



**HAL**  
open science

## Note on Gagliardo's theorem " $\text{tr } W^{1,1} = L^1$ "

Petru Mironescu

► **To cite this version:**

Petru Mironescu. Note on Gagliardo's theorem "  $\text{tr } W^{1,1} = L^1$  ". Annals of the University of Bucharest. Mathematical series, 2015, 6 (LXIV) (1), pp.99–103. hal-01131162

**HAL Id: hal-01131162**

**<https://hal.science/hal-01131162>**

Submitted on 13 Mar 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Note on Gagliardo's theorem “ $\text{tr } W^{1,1} = L^1$ ”

PETRU MIRONESCU

March 13, 2015

**Abstract** - We present a short proof of the fact, originally established by Gagliardo, that every function  $f \in L^1(\mathbb{R}^n)$  is the trace of a function  $u \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$ .

**Key words and phrases** : Sobolev space, trace, Gagliardo

**Mathematics Subject Classification** (2000) : 46E35

## 1 A short proof

The aim of this note is to give the “proof from the book” of the fact that, given a function  $f \in L^1(\mathbb{R}^n)$ , there exists some  $u \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$  whose trace is  $f$ . This classical result goes back to Gagliardo [3]. The impatient reader can go directly to the proof of (2), which is the heart of the note.

We start by recalling some definitions and basic facts. Let  $\Omega = \mathbb{R}^n \times (0, \infty)$ . For  $1 \leq p < \infty$ , we consider the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \frac{\partial u}{\partial x_j} \in L^p(\Omega), j = 1, \dots, n+1 \right\},$$

equipped with the standard norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{j=1}^{n+1} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p}.$$

Here, the partial derivatives are generalized derivatives or, equivalently, derivatives in the distributions sense. Before Gagliardo's work [3], the state of the art concerning existence of traces was the following.

**Fact 1**  $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$  is dense in  $W^{1,p}(\Omega)$ ; see e.g. [2, Corollary 9.8].

**Fact 2** For  $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ , set  $f(x) = u(x, 0)$ ,  $x \in \mathbb{R}^n$ . Then we have the estimate

$$\|f\|_{L^p} \leq \|u\|_{W^{1,p}};$$

see e.g. [2, Lemma 9.9].

**Fact 3** By the first two facts, the mapping  $u \mapsto f$  admits a unique linear continuous extension, the *trace operator*  $\text{tr}$ , from  $W^{1,p}(\Omega)$  into  $L^p(\mathbb{R}^n)$ .

**Fact 4** When  $p = 2$ , Aronszajn [1] discovered that the trace operator is not onto, and that its image is precisely the set

$$\left\{ f \in L^2(\mathbb{R}^n); \int_{\mathbb{R}^n} |\xi| |\widehat{f}(\xi)|^2 d\xi < \infty \right\};$$

here,  $\widehat{\cdot}$  stands for the Fourier transform. The arguments in [1] rely on Fourier transform methods, and are difficult to apply to the case where  $p \neq 2$ .

Gagliardo achieved the more complicated task of characterizing  $\text{tr } W^{1,p}(\Omega)$  for  $p \neq 2$ . He obtained two results with different flavors and proofs.

**Theorem 1.1 (Gagliardo's theorem #1)** For  $1 < p < \infty$ , we have

$$\text{tr } W^{1,p}(\Omega) = W^{1-1/p,p}(\mathbb{R}^n). \quad (1)$$

Here,

$$W^{1-1/p,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n); \underbrace{\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p-1}} dx dy}_{I(f)} < \infty \right\},$$

endowed with the natural norm

$$\|f\|_{W^{1-1/p,p}} = \|f\|_{L^p} + [I(f)]^{1/p}.$$

For a proof of this result using only elementary arguments, see e.g. [4, Chapter 11]. We just mention here the three steps of the proof.

**Step 1** For  $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$  and with  $f(x) = u(x, 0)$ , one proves the “direct” estimate

$$I(f) \leq C \|u\|_{W^{1,p}}^p.$$

Combined with Facts 1 and 2, this leads to the continuity of the trace operator from  $W^{1,p}(\Omega)$  into  $W^{1-1/p,p}(\mathbb{R}^n)$ .

**Step 2** One proves that  $C^\infty(\mathbb{R}^n) \cap W^{1-1/p,p}(\mathbb{R}^n)$  is dense in  $W^{1-1/p,p}(\mathbb{R}^n)$ .

**Step 3** One proves the existence of a linear map

$$C^\infty(\mathbb{R}^n) \cap W^{1-1/p,p}(\mathbb{R}^n) \ni f \mapsto u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$$

such that  $u(x, 0) = f(x)$  and we have the “inverse” estimate

$$\|u\|_{W^{1,p}} \leq C\|f\|_{W^{1-1/p,p}}.$$

By Steps 2 and 3 and by Fact 3, each function  $f \in W^{1-1/p,p}(\mathbb{R}^n)$  is the trace of some  $u \in W^{1,p}(\Omega)$ , and in addition we may choose  $u$  such that the mapping  $f \mapsto u$  is linear continuous.

With moderate work, one may prove that the function

$$f(x) = \begin{cases} 1/(|x|^{n/p} |\ln x|), & \text{if } |x| < 1/2 \\ 0, & \text{if } |x| \geq 1/2 \end{cases}$$

is in  $L^p(\mathbb{R}^n)$  but not in  $W^{1-1/p,p}(\mathbb{R}^n)$ . Thus, for  $1 < p < \infty$ , the image of the trace operator is a strict subspace of  $L^p(\mathbb{R}^n)$ .

This contrasts with the next theorem.

**Theorem 1.2 (Gagliardo’s theorem #2)** *We have  $\text{tr } W^{1,1}(\Omega) = L^1(\mathbb{R}^n)$ .*

Here is our proof of this result.

**Proof.** In view of Fact 3, we have to prove that for each  $f \in L^1(\mathbb{R}^n)$  there exists some  $u \in W^{1,1}(\Omega)$  such that  $\text{tr } u = f$ . We claim that for every  $g \in C_c^\infty(\mathbb{R}^n)$ , there exists some  $v \in C_c^\infty(\overline{\Omega})$  such that  $v(x, 0) = g(x)$  and

$$\|v\|_{W^{1,1}} \leq C\|g\|_{L^1}. \quad (2)$$

Indeed, fix some  $\zeta \in C_c^\infty([0, \infty))$  such that  $\zeta(0) = 1$ , and let

$$v(x, x_{n+1}) = v^\delta(x, x_{n+1}) = g(x)\zeta(x_{n+1}/\delta), \quad \text{with } \delta > 0.$$

By straightforward calculations, we have

$$\lim_{\delta \rightarrow 0} \|v^\delta\|_{L^1} = 0, \quad (3)$$

$$\lim_{\delta \rightarrow 0} \left\| \frac{\partial v^\delta}{\partial x_j} \right\|_{L^1} = 0, \quad j = 1, \dots, n, \quad (4)$$

and

$$\left\| \frac{\partial v^\delta}{\partial x_{n+1}} \right\|_{L^1} = C\|g\|_{L^1}. \quad (5)$$

By (3)–(5), for sufficiently small  $\delta$ ,  $v$  satisfies (2).

We complete the proof of the theorem as follows. Let  $f \in L^1(\mathbb{R}^n)$ . Then there exists a sequence  $(f_j) \subset C_c^\infty(\mathbb{R}^n)$  such that  $\sum_j f_j = f$  in  $L^1$  and

$$\sum_j \|f_j\|_{L^1} \leq 2\|f\|_{L^1}.^1 \quad (6)$$

For each  $j$ , consider (in virtue of (2))  $v_j \in C_c^\infty(\overline{\Omega})$  satisfy  $v_j(x, 0) = f_j(x)$  and  $\|v_j\|_{W^{1,1}} \leq C\|f_j\|_{L^1}$ . Set  $v = \sum_j v_j$ , so that (by (2) and (6)) we have  $v \in W^{1,1}(\Omega)$  and  $\|v\|_{W^{1,1}} \leq C\|f\|_{L^1}$ . By Fact 3, we have  $\text{tr } v = f$ .  $\square$

## 2 Other stories

If we examine the proofs of Theorems 1.1 and 1.2, we see that, in the proof of Theorem 1.1, the map  $f \mapsto u$  is linear, while in the case of Theorem 1.2 this is not the case.<sup>2</sup> This is not an artefact of the proof. Peetre [6] proved that there was no linear continuous map

$$L^1(\mathbb{R}^n) \ni f \mapsto u \in W^{1,1}(\Omega)$$

such that  $\text{tr } u = f$ . For a relatively simple proof of this result, see [7, Section 5].

Another unexpected fact arises when we consider higher order spaces. If we set

$$W^{2,p}(\Omega) = \left\{ u \in L^p(\Omega); \frac{\partial u}{\partial x_j} \in W^{1,p}(\Omega), j = 1, \dots, n+1 \right\},$$

then (as suggested by Theorem 1.1) for  $1 < p < \infty$  we have

$$\text{tr } W^{2,p}(\Omega) = \left\{ f \in L^p(\mathbb{R}^n); \frac{\partial u}{\partial x_j} \in W^{1-1/p,p}(\mathbb{R}^n), j = 1, \dots, n+1 \right\};$$

see e.g. [4, Chapter 11].

On the other hand, Theorem 1.2 suggests that

$$\text{tr } W^{2,1}(\Omega) = W^{1,1}(\mathbb{R}^n).$$

But this is not true! Uspenskiĭ found the right answer.<sup>3</sup> His discovery was a significant achievement of another type of trace theory, the one of the weighted Sobolev spaces. For a modern treatment of this theory, see [5].

<sup>1</sup>This is a special case of the following trivial fact: if  $(X, \|\cdot\|)$  is a normed space and if  $Y$  is a dense linear subspace of  $X$ , then for each  $x \in X$  there exists a sequence  $(y_j) \subset Y$  such that  $\sum_j y_j = x$  in  $X$  and  $\sum_j \|y_j\| \leq 2\|x\|$ .

<sup>2</sup>Indeed, in (2) the map  $v$  depends not only on  $g$ , but also on  $\delta$ , which need not be the same for every  $g$ .

<sup>3</sup>The trace of  $W^{2,1}(\Omega)$  is the Besov space  $B_{1,1}^1(\mathbb{R}^n)$ . But this goes beyond the scope of this note.

## References

- [1] N. Aronszajn. Boundary values of functions with finite Dirichlet integral. *Techn. Report Univ. of Kansas*, 14:77–94, 1955.
- [2] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [3] E. Gagliardo. Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in  $n$  variabili. *Rend. Sem. Mat. Univ. Padova*, 27:284–305, 1957.
- [4] P. Mironescu. Fine properties of functions: an introduction. *Scoala Normala Superioara din Bucuresti, 2005*, pp.80., <https://cel.archives-ouvertes.fr/cel-00747696>.
- [5] P. Mironescu and E. Russ. Traces of weighted sobolev spaces. Old and new. *Nonlinear Analysis TMA*, to appear, hal-01064025.
- [6] J. Peetre. A counterexample connected with Gagliardo’s trace theorem. *Comment. Math. Special Issue*, 2:277–282, 1979. Special issue dedicated to Wladyslaw Orlicz on the occasion of his seventy-fifth birthday.
- [7] A. Pełczyński, M. Wojciechowski. Spaces of functions with bounded variation and Sobolev spaces without local unconditional structure. *J. Reine Angew. Math.*, 558:109–157, 2003.
- [8] S.V. Uspenskii. Imbedding theorems for classes with weights. *Trudy Mat. Inst. Steklov.*, 60:282–303, 1961. English translation: *Am. Math. Soc. Transl.*, 87:121–145, 1970.

*Petru Mironescu*

Université de Lyon, CNRS UMR 5208, Université Lyon 1, Institut Camille Jordan  
43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France  
E-mail: [mironescu@math.univ-lyon1.fr](mailto:mironescu@math.univ-lyon1.fr)