Note on Gagliardo’s theorem “$\text{tr} W^{1,1} = L^1$”

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Abstract - We present a short proof of the fact, originally established by Gagliardo, that every function $f \in L^1(\mathbb{R}^n)$ is the trace of a function $u \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$.

Key words and phrases: Sobolev space, trace, Gagliardo

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1 A short proof

The aim of this note is to give the “proof from the book” of the fact that, given a function $f \in L^1(\mathbb{R}^n)$, there exists some $u \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$ whose trace is $f$. This classical result goes back to Gagliardo [3]. The impatient reader can go directly to the proof of (2), which is the heart of the note.

We start by recalling some definitions and basic facts. Let $\Omega = \mathbb{R}^n \times (0, \infty)$. For $1 \leq p < \infty$, we consider the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega); \frac{\partial u}{\partial x_j} \in L^p(\Omega), \; j = 1, \ldots, n+1 \right\},$$

equipped with the standard norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{j=1}^{n+1} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p}.$$

Here, the partial derivatives are generalized derivatives or, equivalently, derivatives in the distributions sense. Before Gagliardo’s work [3], the state of the art concerning existence of traces was the following.

**Fact 1** $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$; see e.g. [2, Corollary 9.8].

**Fact 2** For $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$, set $f(x) = u(x, 0)$, $x \in \mathbb{R}^n$. Then we have the estimate

$$\|f\|_{L^p} \leq \|u\|_{W^{1,p}};$$

see e.g. [2, Lemma 9.9].
Fact 3 By the first two facts, the mapping \( u \mapsto f \) admits a unique linear continuous extension, the **trace operator** \( \text{tr} \), from \( W^{1,p}(\Omega) \) into \( L^p(\mathbb{R}^n) \).

Fact 4 When \( p = 2 \), Aronszajn [1] discovered that the trace operator is not onto, and that its image is precisely the set

\[
\left\{ f \in L^2(\mathbb{R}^n); \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi < \infty \right\};
\]

here, \( \hat{f} \) stands for the Fourier transform. The arguments in [1] rely on Fourier transform methods, and are difficult to apply to the case where \( p \neq 2 \).

Gagliardo achieved the more complicated task of characterizing \( \text{tr} W^{1,p}(\Omega) \) for \( p \neq 2 \). He obtained two results with different flavors and proofs.

**Theorem 1.1 (Gagliardo’s theorem #1)** For \( 1 < p < \infty \), we have

\[
\text{tr} W^{1,p}(\Omega) = W^{1-1/p,p}(\mathbb{R}^n).
\] (1)

Here,

\[
W^{1-1/p,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n); \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p-1}} dxdy < \infty \right\},
\]

endowed with the natural norm

\[
\|f\|_{W^{1-1/p,p}} = \|f\|_{L^p} + [I(f)]^{1/p}.
\]

For a proof of this result using only elementary arguments, see e.g. [4, Chapter 11]. We just mention here the three steps of the proof.

**Step 1** For \( u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega) \) and with \( f(x) = u(x,0) \), one proves the “direct” estimate

\[
I(f) \leq C\|u\|_{W^{1,p}}^p.
\]

Combined with Facts 1 and 2, this leads to the continuity of the trace operator from \( W^{1,p}(\Omega) \) into \( W^{1-1/p,p}(\mathbb{R}^n) \).

**Step 2** One proves that \( C^\infty(\mathbb{R}^n) \cap W^{1-1/p,p}(\mathbb{R}^n) \) is dense in \( W^{1-1/p,p}(\mathbb{R}^n) \).
Step 3 One proves the existence of a linear map
\[ C^\infty(\mathbb{R}^n) \cap W^{1-1/p,p}(\mathbb{R}^n) \ni f \mapsto u \in C^\infty_{c}(\Omega) \cap W^{1,p}(\Omega) \]
such that \( u(x,0) = f(x) \) and we have the “inverse” estimate
\[ \|u\|_{W^{1,p}} \leq C\|f\|_{W^{1-1/p,p}}. \]

By Steps 2 and 3 and by Fact 3, each function \( f \in W^{1-1/p,p}(\mathbb{R}^n) \) is the trace of some \( u \in W^{1,p}(\Omega) \), and in addition we may choose \( u \) such that the mapping \( f \mapsto u \) is linear continuous.

With moderate work, one may prove that the function
\[ f(x) = \begin{cases} 
1/\left(|x|^{n/p} \ln x \right), & \text{if } |x| < 1/2 \\
0, & \text{if } |x| \geq 1/2
\end{cases} \]
is in \( L^p(\mathbb{R}^n) \) but not in \( W^{1-1/p,p}(\mathbb{R}^n) \). Thus, for \( 1 < p < \infty \), the image of the trace operator is a strict subspace of \( L^p(\mathbb{R}^n) \).

This contrasts with the next theorem.

**Theorem 1.2 (Gagliardo’s theorem #2)** We have \( \text{tr } W^{1,1}(\Omega) = L^1(\mathbb{R}^n) \).

Here is our proof of this result.

**Proof.** In view of Fact 3, we have to prove that for each \( f \in L^1(\mathbb{R}^n) \) there exists some \( u \in W^{1,1}(\Omega) \) such that \( \text{tr } u = f \). We claim that for every \( g \in C^\infty_{c}(\mathbb{R}^n) \), there exists some \( v \in C^\infty_{c}(\Omega) \) such that \( v(x,0) = g(x) \) and
\[ \|v\|_{W^{1,1}} \leq C\|g\|_{L^1}. \] (2)

Indeed, fix some \( \zeta \in C^\infty_{c}([0,\infty)) \) such that \( \zeta(0) = 1 \), and let
\[ v(x,x_{n+1}) = v^\delta(x,x_{n+1}) = g(x)\zeta(x_{n+1}/\delta), \text{ with } \delta > 0. \]

By straighforward calculations, we have
\[ \lim_{\delta \to 0} \|v^\delta\|_{L^1} = 0, \quad (3) \]
\[ \lim_{\delta \to 0} \left\| \frac{\partial v^\delta}{\partial x_j} \right\|_{L^1} = 0, \quad j = 1, \ldots, n, \quad (4) \]
and
\[ \left\| \frac{\partial v^\delta}{\partial x_{n+1}} \right\|_{L^1} = C\|g\|_{L^1}. \] (5)

By (3)–(5), for sufficiently small \( \delta \), \( v \) satisfies (2).
We complete the proof of the theorem as follows. Let \( f \in L^1(\mathbb{R}^n) \). Then there exists a sequence \((f_j) \subset C_c^\infty(\mathbb{R}^n)\) such that \( \sum_j f_j = f \) in \( L^1 \) and
\[
\sum_j \|f_j\|_{L^1} \leq 2\|f\|_{L^1}. \tag{6}
\]

For each \( j \), consider (in virtue of (2)) \( v_j \in C_c^\infty(\Omega) \) satisfy \( v_j(x,0) = f_j(x) \) and \( \|v_j\|_{W^{1,1}} \leq C\|f_j\|_{L^1} \). Set \( v = \sum_j v_j \), so that (by (2) and (6)) we have \( v \in W^{1,1}(\Omega) \) and \( \|v\|_{W^{1,1}} \leq C\|f\|_{L^1} \). By Fact 3, we have \( \text{tr} v = f \). \( \square \)

2 Other stories

If we examine the proofs of Theorems 1.1 and 1.2, we see that, in the proof of Theorem 1.1, the map \( f \mapsto u \) is linear, while in the case of Theorem 1.2 this is not the case.\(^2\) This is not an artefact of the proof. Peetre [6] proved that there was no linear continuous map
\[
L^1(\mathbb{R}^n) \ni f \mapsto u \in W^{1,1}(\Omega)
\]
such that \( \text{tr} u = f \). For a relatively simple proof of this result, see [7, Section 5].

Another unexpected fact arises when we consider higher order spaces. If we set
\[
W^{2,p}(\Omega) = \left\{ u \in L^p(\Omega); \frac{\partial u}{\partial x_j} \in W^{1,p}(\Omega), \; j = 1, \ldots, n+1 \right\},
\]
then (as suggested by Theorem 1.1) for \( 1 < p < \infty \) we have
\[
\text{tr} W^{2,p}(\Omega) = \left\{ f \in L^p(\mathbb{R}^n); \frac{\partial u}{\partial x_j} \in W^{1-1/p,p}(\mathbb{R}^n), \; j = 1, \ldots, n+1 \right\};
\]
see e.g. [4, Chapter 11].

On the other hand, Theorem 1.2 suggests that
\[
\text{tr} W^{2,1}(\Omega) = W^{1,1}(\mathbb{R}^n).
\]
But this is not true! Uspenskiĭ found the right answer.\(^3\) His discovery was a significant achievement of another type of trace theory, the one of the weighted Sobolev spaces. For a modern treatment of this theory, see [5].

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1. This is a special case of the following trivial fact: if \((X,\|\|)\) is a normed space and if \(Y\) is a dense linear subspace of \(X\), then for each \(x \in X\) there exists a sequence \((y_j) \subset Y\) such that \( \sum_j y_j = x \in X \) and \( \sum_j \|y_j\| \leq 2\|x\| \).

2. Indeed, in (2) the map \( v \) depends not only on \( g \), but also on \( \delta \), which need not be the same for every \( g \).

3. The trace of \( W^{2,1}(\Omega) \) is the Besov space \( B_{1,1}^1(\mathbb{R}^n) \). But this goes beyond the scope of this note.
References


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