \in \mathfrak{P}_k\}$ .

In [2], the authors study the minimization of the sum of the lowest eigenvalues, which is to say they look for  $\mathfrak{L}_{k,1}(M)$ . We have modified their algorithm to look for  $\mathfrak{L}_{k,p}(M)$  for any  $1 \leq p < \infty$ . To approach the minimal energy  $\mathfrak{L}_k(M)$ , corresponding to the case  $p = \infty$ , we look for the minimal energy  $\mathfrak{L}_{k,p}(M)$  with  $1 < p < \infty$  large enough. This procedure is justified by the following result, proved in [13].

**Proposition 4.2.** Let  $k \in \mathbb{N}^*$ . The minimal energy  $\mathfrak{L}_{k,p}(M)$  is increasing with respect to  $p$  and  $\lim_{p \rightarrow +\infty} \mathfrak{L}_{k,p}(M) = \mathfrak{L}_k(M)$ .

The second step of the relaxation consists in replacing the optimal partition problem, concerning  $k$ -partitions of  $M$ , by a problem concerning  $k$ -tuples of functions, as done in [2].

**Definition 4.3.** For  $k \in \mathbb{N}^*$ , the set of admissible  $k$ -tuples is given by

$$\mathfrak{F}_k(M) = \left\{ \mathcal{F} = (f_1, \dots, f_k); f_i : M \rightarrow [0, 1] \text{ measurable, } \sum_{i=1}^k f_i(x) = 1 \text{ a.e.} \right\}.$$

We associate with such a  $k$ -tuple an energy that is a relaxed version of  $\Lambda_{k,p}$ .

**Definition 4.4.** Let  $f : M \rightarrow [0, 1]$  be a measurable function and  $\varepsilon$  a positive constant. The quantity  $\lambda_1(f, \varepsilon)$  is defined as the first eigenvalue of the operator  $-\Delta + \varepsilon^{-1}(1 - f)$ . To be more precise,  $\lambda_1(f, \varepsilon)$  is defined, using a Rayleigh quotient, as

$$\lambda_1(f, \varepsilon) = \inf_{u \in H_0^1(M) \setminus \{0\}} \frac{\int_M (|\nabla u|^2 + \frac{1}{\varepsilon}(1 - f)u^2) dx dy}{\int_M |u|^2 dx dy}.$$

**Definition 4.5.** Let  $\mathcal{F} = (f_1, \dots, f_k)$  be an admissible  $k$ -tuple, and let us choose  $p \geq 1$  and  $\varepsilon > 0$ . We define the energy of  $\mathcal{F}$  as

$$E_{k,p,\varepsilon}(\mathcal{F}) = \left( \frac{1}{k} \sum_{i=1}^k \lambda_1(f_i, \varepsilon)^p \right)^{\frac{1}{p}}. \quad (4.1)$$

**Definition 4.6.** Let  $k \in \mathbb{N}^*$ ,  $p \geq 1$ , and  $\varepsilon > 0$ . We define the minimal energy as

$$\mathfrak{E}_{k,p,\varepsilon}(M) = \inf_{\mathcal{F} \in \tilde{\mathfrak{F}}_k(M)} E_{k,p,\varepsilon}(\mathcal{F}). \quad (4.2)$$

The parameter  $\varepsilon$  can be seen as penalizing the overlapping of the supports of the functions  $f_i$ . This idea is supported by the following result, proved in [2] in the case  $p = 1$ .

**Proposition 4.7.** Let  $\mathcal{F} = (f_1, \dots, f_k)$  be a  $k$ -tuple that solves the optimization problem (4.2). Then, for all  $i \in \{1, \dots, k\}$ ,  $f_i(x) = 0$  or  $f_i(x) = 1$  for almost every  $x$  in  $M$ .

This suggests the following procedure for determining numerically a minimal  $k$ -partition (in the sense of Definition 2.1), for a given domain  $M$  and integer  $k$ . We choose some  $p \gg 1$  and  $0 < \varepsilon \ll 1$  and compute an approximation of a minimal admissible  $k$ -tuple  $\mathcal{F} = (f_1, \dots, f_k)$  for the optimization problem (4.2). We then consider the sets  $D_i = \{x \in M; f_i(x) = 1\}$  as domains of an approximate minimal partition.

To make this procedure effective, we need to specify how to represent the functions  $f_i$  numerically and how to construct the domains  $D_i$  from this representation. This question is addressed in Subsections 4.2 and 4.3.

## 4.2 Discretization

To perform numerical optimization, we replace Problem (4.2) by a discrete version. We approximate the torus  $T(1, b)$  by a  $m \times n$  grid of points in the rectangle  $[0, 1] \times [0, b]$  (with a constant step in each direction). Let us note  $N = mn$  the total number of points. An admissible  $k$ -tuple of functions is represented, in this setting, by a  $N \times k$  matrix  $\Phi$ , whose columns  $\phi_1, \dots, \phi_k$  contain the values of the functions  $f_1, \dots, f_k$  at each point of the grid. For  $1 \leq i \leq k$  and  $1 \leq I \leq N$ , we will sometime denote the element  $[\Phi]_{I,i}$  by  $[\phi_i]_I$  if we want to focus on the vector  $\phi_i$ . We impose the following constraints on  $\Phi$ :

- $\forall 1 \leq i \leq k, \forall 1 \leq I \leq N, 0 \leq [\Phi]_{I,i}$ ;
- $\forall 1 \leq I \leq N, \sum_{i=1}^k [\Phi]_{I,i} = 1$ .

Let us give a geometric formulation of these conditions. For  $k \geq 1$ , we define the simplex  $\mathcal{S}_k$  in  $\mathbb{R}^k$  by

$$\mathcal{S}_k = \left\{ (x_1, \dots, x_k); 0 \leq x_i, \sum_{j=0}^k x_j = 1 \right\}.$$

We can express the constraints as  $([\Phi]_{I,1}, \dots, [\Phi]_{I,k}) \in \mathcal{S}_k$  for  $1 \leq I \leq N$ .

Let us now define a discrete analogue of the energy (4.1) that we will try to minimize in this discretized problem. With each column  $\phi \in \{\phi_j; 1 \leq j \leq k\}$ , we associate  $\tilde{\lambda}_1(\phi, \varepsilon)$  the lowest eigenvalue of the matrix

$$L(\phi) = L + \frac{1}{\varepsilon} (Id_N - \text{Diag}(\phi)),$$

where  $L$  is the finite difference approximation of  $-\Delta$  on the  $m \times n$  grid with periodic boundary conditions.

Finally, we replace the energy  $E(p, k, \varepsilon, \mathcal{F})$  by its discrete version:

$$\tilde{E}(p, k, \varepsilon, \Phi) = \left( \frac{1}{k} \sum_{i=1}^k \tilde{\lambda}_1(\phi_i, \varepsilon)^p \right)^{\frac{1}{p}}. \quad (4.3)$$

We now want to minimize  $\tilde{E}(p, k, \varepsilon, \Phi)$  with respect to the  $Nk$  variables  $[\Phi]_{i,I}$ , under the constraint  $([\Phi]_{I,1}, \dots, [\Phi]_{I,k}) \in \mathcal{S}_k$  for  $1 \leq I \leq N$ . As seen in [2], the derivatives of  $\tilde{E}(p, k, \varepsilon, \Phi)$  with respect to  $[\Phi]_{I,I}$  can be expressed with the help of eigenvalues and eigenvectors of the matrices  $L(\phi_i)$ . We can therefore compute those derivatives and apply a projected gradient algorithm to the problem.

To make the convergence easier, we adopt the multi-step procedure described in [2], which seems empirically very effective<sup>1</sup>. We begin with applying the projected gradient algorithm on a very small grid, starting from a randomly generated  $\Phi$ . We do not use the orthogonal projection operator but the so-called *simple projection operator*  $S_k$ , which is defined below.

<sup>1</sup>We thank Édouard Oudet for giving us detailed explanations on this point.

**Definition 4.8.** Let  $k \in \mathbb{N}^*$ . The simple projection operator on the simplex  $\mathcal{S}_k$  is defined by

$$[S_k x]_i = \frac{|x_i|}{\sum_{j=1}^k |x_j|},$$

for each  $x \in \mathbb{R}^k$  and  $1 \leq i \leq k$ .

Once this step has converged, we double the number of points of the grid in each direction and extend the optimal  $\Phi$  to the new grid by linear interpolation. We reiterate the same procedure, using the interpolated  $\Phi$  as a starting point. Once we have repeated this step a certain number of times, we perform one last iteration using the orthogonal projection operator.

In practice, coefficients in the final  $\Phi$  are either 0 or 1. There is, as far as we know, no proof of this fact in this discrete version of the problem, but it is consistent with Proposition 4.7. Taking it for granted,  $\Phi$  gives us a partition of the points of the grid into subsets  $\tilde{D}_i$ , with  $1 \leq i \leq k$ . From these subsets, we construct, in Subsection 4.3, a  $k$ -partition  $(D_i)_{1 \leq i \leq k}$  of  $T(1, b)$  and compute its energy.

### 4.3 Construction of a $k$ -partition

The implementation of the iterative gradient algorithm gives us a discrete  $k$ -partition  $(\tilde{D}_i)_{1 \leq i \leq k}$  and a numerical relaxed energy  $\tilde{M}(p, k, \varepsilon, \Phi)$  (see (4.3)). To approximate  $\mathfrak{L}_{k,p}(T(1, b))$ , we have two further steps to perform:

- construct a  $k$ -partition  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  of  $T(1, b)$  from  $(\tilde{D}_i)_{1 \leq i \leq k}$ ;
- compute the associated energy  $\Lambda_{k,p}(\mathcal{D})$ .

Since we want to keep the same size of discretization, and since the torus is represented with a  $m \times n$  grid, we construct the domains  $D_i$  of the new partition as the union of elementary cells, of the form

$$\left[ \frac{r-1}{m}, \frac{p}{m} \right] \times \left[ \frac{s-1}{n} b, \frac{q}{n} b \right],$$

with  $1 \leq r \leq m$  and  $1 \leq s \leq n$ . Then we compute numerically the first eigenvalue of the Dirichlet Laplacian on each domain  $D_i$  by a finite difference method. We can then, for any  $1 \leq p \leq \infty$ , compute an approximation of the energy  $\Lambda_{k,p}(\mathcal{D})$  without relaxation.

In the rest of this section, we use a convenient graphical representation of the data in the matrix  $\Phi$ . Let us explain it on an example. We choose  $m = 7$ ,  $n = 6$  (therefore  $N = 42$ ) and  $k = 3$ . A matrix  $\Phi$

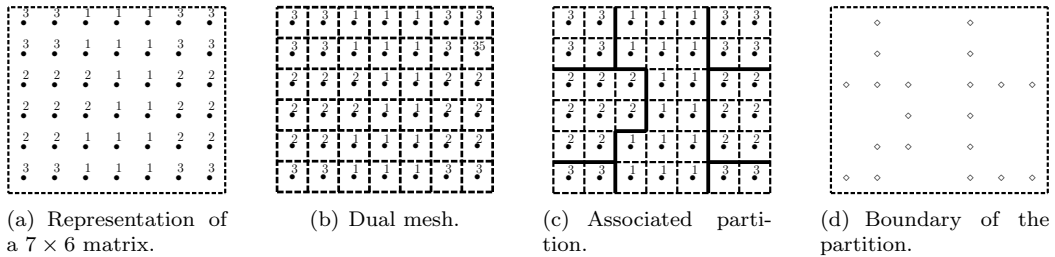


Figure 4: Partition obtained from the matrix  $\Phi$ .

of the type produced by the optimization algorithm is represented in the following way: for  $1 \leq I \leq N$  and  $1 \leq i \leq k$ , if  $[\phi_i]_I = 1$ , we label the point indexed by  $I$  by the integer  $i$  (see Figure 4(a)).

We can imagine each point of the grid to be the center of a rectangular cell, which is of the form

$$\left[ \frac{r-1/2}{m}, \frac{r+1/2}{m} \right] \times \left[ \frac{s-1/2}{n} b, \frac{s+1/2}{n} b \right],$$

with  $0 \leq r \leq m-1$  and  $0 \leq s \leq n-1$ . We obtain a new mesh (which is in some sense dual to the first one), represented on Figure 4(b). The torus can, in our example, be divided into three domains

$D_1, D_2, D_3$ : for each  $i \in \{1, 2, 3\}$ , the domain  $D_i$  is the union of all the cells whose center is labeled by the integer  $i$ . The partition,  $\mathcal{D} = \{D_1, D_2, D_3\}$  is strong in the sense of Definition 2.1. It is represented on Figure 4(c).

We can describe the boundary of this partition. It is a union of some edges of the cells. More precisely, an edge is in the boundary if, and only if, it separates two cells that belong to different domains. To construct the boundary, we consider each cell successively. One of its edges is added to the boundary if the cell on the other side is labeled by a different integer. Actually, since all the cells are considered in turn, it is enough to look at the right and upper edges (for instance) of each cell. Let us finally note that by translating the boundary by the vector  $\mathbf{v} = (-1/2m, -b/2n)$ , we can use the points of the original grid to represent the boundary. In our example, the boundary is represented on Figure 4(d).

To find an approximation of  $\Lambda_k(\mathcal{D})$  we compute

$$\tilde{\Lambda}_k(\mathcal{D}) = \max_{1 \leq i \leq k} \tilde{\lambda}_1(D_i),$$

where  $\tilde{\lambda}_1(D_i)$  is the first eigenvalue of the finite differences approximation of the Dirichlet Laplacian on  $D_i$ .

#### 4.4 Implementation of the method

We recall that we perform a numerical optimization by applying a projected gradient algorithm at several steps, quadrupling the number of discretization points after each step. In the following, we denote by  $J$  the total number of steps. To be complete, we must also specify which criterion is used to terminate the projected gradient algorithm at each step. For  $1 \leq j \leq J$ , we denote by  $\Phi^{(j,r)}$  the matrix  $\Phi$ , at the step  $j$ , after  $t$  iterations of the projected gradient algorithm. In this notation, the projected gradient algorithm stops when one of the two following conditions is satisfied:

- $\frac{1}{kN} \sum_{I,i} \left| \Phi_{I,i}^{(j,t+1)} - \Phi_{I,i}^{(j,t)} \right| < \delta$ , where  $\delta$  is a fixed threshold;
- $t \geq T$ , where  $T$  is the maximal number of iterations, chosen beforehand.

We insist on the fact that this optimization algorithm is not necessarily successful. All the other parameters being equal, it could converge rapidly to a good candidate for some initial data, whereas for others it would terminate without reaching convergence. To overcome this problem, we have made several simulations, starting from different random initial data. We have also tested some particular initial data, for example the results of previous optimizations for different values of  $b$ . In any case, we have not accepted a result without precaution. We have checked whether the algorithm has converged, and have compared the energy of the resulting partition with the results of other runs for the same values of the parameters.

## 5 Numerical results

Let us now present the results obtained with the help of the above algorithm. We have used the following values for the parameters:  $p = 5, \varepsilon = 10^{-4}, J = 5, n^{(1)} = 16, m^{(1)} = 16$ .

Since  $J = 5$ , we have refined four times the initial grid to obtain finally a  $256 \times 256$  grid. We label the domains of the partition  $\mathcal{D} = (D_i)_{1 \leq i \leq k}$  so that  $\Lambda_k(\mathcal{D}) = \lambda_1(D_1) \geq \lambda_1(D_2) \geq \dots \geq \lambda_1(D_k)$ .

### 5.1 3-partitions of the torus $T(1, b)$

We know from Proposition 3.3, that  $b_3 \geq 1/3$ . We have conjectured in Subsection 3.2 that  $b_3 = 1/\sqrt{2} \simeq 0.707$ . If this is true,  $D_3(1, b)$  should be minimal for  $1/3 < b < 1/\sqrt{2}$ . To test this, we have applied the algorithm for  $b \in \{j/50, j = 17, \dots, 34\}$ . As was expected, the lowest energy in these cases is obtained for partitions similar to the one on Figure 5(a).

Let us now study what happens when  $b$  is close to  $1/\sqrt{2}$ . Figure 5 shows our best result for different values of  $b$ . When  $b$  is greater than 0.71, the minimal partition seems to be a tiling of the torus by three isometric domains. These domains are roughly hexagonal, and close to the rectangles appearing in the

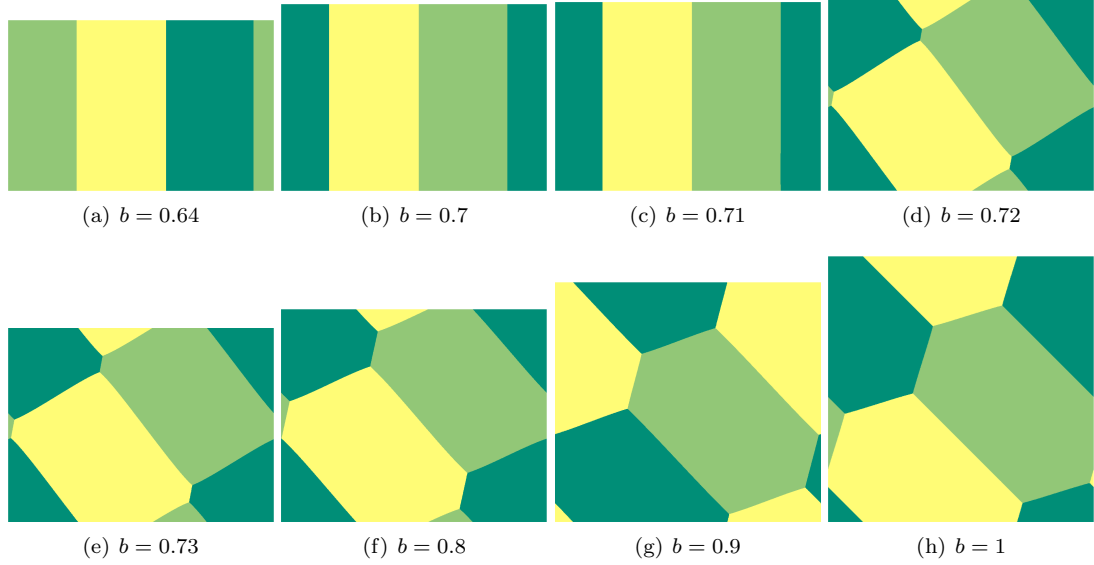


Figure 5: 3-partitions for some values of  $b$ .

partition of Figure 2. For brevity, we will say in the following that a partition with hexagonal domains is *hexagonal*.

For  $b$  close to  $1/\sqrt{2}$ , the final result of the optimization algorithm appears to be very sensitive to the initial data. As a consequence, the partitions of Figures 5(d) and 5(e) were not actually obtained by starting from random initial data. Rather, we ran the algorithm, starting from a random matrix, in the case  $b = 0.81$ , where it produced an hexagonal partition similar to those of Figure 5. We then used as a starting point of the algorithm the matrix obtained after two steps in the case  $b = 0.81$ . Of course, we compared our final results with those of other runs starting from random initial data, and found that they always had a lower energy.

For larger  $b$ 's, up to  $b = 1$ , the best candidates produced by the algorithm are still hexagonal partitions, as seen on Figures 5(f), 5(g), and 5(h).

For each  $b$ , the energy of the best numerical candidate to be a minimal partition of  $T(1, b)$  is an upper bound for  $\mathfrak{L}_3(T(1, b))$ . This upper bound is plotted on Figure 6 as a function of  $b$ , and compared with  $9\pi^2 = \Lambda_3(\mathcal{D}_3(1, b))$ . We obtain a significant improvement for large  $b$ . The third upper bound, represented by a solid line, will be discussed in Section 6.

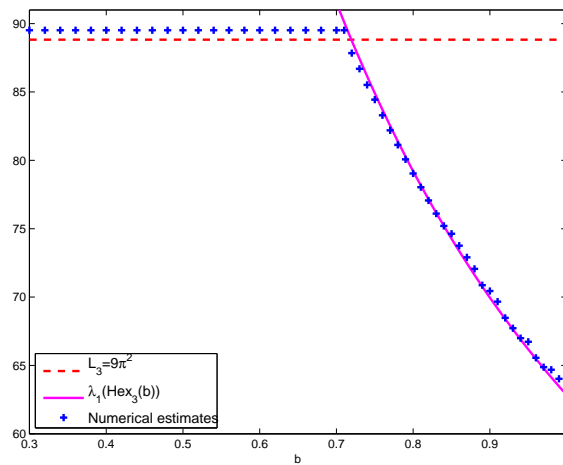


Figure 6: Upper bounds of  $\mathfrak{L}_3(T(1, b))$  for  $b \in \{j/100; j = 30, \dots, 100\}$ .

## 5.2 4-partitions of the torus $T(1, b)$

We know from Proposition 3.3 that  $b_4 = 1/2$ . We are interested in the nature of minimal partitions for  $b$  close to  $b_4$ . Figure 7 shows the best candidates obtained from the algorithm.

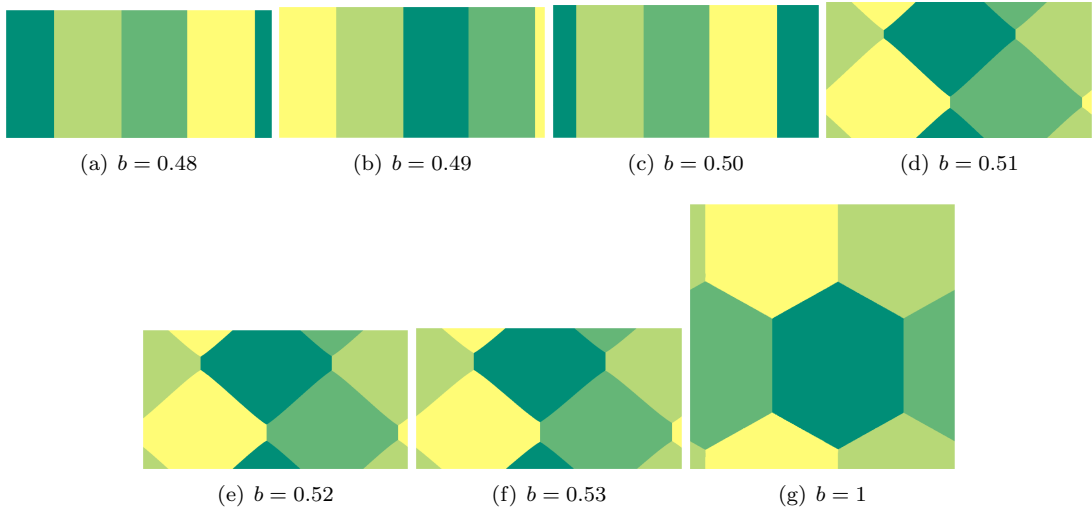


Figure 7: 4-partition for  $b \in \{j/100; j = 48, \dots, 53\}$  and  $b = 1$ .

As in the case  $k = 3$ , we note the apparition of hexagonal partitions, shown on Figures 7(d), 7(e), and 7(f). The hexagonal domains of the partition shown on Figure 7(d) seem close to the square domains of the nodal 4-partition shown on Figure 1. This suggests that the partition of  $T(1, 1/2)$  into four squares, shown on Figure 1, is the starting point for the apparition of non-nodal 4-partitions of  $T(1, b)$ , when  $b$  becomes greater than  $1/2$ .

For larger values of  $b$ , up to  $b = 1$ , the minimal partitions are apparently still hexagonal. Figure 7(g) shows for instance the best candidate for a minimal 4-partition of  $T(1, 1)$ . The energy of the best candidates gives us an upper bound for  $\mathfrak{L}_4(T(1, b))$ , represented on Figure 8 as a function of  $b$ .

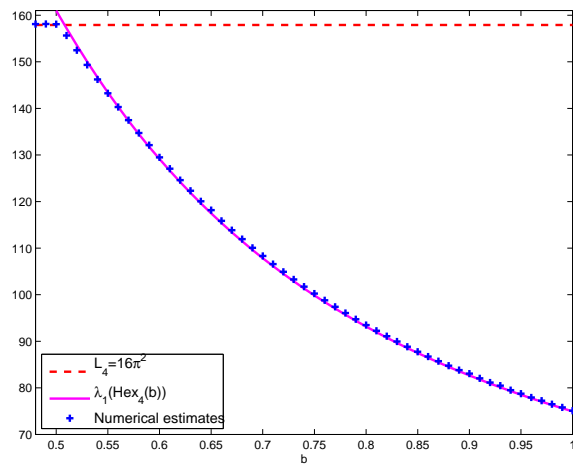


Figure 8: Upper bounds of  $\mathfrak{L}_4(T(1, b))$  for  $b \in \{j/100; j = 48, \dots, 100\}$ .

### 5.3 5-partitions of the torus $T(1, b)$

We have conjectured that  $b_5 = 1/\sqrt{6} \simeq 0.408$ . The first images of Figure 9 presents the best candidates obtained numerically when  $b$  is close to this conjectured  $b_5$ . They seem to support the conjecture. Furthermore, for  $b$  larger than  $1/\sqrt{6}$ , minimal partitions seem to be hexagonal, with domains close to the rectangles appearing in the partition on Figure 3.

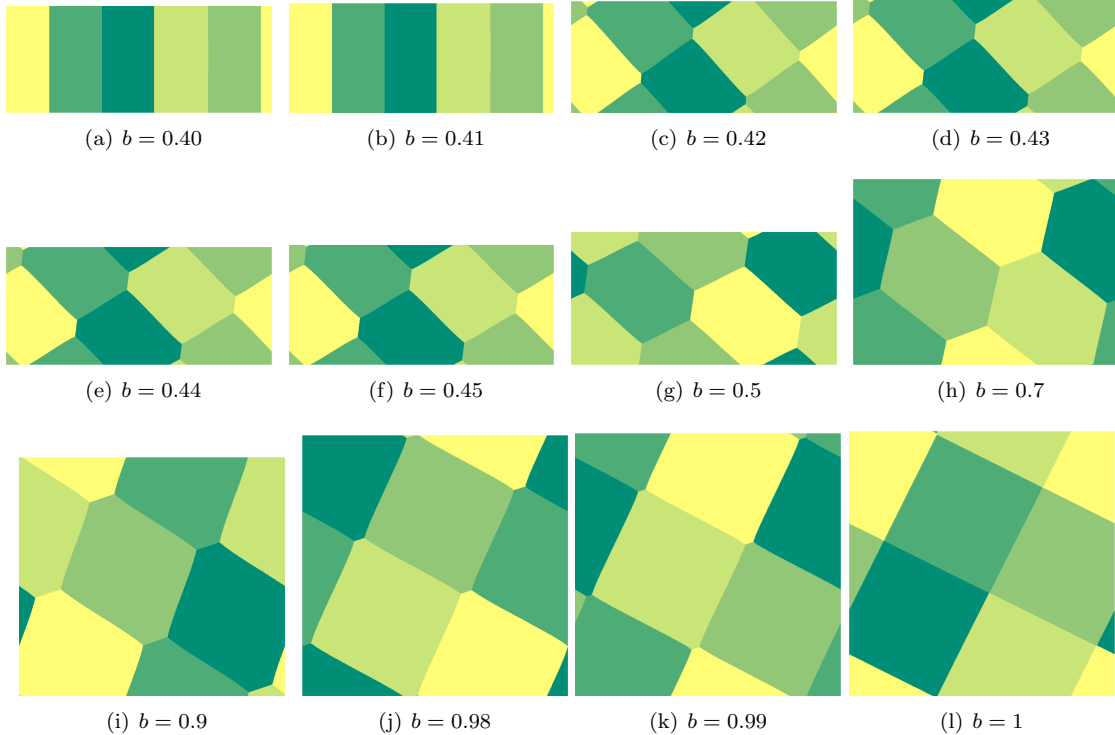


Figure 9: 5-partitions for some values of  $b$ .

For  $b$  between  $1/\sqrt{6}$  and 1, minimal partitions seem to be hexagonal. However, pairs of singular points in the boundary of these hexagonal partitions seem to merge when  $b$  approaches 1, and for  $b = 1$ , the best candidate produced by the algorithm is a partition of  $T(1, 1)$  into five equal squares. This process is shown on Figures 9(j), 9(k), and 9(l).

We also obtain an upper bound for  $\mathcal{L}_5(T(1, b))$  as a function of  $b$ , plotted on Figure 10.

## 6 Examples of partitions

### 6.1 Tilings of $T(1, b)$

The results of Section 4 suggest that, at least for some values of  $k$  and  $b$ , the domains of minimal  $k$ -partitions of  $T(1, b)$  are isometric polygons. In fact, except when  $k = 5$  and  $b = 1$ , these polygons seem to be hexagons. On the other hand, according to Theorem 2.6, any minimal partition satisfies the equal angle meeting property.

This suggests the existence of partitions with a low energy, possibly minimal, that are tilings of the torus  $T(1, b)$  by  $k$  identical hexagons satisfying the equal angle meeting property. Let us note that in this case, the equal angle meeting property is satisfied if, and only if, the interior angle at each vertex of the hexagon is  $2\pi/3$ .

Finding these tilings is a purely geometrical problem. The task of enumerating, for given  $b$  and  $k$ , all admissible tilings seems quite complicated, and we have not been able to complete it. We can however construct families of tilings, depending on  $k$  and  $b$ , that seem close to the partitions produced by the algorithm. This is the object of Appendix A, where we prove in particular the following result.



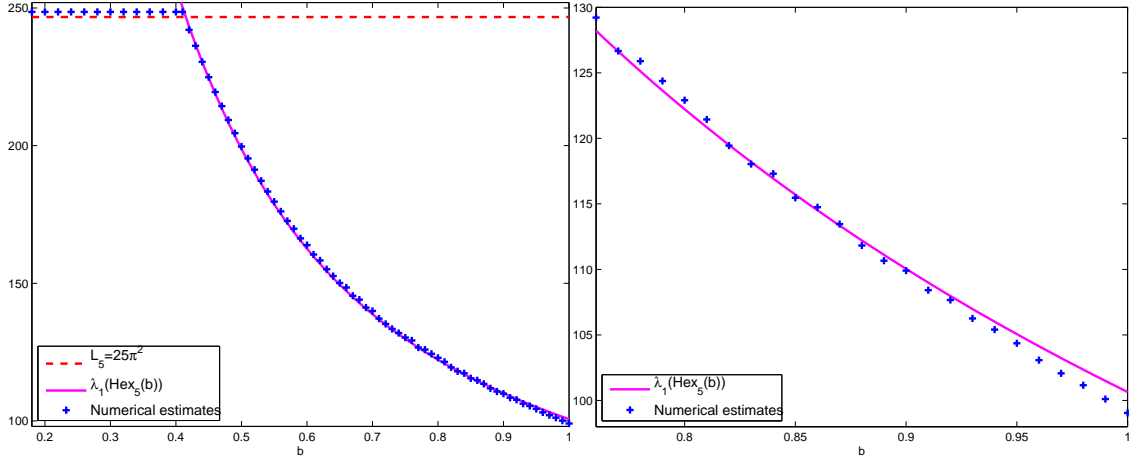


Figure 10: Upper bounds of  $\mathfrak{L}_5(T(1, b))$  for  $b \in \{j/100; j = 18, \dots, 100\}$ .

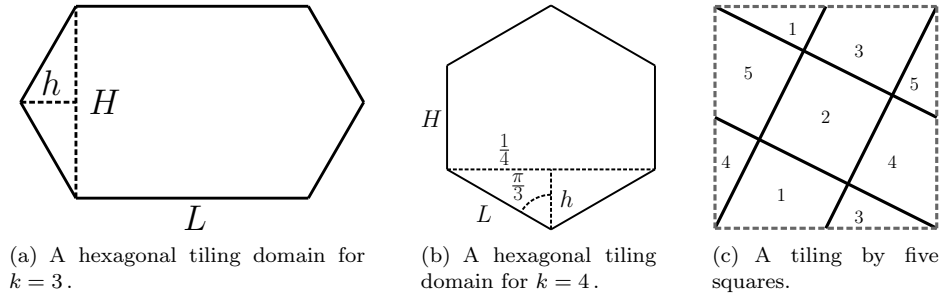


Figure 11: The case of  $T(1, 1)$ .

**Proposition 6.1.** *For  $k \in \{3, 4, 5\}$ , there exists  $b_k^c \in (0, 1)$  such that, for any  $b \in (b_k^c, 1]$ , there exists a tiling of  $T(1, b)$  by  $k$  hexagons that satisfies the equal angle meeting property. We denote by  $Hex_k(b)$  the tiling domain.*

We have computed  $\lambda_1(Hex_k(b))$  with high precision using the finite elements library MELINA (see [15]). We have obtained in this way the upper bounds for  $\mathfrak{L}_k(T(1, b))$  represented by solid lines on Figures 6, 8, and 10.

For  $b \in (b_k^c, 1]$  and  $k \in \{3, 4, 5\}$ , we can therefore bound  $\mathfrak{L}_k(T(1, b))$  from above either by the energy of  $\mathcal{D}_k(1, b)$  or by the energy of our hexagonal tiling, as summarized in the following statement.

**Proposition 6.2.** *For  $k \in \{3, 4, 5\}$  and  $b \in (b_k^c, 1]$ ,  $\mathfrak{L}_k(T(1, b)) \leq \min(k^2 \pi^2, \lambda_1(Hex_k(b)))$ .*

## 6.2 The case $b = 1$

Let us consider the special case  $b = 1$ , in which the tiling domains can be described simply, and let us formulate the corresponding conjectures on  $\mathfrak{L}_k(T(1, b))$  for  $k \in \{3, 4, 5\}$ .

### 6.2.1 3-partition

The tiling domain  $Hex_3(1)$  is an hexagon with two axes of symmetry, described on Figure 11(a), with  $H = \frac{\sqrt{2}}{3}$ ,  $h = \frac{1}{3\sqrt{6}}$ , and  $L = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{6}}$ .

**Conjecture 6.3.**  $\mathfrak{L}_3(T(1, 1)) = \lambda_3(Hex_3(1)) \simeq 62.8389$ .

### 6.2.2 4-partition

The tiling domain  $Hex_4(1)$  is also an hexagon with two axes of symmetry, described on Figure 11(b), with  $h = \frac{1}{4\sqrt{3}}$ ,  $L = \frac{1}{2\sqrt{3}}$ , and  $H = \frac{1}{2} - \frac{1}{4\sqrt{3}}$ .

**Conjecture 6.4.**  $\mathfrak{L}_4(T(1, 1)) = \lambda_4(Hex_4(1)) \simeq 74.9467$ .

### 6.2.3 5-partition

We can construct a 5-partition of  $T(1, 1)$  by identical hexagons with interior angles  $2\pi/3$ . However, numerical computations show that we obtain a lower energy when we divide  $T(1, 1)$  into five identical squares, as represented on Figure 11(c).

**Conjecture 6.5.**  $\mathfrak{L}_5(T(1, 1)) = 10\pi^2 \simeq 98.6960$ .

## 6.3 Comparison with the numerical results

One might think that a minimal  $k$ -partition of  $T(1, b)$ , with  $k \in \{3, 4, 5\}$  and  $b \in (0, 1]$ , is either  $\mathcal{D}_k(b)$  or a tiling of  $T(1, b)$  by hexagons or squares. Closer inspection of the numerical results indicates that this is not the case.

Let us for instance focus on the case  $k = 3$ , for  $b$  close to  $1/\sqrt{2}$ . Figure 6 shows that there exists a  $b^* > 1/\sqrt{2}$  such that, for  $b \in [\frac{1}{2}, b^*)$ ,  $\lambda_1(Hex_3(b)) > 9\pi^2$ . Therefore, for  $b \in [\frac{1}{2}, b^*)$ , the tiling of  $T(1, b)$  by three hexagons with straight edges that we have constructed is not minimal.

This is consistent with the idea that there is some continuity of the minimal partitions with respect to  $b$ , and with the conjecture that the projection on  $T(1, 1/\sqrt{2})$  of the nodal partition of  $T(2, 1/\sqrt{2})$  shown on Figure 2 is minimal. Indeed, if we try to deform this latest partition by splitting each singular point of order four into a pair of singular points of order three, while keeping each domain close the original rectangle, the resulting partition cannot satisfy the equal angle meeting property if all the regular parts of the boundary remain straight lines. This suggests that the boundary of the partition should be curved in the neighborhood of the singular points in such a way that the we keep the equal angle meeting property. This seems to be the case for the partitions represented on Figure 5.

The same phenomenon occurs for 4-partitions of  $T(1, b)$  when  $b$  is close to  $1/2$ , as seen on Figures 8 and 7, and for 5-partitions of  $T(1, b)$  when  $b$  is close to  $1/\sqrt{6}$  or 1, as seen on Figures 10 and 9.

## 7 Conclusion

We have studied in this work the transition values  $b_k$  defined in Subsection 3.1, when  $k$  is an odd integer. We have presented partial results that lead to the conjecture

$$b_k = \frac{2}{\sqrt{k^2 - 1}}.$$

Our numerical simulations support this conjecture when  $k = 3$  and  $k = 5$ .

We have also proposed a mechanism, inspired by [1], that describes how a minimal 3-partition with singular points appears when  $b$  becomes larger than  $1/\sqrt{2}$ . For  $b = 1/\sqrt{2}$ , we have constructed examples of 3-partitions of  $T(1, b)$  with energy  $9\pi^2$  that are projections of nodal 6-partitions of  $T(2, b)$ . We have found partitions whose boundary contains singular points of order four, which can split into two singular points of order three when  $b$  increases. Our numerical simulations singled out a partition of  $T(1, b)$  into three rectangles, which seems, when  $b$  increases, to change into a 3-partition with six singular points and three isometric, hexagonal domains, with slightly curved sides. We have proposed similar deformation mechanisms when  $k = 4$  and  $k = 5$ . They also seem to be supported by our numerical simulations.

For  $k \in \{3, 4, 5\}$ , we have improved the formerly known upper bounds of  $\mathfrak{L}_k(T(1, b))$  by constructing explicit examples of partitions: hexagonal tilings of  $T(1, b)$  (see Figures 6, 8, and 10 for numerical computations of the new upper bounds). However, these tilings do not seem to be minimal partitions when  $k = 3$  and  $b$  is close to  $1/\sqrt{2}$ , when  $k = 4$  and  $b$  is close to  $1/2$ , nor when  $k = 5$  and  $b$  is close to  $1/\sqrt{6}$  or 1. Generally speaking, it seems interesting to investigate further for which values of  $k$  and  $b$  (if any) such tilings are minimal partitions.

## A Tilings of a family of tori by hexagons

The purpose of this appendix is to describe, for  $b \in (0, 1]$ , explicit examples of tilings of the torus  $T(1, b)$  by isometric hexagons that satisfy the equal angle meeting property of Definition 2.5. We show that such tilings can be described by a  $2 \times 2$  matrix with integer coefficients. We explain how to find conditions on  $b$  that allow the existence of a tiling described by a given matrix. We finally apply these results to  $k$ -partitions, with  $k \in \{3, 4, 5\}$ , in order to prove Proposition 6.1.

### A.1 Tilings of the plane

To study tilings of the torus  $T(1, b)$  by hexagons, it will be useful to consider tilings of the plane. We recall that we consider  $T(1, b)$  as the quotient  $T(1, b) = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$ , with the natural projection map

$$\begin{aligned} \Pi : \mathbb{R}^2 &\rightarrow T(1, b) \\ (x, y) &\mapsto (x \bmod 1, y \bmod b). \end{aligned}$$

We will denote by  $(\mathbf{e}_1, \mathbf{e}_2)$  the canonical basis of  $\mathbb{R}^2$  with  $\mathbf{e}_1 = (1, 0)$  and  $\mathbf{e}_2 = (0, 1)$ .

Let us consider a regular  $k$ -partition  $\mathcal{D} = \{D_1, \dots, D_k\}$  of the torus  $T(1, b)$ , such that all the  $D_i$ 's are isometric to an hexagon that we denote by  $Hex$ . Let us note that, since  $\mathcal{D}$  is strong, the area of  $Hex$  is  $b/k$  and, since  $\mathcal{D}$  satisfies the equal angle meeting property, all the interior angles of  $Hex$  are  $2\pi/3$ . Let us then consider, for any  $i \in \{1, \dots, k\}$ , the open set  $\Pi^{-1}(D_i)$ . It has an infinite number of connected components, each one being isometric to  $Hex$ . The set of all the connected components of all the sets  $\Pi^{-1}(D_i)$ , for  $i \in \{1, \dots, k\}$ , is a tiling of the plane  $\mathbb{R}^2$  by the hexagon  $Hex$ . This tiling is invariant under the translations associated with the vectors  $\mathbf{e}_1$  and  $b\mathbf{e}_2$ .

We can see that, conversely, the image by  $\Pi$  of an tiling  $\mathcal{T}$  of  $\mathbb{R}^2$  by an hexagon  $Hex$  is a regular  $k$ -partition of  $T(1, b)$  into domains isometric to  $Hex$  if  $\mathcal{T}$  satisfies the following properties.

- (i)  $\mathcal{T}$  is invariant under the translations associated with the vectors  $\mathbf{e}_1$  and  $b\mathbf{e}_2$ .
- (ii) The area of  $Hex$  is  $\frac{b}{k}$ .
- (iii) All the interior angles of  $Hex$  are  $2\pi/3$ .

We have therefore reformulated the original problem. We now look for the tilings of  $\mathbb{R}^2$  that satisfy properties i, ii, and iii, and, if possible, for an algorithm to construct those tilings.

### A.2 Change of basis

Let  $\mathcal{T}$  be an hexagonal tiling of  $\mathbb{R}^2$ , invariant under the translations associated with the vectors  $\mathbf{e}_1$  and  $b\mathbf{e}_2$ , with a tiling domain  $Hex$  that satisfies properties ii and iii of Subsection A.1. We introduce the following definition, which will help us in our study.

**Definition A.1.** *We say that a basis of  $\mathbb{R}^2$  is  $\mathcal{T}$ -adapted if its vectors connect the center of a tiling domain to the centers of two neighboring domains, with these two neighboring domains having a common side (see Figure 12(b)).*

Let us now denote by  $(\mathbf{u}_1, \mathbf{u}_2)$  a  $\mathcal{T}$ -adapted basis.

**Proposition A.2.** *There exists a  $2 \times 2$  matrix  $V = (v_{i,j})$  with integer coefficients such that*

$$\begin{cases} \mathbf{u}_1 &= \frac{v_{1,1}}{k} \mathbf{e}_1 + \frac{v_{2,1}}{k} (b\mathbf{e}_2), \\ \mathbf{u}_2 &= \frac{v_{1,2}}{k} \mathbf{e}_1 + \frac{v_{2,2}}{k} (b\mathbf{e}_2), \end{cases}$$

and  $\det(V) = \pm k$ .

*Proof.* Since  $\mathcal{T}$  is invariant under the translations associated with the vectors  $\mathbf{e}_1$  and  $b\mathbf{e}_2$ , there exist integers  $s_{1,1}$ ,  $s_{2,1}$ ,  $s_{1,2}$  and  $s_{2,2}$  such that

$$\begin{cases} \mathbf{e}_1 &= s_{1,1} \mathbf{u}_1 + s_{2,1} \mathbf{u}_2; \\ b\mathbf{e}_2 &= s_{1,2} \mathbf{u}_1 + s_{2,2} \mathbf{u}_2. \end{cases}$$

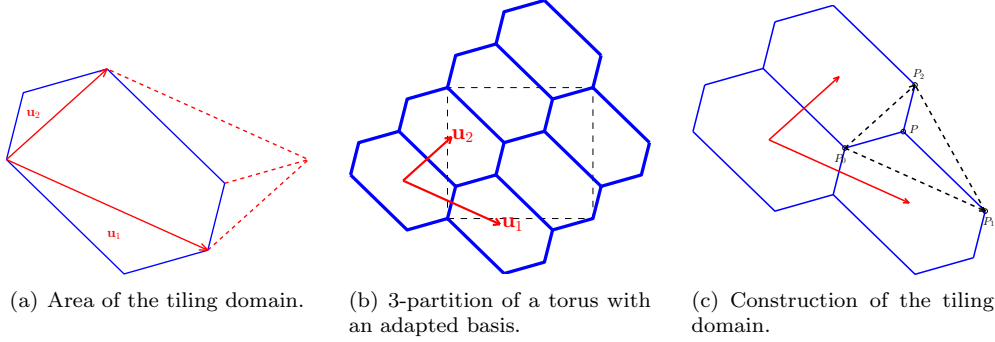


Figure 12: Hexagonal Tiling.

We use the notation  $S = (s_{i,j})$ ,  $\mathbf{u}_1 = (u_{1,1}, u_{2,1})$ ,  $\mathbf{u}_2 = (u_{1,2}, u_{2,2})$ , and  $U = (u_{i,j})$ . We have

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = US$$

and therefore  $\det(U) \det(S) = b$ . But  $|\det(U)|$  is the area of the tiling domain *Hex* (see Figure 12(a)). Therefore  $\det(U) = \pm b/k$  and  $\det(S) = \pm k$ . We then obtain the desired result by setting  $V = kS^{-1}$ .  $\square$

One can give a geometrical interpretation of the coefficients in the matrix  $V$  in the following way. Let us go back to the torus  $T(1, b)$ . Let us assume that the matrix  $V = (v_{i,j})$  has been obtained in the  $\mathcal{T}$ -adapted basis  $(\mathbf{u}_1, \mathbf{u}_2)$  (see Figure 12(b) for an example).

If we start from some hexagonal domain and translate it  $k$  times in the  $\mathbf{u}_1$  direction, it returns to its original position after turning  $v_{1,1}$  times around the torus in the horizontal direction and  $v_{2,1}$  times in the vertical direction. Similarly, if we translate the domain  $k$  times in the  $\mathbf{u}_2$  direction, it returns to its original position after turning  $v_{1,2}$  times around the torus in the horizontal direction and  $v_{2,2}$  times in the vertical direction.

We can therefore say that the matrix  $V$  describes how the hexagonal tiling  $\mathcal{T}$  wraps around the torus  $T(1, b)$ .

Let us now choose some domain of the tiling  $\mathcal{T}$  and denote by  $P_0$  the vertex that connects the sides of this domain associated with  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . Let us note  $P_1 = P_0 + \mathbf{u}_1$  and  $P_2 = P_0 + \mathbf{u}_2$  (see Figure 12(c)). Since the basis  $(\mathbf{u}_1, \mathbf{u}_2)$  is  $\mathcal{T}$ -adapted, both  $P_1$  and  $P_2$  are vertices of the tiling. Additionally, there is a vertex  $P$  of the tiling that is contained in the triangle  $P_1P_2P_3$  and is connected to the points  $P_0$ ,  $P_1$ , and  $P_2$ . The segments  $P_0P$ ,  $P_1P$ , and  $P_2P$  thus meet at  $P$  with equal angles.

Conversely, let us consider  $(\mathbf{u}_1, \mathbf{u}_2)$  a basis of  $\mathbb{R}^2$  such that there exists a  $2 \times 2$  matrix  $V = (v_{i,j})$  with integer coefficients that satisfies

$$\begin{cases} \mathbf{u}_1 &= \frac{v_{1,1}}{k} \mathbf{e}_1 + \frac{v_{2,1}}{k} (b\mathbf{e}_2), \\ \mathbf{u}_2 &= \frac{v_{1,2}}{k} \mathbf{e}_1 + \frac{v_{2,2}}{k} (b\mathbf{e}_2), \end{cases}$$

and  $\det(V) = \pm k$ .

Let  $P_0$  be some point in  $\mathbb{R}^2$  and let us note  $P_1 = P_0 + \mathbf{u}_1$  and  $P_2 = P_0 + \mathbf{u}_2$ . Let us now assume that there exists a point  $P$  in the triangle  $P_0P_1P_2$  such that the segments  $P_0P$ ,  $P_1P$ , and  $P_2P$  meet at  $P$  with equal angles. After translating the three segments  $P_0P$ ,  $P_1P$ , and  $P_2P$  according to all the vectors in  $\mathbb{Z}\mathbf{u}_1 + \mathbb{Z}\mathbf{u}_2$ , we obtain the boundary of a tiling  $\mathcal{T}$  which satisfies the three properties stated in Subsection A.1. Furthermore, the basis  $(\mathbf{u}_1, \mathbf{u}_2)$  is  $\mathcal{T}$ -adapted.

### A.3 Reconstruction of the tiling domain

Let us now recall, without proof, a very classical geometrical result (see for instance [10]).

**Theorem A.3.** *Let  $A_1$ ,  $A_2$ , and  $A_3$  be three non-collinear points in  $\mathbb{R}^2$  (see Figure 13(a)). One of the two following situations occurs.*

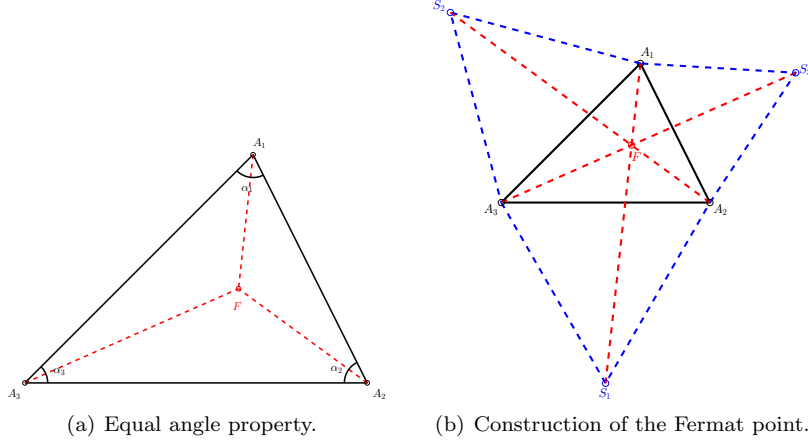


Figure 13: Fermat point of a triangle.

- (i) If all three angles  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  of the triangle  $A_1A_2A_3$  are smaller than  $\frac{2\pi}{3}$ , there is a unique point  $F$  belonging to the interior of the triangle  $A_1A_2A_3$  such that the segments  $A_1F$ ,  $A_2F$  and  $A_3F$  meet with equal angles at  $F$ . The point  $F$  is called the Fermat point of the triangle  $A_1A_2A_3$ . It is the point of minimum for the function  $P \mapsto PA_1 + PA_2 + PA_3$ .
- (ii) If  $\alpha_i \geq \frac{2\pi}{3}$  for some  $i \in \{1, 2, 3\}$ , then there is no point in the interior of  $A_1A_2A_3$  at which the segments from the vertices meet with equal angles. In that case, the function  $P \mapsto PA_1 + PA_2 + PA_3$  reaches its minimum at  $A_i$ .

Using the vocabulary of Theorem A.3, we can summarize the above discussion in the following statement.

**Proposition A.4.** Let  $V = (v_{i,j})_{1 \leq i,j \leq 2}$  be an invertible matrix with integer coefficients and let  $b \in (0, 1]$ . Let us note  $k = |\det(V)|$ ,

$$\mathbf{u}_1 = \frac{v_{1,1}}{k} \mathbf{e}_1 + \frac{v_{2,1}}{k} (b\mathbf{e}_2),$$

and

$$\mathbf{u}_2 = \frac{v_{1,2}}{k} \mathbf{e}_1 + \frac{v_{2,2}}{k} (b\mathbf{e}_2).$$

There exists a regular  $k$ -partition of  $T(1, b)$  by isometric hexagons for which  $(\mathbf{u}_1, \mathbf{u}_2)$  is an adapted basis if, and only if, the triangle spanned by  $\mathbf{u}_1$  and  $\mathbf{u}_2$  has a Fermat point.

The following result gives an easy criterion for the existence of a Fermat point.

**Proposition A.5.** Let  $\mathbf{u}_1$  and  $\mathbf{u}_2$  be two non-zero vectors in  $\mathbb{R}^2$ . Let  $A_3$  be a point in  $\mathbb{R}^2$ . We set  $A_2 = A_3 + \mathbf{u}_1$  and  $A_1 = A_3 + \mathbf{u}_2$ . We use the notation  $p = \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{\|\mathbf{u}_1\| \|\mathbf{u}_2\|}$  and  $r = \frac{\|\mathbf{u}_1\|}{\|\mathbf{u}_2\|}$ . The triangle  $A_1A_2A_3$  has a Fermat point if, and only if,

$$p \in \left(-\frac{1}{2}, \frac{1}{2}\right] \quad \text{or} \quad p \in \left(\frac{1}{2}, 1\right) \text{ and } p - \sqrt{\frac{1-p^2}{3}} < r < \frac{1}{p - \sqrt{\frac{1-p^2}{3}}}. \quad (\text{A.1})$$

*Proof.* We compute  $\cos(\alpha_i)$  for  $i \in \{1, 2, 3\}$ , as a function of  $p$  and  $r$ . Writing down the condition

$$\forall i \in \{1, 2, 3\}, \quad \cos(\alpha_i) \in \left(-\frac{1}{2}, 1\right), \quad (\text{A.2})$$

we show that it is equivalent to (A.1). □

## A.4 Algorithm

The results stated above give us an algorithm to build a tiling of the torus  $T(1, b)$  by  $k$  hexagons:

- choose a  $2 \times 2$  matrix  $V$  with integer coefficients such that  $\det(V) = \pm k$ ;
- check whether the triangle generated by the vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  (defined from  $V$  as in Proposition A.4) has a Fermat point, using the conditions (A.1);
- if the triangle has a Fermat point, compute its coordinates;
- use the coordinates of the Fermat point to build the tiling domain.

Let us describe in more details how we perform the last two steps. Let us first recall a geometric construction of the Fermat point (see for instance [10]).

**Theorem A.6.** *Let  $A_1A_2A_3$  be a triangle in  $\mathbb{R}^2$  such that each of the angles  $\alpha_i$ ,  $i \in \{1, 2, 3\}$ , is smaller than  $\frac{2\pi}{3}$ . Let us consider the three equilateral triangles lying outside of  $A_1A_2A_3$  and having one side in common with it. For each of these triangles, let us consider the line passing through the outer vertex and the vertex of  $A_1A_2A_3$  that does not belong to it (see Figure 13(b)). The three lines meet at the Fermat point  $F$ .*

Let us assume that we have performed the first two steps of the algorithm. We now have two vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  such that the basis  $(\mathbf{u}_1, \mathbf{u}_2)$  is adapted to some tiling  $\mathcal{T}$ . We chose (arbitrarily) some point  $P_0$  in  $\mathbb{R}^2$ . We can then build the points  $P_1$  and  $P_2$  of Figure 12(c). Using the construction of Theorem A.6, we can find the coordinates of the Fermat point  $P$  of the triangle  $P_0P_1P_2$ . The vectors  $\mathbf{c}_1 = P_0P$ ,  $\mathbf{c}_2 = PP_2$ , and  $\mathbf{c}_3 = P_1P$  then define three successive sides of the tiling domain, which is enough to construct the tiling domain itself.

## A.5 Examples

We have shown that for any given  $b \in (0, 1]$  and  $k \in \mathbb{N}^*$ , finding all the tilings of  $T(1, b)$  by hexagons which are regular  $k$ -partitions is equivalent to finding all the matrices  $V$  with integer coefficients that satisfy the following properties.

- $\det(V) = \pm k$ .
- The vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$ , defined from  $V$  as in Proposition A.4, generate a triangle that has a Fermat point.

We have not been able to solve this general problem. We have nonetheless looked for examples of tilings with given matrices  $V$ , suggested by the numerical simulations. The following results can be proved using the criterion (A.1).

**Proposition A.7.** *There exists a tiling of the torus  $T(1, b)$  with an associated matrix*

$$V = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

*if, and only if,  $b > b_3^c = (\sqrt{11} - \sqrt{3})/4$ . If it exists, this tiling is a 3-partition of  $T(1, b)$ , and we denote by  $Hex_3(b)$  its tiling domain.*

**Proposition A.8.** *There exists a tiling of the torus  $T(1, b)$  with an associated matrix*

$$V = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$$

*if, and only if,  $b > b_4^c = 1/2\sqrt{3}$ . If it exists, this tiling is a 4-partition of  $T(1, b)$ , and we denote by  $Hex_4(b)$  its tiling domain.*

**Proposition A.9.** *There exists a tiling of the torus  $T(1, b)$  with an associated matrix*

$$V = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

*if, and only if,  $b > b_5^c = (\sqrt{291} - 5\sqrt{3})/36$ . If it exists, this tiling is a 5-partition of  $T(1, b)$ , and we denote by  $Hex_5(b)$  its tiling domain.*

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