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Quotients of the Artin braid groups and crystallographic groups

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Abstract

Let $n \geq 3$. In this paper, we study the quotient group $B_n/[P_n, P_n]$ of the Artin braid group B_n by the commutator subgroup of its pure Artin braid group P_n . We show that $B_n/[P_n, P_n]$ is a crystallographic group, and in the case $n = 3$, we analyse explicitly some of its subgroups. We also prove that $B_n/[P_n, P_n]$ possesses torsion, and we show that there is a one-to-one correspondence between the conjugacy classes of the finite-order elements of $B_n/[P_n, P_n]$ with the conjugacy classes of the elements of odd order of the symmetric group S_n , and that the isomorphism class of any Abelian subgroup of odd order of S_n is realised by a subgroup of $B_n/[P_n, P_n]$. Finally, we discuss the realisation of non-Abelian subgroups of S_n of odd order as subgroups of $B_n/[P_n, P_n]$, and we show that the Frobenius group of order 21, which is the smallest non-Abelian group of odd order, embeds in $B_n/[P_n, P_n]$ for all $n \geq 7$.

1 Introduction

Let $n \in \mathbb{N}$. Quotients of the Artin braid group B_n has been studied in various contexts, and may be used to study properties of B_n itself. It is well known that one such quotient is the symmetric group S_n , which may be expressed in the form $B_n / \langle \sigma_1^2 \rangle^{B_n}$, where $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of B_n (see Section 3), and $\langle X \rangle^{B_n}$ denotes the normal closure subgroup of a subset X of B_n . Similar quotients of the form $B_n / \langle \sigma_1^m \rangle^{B_n}$, where $m \in \mathbb{N}$, were analysed by Coxeter in [Co], who showed that this quotient is finite if and only if $(n, m) \in \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}$, and computed the quotient groups in each case, and by Marin in [Ma1] in the case $(n, m) = (5, 3)$ with the aim of studying cubic Hecke algebras. The Brunnian braid groups Brun_n have been studied in connection with homotopy groups of the 2-sphere \mathbb{S}^2 [BCWW, LW, O] by considering quotients of B_n . For example, for all $n \geq 3$, there exists a subgroup G_n of Brun_n that is normal in the Artin pure braid group P_n such that the centre of P_n/G_n is isomorphic to the direct product $\pi_n(\mathbb{S}^2) \times \mathbb{Z}$ (see [LW, Theorem 1] and [O, Theorem 4.3.4]).

In this paper, we study the quotient $B_n/[P_n, P_n]$ of B_n for $n \geq 3$, where $[P_n, P_n]$ is the commutator subgroup of P_n . Our initial motivation emanates from the observation that $B_3/[P_3, P_3]$ is isomorphic to B_3/Brun_3 (see [O, Corollary 2.1.4] as well as [O, Section 5.2] for other results about B_3/Brun_3 , and [LW, Proposition 3.9] and [O, Proposition 4.3.10(1)] for a presentation of B_3/Brun_3). The quotient $B_n/[P_n, P_n]$ belongs to a family of groups known as *enhanced symmetric groups* (see [Ma2, page 201]) and analysed in [T]. It also arises in the study of pseudo-symmetric braided categories by Panaite and Staic. They consider the quotient, denoted by PS_n , of B_n by the normal subgroup generated by the relations $\sigma_i \sigma_{i+1}^{-1} \sigma_i = \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}$ for $i = 1, 2, \dots, n-2$, and they show that it is isomorphic to $B_n/[P_n, P_n]$ [PS]. The results that we obtain in this paper for $B_n/[P_n, P_n]$ are different in nature to those of [PS], with the exception of some basic properties.

Crystallographic groups play an important rôle in the study of the groups of isometries of Euclidean spaces (see Section 2 for precise definitions, as well as [Ch, D, W] for more details). As we shall prove in Proposition 1, another reason for studying the quotient $B_n/[P_n, P_n]$ is the fact that it is a crystallographic group:

PROPOSITION 1. *Let $n \geq 2$. There is a short exact sequence:*

$$1 \longrightarrow \mathbb{Z}^{n(n-1)/2} \longrightarrow B_n/[P_n, P_n] \xrightarrow{\bar{\sigma}} S_n \longrightarrow 1,$$

and the middle group $B_n/[P_n, P_n]$ is a crystallographic group.

The aim of this paper is to analyse $B_n/[P_n, P_n]$ in more detail, notably its torsion, the conjugacy classes of its finite-order elements, and the realisation of abstract finite groups as subgroups of $B_n/[P_n, P_n]$. Since B_1 is trivial and B_2 is isomorphic to \mathbb{Z} , in what follows we shall suppose that $n \geq 3$. In Section 2, we recall the basic definitions and some results about crystallographic and Bieberbach groups. In Section 3, we recall some standard information about B_n and P_n , and using the fact that the quotient B_n/P_n is isomorphic to S_n , we shall see that $B_n/[P_n, P_n]$ is an extension of the free Abelian group $P_n/[P_n, P_n]$ by S_n , and we shall compute the associated action, which will enable us to prove it is crystallographic. By analysing the action in more detail, we prove that the torsion of $B_n/[P_n, P_n]$ is odd:

THEOREM 2. *If $n \geq 3$ then the quotient group $B_n/[P_n, P_n]$ has no finite-order element of even order.*

By restricting the short exact sequence involving $B_n/[P_n, P_n]$, $P_n/[P_n, P_n]$ and S_n to 2-subgroups of the latter (see equation (9)), we are able to construct Bieberbach groups of dimension $n(n-1)/2$ (which is the rank of $P_n/[P_n, P_n]$), and show that there exist flat manifolds of the same dimension whose holonomy group is the given 2-subgroup (see Theorem 16).

In Section 4, we analyse the torsion of $B_n/[P_n, P_n]$ in more detail. In order to do so, we shall make use of the induced action of certain elements $\alpha_{0,r}$ of $B_n/[P_n, P_n]$, where $2 \leq r \leq n$, on the basis $(A_{i,j})_{1 \leq i < j \leq n}$ of $P_n/[P_n, P_n]$. The structure of the corresponding orbits is very rigid, and allows us to express the existence of elements of $B_n/[P_n, P_n]$ of order n in terms of the existence of solutions of a certain linear system. It will follow from this that $B_n/[P_n, P_n]$ has infinitely many elements of order n (see Proposition 19). We then show that if $1 \leq n \leq m$, the standard injective homomorphism of B_n in B_m induces an injective homomorphism of $B_n/[P_n, P_n]$ in $B_m/[P_m, P_m]$:

THEOREM 3. *Let m and n be integers such that $2 \leq n \leq m$.*

(a) *Consider the injective homomorphism $\iota: B_n \rightarrow B_m$ defined by $\iota(\sigma_i) = \sigma_i$ for all $1 \leq i \leq n-1$. Then the induced homomorphism $\bar{\iota}: B_n/[P_n, P_n] \rightarrow B_m/[P_m, P_m]$ of the corresponding quotient groups is injective.*

(b) *If $n \geq 3$ and n is odd then $B_m/[P_m, P_m]$ possesses elements of order n . Further, there exists such an element whose permutation is an n -cycle.*

(c) *Let n_1, n_2, \dots, n_t be odd integers greater than or equal to 3 for which $\sum_{i=1}^t n_i \leq m$. Then $B_m/[P_m, P_m]$ possesses elements of order $\text{lcm}(n_1, \dots, n_t)$. Further, there exists such an element whose cycle type is (n_1, \dots, n_t) .*

Part (b) follows from part (a) and Proposition 19. In the course of the proof of Theorem 3, we shall see that the direct product of the groups of the form $B_{n_i}/[P_{n_i}, P_{n_i}]$ injects into $B_m/[P_m, P_m]$, which will enable us to prove part (c). One consequence of Theorems 2 and 3 is the characterisation of the torsion of $B_n/[P_n, P_n]$ as that of the odd torsion of the symmetric group S_n :

COROLLARY 4. *Let $n \geq 3$. The torsion of the quotient $B_n/[P_n, P_n]$ is equal to the odd torsion of the symmetric group S_n . Moreover, given an element $\theta \in S_n$ of odd order r , there exists $\beta \in B_n/[P_n, P_n]$ of order r such that $\bar{\sigma}(\beta) = \theta$. So given any cyclic subgroup H of S_n of odd order r , there exists a finite-order subgroup \tilde{H} of $B_n/[P_n, P_n]$ such that $\bar{\sigma}(\tilde{H}) = H$.*

In Section 5, we focus on the simplest non-trivial case, that of $B_3/[P_3, P_3]$, and we describe the structure of the preimages of the subgroups of S_3 under the induced homomorphism $B_3/[P_3, P_3] \rightarrow S_3$. In the cases where these preimages are Bieberbach groups, we describe the corresponding flat 3-manifold. We also carry out this analysis for the group $B_3/[P_3, P_3]$ itself, and identify it in the international tables of crystallographic groups given in [BBNWS, HL], as well as for the quotient of $B_3/[P_3, P_3]$ by the subgroup generated by the class of the full-twist braid.

In Section 6, we study the conjugacy classes of the elements and the cyclic subgroups of $B_n/[P_n, P_n]$. This is achieved in Propositions 27, 28 and 29 by studying in detail the action of certain elements $\delta_{r,k}$ and $\alpha_{r,k}$ (the latter being a generalisation of $\alpha_{r,0}$) on the

basis $(A_{i,j})_{1 \leq i < j \leq n}$ of $P_n/[P_n, P_n]$. It is straightforward to see that if any two elements of $B_n/[P_n, P_n]$ are conjugate then their permutations have the same cycle type, and the use of these propositions and a specific product of certain $\delta_{r,k}$ enables us to prove the converse:

THEOREM 5. *Let $n \geq 3$, and let $k \geq 3$ be odd. Two elements of $B_n/[P_n, P_n]$ of order k are conjugate if and only if their permutations have the same cycle type. Thus two finite cyclic subgroups of $B_n/[P_n, P_n]$ of order k are conjugate if and only if their images under $\bar{\sigma}$ are conjugate in S_n .*

Consequently, given $n \geq 3$, we may determine the number of conjugacy classes of elements of odd order k in $B_n/[P_n, P_n]$.

From Lemma 9, it follows that the set of isomorphism classes of the finite subgroups of $B_n/[P_n, P_n]$ is contained in the corresponding set of finite subgroups of S_n of odd order. One may ask whether this inclusion is strict or not. As we shall see in Corollary 4, any cyclic subgroup of S_n of odd order is realised as a subgroup of $B_n/[P_n, P_n]$. Combining Theorem 3(c) with a result of [Ho], this result may be extended to the Abelian subgroups of S_n :

THEOREM 6. *Let $n \geq 3$. Then there is a one-to-one correspondence between the isomorphism classes of the finite Abelian subgroups of $B_n/[P_n, P_n]$ and the isomorphism classes of the Abelian subgroups of S_n of odd order.*

In Section 7, we turn our attention to what is probably a more difficult open problem, namely the realisation of finite non-Abelian groups of S_n as subgroups of the group $B_n/[P_n, P_n]$. As a initial experiment, we consider the smallest value of n , $n = 7$, for which S_n possesses a non-Abelian subgroup of odd order. This subgroup is isomorphic to the Frobenius group of order 21 that we denote by \mathcal{F} . We show that \mathcal{F} is indeed realised as a subgroup of $B_7/[P_7, P_7]$.

THEOREM 7. *The quotient group $B_7/[P_7, P_7]$ possesses a subgroup isomorphic to the Frobenius group \mathcal{F} .*

It then follows from Theorem 3 that \mathcal{F} is realised as a subgroup of $B_n/[P_n, P_n]$ for all $n \geq 7$. In Proposition 35, we prove that $B_7/[P_7, P_7]$ admits a single conjugacy class of subgroups isomorphic to \mathcal{F} . We remark that we do not currently know of an example of a subgroup of odd order of S_n whose isomorphism class is not represented by a subgroup of $B_n/[P_n, P_n]$.

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2 Crystallographic and Bieberbach groups

In this section, we recall briefly the definition of crystallographic and Bieberbach groups, and we characterise crystallographic groups in terms of a representation that arises from certain group extensions whose kernel is a free Abelian group of finite rank and whose quotient is finite. We also review some results concerning Bieberbach groups and the fundamental groups of flat Riemannian manifolds. For more details, see [Ch, Section I.1.1], [D, Section 2.1] or [W, Chapter 3]. From now on, we identify $\text{Aut}(\mathbb{Z}^n)$ with $\text{GL}(n, \mathbb{Z})$.

DEFINITION. A discrete and uniform subgroup Π of $\mathbb{R}^n \rtimes \text{O}(n, \mathbb{R}) \subseteq \text{Aff}(\mathbb{R}^n)$ is said to be a *crystallographic group of dimension n* . If in addition Π is torsion free then Π is called a *Bieberbach group of dimension n* .

DEFINITION. Let Φ be a group. An *integral representation of rank n of Φ* is defined to be a homomorphism $\Theta: \Phi \rightarrow \text{Aut}(\mathbb{Z}^n)$. Two such representations are said to be *equivalent* if their images are conjugate in $\text{Aut}(\mathbb{Z}^n)$. We say that Θ is a *faithful representation* if it is injective.

The following characterisation of crystallographic groups seems to be well known to the experts in the field. Since we did not find a suitable reference, we give a short proof.

LEMMA 8. *Let Π be a group. Then Π is a crystallographic group if and only if there exist an integer $n \in \mathbb{N}$ and a short exact sequence*

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Pi \xrightarrow{\zeta} \Phi \longrightarrow 1 \quad (1)$$

such that:

- (a) Φ is finite, and
- (b) the integral representation $\Theta: \Phi \rightarrow \text{Aut}(\mathbb{Z}^n)$, induced by conjugation on \mathbb{Z}^n and defined by $\Theta(\varphi)(x) = \pi x \pi^{-1}$, where $x \in \mathbb{Z}^n$, $\varphi \in \Phi$ and $\pi \in \Pi$ is such that $\zeta(\pi) = \varphi$, is faithful.

DEFINITION. If Π is a crystallographic group, the integer n that appears in the statement of Lemma 8 is called the *dimension* of Π , the finite group Φ is called the *holonomy group* of Π , and the integral representation $\Theta: \Phi \rightarrow \text{Aut}(\mathbb{Z}^n)$ is called the *holonomy representation* of Π .

Proof of Lemma 8. Let Φ and Π be groups, and suppose that there exist $n \in \mathbb{N}$ and a short exact sequence of the form (1) such that conditions (a) and (b) hold. Assume on the contrary that Π is not crystallographic. The characterisation of [D, Theorem 2.1.4] implies that \mathbb{Z}^n is not a maximal Abelian subgroup of Π , in other words, there exists an Abelian group A for which $\mathbb{Z}^n \subsetneq A \subseteq \Pi$. Let $a \in A \setminus \mathbb{Z}^n$. Then $\zeta(a) \neq 1$ and

$\Theta(\zeta(a))(x) = axa^{-1} = x$ for all $x \in \mathbb{Z}^n$. Hence $\Theta(\zeta(a)) = \text{Id}_{\mathbb{Z}^n}$, which contradicts the hypothesis that Θ is injective. We conclude that Π is a crystallographic group of dimension n with holonomy Φ .

The converse follows from the paragraph preceding [Ch, Definition I.6.2], since the short exact sequence (1) gives rise to an integral representation $\Theta: \Phi \rightarrow \text{Aut}(\mathbb{Z}^n)$ that is faithful by [Ch, Proposition I.6.1]. \square

The following lemma will be very useful in what follows.

LEMMA 9. *Let G, G' be groups, and let $f: G \rightarrow G'$ be a homomorphism whose kernel is torsion free. If K is a finite subgroup of G then the restriction $f|_K: K \rightarrow f(K)$ of f to K is an isomorphism. In particular, with the notation of the statement of Lemma 8, if Π is a crystallographic group then the restriction $\zeta|_K: K \rightarrow \zeta(K)$ of ζ to any finite subgroup K of Π is an isomorphism.*

Proof. Since $\text{Ker}(f)$ is torsion free, the restriction of f to the finite subgroup K is injective, which yields the first part, and the second part then follows directly. \square

COROLLARY 10. *Let Π be a crystallographic group of dimension n and holonomy group Φ , and let H be a subgroup of Φ . Then there exists a crystallographic subgroup of Π of dimension n with holonomy group H .*

Proof. The result follows by considering the short exact sequence (1), and by applying Lemma 8 to the subgroup $\zeta^{-1}(H)$ of Π . \square

DEFINITION. A Riemannian manifold M is called *flat* if it has zero curvature at every point.

As a consequence of the first Bieberbach Theorem, there is a correspondence between Bieberbach groups and fundamental groups of closed flat Riemannian manifolds (see [D, Theorem 2.1.1] and the paragraph that follows it). We recall that the flat manifold determined by a Bieberbach group Π is orientable if and only if the integral representation $\Theta: \Phi \rightarrow \text{GL}(n, \mathbb{Z})$ satisfies $\text{Im}(\Theta) \subseteq \text{SO}(n, \mathbb{Z})$. This being the case, we say that Π is an *orientable Bieberbach group*. By [W, Corollary 3.4.6], the holonomy group of a flat manifold M is isomorphic to the group Φ .

It is a natural problem to classify the finite groups that are the holonomy group of a flat manifold. The answer was given by L. Auslander and M. Kuranishi in 1957.

THEOREM 11 (Auslander and Kuranishi [W, Theorem 3.4.8], [Ch, Theorem III.5.2]). *Any finite group is the holonomy group of some flat manifold.*

3 Artin braid groups and crystallographic groups

In this section we prove Proposition 1 and Theorem 2, namely that if $n \geq 3$ then the quotient group $B_n/[P_n, P_n]$ of the Artin braid group B_n by the commutator subgroup $[P_n, P_n]$ of its pure braid subgroup P_n is crystallographic and does not have 2-torsion. As we shall see in Section 4, $B_n/[P_n, P_n]$ possesses (odd) torsion. We first recall some facts about the Artin braid group B_n on n strings. We refer the reader to [Ha] for more

details. It is well known that B_n possesses a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ that are subject to the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for all } 1 \leq i < j \leq n-1 \quad (2)$$

$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \text{ for all } 1 \leq i \leq n-2. \quad (3)$$

Let $\sigma: B_n \rightarrow S_n$ be the homomorphism defined on the given generators of B_n by $\sigma(\sigma_i) = (i, i+1)$ for all $1 \leq i \leq n-1$. Just as for braids, we read permutations from left to right so that if $\alpha, \beta \in S_n$ then their product is defined by $\alpha \cdot \beta(i) = \beta(\alpha(i))$ for $i = 1, 2, \dots, n$. The pure braid group P_n on n strings is defined to be the kernel of σ , from which we obtain the following short exact sequence:

$$1 \rightarrow P_n \rightarrow B_n \xrightarrow{\sigma} S_n \rightarrow 1. \quad (4)$$

A generating set of P_n is given by $\{A_{i,j}\}_{1 \leq i < j \leq n}$, where:

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}. \quad (5)$$

Relations (2) and (3) may be used to show that:

$$A_{i,j} = \sigma_i^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^2 \sigma_{j-2} \cdots \sigma_i. \quad (6)$$

It follows from the presentation of P_n given in [Ha] that $P_n/[P_n, P_n]$ is isomorphic to $\mathbb{Z}^{n(n-1)/2}$, and that a basis is given by the $A_{i,j}$, where $1 \leq i < j \leq n$, and where by abuse of notation, the $[P_n, P_n]$ -coset of $A_{i,j}$ will also be denoted by $A_{i,j}$. Using equation (4), we obtain the following short exact sequence:

$$1 \rightarrow P_n/[P_n, P_n] \rightarrow B_n/[P_n, P_n] \xrightarrow{\bar{\sigma}} S_n \rightarrow 1, \quad (7)$$

where $\bar{\sigma}: B_n/[P_n, P_n] \rightarrow S_n$ is the homomorphism induced by σ . This short exact may also be found in [PS, Proposition 3.2].

Since B_1 is the trivial group and $B_2/[P_2, P_2] \cong \mathbb{Z}$, we shall suppose in most of this paper that $n \geq 3$. We shall be interested in the action by conjugation of B_n on P_n and on $P_n/[P_n, P_n]$. Recall from [LW, Lemma 3.1] (see also [MK, Proposition 3.7, Chapter 3]) that for all $1 \leq k \leq n-1$ and for all $1 \leq i < j \leq n$,

$$\sigma_k A_{i,j} \sigma_k^{-1} = \begin{cases} A_{i,j} & \text{if } k \neq i-1, i, j-1, j \\ A_{i,k+1} & \text{if } j = k \\ A_{i,k+1}^{-1} A_{i,k} A_{i,k+1} & \text{if } j = k+1 \text{ and } i < k \\ A_{k,k+1} & \text{if } j = k+1 \text{ and } i = k \\ A_{i+1,j} & \text{if } i = k < j-1 \\ A_{k+1,j}^{-1} A_{k,j} A_{k+1,j} & \text{if } i = k+1. \end{cases}$$

So the induced action of B_n on $P_n/[P_n, P_n]$ is given by

$$\sigma_k A_{i,j} \sigma_k^{-1} = \begin{cases} A_{i,j} & \text{if } k \neq i-1, i, j-1, j \\ A_{i,k+1} & \text{if } j = k \\ A_{i,k} & \text{if } j = k+1 \text{ and } i < k \\ A_{k,k+1} & \text{if } j = k+1 \text{ and } i = k \\ A_{i+1,j} & \text{if } i = k < j-1 \\ A_{k,j} & \text{if } i = k+1. \end{cases} \quad (8)$$

A study of this action now allows us to prove Proposition 1.

Proof of Proposition 1. Suppose first that $n = 2$. Since $B_2 = \mathbb{Z}$ and $[P_2, P_2] = 1$, we obtain $B_2/[P_2, P_2] = B_2 \cong \mathbb{Z}$, and thus the group $B_2/[P_2, P_2]$ is crystallographic. So assume that $n \geq 3$, and consider the short exact sequence (7). We shall show that the induced action $\varphi: S_n \rightarrow \text{Aut}(\mathbb{Z}^{n(n-1)/2})$ is injective, from which it will follow that the group $B_n/[P_n, P_n]$ is crystallographic. Using equation (8), if $1 \leq i < j \leq n$, the automorphisms induced by the elements σ_1 and σ_2 of B_n are given by:

$$\sigma_1 A_{i,j} \sigma_1^{-1} = \begin{cases} A_{2,j} & \text{if } i = 1 \text{ and } j \geq 3 \\ A_{1,j} & \text{if } i = 2 \text{ and } j \geq 3 \\ A_{i,j} & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_2 A_{i,j} \sigma_2^{-1} = \begin{cases} A_{3,j} & \text{if } i = 2 \text{ and } j \geq 4 \\ A_{2,j} & \text{if } i = 3 \text{ and } j \geq 4 \\ A_{1,3} & \text{if } i = 1 \text{ and } j = 2 \\ A_{1,2} & \text{if } i = 1 \text{ and } j = 3 \\ A_{i,j} & \text{otherwise.} \end{cases}$$

These automorphisms are distinct and non trivial, so the image $\text{Im}(\varphi)$ of φ possesses at least three elements. Applying the First Isomorphism Theorem, the kernel $\text{Ker}(\varphi)$ of φ is thus a normal subgroup of S_n whose order is bounded above by $n!/3$. If $n \neq 4$, the only normal subgroups of S_n are the trivial subgroup, the alternating subgroup A_n and S_n itself, from which we conclude that $\text{Ker}(\varphi)$ is trivial as required. Now suppose that $n = 4$. The same argument applies, but additionally, $\text{Ker}(\varphi)$ may be isomorphic to the Klein group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, in which case $\text{Im}(\varphi)$ is of order 6. But by equation (8), the action of $\sigma_3 \sigma_2 \sigma_1$ on the basis elements of $P_4/[P_4, P_4]$ is given by:

$$A_{1,2} \mapsto A_{1,4} \mapsto A_{3,4} \mapsto A_{2,3} \mapsto A_{1,2} \quad \text{and} \quad A_{1,3} \mapsto A_{2,4} \mapsto A_{1,3}.$$

Thus $\varphi(\sigma_3 \sigma_2 \sigma_1)$ is an element of $\text{Im}(\varphi)$ of order 4, so $\text{Im}(\varphi)$ cannot be of order 6. Once more we see that $\text{Ker}(\varphi)$ is trivial, and thus the associated integral representation is faithful for all $n \geq 3$. It then follows from Lemma 8 that the group $B_n/[P_n, P_n]$ is crystallographic. \square

Using Proposition 1 and Corollary 10, we may produce other crystallographic groups as follows. Let H be a subgroup of S_n , and consider the following short exact sequence:

$$1 \longrightarrow \frac{P_n}{[P_n, P_n]} \longrightarrow \tilde{H}_n \xrightarrow{\bar{\sigma}} H \longrightarrow 1 \quad (9)$$

induced by that of equation (7), where \tilde{H}_n is defined by:

$$\tilde{H}_n = \frac{\sigma^{-1}(H)}{[P_n, P_n]}. \quad (10)$$

The following corollary is then a consequence of Corollary 10 and Proposition 1.

COROLLARY 12. *Let $n \geq 3$, and let H be a subgroup of S_n . Then the group \tilde{H}_n defined by equation (10) is a crystallographic group of dimension $n(n-1)/2$ with holonomy group H .*

Our next goal is to prove Theorem 2, that the quotient groups $B_n/[P_n, P_n]$ do not have 2-torsion.

Proof of Theorem 2. Let $n \geq 3$. Suppose on the contrary that there exists $\beta \in B_n$ whose $[P_n, P_n]$ -coset, which we also denote by β , is of even order in $B_n/[P_n, P_n]$. By taking a power of β if necessary, we may suppose that β is of order 2 in $B_n/[P_n, P_n]$. Since $P_n/[P_n, P_n]$ is torsion free, it follows that $\beta \in B_n \setminus P_n$. Conjugating β by an element of B_n if necessary, we may suppose that $\sigma(\beta) = (1, 2)(3, 4) \cdots (k, k+1)$, where $1 \leq k \leq n-1$ and k is odd. Thus $\beta^2 \in P_n$. Let $\alpha = \sigma_1 \sigma_3 \cdots \sigma_{k-2} \sigma_k$. Then $\sigma(\beta) = \sigma(\alpha)$, thus $N = \beta \alpha^{-1}$ belongs to P_n , and so:

$$\beta^2 = (N\alpha)^2 = N \cdot \alpha N \alpha^{-1} \cdot \alpha^2 \quad (11)$$

in $P_n/[P_n, P_n]$ because $\beta^2 \in P_n$. Further, $\alpha^2 = A_{1,2} A_{3,4} \cdots A_{k,k+1}$, $\alpha A_{1,2} \alpha^{-1} = A_{1,2}$ by equation (8), and if $1 \leq r < s \leq n$ then by equation (8), $\alpha A_{r,s} \alpha^{-1} = A_{1,2}$ in $B_n/[P_n, P_n]$ if and only if $(r, s) = (1, 2)$. In particular, if we express N (considered as an element of $P_n/[P_n, P_n]$) using the basis $\{A_{i,j}\}_{1 \leq i < j \leq n}$ and if r is the coefficient of $A_{1,2}$ in this expression then the coefficient of $A_{1,2}$ in the expression for β^2 in equation (11) is equal to $2r + 1$, which contradicts the fact that β^2 is trivial in $P_n/[P_n, P_n]$, and the result follows. \square

REMARKS 13. Let $n \geq 3$.

- (a) Theorem 2 generalises [PS, Proposition 3.6], where it is shown that there is no element $B_n/[P_n, P_n]$ of order two whose image under $\bar{\sigma}$ is the transposition $(1, 2)$.
- (b) Theorem 2 implies that any finite-order subgroup of $B_n/[P_n, P_n]$ is of odd order.
- (c) Applying Proposition 1, Theorem 2 and Lemma 9 to the short exact sequence (7), the restriction $\bar{\sigma}|_K : K \rightarrow \bar{\sigma}(K)$ of $\bar{\sigma}$ to any finite subgroup K of $B_n/[P_n, P_n]$ is an isomorphism. In particular, the set of isomorphism classes of the finite subgroups of $B_n/[P_n, P_n]$ is contained in the set of isomorphism classes of the odd-order subgroups of S_n .

As we shall now see, by choosing H appropriately, we may use Corollary 12 to construct Bieberbach groups of dimension $n(n-1)/2$ in $B_n/[P_n, P_n]$. In Theorem 16, we will give a statement for $B_n/[P_n, P_n]$ analogous to that of Theorem 11 in the case that the holonomy group is a finite 2-group.

LEMMA 14. Let $n \geq 3$, and let H be a 2-subgroup of S_n . Then the group \tilde{H}_n given by equation (10) is a Bieberbach group of dimension $n(n-1)/2$.

Proof. Let $n \geq 3$, and let H be a 2-subgroup of S_n . Consider the short exact sequences (7) and (9). By Corollary 12, \tilde{H}_n is a crystallographic group of dimension $n(n-1)/2$ with holonomy group H . Since the kernel is torsion free, $\bar{\sigma}$ respects the order of the torsion elements of \tilde{H}_n [GG, Lemma 13]. In particular, the fact that H is a 2-group implies that the order of any non-trivial torsion element of \tilde{H}_n is a positive power of 2. On the other hand, by Theorem 2, the group $B_n/[P_n, P_n]$ has no such torsion elements, so the same is true for \tilde{H}_n . It follows that \tilde{H}_n is torsion free, hence it is a Bieberbach group of dimension $n(n-1)/2$ because $P_n/[P_n, P_n] \cong \mathbb{Z}^{n(n-1)/2}$. \square

REMARK 15. It is not clear to us whether the family of groups that satisfy the conclusions of Corollary 12 (resp. of Lemma 14) contains all of the isomorphism classes of crystallographic (resp. Bieberbach) subgroups of $B_n/[P_n, P_n]$ of dimension $n(n-1)/2$.

Lemma 14 enables us to give an alternative proof of Theorem 11 in the case that the finite group in question is a 2-group, and to estimate the dimension of the resulting flat manifold.

THEOREM 16. *Let H be a finite 2-group. Then H is the holonomy group of some flat manifold M . Further, the dimension of M may be chosen to be $n(n-1)/2$, where n is an integer for which H embeds in the symmetric group S_n , and the fundamental group of M is isomorphic to a subgroup of $B_n/[P_n, P_n]$.*

Proof. Let H be a finite 2-group. Cayley's Theorem implies that there exists an integer $n \geq 3$ such that H is isomorphic to a subgroup of S_n . From Lemma 14, \tilde{H}_n is a Bieberbach group of dimension $n(n-1)/2$ with holonomy group H and is a subgroup of $B_n/[P_n, P_n]$. By the first Bieberbach Theorem, there exists a flat manifold M of dimension $n(n-1)/2$ with holonomy group H such that $\pi_1(M) = \tilde{H}_n$ (see [D, Theorem 2.1.1] and the paragraph that follows it). \square

For a given finite group H , it is natural to ask what is the minimal dimension of a flat manifold whose holonomy group is H . Theorem 16 provides an upper bound for this minimal dimension when H is a 2-group. This upper bound is not sharp in general, for example if $H = \mathbb{Z}_2$.

4 The torsion of the group $B_n/[P_n, P_n]$

Let $n \geq 3$. In this section we study the torsion elements of the group $B_n/[P_n, P_n]$. The main aim is to show that if $\theta \in S_n$ is of odd order r then there exists $\beta \in B_n$ whose $[P_n, P_n]$ -coset projects to θ in S_n and is of order r in $B_n/[P_n, P_n]$ (see Corollary 4). We begin by showing that if r is an odd number such that S_n possesses an element of order r then $B_n/[P_n, P_n]$ also has an element of order r . By abuse of notation let $\sigma_k = q_n(\sigma_k)$, and let $A_{i,j} = q_n(A_{i,j})$, where $q_n: B_n \rightarrow B_n/[P_n, P_n]$ is the natural projection.

PROPOSITION 17. *Let $n \geq 3$, let $1 \leq i < j \leq r \leq n$, and let $\alpha_{0,r} = \sigma_1 \sigma_2 \cdots \sigma_{r-1} \in B_n/[P_n, P_n]$. The following relations hold in $B_n/[P_n, P_n]$:*

$$\alpha_{0,r} A_{i,j} \alpha_{0,r}^{-1} = \begin{cases} A_{i+1,j+1} & \text{if } j \leq r-1 \\ A_{1,i+1} & \text{if } j = r. \end{cases} \quad (12)$$

$$\alpha_{0,r} A_{i,j} \alpha_{0,r}^{-1} = \begin{cases} A_{i+1,j+1} & \text{if } j \leq r-1 \\ A_{1,i+1} & \text{if } j = r. \end{cases} \quad (13)$$

Proof. We first prove equation (12). If $1 \leq k \leq r-2$ then

$$\begin{aligned} \alpha_{0,r} \sigma_k \alpha_{0,r}^{-1} &= \sigma_1 \cdots \sigma_{r-1} \sigma_k \sigma_{r-1}^{-1} \cdots \sigma_1^{-1} = \sigma_1 \cdots \sigma_k \sigma_{k+1} \sigma_k \sigma_{k+1}^{-1} \sigma_k^{-1} \cdots \sigma_1^{-1} \\ &= \sigma_1 \cdots \sigma_{k-1} \sigma_{k+1} \sigma_k \sigma_{k+1} \sigma_{k+1}^{-1} \sigma_k^{-1} \sigma_{k-1}^{-1} \cdots \sigma_1^{-1} = \sigma_{k+1}. \end{aligned}$$

So if $1 \leq i < j \leq r-1$ then $\alpha_{0,r} A_{i,j} \alpha_{0,r}^{-1} = A_{i+1,j+1}$ by equation (5), which proves equation (12). So suppose that $j = r$. Let $\gamma = \sigma_{i+1} \cdots \sigma_{r-2} \sigma_{r-1}^2 \sigma_{r-2} \cdots \sigma_{i+1}$. Since $\gamma \in P_n/[P_n, P_n]$, we have $\alpha_{0,i+1} \gamma \alpha_{0,i+1}^{-1} \in P_n/[P_n, P_n]$, and thus $\alpha_{0,i+1} \gamma \alpha_{0,i+1}^{-1}$ and $\alpha_{0,i+1} \sigma_i^2 \alpha_{0,i+1}^{-1}$ commute pairwise in $P_n/[P_n, P_n]$. Hence:

$$\begin{aligned} \alpha_{0,r} A_{i,r} \alpha_{0,r}^{-1} &= \sigma_1 \cdots \sigma_{r-1} \sigma_{r-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{r-1}^{-1} \sigma_{r-1}^{-1} \cdots \sigma_1^{-1} = \alpha_{0,i+1} \gamma \sigma_i^2 \gamma^{-1} \alpha_{0,i+1}^{-1} \\ &= (\alpha_{0,i+1} \gamma \alpha_{0,i+1}^{-1}) (\alpha_{0,i+1} \sigma_i^2 \alpha_{0,i+1}^{-1}) (\alpha_{0,i+1} \gamma^{-1} \alpha_{0,i+1}^{-1}) = \alpha_{0,i+1} \sigma_i^2 \alpha_{0,i+1}^{-1} \\ &= \sigma_1 \cdots \sigma_i \sigma_i^2 \sigma_i^{-1} \cdots \sigma_1^{-1} \\ &= (\sigma_1 \cdots \sigma_i \sigma_i \cdots \sigma_1) (\sigma_1^{-1} \cdots \sigma_{i-1}^{-1} \sigma_i^2 \sigma_{i-1} \cdots \sigma_1) (\sigma_1 \cdots \sigma_i \sigma_i \cdots \sigma_1)^{-1} \end{aligned}$$

$$= \sigma_1^{-1} \cdots \sigma_{i-1}^{-1} \sigma_i^2 \sigma_{i-1} \cdots \sigma_1 = A_{1,i+1}$$

by equation (6), since the three bracketed terms in the penultimate line belong to the quotient $P_n/[P_n, P_n]$ and so commute pairwise. This proves equation (13). \square

We now apply Proposition 17 to determine the orbits in $P_n/[P_n, P_n]$ for the action of conjugation by the element $\alpha_{0,n} \in B_n/[P_n, P_n]$. If $x \in \mathbb{R}$, $\lfloor x \rfloor$ shall denote the largest integer less than or equal to x .

COROLLARY 18. *Let $n \geq 3$. The set $\{A_{i,j} \in P_n/[P_n, P_n] \mid 1 \leq i < j \leq n\}$ is invariant under the action of conjugation by the element $\alpha_{0,n}$, and there are $\lfloor \frac{n-1}{2} \rfloor$ orbits each of length n given by:*

$$A_{1,j+1} \xrightarrow{\alpha_{0,n}} A_{2,j+2} \xrightarrow{\alpha_{0,n}} \cdots \xrightarrow{\alpha_{0,n}} A_{n-j,n} \xrightarrow{\alpha_{0,n}} A_{1,n-j+1} \xrightarrow{\alpha_{0,n}} A_{2,n-j+2} \xrightarrow{\alpha_{0,n}} \cdots \xrightarrow{\alpha_{0,n}} A_{j,n} \xrightarrow{\alpha_{0,n}} A_{1,j+1} \quad (14)$$

for $j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$. If n is even then there is an additional orbit of length $n/2$ given by:

$$A_{1, \frac{n+2}{2}} \xrightarrow{\alpha_{0,n}} A_{2, \frac{n+4}{2}} \xrightarrow{\alpha_{0,n}} \cdots \xrightarrow{\alpha_{0,n}} A_{\frac{n}{2}, n} \xrightarrow{\alpha_{0,n}} A_{1, \frac{n+2}{2}}.$$

Corollary 18 plays an important rôle in the proof of the following proposition, which states that if n is odd then the crystallographic group $B_n/[P_n, P_n]$ possesses elements of order n .

PROPOSITION 19. *If $n \geq 3$ is odd then $B_n/[P_n, P_n]$ possesses infinitely many elements of order n .*

Proof. Let $n \geq 3$ be odd. For $1 \leq i \leq (n-1)/2$ and $1 \leq j \leq n$, let

$$e_{i,j} = \begin{cases} A_{j,i+j} & \text{if } i+j \leq n \\ A_{i+j-n,j} & \text{if } i+j > n. \end{cases} \quad (15)$$

By equation (14), the action by conjugation of $\alpha_{0,n}$ on the $e_{i,j}$ is given by:

$$e_{i,1} \xrightarrow{\alpha_{0,n}} e_{i,2} \xrightarrow{\alpha_{0,n}} \cdots \xrightarrow{\alpha_{0,n}} e_{i,n-1} \xrightarrow{\alpha_{0,n}} e_{i,n} \xrightarrow{\alpha_{0,n}} e_{i,1} \text{ for } i = 1, \dots, (n-1)/2. \quad (16)$$

In particular, the set $\{e_{i,j}\}_{1 \leq i \leq (n-1)/2, 1 \leq j \leq n}$ is a basis for $P_n/[P_n, P_n]$. The full-twist braid of B_n may be written as $(\sigma_1 \cdots \sigma_{n-1})^n$, or alternatively as the product $\prod_{j=2}^n \left(\prod_{i=1}^j A_{i,j} \right)$. This expression contains each of the $A_{i,j}$ exactly once, and so $\alpha_{0,n}^n = \sum_{\substack{1 \leq i \leq (n-1)/2 \\ 1 \leq j \leq n}} e_{i,j}$ in

$P_n/[P_n, P_n]$, using additive notation for this group. Let $N \in P_n/[P_n, P_n]$, and for $1 \leq i \leq (n-1)/2$ and $1 \leq j \leq n$, let $a_{i,j} \in \mathbb{Z}$ be such that:

$$N = \sum_{\substack{1 \leq i \leq (n-1)/2 \\ 1 \leq j \leq n}} a_{i,j} e_{i,j}. \quad (17)$$

It follows from equation (16) that for all $k = 0, 1, \dots, n-1$,

$$\alpha_{0,n}^k N \alpha_{0,n}^{-k} = \sum_{\substack{1 \leq i \leq (n-1)/2 \\ 1 \leq j \leq n}} a_{i,j} e_{i,j+k},$$

where the second index of $e_{i,j+k}$ is taken modulo n . Hence:

$$\begin{aligned} (N \cdot \alpha_{0,n})^n &= N + \alpha_{0,n} N \alpha_{0,n}^{-1} + \alpha_{0,n}^2 N \alpha_{0,n}^{-2} + \dots + \alpha_{0,n}^{n-1} N \alpha_{0,n}^{1-n} + \alpha_{0,n}^n \\ &= \sum_{i=1}^{n(n-1)/2} \left(\sum_{j=1}^n a_{i,j} \right) \left(\sum_{j=1}^n e_{i,j} \right) + \sum_{\substack{1 \leq i \leq (n-1)/2 \\ 1 \leq j \leq n}} e_{i,j} \\ &= \sum_{i=1}^{n(n-1)/2} \left(\left(\sum_{j=1}^n a_{i,j} \right) + 1 \right) \left(\sum_{j=1}^n e_{i,j} \right). \end{aligned}$$

Thus $(N \cdot \alpha_{0,n})^n$ is equal to the trivial element of $P_n/[P_n, P_n]$ if and only if:

$$\left(\sum_{j=1}^n a_{i,j} \right) + 1 = 0 \text{ for all } i = 1, \dots, (n-1)/2. \quad (18)$$

This system of equations admits infinitely many solutions in \mathbb{Z} . For each such solution, $N \alpha_{0,n}$ is of finite order, and its order divides n . On the other hand, since $\bar{\sigma}(N \alpha_{0,n}) = (1, n, n-1, \dots, 2)$, the order of $N \alpha_{0,n}$ is at least n . We thus conclude that for any $N \in P_n/[P_n, P_n]$ given by the expression (17) whose coefficients satisfy the system (18), the element $N \alpha_{0,n}$ is of order n in $B_n/[P_n, P_n]$. \square

As we shall now see, Proposition 19 implies part of Theorem 3, namely that if $3 \leq n \leq m$ and n is odd then $B_m/[P_m, P_m]$ possesses elements of order n .

Proof of Theorem 3. Let m and n be integers such that $2 \leq n \leq m$.

(a) By equation (5), ι restricts to an injective homomorphism $\iota|_{P_n} : P_n \rightarrow P_m$ given by $\iota|_{P_n}(A_{i,j}) = A_{i,j}$ for all $1 \leq i < j \leq n$. We wish to prove that the induced homomorphism $\bar{\iota} : B_n/[P_n, P_n] \rightarrow B_m/[P_m, P_m]$ is injective. Since $\{A_{i,j}\}_{1 \leq i < j \leq n}$ is a subset of the basis $\{A_{i,j}\}_{1 \leq i < j \leq m}$ of $P_m/[P_m, P_m]$, by regarding the $A_{i,j}$ as elements of the quotient $P_n/[P_n, P_n]$, we see that the restriction $\bar{\iota}|_{P_n/[P_n, P_n]} : P_n/[P_n, P_n] \rightarrow P_m/[P_m, P_m]$ is also injective. Using the short exact sequence (7) and the fact that the homomorphism ι induces an inclusion of S_n in S_m , we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \frac{P_n}{[P_n, P_n]} & \longrightarrow & \frac{B_n}{[P_n, P_n]} & \longrightarrow & S_n \longrightarrow 1 \\ & & \downarrow \bar{\iota}|_{P_n/[P_n, P_n]} & & \downarrow \bar{\iota} & & \downarrow \\ 1 & \longrightarrow & \frac{P_m}{[P_m, P_m]} & \longrightarrow & \frac{B_m}{[P_m, P_m]} & \longrightarrow & S_m \longrightarrow 1. \end{array}$$

The injectivity of $\bar{\iota}$ is then a consequence of the 5-Lemma.

(b) Suppose that $3 \leq n \leq m$ and that n is odd. Proposition 19 implies that $B_n/[P_n, P_n]$ possesses elements of order n . By part (a), $\bar{\iota}$ is injective, and so $B_m/[P_m, P_m]$ also has elements of order n , which proves the first part of the statement. The second part follows from the construction given in the proof of Proposition 19.

(c) Let n_1, \dots, n_t be odd integers greater than or equal to 3 such that $\sum_{i=1}^t n_i \leq m$, let $\sigma: B_{\sum_{i=1}^t n_i} \rightarrow S_{\sum_{i=1}^t n_i}$ denote the usual homomorphism that to a braid associates its permutation, and let B_{n_1, \dots, n_t} denote the corresponding mixed braid group, namely the preimage under σ of the subgroup $S_{n_1} \times \dots \times S_{n_t}$ of $S_{\sum_{i=1}^t n_i}$. We first prove that the group $B_m/[P_m, P_m]$ possesses elements of order $\text{lcm}(n_1, \dots, n_t)$. For $1 \leq i \leq t$, let $\varphi_i: B_{n_i} \rightarrow B_{n_1, \dots, n_t}$ denote the embedding of B_{n_i} into the i^{th} factor of B_{n_1, \dots, n_t} . Since $\varphi_i([P_{n_i}, P_{n_i}]) \subset [P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]$, the homomorphism φ_i induces a homomorphism $\overline{\varphi}_i: \frac{B_{n_i}}{[P_{n_i}, P_{n_i}]} \rightarrow \frac{B_{n_1, n_2, \dots, n_t}}{[P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]}$. Now let $\psi_i: B_{n_1, \dots, n_t} \rightarrow \frac{B_{n_i}}{[P_{n_i}, P_{n_i}]}$ be the composition of the projection onto the i^{th} factor of B_{n_1, \dots, n_t} , followed by the canonical projection $B_{n_i} \rightarrow \frac{B_{n_i}}{[P_{n_i}, P_{n_i}]}$. Under this composition, the normal subgroup $P_{\sum_{i=1}^t n_i}$ of B_{n_1, \dots, n_t} is sent to $\frac{P_{n_i}}{[P_{n_i}, P_{n_i}]}$, hence the normal subgroup $[P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]$ of B_{n_1, \dots, n_t} is sent to the trivial element of $\frac{B_{n_i}}{[P_{n_i}, P_{n_i}]}$, from which it follows that ψ_i induces a homomorphism $\overline{\psi}_i: \frac{B_{n_1, \dots, n_t}}{[P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]} \rightarrow \frac{B_{n_i}}{[P_{n_i}, P_{n_i}]}$. From the constructions of φ_i and ψ_i , we see that $\overline{\psi}_i \circ \overline{\varphi}_i = \text{Id}_{B_{n_i}/[P_{n_i}, P_{n_i}]}$ for all $1 \leq i \leq t$, and so the composition

$$\frac{B_{n_1}}{[P_{n_1}, P_{n_1}]} \times \dots \times \frac{B_{n_t}}{[P_{n_t}, P_{n_t}]} \xrightarrow{\overline{\varphi}_1 \times \dots \times \overline{\varphi}_t} \frac{B_{n_1, n_2, \dots, n_t}}{[P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]} \xrightarrow{\overline{\psi}_1 \times \dots \times \overline{\psi}_t} \frac{B_{n_1}}{[P_{n_1}, P_{n_1}]} \times \dots \times \frac{B_{n_t}}{[P_{n_t}, P_{n_t}]}$$

is the identity. Thus $\overline{\varphi}_1 \times \dots \times \overline{\varphi}_t$ is injective, and the composition

$$\frac{B_{n_1}}{[P_{n_1}, P_{n_1}]} \times \dots \times \frac{B_{n_t}}{[P_{n_t}, P_{n_t}]} \xrightarrow{\overline{\varphi}_1 \times \dots \times \overline{\varphi}_t} \frac{B_{n_1, n_2, \dots, n_t}}{[P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]} \rightarrow \frac{B_{\sum_{i=1}^t n_i}}{[P_{\sum_{i=1}^t n_i}, P_{\sum_{i=1}^t n_i}]} \rightarrow \frac{B_m}{[P_m, P_m]}$$

which we denote by Φ , is injective by part (a) and by the injectivity of the homomorphism $B_{n_1, \dots, n_t} \hookrightarrow B_{\sum_{i=1}^t n_i}$. For $1 \leq i \leq t$, let $\gamma_i \in \frac{B_{n_i}}{[P_{n_i}, P_{n_i}]}$ be an element of order n_i whose permutation is an n_i -cycle; the existence of γ_i is guaranteed by part (b). Then $\gamma = (\gamma_1, \dots, \gamma_t) \in \frac{B_{n_1}}{[P_{n_1}, P_{n_1}]} \times \frac{B_{n_2}}{[P_{n_2}, P_{n_2}]} \times \dots \times \frac{B_{n_t}}{[P_{n_t}, P_{n_t}]}$ is of order $\text{lcm}(n_1, \dots, n_t)$, and the injectivity of Φ implies that $\Phi(\gamma) \in \frac{B_m}{[P_m, P_m]}$ is also of order $\text{lcm}(n_1, \dots, n_t)$. The second part of the statement follows also. \square

As a consequence, we are able to prove Corollary 4, which says that the torsion of $B_n/[P_n, P_n]$ is equal to the odd torsion of the symmetric group S_n , and that the map induced by $\bar{\sigma}$ from the set of finite cyclic subgroups of $B_n/[P_n, P_n]$ to the set of cyclic subgroups of S_n of odd order is surjective.

Proof. Let $\beta \in B_n/[P_n, P_n]$ be a non-trivial element of finite order r . By Theorem 2, r is odd. Lemma 9 implies that $\bar{\sigma}(\beta)$ is also of order r , so the torsion of $B_n/[P_n, P_n]$ is contained in the odd torsion of the symmetric group S_n . Conversely, suppose that θ is an element of S_n of odd order $r \geq 3$, and let $\theta = \theta_1\theta_2 \cdots \theta_t$ be a product of disjoint non-trivial cycles, where θ_i is an n_i -cycle for all $i = 1, \dots, t$. Then $r = \text{lcm}(n_1, \dots, n_t)$, the n_i are odd and greater than or equal to 3, and $\sum_{i=1}^t n_i \leq n$ since the θ_i are disjoint. By Theorem 3(c), $B_n/[P_n, P_n]$ possesses an element γ of order r whose permutation has cycle type (n_1, \dots, n_t) . So $\bar{\sigma}(\gamma)$ is conjugate to θ , and thus a suitable conjugate of γ is an element of order r whose permutation is equal to θ . The last part of the statement follows in a straightforward manner. \square

REMARK 20. In order to study the conjugacy classes of finite-order elements of the group $B_n/[P_n, P_n]$, we will describe some of these elements in more detail in Section 6.

5 A study of some crystallographic subgroups of dimension 3 of $B_3/[P_3, P_3]$

As we saw in Section 3, the group $B_3/[P_3, P_3]$ is crystallographic and has no 2-torsion. In this section, we further analyse this quotient and we study some of the crystallographic subgroups of $B_3/[P_3, P_3]$ of dimension 3, of the form $\bar{\sigma}^{-1}(H)$, where H is a subgroup of S_3 . In order to study these subgroups, it suffices to consider a representative of each conjugacy class of subgroups of S_3 . We shall also comment on some other subgroups of $B_3/[P_3, P_3]$.

PROPOSITION 21. *Let H be a subgroup of S_3 , and let \tilde{H}_3 be given by equation (10).*

- (a) *Let $H = \{1\}$. The crystallographic group \tilde{H}_3 admits a presentation whose generators are $A_{1,2}, A_{1,3}, A_{2,3}$, with defining relations $[A_{1,2}, A_{1,3}] = 1$, $[A_{1,2}, A_{2,3}] = 1$ and $[A_{1,3}, A_{2,3}] = 1$.*
(b) *Let $H = \langle (1, 3, 2) \rangle$. The crystallographic group \tilde{H}_3 is normal in $B_3/[P_3, P_3]$ and admits a presentation given by:*

- *generators:* $A_{1,2}, A_{2,3}, A_{1,3}, \alpha_{0,3}$, where $\alpha_{0,3} = \sigma_1\sigma_2 \in B_3/[P_3, P_3]$.

- *relations:*

(i) $[A_{1,2}, A_{1,3}] = 1, [A_{1,2}, A_{2,3}] = 1, [A_{1,3}, A_{1,3}] = 1.$

(ii) $\alpha_{0,3}^3 = \Delta_3^2 = A_{1,2}A_{1,3}A_{2,3}$ (Δ_3^2 is the class of the full-twist braid in $P_3/[P_3, P_3]$).

(iii) $\alpha_{0,3}A_{1,2}\alpha_{0,3}^{-1} = A_{2,3}, \alpha_{0,3}A_{1,3}\alpha_{0,3}^{-1} = A_{1,2}, \alpha_{0,3}A_{2,3}\alpha_{0,3}^{-1} = A_{1,3}.$

The Abelianisation $(\tilde{H}_3)_{Ab}$ of \tilde{H}_3 is given by:

$$(\tilde{H}_3)_{Ab} = \langle A_{1,2}, \alpha_{0,3} \mid [A_{1,2}, \alpha_{0,3}] = 1, \alpha_{0,3}^3 = A_{1,2}^3 \rangle,$$

and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$, where the factors are generated by $A_{1,2}$ and $A_{1,2}\alpha_{0,3}^{-1}$.

- (c) *Let $H = \langle (1, 2) \rangle$. The crystallographic group \tilde{H}_3 admits a presentation given by:*

- *generators:* $A_{1,2}, A_{2,3}, A_{1,3}, \sigma_1.$

- *relations:*

- (i) $[A_{1,2}, A_{1,3}] = 1, [A_{1,2}, A_{2,3}] = 1, [A_{1,3}, A_{1,3}] = 1.$
- (ii) $\sigma_1^2 = A_{1,2}.$
- (iii) $\sigma_1 A_{1,2} \sigma_1^{-1} = A_{1,2}, \sigma_1 A_{1,3} \sigma_1^{-1} = A_{2,3}, \sigma_1 A_{2,3} \sigma_1^{-1} = A_{1,3}.$

We have:

$$\left(\tilde{H}_3\right)_{Ab} = \left\langle A_{1,2}, A_{1,3}, \sigma_1 \mid [A_{1,2}, \sigma_1] = 1, [A_{1,3}, \sigma_1] = 1, \sigma_1^2 = A_{1,2} \right\rangle \cong \mathbb{Z} \oplus \mathbb{Z}.$$

(d) Let $H_3 = S_3$. The crystallographic group $\tilde{H}_3 = B_3/[P_3, P_3]$ admits a presentation whose generators are σ_1, σ_2 , with defining relations $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ and $(\sigma_1^{-1} \sigma_2)^3 = 1$. We have $\left(\tilde{H}_3\right)_{Ab} = \langle \sigma_1 \rangle \cong \mathbb{Z}.$

Proof. Part (a) follows from the fact that $\tilde{H}_3 = P_3/[P_3, P_3]$ if H is trivial. The presentations given in parts (b) and (c) may be obtained by applying the method of presentations of group extensions given in [J, Section 10.2]. In part (b), the normality of \tilde{H}_3 follows from that of \mathbb{Z}_3 in S_3 .

By [O, Lemma 4.3.9], the commutator subgroup $[P_3, P_3]$ is equal to the normal closure of $[A_{1,2}, A_{2,3}]$ in B_3 . Since $[A_{1,2}, A_{2,3}] = (\sigma_1^{-1} \sigma_2)^3$ and $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$, we thus obtain the presentation given in part (d). In each case, $\left(\tilde{H}_3\right)_{Ab}$ is obtained in a straightforward manner from the presentation of \tilde{H}_3 . \square

REMARK 22. The presentation of $B_3/[P_3, P_3]$ of Proposition 21(d) also appeared in [O, Proposition 4.3.10] and in [LW, Proposition 3.9].

THEOREM 23. Let H be a subgroup of S_3 , and let \tilde{H}_3 be given by equation (10).

- (a) Let $H = \{1\}$. Then \tilde{H}_3 is isomorphic to the quotient $P_3/[P_3, P_3]$, which is isomorphic to \mathbb{Z}^3 . The corresponding flat manifold is the 3-torus.
- (b) Let $H = \langle (1, 2) \rangle$. Then \tilde{H}_3 is a Bieberbach group of dimension 3 with holonomy group \mathbb{Z}_2 . The corresponding flat Riemannian manifold is diffeomorphic to the non-orientable manifold \mathcal{B}_2 that appears in the classification of flat Riemannian 3-manifolds given in [W, Corollary 3.5.10].
- (c) Let $H = \langle (1, 3, 2) \rangle$. Then \tilde{H}_3 is isomorphic to the semi-direct product $\mathbb{Z}^3 \rtimes \mathbb{Z}_3$, where the action is given by the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with respect to the basis $(A_{1,2}, A_{2,3}, A_{1,3})$ of $P_3/[P_3, P_3]$, this quotient being identified with \mathbb{Z}^3 .

Proof. Part (a) is clear, so let us prove part (b). Theorem 2 implies that $B_3/[P_3, P_3]$ has no 2-torsion, and so the subgroup \tilde{H}_3 is a Bieberbach group of dimension 3 by Lemma 14. Let X be the flat Riemannian manifold uniquely determined by \tilde{H}_3 , so that $\pi_1(X) = \tilde{H}_3$. The holonomy representation of \tilde{H}_3 is a homomorphism of the form $\mathbb{Z}_2 \rightarrow \text{Aut}(\mathbb{Z}^3)$, where we identify $P_3/[P_3, P_3]$ with \mathbb{Z}^3 . Relative to the basis $(A_{1,2}, A_{1,3}, A_{2,3})$ of $P_3/[P_3, P_3]$, by equation (8), the image of the generator of \mathbb{Z}_2 by this homomorphism is given by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ whose determinant is equal to -1 . Thus X is a non-orientable flat Riemannian 3-manifold with holonomy group \mathbb{Z}_2 . Up to affine diffeomorphism, X is one of the two manifolds \mathcal{B}_1 or \mathcal{B}_2 described in [W, Theorem 3.5.9]. Using the presentation of \tilde{H}_3 given in Proposition 21(b) we have $H_1(X; \mathbb{Z}) \cong \mathbb{Z}^2$, and from the table in [W, Corollary 3.5.10], we conclude that $X = \mathcal{B}_2$.

Finally we prove part (c). The following short exact sequence:

$$1 \longrightarrow P_3/[P_3, P_3] \longrightarrow \tilde{H}_3 \longrightarrow H \longrightarrow 1$$

admits a section given by sending the generator $(1, 3, 2)$ of H to the element $\sigma_1^{-1}\sigma_2$ of \tilde{H}_3 , and so \tilde{H}_3 is isomorphic to the semi-direct product of the form $\mathbb{Z}^3 \rtimes \mathbb{Z}_3$. Relative to the basis $(A_{1,2}, A_{2,3}, A_{1,3})$ of $P_3/[P_3, P_3]$, the matrix of the associated action is equal to $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. \square

REMARKS 24.

(a) The subgroup of $B_3/[P_3, P_3]$ generated by the class of the full-twist braid $A_{1,2}A_{1,3}A_{2,3}$, given by $(1, 1, 1)$ in terms of the basis $(A_{1,2}, A_{1,3}, A_{2,3})$ of $P_3/[P_3, P_3]$, is a normal subgroup of $B_3/[P_3, P_3]$. The associated quotient group admits the following presentation that is obtained from a presentation of $B_3/[P_3, P_3]$:

$$\left\langle \sigma_1, \sigma_2 \mid \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1, (\sigma_1^{-1}\sigma_2)^3 = 1, A_{1,2}A_{1,3}A_{2,3} = 1 \right\rangle.$$

The group G_3^1 given in the first theorem of [Ly, page 73] is generated by the set $\{\alpha, \beta, \sigma, \rho\}$, with relations:

$$[\alpha, \beta] = [\rho, \alpha] = \sigma^3 = \rho^2 = (\sigma\rho)^2 = 1, \sigma\alpha\sigma^{-1} = \alpha^{-1}\beta, \sigma\beta\sigma^{-1} = \alpha^{-1}, \rho\beta\rho^{-1} = \alpha\beta^{-1}.$$

A routine calculation shows that the map that sends $A_{1,2}$ (resp. $A_{1,3}^{-1}, A_{1,3}\sigma_1, \sigma_1^{-1}\sigma_2$) to α (resp. β, ρ, σ) extends to an isomorphism of the two groups.

(b) The group $B_3/[P_3, P_3]$ is the three-dimensional crystallographic group that appears as 5/4/1:SPGR:02 of [BBNWS, page 71], and that corresponds to IT 161; OBT 1 in the international table [HL].

(c) Let L be a crystallographic subgroup of $B_3/[P_3, P_3]$ of dimension 3, and consider the subgroup $\bar{\sigma}(L)$ of S_3 . If $\bar{\sigma}(L) = \{\text{Id}\}$ then clearly L is isomorphic to \mathbb{Z}^3 . If $\bar{\sigma}(L) = \langle(1, 2)\rangle$ then L is a Bieberbach group. If $\bar{\sigma}(L) = \langle(1, 3, 2)\rangle$ then the group L may be Bieberbach or not, with holonomy \mathbb{Z}_3 . For example, if L is the subgroup generated by $\{\sigma_1^{-1}\sigma_2, A_{1,2}^2, A_{1,3}^2, A_{2,3}^2\}$ then L is a proper crystallographic subgroup of $\bar{\sigma}^{-1}(\langle(1, 3, 2)\rangle)$ of dimension 3 with holonomy \mathbb{Z}_3 and that admits torsion elements, $\sigma_1^{-1}\sigma_2$ for example.

On the other hand, if L is the subgroup generated by $\{A_{1,2}\sigma_1^{-1}\sigma_2, A_{1,2}^3, A_{1,3}^3, A_{2,3}^3\}$ then L is a proper subgroup of $\bar{\sigma}^{-1}(\langle(1, 3, 2)\rangle)$, and is a Bieberbach group of dimension 3 with holonomy \mathbb{Z}_3 . To see this, let $L_1 = L \cap \text{Ker}(\bar{\sigma}) = L \cap \langle A_{1,2}, A_{1,3}, A_{2,3} \rangle$. Clearly L_1 is a free Abelian group, so is torsion free. Using equation (8), we see that $(A_{1,2}\sigma_1^{-1}\sigma_2)^3 = A_{1,2}A_{1,3}A_{2,3}$, and since $\{(A_{1,2}, A_{1,3}, A_{2,3})^j\}_{j \in \{0,1,2\}}$ is a set of coset representatives of L_1 in L , it follows that L_1 is generated by $\{A_{1,2}A_{1,3}A_{2,3}, A_{1,2}^3, A_{1,3}^3, A_{2,3}^3\}$. Note then that $\{A_{1,2}A_{1,3}A_{2,3}, A_{1,3}^3, A_{2,3}^3\}$ is a basis of L_1 . Suppose that w is a non-trivial torsion element of L . By Lemma 9, w must be of order 3. Now $w \notin L_1$, so there exist $\theta \in L_1$ and $j \in \{1, 2\}$ such that $w = \theta(A_{1,2}\sigma_1^{-1}\sigma_2)^j$. Since $\theta \in L_1$, there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$ such that $\theta = (A_{1,2}A_{1,3}A_{2,3})^{\lambda_1} A_{1,3}^{3\lambda_2} A_{2,3}^{3\lambda_3}$, and hence:

$$1 = w^3 = \theta(A_{1,2}\sigma_1^{-1}\sigma_2)^j \theta(A_{1,2}\sigma_1^{-1}\sigma_2)^{-j} \cdot (A_{1,2}\sigma_1^{-1}\sigma_2)^{2j} \theta(A_{1,2}\sigma_1^{-1}\sigma_2)^{-2j} \cdot (A_{1,2}\sigma_1^{-1}\sigma_2)^{3j}.$$

Using once more equation (8), the relation $(A_{1,2}\sigma_1^{-1}\sigma_2)^3 = A_{1,2}A_{1,3}A_{2,3}$, and comparing the coefficients of $A_{1,2}$, $A_{1,3}$ and $A_{2,3}$, we obtain the equality $3(\lambda_1 + \lambda_2 + \lambda_3) + j = 0$, which has no solution in \mathbb{Z} . It follows that L is torsion free, and so is a Bieberbach group of dimension 3 with holonomy \mathbb{Z}_3 .

(d) There is no Bieberbach subgroup of $B_3/[P_3, P_3]$ of dimension 3 that projects to S_3 , since none of the ten flat Riemannian 3-manifolds have fundamental group with holonomy S_3 (see [W, Theorems 3.5.5 and 3.5.9]).

6 Conjugacy classes of finite-order elements of $B_n/[P_n, P_n]$

In this section, we study the conjugacy classes of finite-order elements of $B_n/[P_n, P_n]$. The aim is to prove Theorem 5, which states that there is a bijection between the conjugacy classes of cyclic subgroups of odd order of $B_n/[P_n, P_n]$ and the set of conjugacy classes of cyclic subgroups of odd order of the symmetric group S_n .

We begin with an elementary fact about conjugacy classes that will help to simplify the study of our problem.

LEMMA 25. *Let $\alpha, \beta \in B_n/[P_n, P_n]$ be two conjugate elements of finite order. Then $\bar{\sigma}(\alpha)$ and $\bar{\sigma}(\beta)$ are permutations of odd order and have the same cycle type.*

Proof. Since α, β are of finite order, their common order is odd by Theorem 2. The fact that α and β are conjugate in $B_n/[P_n, P_n]$ implies that the permutations $\bar{\sigma}(\alpha)$ and $\bar{\sigma}(\beta)$ are conjugate in S_n . The result then follows since two permutations are conjugate in S_n if and only they have the same cycle type. \square

In order to analyse the conjugacy classes of elements of finite order, Lemma 25 implies that it suffices to choose a single representative permutation for each conjugacy class of S_n of odd order and to study the conjugacy classes of elements of $B_n/[P_n, P_n]$ of finite order that project to the chosen permutation.

Let us consider the action by conjugation of certain elements of $B_n/[P_n, P_n]$ on the group $P_n/[P_n, P_n]$. If $k, n \geq 3$ and $r \geq 0$ are integers such that $r + k \leq n$, define $\delta_{r,k}, \alpha_{r,k} \in B_n/[P_n, P_n]$ by:

$$\delta_{r,k} = \sigma_{r+k-1} \cdots \sigma_{r+\frac{k+1}{2}} \sigma_{r+\frac{k-1}{2}}^{-1} \cdots \sigma_{r+1}^{-1} \text{ and } \alpha_{r,k} = \sigma_{r+1} \cdots \sigma_{r+k-1}. \quad (19)$$

LEMMA 26. *Let $n, k \geq 3$ and $r \geq 0$ be integers such that k is odd and $r + k \leq n$. Then $\delta_{r,k}$ is of order k in $B_n/[P_n, P_n]$, and satisfies:*

$$\delta_{r,k} = \left(A_{r+\frac{k+1}{2}, r+k} A_{r+\frac{k+3}{2}, r+k} \cdots A_{r+k-1, r+k} \right) \alpha_{r,k}^{-1}. \quad (20)$$

Furthermore, the action of conjugation by $\alpha_{r,k}$ on the basis elements $A_{i,j}$ of $P_n/[P_n, P_n]$ is given

by:

$$\alpha_{r,k} A_{i,j} \alpha_{r,k}^{-1} = \begin{cases} A_{i,j} & \text{if } i, j \notin \{r+1, \dots, r+k\} \\ A_{i+1, j+1} & \text{if } r+1 \leq i < j \leq r+k-1 \\ A_{r+1, i+1} & \text{if } r+1 \leq i < j = r+k \\ A_{i, j+1} & \text{if } i < r+1 \leq j \leq r+k-1 \\ A_{i, r+1} & \text{if } i < r+1 \text{ and } j = r+k \\ A_{i+1, j} & \text{if } r+1 \leq i \leq r+k-1 \text{ and } r+k < j \leq n \\ A_{r+1, j} & \text{if } i = r+k < j \leq n, \end{cases} \quad (21)$$

and the action of conjugation by $\delta_{r,k}$ is the inverse action of $\alpha_{r,k}$ and is given by:

$$\delta_{r,k} A_{i,j} \delta_{r,k}^{-1} = \begin{cases} A_{i,j} & \text{if } i, j \notin \{r+1, \dots, r+k\} \\ A_{i-1, j-1} & \text{if } r+2 \leq i < j \leq r+k \\ A_{j-1, r+k} & \text{if } r+1 = i < j \leq r+k \\ A_{i, j-1} & \text{if } i < r+1 < j \leq r+k \\ A_{i, r+k} & \text{if } i < r+1 \text{ and } j = r+1 \\ A_{i-1, j} & \text{if } r+1 < i \leq r+k \text{ and } r+k < j \leq n \\ A_{r+k, j} & \text{if } r+1 = i \text{ and } r+k < j \leq n. \end{cases} \quad (22)$$

Proof. Let $n \geq 3$, $k \geq 3$ and $r \geq 0$ be integers such that k is odd and $r+k \leq n$. We start by proving equation (20) and by showing that $\delta_{r,k} \in B_n/[P_n, P_n]$ is of order k . First let $r = 0$. Then:

$$\begin{aligned} \delta_{0,k} \alpha_{0,k} &= \sigma_{k-1} \cdots \sigma_{\frac{k+1}{2}} \sigma_{\frac{k-1}{2}}^{-1} \cdots \sigma_1^{-1} \sigma_1 \cdots \sigma_{\frac{k-1}{2}} \sigma_{\frac{k+1}{2}} \cdots \sigma_{k-1} \\ &= \sigma_{k-1} \cdots \sigma_{\frac{k+3}{2}} \sigma_{\frac{k+1}{2}}^2 \sigma_{\frac{k+3}{2}} \cdots \sigma_{k-1} = A_{\frac{k+1}{2}, k} A_{\frac{k+3}{2}, k} \cdots A_{k-1, k}, \end{aligned} \quad (23)$$

which yields the equality (20). Set $N = (A_{\frac{k+1}{2}, k} A_{\frac{k+3}{2}, k} \cdots A_{k-1, k})^{-1} \in P_n/[P_n, P_n]$. Then $\alpha_{0,k}^{-1} \delta_{0,k}^{-1} \alpha_{0,k} = N \alpha_{0,k}$ by equation (23), and so $\delta_{0,k}$ and $N \alpha_{0,k}$ are of the same order. Considering N and $\alpha_{0,k}$ to be elements of $B_k/[P_k, P_k]$ for a moment, and using (15) and (17),

we have $N = - \sum_{i=1}^{\frac{k-1}{2}} e_{i, k-i}$, and so N satisfies the system of equations (18) (taking $n = k$

in that system). It follows from the proof of Proposition 19 that $N \alpha_{0,k}$ is of order k in $B_k/[P_k, P_k]$, and so $\delta_{0,k}$ is of order k in $B_k/[P_k, P_k]$. Since $k \leq n$, we deduce from Theorem 3(a) that $\delta_{0,k}$, considered as an element of $B_n/[P_n, P_n]$, is also of order k .

Now assume that $r \geq 1$. Let ψ denote the composition of the following homomorphisms:

$$\frac{B_k}{[P_k, P_k]} \longrightarrow \frac{B_{r,k,n-r-k}}{[P_n, P_n]} \longrightarrow \frac{B_n}{[P_n, P_n]}'$$

where the first homomorphism is induced by the inclusion $B_k \hookrightarrow B_{r,k,n-r-k}$ of B_k in the middle block of the mixed braid group $B_{r,k,n-r-k}$, and the second homomorphism is induced by the inclusion $B_{r,k,n-r-k} \hookrightarrow B_n$. In a manner similar to that for \bar{i} in the proof of Theorem 3(a), the homomorphism ψ may be seen to be injective. For all $1 \leq i \leq n-1$, $\psi(\sigma_i) = \sigma_{r+i}$, hence $\psi(\delta_{0,k}) = \delta_{r,k}$ and $\psi(\alpha_{0,k}) = \alpha_{r,k}$ by equation (19). The

injectivity of ψ implies that $\delta_{r,k}$ is of order k in $\frac{B_n}{[P_n, P_n]}$. Moreover, for all $1 \leq i < j \leq n$, $\psi(A_{i,j}) = A_{r+i,r+j}$ by equation (5), thus:

$$\begin{aligned} \delta_{r,k} &= \psi(\delta_{0,k}) = \psi \left(\left(A_{\frac{k+1}{2},k} A_{\frac{k+3}{2},k} \cdots A_{k-1,k} \right) \alpha_{0,k}^{-1} \right) \quad \text{by equation (23)} \\ &= \left(A_{r+\frac{k+1}{2},r+k} A_{r+\frac{k+3}{2},r+k} \cdots A_{r+k-1,r+k} \right) \alpha_{r,k}^{-1}, \end{aligned}$$

which is equation (20). This proves the first part of the statement. It remains to establish equations (21) and (22). The first relation of (21) holds clearly. Applying ψ to both sides of equations (12) and (13) (and taking $r = k$) gives rise to the second and third relations of (21). Finally, equation (8) yields the four remaining relations of (21). To obtain (22), by equation (20), conjugation by $\alpha_{r,k} \delta_{r,k}$ in $P_n/[P_n, P_n]$ is conjugation by an element of $P_n/[P_n, P_n]$, which gives rise to the trivial action. So the actions by conjugation of $\alpha_{r,k}$ and $\delta_{r,k}$ on $P_n/[P_n, P_n]$ are mutual inverses. Equation (21) then implies equation (22). \square

Another corollary of Theorem 3(c), which we now prove, is Theorem 6, which states that there is a one-to-one correspondence between the finite Abelian subgroups of $B_n/[P_n, P_n]$ and the Abelian subgroups of S_n of odd order.

Proof of Theorem 6. First, it follows from Remarks 13(c) that the isomorphism class of a finite Abelian subgroup of $B_n/[P_n, P_n]$ is realised by a subgroup of S_n (of odd order). Conversely, let H be an Abelian subgroup of S_n of odd order. Then H is isomorphic to a direct product of the form $\mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_r}$, where for $i = 1, \dots, r$, k_i is a power of an odd prime number. By [Ho], $\sum_{i=1}^r k_i \leq n$. Let $k_0 = 0$. Then for $l = 1, \dots, r$, the element $\delta_{\sum_{j=0}^{l-1} k_j, k_l}$ belongs to $B_n/[P_n, P_n]$ and is of order k_l by Lemma 26. By construction, the

$\delta_{\sum_{j=1}^{l-1} k_j, k_l}$ commute pairwise. The subgroup $\left\langle \delta_{0,k_1}, \dots, \delta_{\sum_{j=1}^{r-1} k_j, k_r} \right\rangle$ is then isomorphic to H since $\bar{\sigma} \left(\delta_{\sum_{j=1}^{l-1} k_j, k_l} \right)$ is a k_l -cycle in S_n , and the supports of such cycles are pairwise disjoint. \square

The following two propositions are immediate consequences of Lemma 26.

PROPOSITION 27. *Let $n, k \geq 3$ and $r \geq 0$ be integers such that k is odd, and suppose that $3 \leq r + k \leq n$. Then the action of conjugation by $\delta_{r,k}$ on $P_n/[P_n, P_n]$ restricts to an action on the set $A = \{A_{i,j} \mid r+1 \leq i < j \leq r+k\}$. The orbits of this action partition the set A into $\frac{k-1}{2}$ orbits each of length k and given by:*

$$\begin{aligned} &A_{r+1,r+i+1} \xrightarrow{\delta_{r,k}} A_{r+i,r+k} \xrightarrow{\delta_{r,k}} A_{r+i-1,r+k-1} \xrightarrow{\delta_{r,k}} \cdots \xrightarrow{\delta_{r,k}} A_{r+2,r+k-i+2} \xrightarrow{\delta_{r,k}} \\ &A_{r+1,r+k-i+1} \xrightarrow{\delta_{r,k}} A_{r+k-i,r+k} \xrightarrow{\delta_{r,k}} \cdots \xrightarrow{\delta_{r,k}} A_{r+2,r+i+2} \xrightarrow{\delta_{r,k}} A_{r+1,r+i+1} \quad \text{for } i = 1, \dots, \frac{k-1}{2}. \end{aligned}$$

Proof. The result follows from the second and third lines of equation (22). \square

PROPOSITION 28. *Let $n, k \geq 3$ and $r \geq 0$ be integers such that k is odd, and suppose that $3 \leq r + k \leq n$. Then for each $j > r + k$, the action of conjugation by $\delta_{r,k}$ on $P_n/[P_n, P_n]$ restricts to a transitive action on the set $\{A_{i,j} \mid r+1 \leq i \leq r+k\}$, whose orbit of length k is given by:*

$$A_{r+k,j} \xrightarrow{\delta_{r,k}} A_{r+k-1,j} \xrightarrow{\delta_{r,k}} \cdots \xrightarrow{\delta_{r,k}} A_{r+2,j} \xrightarrow{\delta_{r,k}} A_{r+1,j} \xrightarrow{\delta_{r,k}} A_{r+k,j}. \quad (24)$$

Similarly, if $i < r + 1$, the action of conjugation by $\delta_{r,k}$ on $P_n/[P_n, P_n]$ restricts to a transitive action on the set $A = \{A_{i,j} \mid r + 1 \leq j \leq r + k\}$, whose orbit of length k is given by:

$$A_{i,r+k} \xrightarrow{\delta_{r,k}} A_{i,r+k-1} \xrightarrow{\delta_{r,k}} \cdots \xrightarrow{\delta_{r,k}} A_{i,r+2} \xrightarrow{\delta_{r,k}} A_{i,r+1} \xrightarrow{\delta_{r,k}} A_{i,r+k}. \quad (25)$$

Proof. Equation (24) (resp. equation (25)) follows from the 4th and 5th lines (resp. the 6th and 7th lines) of equation (22). \square

PROPOSITION 29. Let $n, k, l \geq 3$ and $r, s \geq 0$ be integers such that k and l are odd, $3 \leq r + k < s + 1$ and $s + l \leq n$, and let $\ell_0 = \text{lcm}(k, l)$. The action of conjugation by $\delta_{r,k}\delta_{s,l}$ on $P_n/[P_n, P_n]$ restricts to an action on the set $\{A_{i,j} \mid r + 1 \leq i \leq r + k \text{ and } s + 1 \leq j \leq s + l\}$, given by:

$$\delta_{r,k}\delta_{s,l}A_{i,j}(\delta_{r,k}\delta_{s,l})^{-1} = \begin{cases} A_{i-1,j-1} & \text{if } r + 1 < i \leq r + k \text{ and } s + 1 < j \leq s + l \\ A_{r+k,j-1} & \text{if } i = r + 1 \text{ and } s + 1 < j \leq s + l \\ A_{i-1,s+l} & \text{if } r + 1 < i \leq r + k \text{ and } j = s + 1 \\ A_{r+k,s+l} & \text{if } i = r + 1 \text{ and } j = s + 1. \end{cases}$$

The orbits of the action partition this set into kl/ℓ_0 orbits of length ℓ_0 given by combining (24) and (25).

Proof. The result follows by applying Propositions 27 and 28 to the action of $\delta_{r,k}$ and $\delta_{s,l}$ on the elements of the set $\{A_{i,j} \mid r + 1 \leq i \leq r + k \text{ and } s + 1 \leq j \leq s + l\}$. \square

PROPOSITION 30. Let $n \geq 3$, and let β be an element of $B_n/[P_n, P_n]$. If $m \geq 0$ is such that $\bar{\sigma}(\beta)$ belongs to the subgroup of S_n isomorphic to S_q on the symbols $\{m + 1, m + 2, \dots, m + q\}$ then the action of β on the set $\{A_{i,j} \mid 1 \leq i < j \leq m \text{ or } m + q + 1 \leq i < j \leq n\}$ is trivial.

Proof. We just prove the claim for $m = 0$ since the remaining cases are similar. Let $1 \leq q < n - 1$. By Theorem 3(a), the inclusion $\iota: B_q \rightarrow B_n$ induces an injective homomorphism $\bar{\iota}: B_q/[P_q, P_q] \rightarrow B_n/[P_n, P_n]$. Since $\bar{\sigma}(\beta) \in S_q$, there exists $\tau \in B_q/[P_q, P_q]$ such that $\bar{\sigma}(\beta) = \bar{\sigma}(\bar{\iota}(\tau))$ is the identity permutation, and so there exists $\beta' \in P_n/[P_n, P_n]$ such that $\beta = \bar{\iota}(\tau)\beta'$. The result follows from equation (8) and the fact that β' is central in $P_n/[P_n, P_n]$. \square

Let $n \geq 3$, let $k_0 = 0$, let $3 \leq k_1 \leq k_2 \leq \dots \leq k_s$ be odd, and suppose that $\sum_{j=1}^s k_j \leq n$. We define:

$$\delta = \delta(k_0, \dots, k_s) = \delta_{k_0, k_1} \delta_{k_1, k_2} \delta_{k_1+k_2, k_3} \cdots \delta_{\sum_{j=1}^{s-1} k_j, k_s}, \quad (26)$$

where for all $0 \leq l \leq s - 1$, the element $\delta_{\sum_{j=1}^l k_j, k_{l+1}}$ is given by equation (19). Since the $\delta_{\sum_{j=1}^l k_j, k_{l+1}}$ commute pairwise and $\delta_{\sum_{j=1}^l k_j, k_{l+1}}$ is of order k_{l+1} by Lemma 26, it follows that δ is of order $\text{lcm}(k_1, \dots, k_s)$ in $B_n/[P_n, P_n]$ and:

$$\theta = \bar{\sigma}(\delta) = \theta_1 \cdots \theta_s, \quad (27)$$

where for $i = 1, \dots, s$, θ_i is the k_i -cycle defined by:

$$\theta_i = \left(\sum_{j=1}^{i-1} k_j + 1, \sum_{j=1}^{i-1} k_j + 2, \dots, \sum_{j=1}^i k_j \right). \quad (28)$$

The order of the permutation θ is also equal to $\text{lcm}(k_1, k_2, \dots, k_s)$. Using Propositions 27, 28, 29 and 30, we shall now describe the orbits given by the action of conjugation by δ on the basis $\{A_{i,j} \mid 1 \leq i < j \leq n\}$ of $P_n/[P_n, P_n]$. The associated partition will be useful when it comes to proving Theorem 5.

THEOREM 31. *Let $n \geq 3$, let $k_0 = 0$, let $3 \leq k_1 \leq k_2 \leq \dots \leq k_s$ be odd such that $\sum_{j=1}^s k_j \leq n$, and let $\delta \in B_n/[P_n, P_n]$ be defined by equation (26). The following sets are disjoint and invariant under the action by conjugation of δ on the set of basis elements $\{A_{i,j} \mid 1 \leq i < j \leq n\}$ of $P_n/[P_n, P_n]$:*

(a) $\left\{A_{i,j} \mid \sum_{l=1}^{r-1} k_l + 1 \leq i < j \leq \sum_{l=1}^r k_l\right\}$, where $1 \leq r \leq s$. Under the given action, the orbits of this set are obtained from the relations $e_{r,h,t} \xrightarrow{\delta} e_{r,h,t+1}$, where $1 \leq h \leq \frac{k_r-1}{2}$, the index t is taken modulo k_r , and

$$e_{r,h,t} = \begin{cases} A_{\sum_{l=1}^{r-1} k_l + h - t + 1, \sum_{l=1}^r k_l - t + 1} & \text{if } t \in \{1, \dots, h\} \\ A_{\sum_{l=1}^r k_l - t + 1, \sum_{l=1}^{r-1} k_l - t + 1 + h} & \text{if } t \in \{h+1, \dots, k_r\}. \end{cases}$$

(b) $\left\{A_{i,j} \mid \sum_{l=1}^{r-1} k_l + 1 \leq i \leq \sum_{l=1}^r k_l \text{ and } \sum_{l=1}^s k_l < j \leq n\right\}$, where $1 \leq r \leq s$. Under the given action, the orbits of this set are obtained from the relations $e_{r,j,t} \xrightarrow{\delta} e_{r,j,t+1}$, where the index t is taken modulo k_r , and $e_{r,j,t} = A_{\sum_{l=1}^{r-1} k_l + t, j}$.

(c) $\left\{A_{i,j} \mid \sum_{l=1}^{p-1} k_l + 1 \leq i \leq \sum_{l=1}^p k_l \text{ and } \sum_{l=1}^{q-1} k_l + 1 \leq j \leq \sum_{l=1}^q k_l\right\}$, where $1 \leq p < q \leq s$. Under the given action, the orbits of this set are obtained from the relations $e_{p,q,v,t} \xrightarrow{\delta} e_{p,q,v,t+1}$, where $1 \leq v \leq \frac{k_p \cdot k_q}{\text{lcm}(k_p, k_q)}$, $1 \leq t \leq \text{lcm}(k_p, k_q)$, and

$$e_{p,q,v,t} = A_{\sum_{l=1}^{p-1} k_l + [2-t]_{k_p}, \sum_{l=1}^{q-1} k_l + [1-t+v]_{k_q}}$$

where the notation $[x]_n$ means the positive integer between 1 and n that is congruent to x modulo n .

(d) $\{A_{i,j} \mid \sum_{l=1}^s k_l < i < j \leq n\}$. Under the given action, each $A_{i,j}$ is fixed.

Proof. Parts (a)–(d) follow from Propositions 27, 28, 29 and 30 respectively. \square

Let k_1, \dots, k_s be as in the statement of Theorem 31, and let \mathcal{B} denote the basis of $P_n/[P_n, P_n]$ that consists of the following elements:

(a) $e_{r,h,t}$, where $1 \leq r \leq s$, $1 \leq h \leq \frac{k_r-1}{2}$ and $1 \leq t \leq k_r$.

(b) $e_{r,j,t}$, where $1 \leq r \leq s$, $\sum_{l=1}^s k_l < j < n$ and $1 \leq t \leq k_r$.

(c) $e_{p,q,v,t}$, where $1 \leq p < q \leq s$, $1 \leq v \leq \frac{k_p \cdot k_q}{\text{lcm}(k_p, k_q)}$ and $1 \leq t \leq \text{lcm}(k_p, k_q)$.

(d) $A_{i,j}$, where $\sum_{l=1}^s k_l < i < j \leq n$.

An element of \mathcal{B} will then be said to be of type (a), (b) (c) or (d) respectively. If $A \in P_n/[P_n, P_n]$, it may thus be written uniquely in the following form:

$$A = \prod_{\substack{1 \leq r \leq s \\ 1 \leq h \leq \frac{k_r-1}{2} \\ 1 \leq t \leq k_r}} e_{r,h,t}^{m_{r,h,t}} \prod_{\substack{1 \leq r \leq s \\ \sum_{j=1}^s k_j + 1 \leq j \leq n \\ 1 \leq t \leq k_r}} e_{r,j,t}^{m_{r,j,t}} \prod_{\substack{1 \leq p < q \leq s \\ 1 \leq v \leq \frac{k_p \cdot k_q}{\text{lcm}(k_p, k_q)} \\ 1 \leq t \leq \text{lcm}(k_p, k_q)}} e_{p,q,v,t}^{m_{p,q,v,t}} \prod_{\sum_{j=1}^s k_j < i < j \leq n} A_{i,j}^{m_{i,j}}. \quad (29)$$

The following proposition allows us to decide whether $B_n/[P_n, P_n]$ possesses elements of order $\text{lcm}(k_1, \dots, k_s)$.

PROPOSITION 32. *Let $n \geq 3$, let $k_0 = 0$, let $3 \leq k_1 \leq k_2 \leq \dots \leq k_s$ be odd such that $\sum_{j=1}^s k_j \leq n$, let $\delta \in B_n/[P_n, P_n]$ be defined by equation (26), and let $A \in P_n/[P_n, P_n]$ be given by equation (29). Then the element $A\delta$ is of order $\text{lcm}(k_1, \dots, k_s)$ if and only if the following system of equations is satisfied:*

$$\left\{ \begin{array}{l} \sum_{1 \leq t \leq k_r} m_{r,h,t} = 0 \quad \text{for all } 1 \leq r \leq s \text{ and } 1 \leq h \leq \frac{k_r - 1}{2} \\ \sum_{1 \leq t \leq k_r} m_{r,j,t} = 0 \quad \text{for all } 1 \leq r \leq s \text{ and } \sum_{j=1}^s k_j + 1 \leq j \leq n \\ \sum_{1 \leq t \leq \text{lcm}(k_p, k_q)} m_{p,q,v,t} = 0 \quad \text{for all } 1 \leq p < q \leq s \text{ and } 1 \leq v \leq \frac{k_p \cdot k_q}{\text{lcm}(k_p, k_q)} \\ m_{i,j} = 0 \quad \text{for all } \sum_{j=1}^s k_j < i < j \leq n. \end{array} \right. \quad (30)$$

Proof. The argument is similar to that of the proof of Proposition 19. Let A be written in the form of equation (29), and let $\ell = \text{lcm}(k_1, \dots, k_s)$. Since $A \in P_n/[P_n, P_n]$, $\bar{\sigma}(A\delta) = \bar{\sigma}(\delta) = \theta$, where θ is as defined in equation (27). The fact that θ is of order ℓ implies that the order of $A\delta$, if it is finite, cannot be less than ℓ . Since δ is of order ℓ by Lemma 26, it follows that:

$$(A\delta)^\ell = \prod_{j=0}^{\ell-1} \delta^j A \delta^{-j}. \quad (31)$$

Let $w_1 \in \mathcal{B}$, and let q denote the length of the orbit of w_1 under the action of conjugation by δ . By Theorem 31, $q = k_r$ if w_1 is of type (a) or (b), $q = \text{lcm}(k_p, k_q)$ if w_1 is of type (c), and $q = 1$ if w_1 is of type (d). For $i = 1, \dots, q$, let $w_i = \delta^{i-1} w_1 \delta^{-(i-1)}$ be the (distinct) elements of the orbit of w_1 . So $\delta^q w_i \delta^{-q} = w_i$, and since q divides ℓ , we have:

$$\prod_{j=0}^{\ell-1} \delta^j w_i \delta^{-j} = \prod_{\substack{0 \leq j \leq q-1 \\ 0 \leq k \leq \frac{\ell}{q}-1}} \delta^{kq+j} w_i \delta^{-(kq+j)} = \left(\prod_{0 \leq j \leq q-1} \delta^j w_i \delta^{-j} \right)^{\ell/q} = (w_1 \cdots w_q)^{\ell/q}.$$

If $m_i \in \mathbb{Z}$ then for $i = 1, \dots, q$, we have:

$$\left(\prod_{i=1}^q w_i^{m_i} \delta \right)^\ell = \prod_{j=0}^{\ell-1} \delta^j \left(\prod_{i=1}^q w_i^{m_i} \right) \delta^{-j} = \prod_{i=1}^q (w_1 \cdots w_q)^{m_i \ell/q} = (w_1 \cdots w_q)^{\frac{\ell}{q} \sum_{i=1}^q m_i}. \quad (32)$$

Combining equations (31) and (32) and using the fact that the orbits of the elements of \mathcal{B} are invariant under conjugation by δ , it follows that $(A\delta)^\ell = 1$ if and only if

$$\sum_{i=1}^q m_i = 0 \text{ for all } w_1 \in \mathcal{B}, \quad (33)$$

where for $i = 1, \dots, q$, m_i is the coefficient of w_i that appears in equation (29). Taking w_1 to be successively the element $e_{r,h,1}$ of type (a), the element $e_{r,j,1}$ of type (b), the element $e_{p,q,v,1}$ of type (c), and the element $A_{i,j}$ of type (d), we conclude that $(A\delta)^\ell = 1$ if and only if the system of equations (30) is satisfied, and this completes the proof of the proposition. \square

We now prove Theorem 5 that concerns the conjugacy classes of finite-order elements of $B_n/[P_n, P_n]$, and which is the main result of this section.

Proof of Theorem 5. Let $\theta \in S_n$ be of order k . Conjugating θ if necessary, we may suppose that there exist odd integers $3 \leq k_1 \leq \dots \leq k_s$ such that $\sum_{i=1}^s k_i \leq n$ and $k = \text{lcm}(k_1, \dots, k_s)$ for which θ is of the form given by equation (27), and where the elements θ_i of that equation are defined by equation (28). Let $\delta \in B_n/[P_n, P_n]$ be defined by equation (26), which we know to be of order k using Lemma 26. Now let $\beta \in B_n/[P_n, P_n]$ be an element of finite order such that $\bar{\sigma}(\beta) = \theta$. By Lemma 9, β is of order k . To prove Theorem 5, it suffices to show that β and δ are conjugate. Since they have the same permutation, there exists $A \in B_n/[P_n, P_n]$ such that $\beta = A\delta$, and we may write A in the form of equation (29). With the notation of the proof of Proposition 32, equation (33) holds by that proposition because $A\delta$ is of order $\text{lcm}(k_1, \dots, k_s)$. To prove the theorem, it suffices to show that $A\delta$ and δ are conjugate. To do so, we will exhibit $X \in P_n/[P_n, P_n]$ for which $XA\delta X^{-1} = \delta$. This is equivalent to the following relation:

$$XA\delta X^{-1} \delta^{-1} = 1 \text{ in } P_n/[P_n, P_n]. \quad (34)$$

We start by writing X in the form of equation (29) as follows:

$$X = \prod_{\substack{1 \leq r \leq s \\ 1 \leq h \leq \frac{k_r-1}{2} \\ 1 \leq t \leq k_r}} e_{r,h,t}^{x_{r,h,t}} \prod_{\substack{1 \leq r \leq s \\ \sum_{j=1}^s k_j+1 \leq j \leq n \\ 1 \leq t \leq k_r}} e_{r,j,t}^{x_{r,j,t}} \prod_{\substack{1 \leq p < q \leq s \\ 1 \leq v \leq \frac{k_p \cdot k_q}{\text{lcm}(k_p, k_q)} \\ 1 \leq t \leq \text{lcm}(k_p, k_q)}} e_{p,q,v,t}^{x_{p,q,v,t}} \prod_{\sum_{j=1}^s k_j < i < j \leq n} A_{i,j}^{x_{i,j}}, \quad (35)$$

where the exponents are the coefficients of the elements of \mathcal{B} . As we saw in the proof of Proposition 32, it suffices to study the subsystems obtained from equation (34) that correspond to the orbits of the action of conjugation by δ . In particular, if $w_1 \in \mathcal{B}$ and $w_i = \delta^{i-1} w_1 \delta^{-(i-1)}$ are the elements of the orbit of w_1 , where $i = 1, \dots, q$, then it follows from equations (29), (34) and (35) that:

$$\left(\prod_{i=1}^q w_i^{x_i} \right) \left(\prod_{i=1}^q w_i^{m_i} \right) \left(\prod_{i=1}^q w_i^{-x_{i-1}} \right) = 1 \text{ in } P_n/[P_n, P_n],$$

where m_i (resp. x_i) is the coefficient of w_i appearing in equation (29) (resp. in equation (35)), and $x_0 = x_q$. We conclude that:

$$x_{i-1} - x_i = m_i \text{ for all } i = 1, \dots, q \text{ and for all choices of } w_1 \in \mathcal{B}. \quad (36)$$

Choosing $x_q \in \mathbb{Z}$ arbitrarily, the solution of the subsystem of equations obtained by taking $i = 2, \dots, q$ in equation (36) is given by $x_{i-1} = x_q + \sum_{j=i}^q m_j$, which determines x_l for all $l = 1, \dots, q$. The remaining equation, corresponding to $i = 1$, is satisfied, because:

$$x_q - x_1 = - \sum_{j=2}^q m_j = m_1 \text{ by equation (33).}$$

Hence the system of equations (36) possesses solutions for all choices of $w_1 \in \mathcal{B}$, and so equation (34) admits solutions, from which it follows that $A\delta$ is conjugate to δ by an element of $P_n/[P_n, P_n]$. This proves the first part of the statement. The second part is then a direct consequence. \square

REMARKS 33.

(a) The number of conjugacy classes of permutations of order k in S_n is equal to the number of partitions (n_1, \dots, n_r) of n , where $n_i \in \mathbb{N}$, $n_1 \leq n_2 \leq \dots \leq n_r$, $\sum_{i=1}^r n_i = n$ and $\text{lcm}(n_1, \dots, n_r) = k$.

(b) It follows from Corollary 4 and Theorem 5 that if k is odd, $\bar{\sigma}$ induces a bijection between the set of conjugacy classes of elements of order k in $B_n/[P_n, P_n]$ and the set of conjugacy classes of elements of order k in S_n . The same result also holds for finite cyclic subgroups.

(c) Given an Abelian subgroup H of finite odd order of S_n , we saw in Theorem 6 that $B_n/[P_n, P_n]$ contains a subgroup G isomorphic to H . An open and more difficult question is whether $B_n/[P_n, P_n]$ contains a subgroup G such that $\bar{\sigma}(G) = H$.

7 Finite non-Abelian subgroups of $B_n/[P_n, P_n]$

As we saw in Theorem 2 and Lemma 9, any finite subgroup of $B_n/[P_n, P_n]$ is of odd order, and embeds in S_n . Following the discussion of the previous sections, it is natural to try to characterise the isomorphism classes of the finite subgroups of $B_n/[P_n, P_n]$ as well as their conjugacy classes. For the question of isomorphism classes, this was achieved for finite Abelian subgroups in Theorem 6, and for that of conjugacy classes, was solved in Theorem 5 and Corollary 4 for cyclic groups. Going a step further, we may also ask whether $B_n/[P_n, P_n]$ possesses finite non-Abelian subgroups. Since any group of order 9 or 15 is Abelian, the smallest non-Abelian group of odd order is the *Frobenius group* of order 21, which we denote by \mathcal{F} . It admits the following presentation:

$$\mathcal{F} = \langle s, t \mid s^3 = t^7 = 1, sts^{-1} = t^2 \rangle. \quad (37)$$

The group \mathcal{F} is thus a semi-direct product of the form $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and it possesses six (resp. fourteen) elements of order 7 (resp. of order 3). As we shall see in Lemma 34, \mathcal{F} embeds in S_7 , and as a first step in deciding whether $B_n/[P_n, P_n]$ possesses finite non-Abelian subgroups, one may ask whether \mathcal{F} embeds in $B_7/[P_7, P_7]$. The main result of this section, Theorem 7, shows that the answer is positive. Theorem 3(a) then implies that \mathcal{F} embeds in $B_n/[P_n, P_n]$ for all $n \geq 7$. In Theorem 38, we show that in $B_7/[P_7, P_7]$, there is a single conjugacy class of subgroups isomorphic to \mathcal{F} . The general questions regarding the embedding in $B_n/[P_n, P_n]$ of an arbitrary finite non-Abelian group of odd order (for large enough n) and the number of its conjugacy classes remain open.

We first exhibit a subgroup \mathcal{F}_0 of S_7 that is isomorphic to \mathcal{F} . We shall see later in Proposition 35 that any subgroup of S_7 that is isomorphic to \mathcal{F} is conjugate to \mathcal{F}_0 . In what follows, we consider the following elements of S_7 :

$$\alpha = (1, 3, 4, 2, 5, 6, 7) \text{ and } \beta = (1, 2, 3)(4, 5, 6). \quad (38)$$

Let \mathcal{F}_0 denote the subgroup of S_7 generated by $\{\alpha, \beta\}$. As noted previously, we read our permutations from left to right, to coincide with our convention for the composition of braids.

LEMMA 34. The subgroup \mathcal{F}_0 of S_7 is isomorphic to \mathcal{F} . Further, if G is a subgroup of S_7 that is isomorphic to \mathcal{F} then G is generated by two elements α' and β' , where α' is a 7-cycle, the cycle type of β' is $(3, 3, 1)$, and $\beta'\alpha'\beta'^{-1} = \alpha'^2$.

Proof. The first part of the statement is obtained from a straightforward computation using equations (37) and (38). For the second part, if G is a subgroup of S_7 that is isomorphic to \mathcal{F} then it possesses a generating set $\{\alpha', \beta'\}$, where α' is a 7-cycle, β' is of order 3, and $\beta'\alpha'\beta'^{-1} = \alpha'^2$. The cycle type of β' is either $(3, 3, 1)$ or $(3, 1, 1, 1, 1)$. Suppose that we are in the second case. Then $\beta' = (n_1, n_2, n_3)$ where n_1, n_2 and n_3 are distinct elements of $\{1, \dots, 7\}$. Hence the remaining four elements m_1, m_2, m_3 and m_4 of $\{1, 2, 3, 4, 5, 6, 7\} \setminus \{n_1, n_2, n_3\}$ are fixed by β' . So there are two consecutive elements of the 7-cycle α' , denoted by m_j, m_k , that belong to $\{m_1, m_2, m_3, m_4\}$. Since $\beta'\alpha'\beta'^{-1} = \alpha'^2$, we have $\beta\alpha\beta^{-1}(m_j) = \alpha\beta^{-1}(m_j) = \beta^{-1}(m_k) = m_k$, but this is different from $\alpha^2(m_j) = \alpha(m_k)$. This yields a contradiction, and shows that the cycle type of β' is $(3, 3, 1)$. \square

The rest of this section is devoted to proving that \mathcal{F} embeds in $B_7/[P_7, P_7]$ and to showing that in $B_7/[P_7, P_7]$, there is a single conjugacy class of subgroups isomorphic to \mathcal{F} . In this quotient, we define:

$$x = \sigma_2\sigma_1^{-1}\sigma_5\sigma_4^{-1} \text{ and } y = \sigma_2\sigma_3\sigma_6\sigma_5\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}. \quad (39)$$

Then $\bar{\sigma}(y) = \alpha$ and using the notation of equation (19), $y = \sigma_2\sigma_3\delta_{0,7}\sigma_3^{-1}\sigma_2^{-1}$. So y is of order 7 by Lemma 26. Similarly, $\bar{\sigma}(x) = \beta$, $x = \delta_{0,3}\delta_{3,3}$, and x is of order 3 (see the discussion on page 20 just after equation (26)). We now prove Theorem 7 that asserts the existence of a subgroup of $B_7/[P_7, P_7]$ isomorphic to the Frobenius group \mathcal{F} .

Proof of Theorem 7. Consider the subgroup H of $B_7/[P_7, P_7]$ generated by $\{x, y\}$. By the above comments, we know that $\bar{\sigma}(y) = \alpha$ and $\bar{\sigma}(x) = \beta$, therefore $\bar{\sigma}(H) = \mathcal{F}_0$. Let

$$\gamma = \sigma_2^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_6^{-1}\sigma_5^{-1}\sigma_1\sigma_2^{-1}.$$

Using the Artin relations (2) and (3), and equation (8), we have:

$$\begin{aligned} \gamma x y x^{-1} y^{-2} \gamma^{-1} &= \sigma_2^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_6^{-1}\sigma_5^{-1}\sigma_1\sigma_2^{-1} \cdot \sigma_2\sigma_1^{-1}\sigma_5\sigma_4^{-1} \cdot \sigma_2\sigma_3\sigma_6\sigma_5\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1} \cdot \\ &\quad \sigma_4\sigma_5^{-1}\sigma_1\sigma_2^{-1} \cdot \sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_6^{-1}\sigma_3^{-1}\sigma_2^{-1} \cdot \\ &\quad \sigma_2\sigma_1^{-1}\sigma_5\sigma_6\sigma_4^{-1}\sigma_5^{-1}\sigma_4\sigma_2 \\ &= \sigma_4^{-1}\sigma_5^2\sigma_4^{-1}\sigma_3\sigma_4\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_4\sigma_5^{-1}\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_4^{-1}\sigma_5^{-1}\sigma_1\sigma_2\sigma_3\sigma_4^{-1}\sigma_5^{-1} \cdot \\ &\quad \sigma_6^{-1}\sigma_5^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_5\sigma_6\sigma_5\sigma_4^{-1}\sigma_5^{-1}\sigma_2 \\ &= A_{4,6}A_{4,5}^{-1} \cdot A_{4,5} \cdot \sigma_3\sigma_4\sigma_2^{-1}\sigma_1^{-1}\sigma_4\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_2\sigma_4^{-1}\sigma_5^{-1}\sigma_1\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_4\sigma_1^{-1} \cdot \\ &\quad \sigma_4^{-1}\sigma_5^{-1}\sigma_2 \\ &= A_{4,6} \cdot \sigma_3\sigma_4^2\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_4^{-1}\sigma_5^{-1}\sigma_2\sigma_4^{-1}\sigma_5^{-1}\sigma_1\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_2 \\ &= A_{4,6}A_{3,5} \cdot \sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_4^{-2}\sigma_5^{-1}\sigma_4^{-1}\sigma_1\sigma_2\sigma_1\sigma_4^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_2 \\ &= A_{4,6}A_{3,5}A_{2,5}^{-1} \cdot \sigma_3\sigma_2^{-1}\sigma_3\sigma_5^{-1}\sigma_4^{-2}\sigma_2\sigma_3^{-1}\sigma_5^{-1}\sigma_2 \\ &= A_{4,6}A_{3,5}A_{2,5}^{-1}A_{2,6}^{-1} \cdot \sigma_3\sigma_2^{-1}\sigma_3\sigma_5^{-2}\sigma_2\sigma_3^{-1}\sigma_2 \\ &= A_{4,6}A_{3,5}A_{2,5}^{-1}A_{2,6}^{-1}A_{5,6}^{-1} \cdot \sigma_3\sigma_2^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_2 \end{aligned}$$

$$= A_{4,6}A_{3,5}A_{2,5}^{-1}A_{2,6}^{-1}A_{5,6}^{-1}A_{3,4}A_{2,3}A_{2,4}^{-1}.$$

Using equation (8) once more, we obtain:

$$\begin{aligned} xyx^{-1}y^{-2} &= \gamma^{-1}A_{4,6}A_{3,5}A_{2,5}^{-1}A_{2,6}^{-1}A_{5,6}^{-1}A_{3,4}A_{2,3}A_{2,4}^{-1}\gamma \\ &= A_{4,7}A_{1,7}A_{1,6}A_{2,7}^{-1}A_{2,6}^{-1}A_{2,4}^{-1}A_{1,2}A_{4,6}^{-1}. \end{aligned} \quad (40)$$

Note that this shows that $xyx^{-1}y^{-2}$ is non trivial in the free Abelian group $P_7/[P_7, P_7]$, which implies that H is not isomorphic to \mathcal{F} . We now look for an element $N \in P_7/[P_7, P_7]$ such that if $v = Ny$ then the subgroup $\langle x, v \rangle$ is isomorphic to \mathcal{F} , where v is of order 7, $\bar{\sigma}(v) = \alpha$ and $xvx^{-1} = v^2$. This last equality gives rise to the following equivalences:

$$xvx^{-1} = v^2 \iff xNyx^{-1} = NyNy \iff xyx^{-1}y^{-2} = xN^{-1}x^{-1} \cdot N \cdot yNy^{-1}. \quad (41)$$

We seek solutions $N \in P_7/[P_7, P_7]$ of equation (41) taking into account equation (40) and the fact that Ny is of order 7. In order to do so, we use additive notation, and we write N in terms of the basis $\{A_{i,j}\}_{1 \leq i < j \leq 7}$ of $P_7/[P_7, P_7]$ as follows:

$$N = \sum_{1 \leq i < j \leq 7} p_{i,j}A_{i,j}, \quad (42)$$

where $p_{i,j} \in \mathbb{Z}$ for all $1 \leq i < j \leq 7$. By equation (8) and Proposition 27, we see that the orbits under the action of conjugation by y are of the form:

$$\begin{cases} A_{1,2} \mapsto A_{4,7} \mapsto A_{3,6} \mapsto A_{1,5} \mapsto A_{2,7} \mapsto A_{4,6} \mapsto A_{3,5} \mapsto A_{1,2} \\ A_{1,3} \mapsto A_{1,7} \mapsto A_{6,7} \mapsto A_{5,6} \mapsto A_{2,5} \mapsto A_{2,4} \mapsto A_{3,4} \mapsto A_{1,3} \\ A_{1,4} \mapsto A_{3,7} \mapsto A_{1,6} \mapsto A_{5,7} \mapsto A_{2,6} \mapsto A_{4,5} \mapsto A_{2,3} \mapsto A_{1,4}, \end{cases} \quad (43)$$

and the orbits under the action of conjugation by x are of the form:

$$\begin{cases} A_{1,2} \mapsto A_{1,3} \mapsto A_{2,3} \mapsto A_{1,2} & A_{4,6} \mapsto A_{5,6} \mapsto A_{4,5} \mapsto A_{4,6} \\ A_{2,7} \mapsto A_{1,7} \mapsto A_{3,7} \mapsto A_{2,7} & A_{4,7} \mapsto A_{6,7} \mapsto A_{5,7} \mapsto A_{4,7} \\ A_{3,6} \mapsto A_{2,5} \mapsto A_{1,4} \mapsto A_{3,6} & A_{1,5} \mapsto A_{3,4} \mapsto A_{2,6} \mapsto A_{1,5} \\ A_{3,5} \mapsto A_{2,4} \mapsto A_{1,6} \mapsto A_{3,5}. \end{cases} \quad (44)$$

The first (resp. second) line of (44) may be obtained by applying Proposition 27 (resp. Proposition 28) and Proposition 30, and the last two lines follow from Proposition 29. Arguing in a manner similar to that of the proof of Proposition 19, and using the fact that y is of order 7, we obtain:

$$0 = (Ny)^7 = \sum_{k=0}^6 y^k Ny^{-k} = \sum_{k=0}^6 y^k \left(\sum_{1 \leq i < j \leq 7} p_{i,j}A_{i,j} \right) y^{-k} = \sum_{1 \leq i < j \leq 7} p_{i,j} \left(\sum_{k=0}^6 y^k A_{i,j} y^{-k} \right),$$

from which it follows that the sum of the coefficients corresponding to the elements of each of the orbits given in (43) is zero:

$$\begin{cases} p_{1,2} + p_{4,7} + p_{3,6} + p_{1,5} + p_{2,7} + p_{4,6} + p_{3,5} = 0 \\ p_{1,3} + p_{1,7} + p_{6,7} + p_{5,6} + p_{2,5} + p_{2,4} + p_{3,4} = 0 \\ p_{1,4} + p_{3,7} + p_{1,6} + p_{5,7} + p_{2,6} + p_{4,5} + p_{2,3} = 0. \end{cases} \quad (45)$$

Using equations (43) and (44) to compute first xNx^{-1} and yNy^{-1} and then equations (40), (41) and (42), we obtain the following systems of equations:

$$\left\{ \begin{array}{ll} A_{1,2}: & -p_{2,3} + p_{1,2} + p_{3,5} = 1 & A_{1,3}: & -p_{1,2} + p_{1,3} + p_{3,4} = 0 \\ A_{4,7}: & -p_{5,7} + p_{4,7} + p_{1,2} = 1 & A_{1,7}: & -p_{2,7} + p_{1,7} + p_{1,3} = 1 \\ A_{3,6}: & -p_{1,4} + p_{3,6} + p_{4,7} = 0 & A_{6,7}: & -p_{4,7} + p_{6,7} + p_{1,7} = 0 \\ A_{1,5}: & -p_{2,6} + p_{1,5} + p_{3,6} = 0 & A_{5,6}: & -p_{4,6} + p_{5,6} + p_{6,7} = 0 \\ A_{2,7}: & -p_{3,7} + p_{2,7} + p_{1,5} = -1 & A_{2,5}: & -p_{3,6} + p_{2,5} + p_{5,6} = 0 \\ A_{4,6}: & -p_{4,5} + p_{4,6} + p_{2,7} = -1 & A_{2,4}: & -p_{3,5} + p_{2,4} + p_{2,5} = -1 \\ A_{3,5}: & -p_{1,6} + p_{3,5} + p_{4,6} = 0 & A_{3,4}: & -p_{1,5} + p_{3,4} + p_{2,4} = 0 \\ A_{1,4}: & -p_{2,5} + p_{1,4} + p_{2,3} = 0 & A_{2,6}: & -p_{3,4} + p_{2,6} + p_{5,7} = -1 \\ A_{3,7}: & -p_{1,7} + p_{3,7} + p_{1,4} = 0 & A_{4,5}: & -p_{5,6} + p_{4,5} + p_{2,6} = 0 \\ A_{1,6}: & -p_{2,4} + p_{1,6} + p_{3,7} = 1 & A_{2,3}: & -p_{1,3} + p_{2,3} + p_{4,5} = 0 \\ A_{5,7}: & -p_{6,7} + p_{5,7} + p_{1,6} = 0. & & \end{array} \right. \quad (46)$$

One may check that the systems equation (45) and (46) together admit a solution, taking for example all of the coefficients to be zero, with the exception of:

$$p_{2,7} = p_{5,7} = -1 \text{ and } p_{3,5} = p_{1,6} = 1.$$

For these values of $p_{i,j}$, we have $N = A_{3,5} + A_{1,6} - A_{2,7} - A_{5,7}$, and it follows from above that the subgroup $\langle x, v \rangle$ of $B_7/[P_7, P_7]$ is isomorphic to \mathcal{F} , which completes the proof of the theorem. \square

We now analyse the conjugacy classes of subgroups isomorphic to \mathcal{F} in $B_7/[P_7, P_7]$. We first show that S_7 possesses a single such conjugacy class.

PROPOSITION 35. *Any two subgroups of S_7 isomorphic to \mathcal{F} are conjugate.*

Proof. Let G be a subgroup of S_7 isomorphic to \mathcal{F} . It suffices to show that G is conjugate to \mathcal{F}_0 . By Lemma 34, G is generated by two elements α' and β' , where α' is a 7-cycle, the cycle type of β' is $(3, 3, 1)$, and $\beta'\alpha\beta'^{-1} = \alpha^2$. Conjugating G if necessary, we may suppose that $\alpha' = \alpha$. Now $\beta'\alpha\beta'^{-1} = \alpha^2$ in G and $\beta\alpha\beta^{-1} = \alpha^2$ in \mathcal{F}_0 , from which it follows that $\beta^{-1}\beta'$ belongs to the centraliser of α . But since α is a complete cycle in S_7 , its centraliser is equal to $\langle \alpha \rangle$. So there exists $k \in \{0, 1, \dots, 6\}$ such that $\beta' = \beta\alpha^k$, and hence $G = \langle \alpha, \beta' \rangle = \langle \alpha, \beta \rangle = \mathcal{F}_0$ as required. \square

REMARK 36. For the purposes of the proof of Proposition 37, we shall study the elements of the form $\varepsilon\alpha\varepsilon^{-1}$, where ε belongs to the centraliser of β in S_7 . This centraliser may be seen to be of order 18, and consists of the elements of the form $\tau^i(123)^j$, where $\tau = (1, 4, 2, 5, 3, 6)$, $0 \leq i \leq 5$ and $0 \leq j \leq 2$. Let:

$$\left\{ \begin{array}{l} \alpha_1 = (1, 3, 2)\alpha(1, 3, 2)^{-1} = (1, 4, 3, 5, 6, 7, 2), \quad \text{and} \\ \alpha_2 = (4, 6, 5)\alpha(4, 6, 5)^{-1} = (1, 3, 5, 2, 6, 4, 7). \end{array} \right. \quad (47)$$

A straightforward computation shows that:

$$(1, 2, 3)^j\alpha(1, 2, 3)^{-j} = \begin{cases} \alpha & \text{if } j = 0 \\ \alpha_2^2 & \text{if } j = 1 \\ \alpha_1 & \text{if } j = 2, \end{cases}$$

and $\tau\alpha\tau^{-1} = \alpha_2^{-2}$, $\tau\alpha_2\tau^{-1} = \alpha^{-1}$ and $\tau\alpha_1\tau^{-1} = \alpha_1^3$ for all $0 \leq i \leq 5$ and $0 \leq j \leq 2$. It then follows that for all $0 \leq i \leq 5$ and $0 \leq j \leq 2$, there exists $z \in \{\alpha, \alpha_1, \alpha_2\}$ such that $\tau^i(123)^j\alpha(123)^{-j}\tau^{-i}$ is a generator of $\langle z \rangle$.

PROPOSITION 37. *Suppose that H is a subgroup of $B_7/[P_7, P_7]$ isomorphic to \mathcal{F} . Then H is conjugate to a subgroup of the form $\langle x, v \rangle$, where x is given by equation (39) and $\bar{\sigma}(v) = \alpha$.*

Proof. Let H be a subgroup of $B_7/[P_7, P_7]$ isomorphic to \mathcal{F} . Since $\bar{\sigma}(H)$ is a subgroup of S_7 isomorphic to \mathcal{F} by Lemma 9, it follows from Proposition 35 that there exists $\rho \in S_7$ such that $\mathcal{F}_0 = \rho\bar{\sigma}(H)\rho^{-1}$. So if $\hat{\rho} \in B_7/[P_7, P_7]$ is such that $\bar{\sigma}(\hat{\rho}) = \rho$ then $H_1 = \hat{\rho}H\hat{\rho}^{-1}$ satisfies $\bar{\sigma}(H_1) = \mathcal{F}_0$. Let $\tilde{x}, \tilde{y} \in H_1$ be such that $\bar{\sigma}(\tilde{x}) = \beta$ and $\bar{\sigma}(\tilde{y}) = \alpha$, where α and β are given by equation (38). Now $\beta = \bar{\sigma}(x)$, and since x and \tilde{x} are of order 3 and have the same permutation, Theorem 5 implies that they are conjugate. So there exists $\lambda_1 \in B_7/[P_7, P_7]$ such that $\lambda_1\tilde{x}\lambda_1^{-1} = x$. Hence $\bar{\sigma}(\lambda_1)\bar{\sigma}(\tilde{x})\bar{\sigma}(\lambda_1)^{-1} = \bar{\sigma}(x)$, and since $\bar{\sigma}(\tilde{x}) = \bar{\sigma}(x) = \beta$, we conclude that $\bar{\sigma}(\lambda_1)$ belongs to the centraliser of β in S_7 . By Remark 36, this centraliser is equal to $\langle \tau, (1, 2, 3) \rangle$, and the fact that $\bar{\sigma}(\tilde{y}) = \alpha$ implies that there exists $z \in \{\alpha, \alpha_1, \alpha_2\}$ such that $\bar{\sigma}(\lambda_1\tilde{y}\lambda_1^{-1})$ is a generator of $\langle z \rangle$. Let:

$$\lambda_2 = \begin{cases} e & \text{if } z = \alpha \\ \sigma_1\sigma_2^{-1} & \text{if } z = \alpha_1 \\ \sigma_4\sigma_5^{-1} & \text{if } z = \alpha_2. \end{cases}$$

Note that λ_2 commutes with x , and by equation (47), $\bar{\sigma}(\lambda_2^{-1}\lambda_1\tilde{y}\lambda_1^{-1}\lambda_2)$ is a generator of $\langle \alpha \rangle$. Taking v to be the element of $\lambda_2^{-1}\lambda_1\langle \tilde{y} \rangle\lambda_1^{-1}\lambda_2$ for which $\bar{\sigma}(v) = \alpha$, the subgroup $\lambda_2^{-1}\lambda_1\hat{\rho}H(\lambda_2^{-1}\lambda_1\hat{\rho})^{-1}$ is then seen to be equal to $\langle x, v \rangle$, which proves the proposition. \square

THEOREM 38. *The group $B_7/[P_7, P_7]$ possesses a unique conjugacy class of subgroups isomorphic to \mathcal{F} .*

Proof. From the proof of Theorem 7, $B_7/[P_7, P_7]$ possesses a subgroup $H_0 = \langle x, v_0 \rangle$ isomorphic to \mathcal{F} , where $v_0 = N_0y$, and $N_0 = A_{1,6}A_{3,5}A_{2,7}^{-1}A_{5,7}^{-1}$. Let H be a subgroup of $B_7/[P_7, P_7]$ isomorphic to \mathcal{F} . By Proposition 37, up to conjugacy, we may suppose that $H = \langle x, v \rangle$, where $\bar{\sigma}(v) = \alpha = \bar{\sigma}(y) = \bar{\sigma}(v_0)$. Thus $v = Ny$, where $N \in P_7/[P_7, P_7]$. Again from the proof of Theorem 7, the coefficients $p_{i,j}$ of N given by equation (42) satisfy the systems of equations (45) and (46), and one may check that the general solution of these two systems is of rank 6, and is given by:

$$\left\{ \begin{array}{ll} p_{1,2} = -r_6 + r_4 + r_3 - r_2 + 1 & p_{1,3} = -r_6 - r_2 \\ p_{4,7} = r_6 - r_3 + r_2 & p_{1,7} = r_6 - r_5 - r_4 - r_3 + r_2 \\ p_{3,6} = r_3 - r_2 & p_{6,7} = r_5 + r_4 \\ p_{1,5} = r_2 & p_{5,6} = -r_6 + r_3 - r_2 - r_1 \\ p_{2,7} = -r_5 - r_4 - r_3 - 1 & p_{2,5} = r_6 + r_1 \\ p_{4,6} = -r_6 + r_5 + r_4 + r_3 - r_2 - r_1 & p_{2,4} = -r_4 - r_3 + r_2 - 1 \\ p_{3,5} = r_6 - r_4 - r_3 + r_2 + r_1 & p_{3,4} = r_4 + r_3 + 1 \\ p_{1,4} = r_6 & p_{2,6} = r_3 \\ p_{3,7} = -r_5 - r_4 - r_3 + r_2 & p_{4,5} = -r_6 - r_2 - r_1 \\ p_{1,6} = r_5 & p_{2,3} = r_1 \\ p_{5,7} = r_4, & \end{array} \right. \quad (48)$$

where $r_1, \dots, r_6 \in \mathbb{Z}$ are arbitrary. So choose the values of the r_i so that $v = Ny$. We claim that there exists $\Theta \in P_7/[P_7, P_7]$ such that:

$$\Theta x \Theta^{-1} = x, \quad \text{and} \quad (49)$$

$$\Theta v_0 \Theta^{-1} = v. \quad (50)$$

This being the case, we have $H = \langle x, v \rangle = \Theta \langle x, v_0 \rangle \Theta^{-1} = \Theta H_0 \Theta^{-1}$, in particular H and H_0 are conjugate in $B_7/[P_7, P_7]$, which proves the statement of the theorem. To prove the claim, let $\Theta = \sum_{1 \leq i < j \leq 7} \theta_{i,j} A_{i,j}$. We must determine the coefficients $\theta_{i,j}$ of Θ that satisfy

equations (49) and (50). By equation (44), equation (49) holds if and only if there exist $s_1, \dots, s_7 \in \mathbb{Z}$ such that:

$$\begin{cases} s_1 = \theta_{1,2} = \theta_{1,3} = \theta_{2,3} & s_2 = \theta_{2,7} = \theta_{1,7} = \theta_{3,7} \\ s_3 = \theta_{3,6} = \theta_{2,5} = \theta_{1,4} & s_4 = \theta_{3,5} = \theta_{2,4} = \theta_{1,6} \\ s_5 = \theta_{4,6} = \theta_{5,6} = \theta_{4,5} & s_6 = \theta_{4,7} = \theta_{6,7} = \theta_{5,7} \\ s_7 = \theta_{1,5} = \theta_{3,4} = \theta_{2,6}. \end{cases} \quad (51)$$

Equation (50) may be written in the form $\Theta \cdot N_0 \cdot y \Theta^{-1} y^{-1} = N$. Using equation (44), we obtain the following system of equations:

$$\begin{cases} p_{1,2} = \theta_{1,2} - \theta_{3,5} = s_1 - s_4 & p_{1,3} = \theta_{1,3} - \theta_{2,3} = s_1 - s_7 \\ p_{4,7} = \theta_{4,7} - \theta_{1,2} = s_6 - s_1 & p_{1,7} = \theta_{1,7} - \theta_{1,3} = s_2 - s_1 \\ p_{3,6} = \theta_{3,6} - \theta_{4,7} = s_3 - s_6 & p_{6,7} = \theta_{6,7} - \theta_{1,7} = s_6 - s_2 \\ p_{1,5} = \theta_{1,5} - \theta_{3,6} = s_7 - s_3 & p_{5,6} = \theta_{5,6} - \theta_{6,7} = s_5 - s_6 \\ p_{2,7} = \theta_{2,7} - \theta_{1,5} - 1 = s_2 - s_7 - 1 & p_{2,5} = \theta_{2,5} - \theta_{5,6} = s_3 - s_5 \\ p_{4,6} = \theta_{4,6} - \theta_{2,7} = s_5 - s_2 & p_{2,4} = \theta_{2,4} - \theta_{2,5} = s_4 - s_3 \\ p_{3,5} = \theta_{3,5} - \theta_{4,6} + 1 = s_4 - s_5 + 1 & p_{3,4} = \theta_{3,4} - \theta_{2,4} = s_7 - s_4 \\ p_{1,4} = \theta_{1,4} - \theta_{3,5} = s_3 - s_1 & p_{2,6} = \theta_{2,6} - \theta_{5,7} = s_7 - s_6 \\ p_{3,7} = \theta_{3,7} - \theta_{1,4} = s_2 - s_3 & p_{4,5} = \theta_{4,5} - \theta_{2,6} = s_5 - s_7 \\ p_{1,6} = \theta_{1,6} - \theta_{3,7} + 1 = s_4 - s_2 + 1 & p_{2,3} = \theta_{2,3} - \theta_{4,5} = s_1 - s_5 \\ p_{5,7} = \theta_{5,7} - \theta_{1,6} - 1 = s_6 - s_4 - 1. \end{cases} \quad (52)$$

It remains to show that by choosing the s_k appropriately, we obtain a system of coefficients that satisfy the equations of system (52). Consider the system:

$$\begin{cases} s_1 - s_4 = -r_6 + r_4 + r_3 - r_2 + 1 \\ s_6 - s_1 = r_6 - r_3 + r_2 \\ s_3 - s_6 = r_3 - r_2 \\ s_7 - s_3 = r_2 \\ s_2 - s_7 = -r_5 - r_4 - r_3 \\ s_5 - s_2 = -r_6 + r_5 + r_4 + r_3 - r_2 - r_1. \end{cases} \quad (53)$$

This system clearly possesses solutions in the s_k in terms of the r_l , obtained for example by taking s_4 to be an arbitrary integer, and by rewriting the other s_k in terms of s_4 and

the r_l . For such a solution, the first six equations of the first column of (52) are satisfied using equation (48). Using just (48) and (53), we now verify the remaining equations of (52). For example:

$$\begin{aligned} s_4 - s_5 + 1 &= -((s_1 - s_4) + (s_6 - s_1) + (s_3 - s_6) + (s_7 - s_3) + (s_2 - s_7) + (s_5 - s_2)) + 1 \\ &= -((-r_6 + r_4 + r_3 - r_2 + 1) + (r_6 - r_3 + r_2) + (r_3 - r_2) + r_2 + \\ &\quad (-r_5 - r_4 - r_3) + (-r_6 + r_5 + r_4 + r_3 - r_2 - r_1)) + 1 \\ &= r_6 - r_4 - r_3 + r_2 + r_1 = p_{3,5}. \end{aligned}$$

In a similar manner, one may check that the right-hand side of each of the equations of the system (52) is equal to the left-hand side, using first (53) to express the s_k in terms of the r_l , and then using (48) to obtain the corresponding $p_{i,j}$. The straightforward computations are left to the reader. So with this choice of s_k , we obtain values of the $\theta_{i,j}$ using equation (51) for which equations (49) and (50) are satisfied. Conversely, given arbitrary $r_1, \dots, r_6 \in \mathbb{Z}$ and s_1, \dots, s_6 satisfying equation (53), we see that if the $p_{i,j}$ are given by equation (52) and the $\theta_{i,j}$ are given by equation (51) then equations (49) and (50) are satisfied, and this completes the proof of the theorem. \square

REMARK 39. We saw in Theorem 7 that the Frobenius group \mathcal{F} embeds in $B_7/[P_7, P_7]$. It is the only finite non-Abelian subgroup of S_7 of odd order. To see this, besides 3×7 , which is the order of \mathcal{F} , the possible orders of non-Abelian subgroups of odd order of S_7 are $3^2 \times 5$, $3^2 \times 7$, $3^2 \times 5 \times 7$ and $3 \times 5 \times 7$. Further, if H is a subgroup of S_n of odd order then it is necessarily a subgroup of A_n . Indeed, any element $h \in H$ may be decomposed as a product of disjoint cycles each of which is of odd length, and so it follows that $h \in A_n$. From the table of maximal subgroups of A_7 given in [CCNPW, page 10], we see that S_7 has no subgroup of order $3^2 \times 5$, $3^2 \times 7$, $3 \times 5 \times 7$ or $3^2 \times 5 \times 7$, and that if S_7 possesses a subgroup K of order $3^2 \times 5$ then K is a subgroup of A_6 . It follows from the corresponding table for A_6 that there is no such subgroup (see [CCNPW, page 4]).

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