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Jonas Kahn

► **To cite this version:**

Jonas Kahn. IMPROPER POISSON LINE PROCESS AS SIRSN IN ANY DIMENSION. 2015. hal-01131107v1

**HAL Id: hal-01131107**

**<https://hal.science/hal-01131107v1>**

Preprint submitted on 12 Mar 2015 (v1), last revised 29 Sep 2016 (v3)

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# IMPROPER POISSON LINE PROCESS AS SIRSIN IN ANY DIMENSION

BY JONAS KAHN

Aldous has introduced a notion of scale-invariant random spatial network (SIRSIN) as a mathematical formalization of road networks. Intuitively, those are random processes that assign a route between each pair of points in Euclidean space, while being invariant under rotation, translation, and change of scale, and such that the routes are not too long and mainly on “main roads”.

The only known example was somewhat artificial since invariance had to be added at the end of the construction. We prove that the network of geodesics in the random metric space generated by a Poisson line process marked by speeds according to a power law is a SIRSIN, in any dimension.

Along the way, we establish bounds comparing Euclidean balls and balls for the random metric space. We also prove that in dimension more than two, the geodesics have “many directions” near each point where they are not straight.

**1. Introduction.** Scale-invariant random spatial networks (SIRSINs) are a class of random networks defined as a route between each pair of points, with three types of properties. Firstly, invariance properties, secondly, guarantees on lengths of routes – in the Euclidean metric – and thirdly guarantees on the length of intersection of the whole truncated network with a compact set. It turns out that these conditions are enough to deeply constrain the network. For example, all SIRSINs have semi-infinite paths whose any subset is included in a route, but no infinite such path.

The only known example is the binary hierarchy model, in two dimensions. It consists in minimum-time paths on a dyadic grid where speed depends on the two-valuation, and uniformly randomly scaled and rotated. The latter feature is somewhat unsatisfying: the model itself is invariant, but a realisation is “far” from it: observation of a small region gives huge information on the network everywhere.

A more “natural” candidate for a SIRSIN is therefore the *Poisson line process model*. Intuitively, lines are thrown uniformly at random in  $\mathbb{R}^d$ , and marked with a speed limit. Slower lines are dense in  $\mathbb{R}^d$ . Then the route

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*MSC 2010 subject classifications:* Primary 60D05; secondary 90B15, 60G55

*Keywords and phrases:* Poisson Line process, SIRSIN, scale-invariant random spatial network, stochastic geometry, spatial network, random metric space,  $\Pi$ -geodesic, many directions

between two points is the minimum-time path on these lines. The whole construction is invariant. The aim of this paper is showing that the Poisson line process is indeed a SIRS<sub>N</sub>.

Historically, Aldous (2014) introduced the notion of a SIRS<sub>N</sub>, and proved a number of their properties, including all those mentioned up to here in this introduction. Aldous and Ganesan (2013) explain the idea in words. The motivation was two-fold.

First, Aldous and Kendall (2008) had proved that it was possible to build a road network that both had routes almost as short as segments, and total road length almost as short as the shortest possible network. However the network was less efficient at small scales. Thanks to their invariance properties, SIRS<sub>N</sub>s have the same efficiency at all scales. It turns out that shorter routes entail longer networks.

The second motivation was to give a mathematical abstraction of road networks and maps, in particular online maps as they are used today. Namely, we may change viewpoint, zoom in or out, and the appearance changes little, as smaller roads are shown and hidden. SIRS<sub>N</sub>s are then defined as invariant under translation, rotation and change of scale. Moreover, we are less interested in the roads than in the routes: how do we drive from  $A$  to  $B$ ? SIRS<sub>N</sub>s are thus defined only through unique routes connecting pairs of points. It turns out that a notion of “main roads” at any scale can be built from the network of routes itself. Similarly, Kalapala et al. (2006) have shown that a number of statistics of road networks do not depend on scale.

Aldous (2014) has proved that the binary hierarchy model was a SIRS<sub>N</sub>, and suggested that two other models might be, among which the Poisson line process model. Kendall (2014) has then proved important properties of the Poisson line process with appropriate speeds: it does yield a random metric space and this space is a geodesic space. Moreover, in dimension two, the geodesics are almost everywhere unique, the geodesics are locally of finite mean-length, and the subnetwork obtained from the routes connecting points of an independent Poisson point process has finite length in a compact set. The latter properties may be seen as weaker forms of the properties of a SIRS<sub>N</sub>, but fall short of the full definition.

As a candidate for a SIRS<sub>N</sub>, Poisson line process model then fall in a large category: networks derived from geodesic spaces. Indeed, from any geodesic space, we may build a spatial network by associating to any two points the geodesic(s) connecting them. It is not obvious to know when such a network is a SIRS<sub>N</sub>.

In Section 2, we give a precise definition of a SIRS<sub>N</sub> and of Poisson line

processes. We also give other notations and definitions, recall some known results, in particular that the Poisson line process with speed limits yield a random metric on  $\mathbb{R}^d$ . We then give a few basic properties of  $\Pi$ -paths, that is paths in this metric space. In Section 3, we give sharp bounds on the random diameter for this metric of a Euclidean ball, with a few generalizations. These estimates will be an important tool in several subsequent proofs. In Section 4, we prove that geodesics between a given pair of points are almost surely unique, in any dimension. Lemma 4.4 will play a central role: we introduce the notion of “many directions”, and the lemma states that geodesics have many directions at relevant points. A consequence is that any path using the same line as a geodesic will contain these points. We will then conclude by noticing that geodesics between the same pair of points almost surely use the same lines (Lemma 4.7). In Section 5, we prove that geodesics have finite mean Euclidean length. Alternatively, we may see the result as a stochastic control of the Euclidean diameter of balls for the metric generated by the Poisson line process. Section 6 contains the last and most important piece proof that the Poisson line process generates a SIRS. Intuitively, we establish a sharp control of the total length of the intersection of all infinitely many geodesics minus a ball around each endpoint, with a given ball. This corresponds to the fact that all these geodesics coalesce before hitting the ball and split after leaving it. Bounds are given using the pigeonhole principle and the fact that relevant geodesics must use the few fast lines that hit the ball. Finally, Section 7 gives a few potential directions of inquiry.

**2. Notations, definitions, basic properties.** We follow Kendall’s (2014) notations whenever possible.

We write  $B(x, r)$  for the ball with center  $x$  and radius  $r$ .

We first give the precise definition of a SIRS. Recalling that a Poisson point process of intensity measure  $\Lambda$  – without atoms – on a measurable space  $(\mathcal{X}, \mathcal{B})$  is a random set of points such that for any  $B \in \mathcal{B}$ , the number of points  $N(B)$  in  $B$  is a Poisson variable with intensity  $\Lambda(B)$ , and the number of points  $N(B_i)$  in disjoint  $B_i$  are independent, we have:

**Definition 2.1.** *A SIRS is a process that associates to any two points  $x_1$  and  $x_2$  in  $\mathbb{R}^d$  random routes such that:*

1. *Between two specified points  $x_1$  and  $x_2$ , there is almost surely a unique route  $\mathcal{R}(x_1, x_2) = \mathcal{R}(x_2, x_1)$ . It is a finite-length path connecting  $x_1$  and  $x_2$ .*
2. *For a finite number of points  $x_1, \dots, x_k$  in  $\mathbb{R}^d$ , consider the subnetwork  $\mathcal{N}(x_1, \dots, x_k)$  formed by the random routes connecting all  $x_i$  and  $x_j$ .*

Then  $\mathcal{N}(x_1, \dots, x_k)$  is statistically invariant under translation, rotation and change of scale: if  $\mathfrak{R}$  is a Euclidean similarity of  $\mathbb{R}^d$ , then  $\mathcal{N}(\mathfrak{R}(x_1), \dots, \mathfrak{R}(x_k))$  has the same distribution as  $\mathcal{N}(x_1, \dots, x_k)$ .

3. Let  $D_1$  be the length of a route between two points at unit distance. Then  $\mathbb{E}[D_1] < \infty$ .
4. Let  $\{\Xi_n, n \in \mathbb{N}^*\}$  be a collection of Poisson processes on  $\mathbb{R}^d$  with intensity  $n$  times Lebesgue, all independent from the SIRS. Suppose they are coupled so that  $\Xi_n \subset \Xi_{n+1}$ . Write  $\Xi = \bigcup_{n \in \mathbb{N}^*} \Xi_n$ . Then the intensity (mean length per unit area)  $p(1)$  of the following long-distance network is finite:

$$\bigcup_{x_1, x_2 \in \Xi} (\mathcal{R}(x_1, x_2)) \setminus (B(x_1, 1) \cup B(x_2, 1)).$$

As a remark, Kendall (2014) uses more conditions in his definition, but the missing properties are implied by property 4. They were useful to define weaker variants of a SIRS.

The use of Poisson processes in property 4 makes it look slightly complex, but this is a technical shortcut: it allows us to study the network through only countably many routes. Morally, we would like the same to be true for the union of all geodesics between all points of  $\mathbb{R}^d$ .

We now turn to the definition of the improper Poisson line process. We first need a measure on lines. More details on this kind of processes may be found in the book by Stoyan, Kendall and Mecke (1996).

Let  $\mathcal{L}^d$  be the space of all lines of  $\mathbb{R}^d$ . A line is “un-sensed”, that is, it is seen as a subset of  $\mathbb{R}^d$ , without a preferred direction. For  $K$  a compact of  $\mathbb{R}^d$ , the *hitting set* of  $K$  is the set of lines that intersect  $K$ , denoted as:

$$[K] = \left\{ l \in \mathcal{L}^d : l \text{ hits } K \right\}.$$

We also denote by  $m_d$  the Hausdorff measure of dimension  $d$ . With these notations:

**Definition 2.2.** *The invariant line measure  $\mu_d(l)$  is the unique measure on the space of lines of  $\mathbb{R}^d$  that is invariant under Euclidean isometries, and normalized by the following requirement: for a compact set  $K$  in  $\mathbb{R}^d$  of non-empty interior, the  $\mu_d$ -measure of the hitting set of  $K$  is half the Hausdorff  $(d - 1)$ -dimensional measure of the boundary of  $K$ :*

$$\mu_d([K]) = \frac{1}{2} m_{d-1}(\partial K).$$

The reason for the normalizing constant  $\frac{1}{2}$  is to ensure that the measure of the hitting set of a flat hypersurface  $A$  is its hyperarea  $m_{d-1}(A)$ .

Since we will often need the hitting set of a ball, we write  $\omega_{d-1}$  for the hyperarea of the unit sphere  $\mathbb{S}^{d-1}$ . So that  $\mu_d([B(x, r)]) = \frac{\omega_{d-1}}{2} r^{d-1}$ .

Consider the following parametrization of a line  $l$ : it is given by a direction and a localization. The direction is an element of the projective space  $P\mathbb{R}^{d-1}$ . It then defines a hyperplane normal to this direction, through a special point – the origin – of  $\mathbb{R}^d$ . The localization is a point on this hyperplane. The line  $l$  is then the line through this point normal to this hyperplane.

Recall that the projective space  $P\mathbb{R}^{d-1}$  may be seen as the sphere  $\mathbb{S}^{d-1}$  with opposite – antipodal – points identified. The projective space then inherits the natural metric on the sphere, namely the distance between two pairs of antipodal points is the angle between the pair of segments joining them, in radians. Up to a null-measure set, the projective space may be more simply viewed as a hemisphere.

With this parametrization, and writing  $\mathbb{B}^d$  for the unit ball in  $\mathbb{R}^d$  and  $\mathcal{H} = \mathbb{R}^{d-1}$  for an hyperplane of  $\mathbb{R}^d$ , we may write  $\mu_d$  as a product measure on  $P\mathbb{R}^{d-1} \otimes \mathcal{H}$ :

$$\mu_d = \frac{1}{m_{d-1}(\mathbb{B}^{d-1})} m_{d-1} \otimes m_{d-1}.$$

To make a clearer reference to it, we write  $\mu_{d-1}^{(o)} = 2m_{d-1}/m_{d-1}(\mathbb{S}^{d-1})$  for the probability measure on  $P\mathbb{R}^{d-1}$ . For a set of lines  $\mathcal{L}$ , we write  $\mu_{d-1}^{(o)}(\mathcal{L})$  for the measure of the set of directions of lines in  $\mathcal{L}$  that go through the origin. In particular, for  $A$  a subset of  $\mathbb{R}^d$  and  $x$  a point of  $\mathbb{R}^d$ , the value  $\mu_{d-1}^{(o)}([A - x])$  will be the proportion of directions that appear in the set of directions of lines through  $A$  and  $x$ . We call it the *solid angle* of  $A$  from  $x$ . Notice that it is not quite the usual definition since points mirrored with respect to  $x$  contribute only once, and since we have normalized to a probability measure.

A special case we shall need is the measure of a cone, that is all the lines with angle less than  $\theta_0$  from a given line through the origin. This measure is proportional to the area on the hemisphere hit by the lines, hence proportional to  $\int_0^{\theta_0} \sin^{d-2} \theta \cos \theta d\theta = \sin^{d-1}(\theta_0)/(d-1)$ . Value for  $\theta_0 = \pi/2$  is one, so that:

$$(1) \quad \mu_{d-1}^{(o)}(\text{cone of aperture } \theta_0) = \sin^{d-1}(\theta_0).$$

The uniform Poisson line process is the image of the Poisson point process on  $\mathcal{L}^d$  with intensity measure  $\mu_d$ .

We define our improper Poisson line process by adding a mark on each line, a speed limit. Namely the improper Poisson line process is the image

of a Poisson point process on  $P\mathbb{R}^{d-1} \otimes \mathcal{H} \otimes \mathbb{R}_*^+$  with intensity measure  $\mu_{d,\gamma}$ , given by the density:

$$d\mu_{d,\gamma}(l, v) = d\mu_d(l)(\gamma - 1)v^{-\gamma}dv$$

for  $\gamma > d$ . Kendall (2014) does define this process for all  $\gamma > 1$ , but the relevant case is with  $\gamma > d$ .

In words, we have more and more slower lines, following a power law. Since  $\int v^{-\gamma}dv$  diverges at zero, the lines are dense in  $\mathbb{R}^d$ . However lines faster than any given speed are not dense. In particular the number of lines faster than  $v_0$  hitting a convex set  $K$  is a Poisson variable with parameter:

$$\begin{aligned} \mu_{d,\gamma}\left((l, v) : l \in [K] \text{ and } v \geq v_0\right) &= \mu_d([K]) \int_{v_0}^{\infty} (\gamma - 1)v^{-\gamma}dv \\ &= \frac{1}{2}m_{d-1}(\partial K)v_0^{-(\gamma-1)}. \end{aligned}$$

We call  $\Pi = \Pi^{d,\gamma}$  the corresponding random process of marked lines  $(l, v)$ . Since the dimension  $d$  and parameter  $\gamma$  will always be clear from context, we drop them in the notation. Notice that the total number of lines is almost surely countable. If  $(l, v) \in \Pi$ , we say that the speed of line  $l$  is  $v$  and denote it  $v(l)$ .

For a subset  $\mathcal{L}$  of lines, we write  $\Pi_{\mathcal{L}}$  for the restriction of  $\Pi$  to these lines, that is  $\Pi_{\mathcal{L}} = \{(l, v) : l \in \mathcal{L}\}$ . In particular, the line process restricted to lines hitting  $A$  but not  $B$  is  $\Pi_{[A] \setminus [B]}$ .

We denote  $\mathcal{S}$  the *silhouette* of  $\Pi$ , that is the random set in  $\mathbb{R}^d$  made of all the lines of  $\Pi$ , that is  $\mathcal{S} = \{x \in \mathbb{R}^d : \exists (l, v) \in \Pi : x \in l\}$ . We also write  $\mathcal{S}_{v_0}$  for the random closed set in  $\mathbb{R}^d$  made of all the lines  $(l, v)$  in  $\Pi$  such that  $v \geq v_0$ .

We may then define  $\Pi$ -paths:

**Definition 2.3.** *A finite-time  $\Pi$ -path is a locally Lipschitz path in  $\mathbb{R}^d$  respecting the speed limits imposed by  $\Pi$ . More precisely, it is a continuous  $\mathbb{R}^d$ -valued function:*

$$\xi : [0, T] \rightarrow \mathbb{R}^d,$$

with  $T$  finite, such that for almost all  $t \in [0, T]$ , either:

- the speed is zero:  $\xi'(t) = 0$ .
- the path follows a line in  $\Pi$ : there is a  $v \geq |\xi'(t)|$  such that  $(\xi(t) + \xi'(t)\mathbb{R}, v) \in \Pi$ .

We call  $T = T(\xi)$  the time length of the path  $\xi$ , or just its time for short. An infinite-time  $\Pi$ -path  $\xi$  is the same, with  $T(\xi) = \infty$ , except that its domain is  $[0, \infty)$ .

A  $\Pi$ -path is a finite or infinite-time  $\Pi$ -path.

Abusively, we write  $\xi \in \Pi$ .

Notice that the image of a  $\Pi$ -path is not necessarily contained in the silhouette  $\mathcal{S}$ . It only needs to have speed zero outside  $\mathcal{S}$ . The remark is especially relevant in dimension at least three, where the lines never cross. However, since the lines are dense in  $\mathbb{R}^d$ , it turns out that there are paths joining any pair of points, without any segment in  $\mathbb{R}^d \setminus \mathcal{S}$ . We give a clearer intuition of their tree-like structure in the proof of Theorem 3.1 and Figure 1.

We write  $\Pi^{ab} = \{\xi \in \Pi : \xi(0) = a \text{ and } \xi(T(\xi)) = b\}$  for any two points  $a$  and  $b$ .

The following theorem is a union of results from Kendall's (2014) paper:

**Theorem 2.4.** *Almost surely, all finite-time  $\Pi$ -paths have finite Euclidean length.*

*Almost surely, there are finite-time  $\Pi$ -paths between each pair of points of  $\mathbb{R}^d$ . Moreover, for any two points  $a$  and  $b$ , the infimum  $T_{ab}$  of time lengths  $T(\xi)$  of  $\Pi$ -paths  $\xi \in \Pi^{ab}$  is attained.*

*Hence  $\mathbb{R}^d$  with the metric  $d(a, b) = T_{ab}$  is a random metric geodesic space.*

We call this metric *time length* or  $\Pi$ -*length*. Time diameters and similar notions are defined in the same way.

We denote by  $\mathcal{N}$  the random network made by all the geodesics connecting all pairs of points in  $\mathbb{R}^d$ . Our aim is to show that  $\mathcal{N}$  is a SIRSN.

It is often possible to define similar metrics on other sets  $\Gamma$  of marked lines, though they might not be geodesic. We then speak of  $\Gamma$ -length. The typical case is when we restrict  $\Pi$  to a subset of lines  $\mathcal{L}$ , yielding  $\Gamma = \Pi_{\mathcal{L}}$  and  $\Pi_{\mathcal{L}}$ -length.

We now introduce some notations and remarks to make easier manipulating paths and geodesics:

- If  $\xi \in \Pi^{ab}$ , we often write  $\xi_{ab}$  instead.
- Concatenation of  $\Pi$ -paths is denoted  $\xi_{ac} = \xi_{ab}\xi_{bc}$ , that is,  $\xi_{ac}(t) = \xi_{ab}(t)$  if  $t < T(\xi_{ab})$ , and  $\xi_{ac}(t) = \xi_{bc}(t - T(\xi_{ab}))$  if  $t \geq T(\xi_{ab})$ .
- We use the letter  $g$  for geodesics, and usually  $g_{ab}$  for a geodesic from  $a$  to  $b$ . We say that  $g_{ab}$  is unique if there is a unique geodesic from  $a$  to  $b$ .



- Abusively, we identify a  $\Pi$ -path  $\xi$  with its image in  $\mathbb{R}^d$  whenever it is clear. Hence, we may write  $x \in \xi$  if there is a time  $t$  such that  $\xi(t) = x$ . Similarly, if  $g_{ab}$  is unique and  $c, d \in g_{ab}$ , we say that  $g_{cd}$  is included  $g_{ab}$ .
- If  $g_{ab}$  is unique and  $c \in g_{ab}$ , then  $g_{ab} = g_{ac}g_{cb}$ .
- $T(g_{ab}) = T_{ab}$ .
- For a line  $l \in \Pi$  and a  $\Pi$ -path  $\xi : [0, T] \rightarrow \mathbb{R}^d$ , we define the *intersection length* of  $\xi$  and  $l$  as  $L_\xi(l) \doteq m_1(l \cap \xi([0, T]))$ .
- If the intersection length of  $\xi$  and  $l$  is not zero, we say that  $l$  is in the *support* of  $\xi$ , or that it is supporting  $\xi$ . We write  $l \sqsubset \xi$ . Moreover we denote the support of  $\xi$  by  $\mathcal{L}_\xi \doteq \{l \in \Pi : L_\xi(l) > 0\}$ .
- We define the *intersection time* of  $l$  and  $\xi$  as  $T_\xi(l) \doteq m_1([0, T] \cap \xi^{-1}(l))$ .
- In particular, almost surely, for all geodesics  $g$ , we have the following equality and decompositions:

$$\begin{aligned}
 L_g(l) &= v(l)T_g(l) && \text{for all } l \in \Pi. \\
 T(g) &= \sum_{l \in \mathcal{L}_g} T_g(l) \\
 (2) \qquad &= \sum_{l \in \mathcal{L}_g} v(l)L_g(l).
 \end{aligned}$$

- If the support of  $\xi$  is included in  $\mathcal{L}$ , that is  $\mathcal{L}_\xi \subset \mathcal{L}$ , we say that  $\xi$  is  $\Pi_{\mathcal{L}}$ -path. Abusively, we write  $\xi \in \Pi_{\mathcal{L}}$ .
- Similarly, we write  $\xi \in \Pi_{\mathcal{L}}^{ab}$  if  $\xi \in \Pi_{\mathcal{L}}$  and  $\xi \in \Pi^{ab}$ .

Finally, a few more generic notations. We call *internal  $\varepsilon$ -net* of a subset  $A$  of  $\mathbb{R}^d$  any maximal subset  $x_1, \dots, x_k$  of  $A$  such that  $|x_i - x_j| \geq \varepsilon$  for all  $i \neq j$ . For a set  $A \subset \mathbb{R}^d$ , we write  $A^r$  for its  $r$ -widening, that is the Minkowski sum  $A^r = A \oplus B(0, r)$ . We denote the maximal speed of a set of lines  $\mathcal{L}$  by  $v_{max}(\mathcal{L})$ . That is  $v_{max}(\mathcal{L}) = \sup_{l \in \mathcal{L}} v(l)$ . Notice that if those lines all hit a compact set  $K$ , this supremum is a maximum. Abusively, for  $A \in \mathbb{R}^d$ , we write  $v_{max}(A) = v_{max}([A])$ . We use  $C, c, c_1, \dots, c_i$  for any positive constant.

**3.  $\Pi$ -diameters of sets.** We start with giving a more quantitative version of Theorem 3.6 in Kendall's (2014) article. Namely, we show that in a given precompact set  $A$ , any two points can be joined in finite time, and that the largest time between two such points, the  $\Pi$ -diameter of  $A$  is not too big with high probability: this random variable has more than an exponential moment. It is also a generalization that we will need later on, by allowing to reject lines hitting forbidden areas  $F$ .

**Theorem 3.1.** *Let  $\alpha_{min} = 2^{(\gamma-1)/(\gamma-d)}$ . Let  $\Omega < \Omega_{max} = (2\alpha_{min})^{1-d}$  and define  $\alpha_{max} = \Omega^{-\frac{1}{d-1}}/2$ . Let  $\alpha$  such that  $\alpha_{min} < \alpha < \alpha_{max}$ . Let  $A$  and  $F$  be two subsets of  $\mathbb{R}^d$ , such that, for some  $r > 0$ :*

- *$A$  is connected.*
- *from any  $x \in A^{r/\alpha(1-\alpha^{-1})}$ , the solid angle  $\mu_{d-1}^{(o)}([F-x])$  of  $F$  is less than  $\Omega$ .*
- *$A$  may be covered by  $\mathring{N}$  balls  $B_i$  of radius  $r$ .*

*Then, there is a  $T_1$  depending only on  $\alpha$ ,  $\Omega$ ,  $\gamma$  and  $d$  such that, for any  $\varepsilon_{max} = 1/(2\mathring{N}(2^d\alpha^d)) > \varepsilon > 0$ , with probability  $1 - \varepsilon/\varepsilon_{max}$ , the diameter of  $A$  on  $\Pi_{[A^{r/\alpha(1-\alpha^{-1})}] \setminus [F]}$  satisfies the bound:*

$$(3) \quad \begin{aligned} T_{A,F} &\hat{=} \sup_{x,y \in A} \inf_{\xi \in \Pi_{[A^{r/\alpha(1-\alpha^{-1})}] \setminus [F]}^{xy}} T(\xi) \\ &\leq T_{r,\mathring{N}} \left( \ln \frac{1}{\varepsilon} \right)^{\frac{1}{\gamma-1}} \quad \text{with} \\ T_{r,\mathring{N}} &= \mathring{N} T_1 r^{\frac{\gamma-d}{\gamma-1}}. \end{aligned}$$

*In particular, this maximal time has more than an exponential moment: for any  $\delta < T_{r,\mathring{N}}^{-\frac{1}{\gamma-1}}$ , we have:*

$$(4) \quad \mathbb{E} \left[ \exp \left( \delta T_{A,F}^{\gamma-1} \right) \right] < \infty.$$

The proof is a slight variation on that of Theorem 3.6 in Kendall's (2014) article.

PROOF. Since  $A$  is connected and covered by  $\mathring{N}$  open balls, we may build a path between any two points of  $A$  by concatenating at most  $\mathring{N}$  paths between two points of  $A$  belonging to the same ball  $B_i$  of the cover.

We now recursively build a path between each pair of points  $x_0$  and  $y_0$  of  $B_i$ , in a binary tree-like fashion. First we specify

$$r_n = r\alpha^{-n}.$$

Then, with  $v_n$  to be chosen later:

- We call  $A_0$  the set of balls  $\{B_i\}_{1 \leq i \leq \mathring{N}}$ .
- To any ball  $B^{(n)} \in A_n$ , we associate an internal  $r_{n+1}$ -net of that ball. It may be viewed as a collection of balls  $B^{(n+1)}$  of radius  $r_{n+1}$ . We then define  $A_{n+1}$  as the set of all these balls  $B^{(n+1)}$  for all  $B^{(n)}$  together.

- We have thus built nested internal  $r_n$ -nets.
- For any two balls  $B_i^{(n+1)}$  and  $B_j^{(n+1)}$  belonging to the internal  $r_{n+1}$ -net of the same ball  $B^{(n)}$  in  $A_n$ , we find a line of speed at least  $v_n$  that hits both  $B_i^{(n+1)}$  and  $B_j^{(n+1)}$ , but not  $F$ . We will have to prove this is possible with high enough probability.
- For  $x_n$  and  $y_n$  both belonging to  $B^{(n)} \in A_n$ , we may then find two points  $x_{n+1}$  and  $y_{n+1}$  such that:  $x_n$  and  $x_{n+1}$  (resp.  $y_n$  and  $y_{n+1}$ ) belong to the same ball  $B_i^{(n+1)} \in A_n$  (resp.  $B_j^{(n+1)}$ ), and  $x_{n+1}$  and  $y_{n+1}$  both belong to a line of speed at least  $v_n$ .
- We may then build a path between  $x_n$  and  $y_n$  as a concatenation of three paths:  $\xi_{x_n y_n} = \xi_{x_n x_{n+1}} \xi_{x_{n+1} y_{n+1}} \xi_{y_{n+1} y_n}$ . The middle one is a segment. The other two are paths between points of the same ball in  $A_{n+1}$ .

As illustrated in Figure 1, we thus obtain a path between  $x_0$  and  $y_0$  that is made of exactly  $2^n$  segments for each  $n$ , each at speed at least  $v_n$  between two balls of the same internal  $r_{n+1}$ -net of a ball of radius  $r_n$ . Moreover, since the points  $x_0$  and  $y_0$  are in  $A$ , all of the segments are between points of the Minkowski sum  $A \bigoplus_{n=1}^{\infty} B(0, r_n) = A^{r/\alpha(1-\alpha^{-1})}$ .

This construction has built a path for each pair of points  $x_0$  and  $y_0$  in  $B_i$ . Since segments between balls of the same internal  $r_{n+1}$ -net of a ball of radius  $r_n$  are at most  $(2r_n + 2r_{n+1})$  long, and are at speed at least  $v_n$ , the  $\Pi_{[A^{r/\alpha(1-\alpha^{-1})}] \setminus [F]}$ -diameter  $T_{A,F}$  of  $A$  is bounded from above by:

$$(5) \quad T_{A,F} \leq \mathring{N} \sum_{n=1}^{\infty} 2^n \frac{2r_n + 2r_{n+1}}{v_n},$$

on the event that this construction is possible.

Now to control the probability of this event, we need:

- a bound on the number of pairs of balls in the same  $r_n$ -net.
- a bound on the probability that the fastest line hitting two such balls but not  $F$  is slower than  $v_n$ .

We obtain the first bound by using the formula  $r_n = \alpha^{-1} r_{n-1}$ . Indeed, each  $r_n$ -net of a  $r_{n-1}$ -ball is then the same as a  $\alpha^{-1}$ -net of a radius 1 ball. By standard arguments, it is then made of at most  $(2\alpha)^d$  balls. So that there are at most  $\mathring{N}(2^d \alpha^d)^n$  balls in  $A_n$ , and at most  $\mathring{N}(2^d \alpha^d)^{n+1}$  pairs of balls in the same  $r_n$ -net.

We now consider the lines hitting two  $r_{n+1}$ -balls  $B(x, r_{n+1})$  and  $B(y, r_{n+1})$  in an internal  $r_{n+1}$ -net of a  $r_n$ -ball, but not  $F$ . We have seen that they were

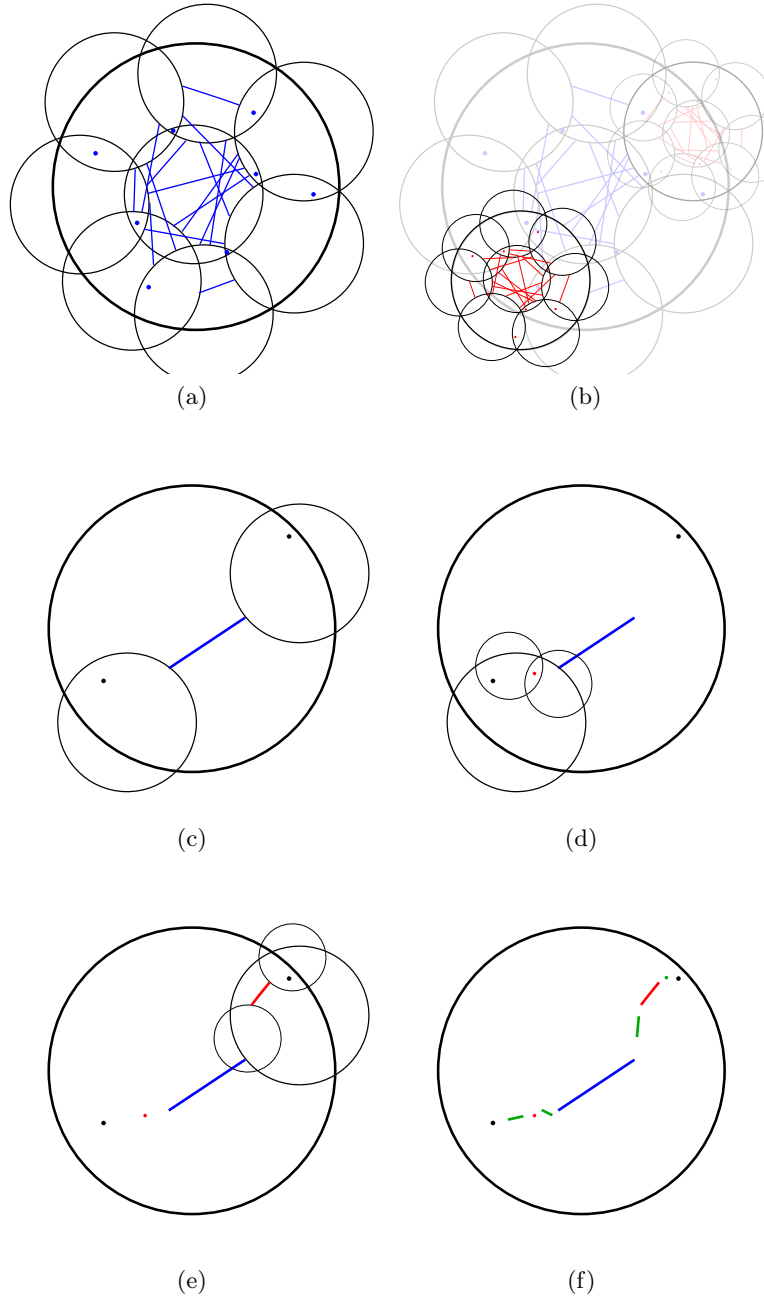


Fig 1: We cover each ball  $B_i$  with a  $r_1$ -net, and connect each pair of balls of the net with a segment, be it a point (a). We then do the same at each scale, for  $r_{n+1}$ -nets of the balls of the  $r_n$ -nets (b). Each point belongs to a ball of the  $r_1$ -net, so we build a path between two points starting from the segment connecting their balls (c). We then (d)(e) connect the points to the segment endpoints with the segment connecting their balls in their respective  $r_2$ -nets. At stage  $n$ , the path (f) is made of  $2^n$  segments at each scale  $n$ .

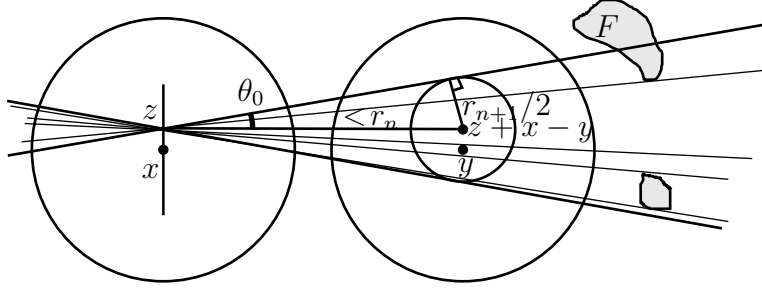


Fig 2: Lines that hit both balls  $B(x, r_{n+1})$  and  $B(y, r_{n+1})$  but not  $F$  (not necessarily connected) contain lines through  $z \in D(x, r_{n+1}/2)$  and  $B(z+x-y, r_{n+1}/2)$ . These are included in the cone of angle  $\theta_0$  through  $z$ .

in  $[A^{r/\alpha(1-\alpha^{-1})}]$ . We may then use the hypothesis of the theorem on the solid angle of  $F$ .

We want a bound on

$$\mu_d([B(x, r_{n+1})] \cap [B(y, r_{n+1})] \cap [F]^c).$$

Now, as illustrated in Figure 2, let  $D(x, r_{n+1}/2)$  be the hyperdisk centered on  $x$  with radius  $r_{n+1}/2$  and included in the hyperplane perpendicular to the vector  $x-y$ . It is included in  $B(x, r_{n+1})$ . Moreover, for any  $z \in D(x, r_{n+1}/2)$ , the ball  $B(z+y-x, r_{n+1}/2)$  is included in  $B(y, r_{n+1})$ .

So that we may write

$$\begin{aligned} & \mu_d([B(x, r_{n+1})] \cap [B(y, r_{n+1})] \cap [F]^c) \\ & \geq m_{d-1}(D(x, r_{n+1}/2)) \times \inf_{z \in D(x, r_{n+1}/2)} \mu_{d-1}^{(o)}([B(y-x, r_{n+1}/2)] \cap [F-z]^c) \\ & \geq m_{d-1}(D(x, r_{n+1}/2)) \times \left( \mu_{d-1}^{(o)}([B(y-x, r_{n+1}/2)]) - \Omega \right), \end{aligned}$$

Now,  $y-x \leq r_n = \alpha r_{n+1}$ . So that the lines in  $[B(y-x, r_{n+1}/2)]$  are in a cone of angle  $\theta_0$  such that  $\sin(\theta_0) \leq (2\alpha)^{-1}$ . So that by Formula 1:

$$(6) \quad \mu_{d-1}^{(o)}([B(y-x, r_{n+1}/2)]) \leq (2\alpha)^{1-d}.$$

Since  $\alpha < \alpha_{max}$ , we have  $(2\alpha)^{1-d} - \Omega > 0$ .

Moreover, with  $\nu_{d-1}$  the  $(d-1)$ -volume of a unit  $(d-1)$ -ball, we have:

$$\begin{aligned} m_{d-1}(D(x, r_{n+1}/2)) &= \nu_{d-1} \left( \frac{r_{n+1}}{2\alpha} \right)^{d-1}, \\ \mu_d([B(x, r_{n+1})] \cap [B(y, r_{n+1})] \cap [F]^c) &\leq \nu_{d-1} \left( (2\alpha)^{1-d} - \Omega \right) \left( \frac{r_{n+1}}{2\alpha} \right)^{d-1} \\ &\hat{=} c_{\alpha, \Omega} r_{n+1}^{d-1}. \end{aligned}$$

using the notation  $c_{\alpha, \Omega}$  to summarize and emphasize that this bound depends on  $\alpha$  and  $\Omega$ , but not on  $n$ ,  $r$ , or any other feature of  $A$  and  $F$ .

We thus get:

$$\mathbb{P}[v_{max}([B(x, r_{n+1})] \cap [B(y, r_{n+1})] \cap [F]^c) \leq v_n] \leq \exp\left(-c_{\alpha, \Omega} \frac{r_{n+1}^{d-1}}{v_n^{\gamma-1}}\right).$$

Multiplying by the number of relevant pairs of ball, we then obtain that the construction is possible except on an event of probability at most:

$$(7) \quad \sum_{n=0}^{\infty} \mathring{N}(2^d \alpha^d)^{n+1} \exp\left(-c_{\alpha, \Omega} \frac{r_{n+1}^{d-1}}{v_n^{\gamma-1}}\right).$$

Taking

$$v_n = \frac{r_{n+1}^{(d-1)/(\gamma-1)}}{((n+1)c_{\alpha, \Omega} \ln(1/\varepsilon))^{1/(\gamma-1)}}$$

and replacing in bound (7), we see that this becomes less than:

$$\sum_{n=0}^{\infty} \mathring{N}(2^d \alpha^d)^{n+1} \varepsilon^{n+1} \leq \frac{\varepsilon}{\varepsilon_{max}},$$

for  $\varepsilon < 1/(2\mathring{N}(2^d \alpha^d)) = \varepsilon_{max}$ , so that  $\varepsilon_{max}$  does not depend on  $A$  or  $F$  except through  $\alpha$  and  $\mathring{N}$ .

Replacing  $v_n$  and  $r_n$  with their value in the bound (5), we get:

$$\begin{aligned} T_{A, F} &\leq \mathring{N} \sum_{n=1}^{\infty} 2^{n+1} (1 + \alpha) \frac{r_{n+1}}{v_n} \\ &= \mathring{N} \left( (1 + \alpha) c_{\alpha, \Omega}^{\frac{1}{\gamma-1}} \sum_{n=1}^{\infty} (2\alpha^{-\frac{\gamma-d}{\gamma-1}})^{n+1} n^{\frac{1}{\gamma-1}} \right) r^{(\gamma-d)/(\gamma-1)} \left( \ln\left(\frac{1}{\varepsilon}\right) \right)^{\frac{1}{\gamma-1}}, \end{aligned}$$

where the first parentheses correspond to  $T_1$  and  $T_1$  is finite since  $\alpha > \alpha_{min}$ . We have thus proved Formula (3) of the theorem.

The moment (4) is simple integration:

$$\begin{aligned} \mathbb{E} \left[ \exp \left( \delta T_{A,F}^{\gamma-1} \right) \right] &\leq C + \frac{1}{\varepsilon_{max}} \int_0^{\varepsilon_{max}} \exp \left( \delta \left( T_{r,\dot{N}} \left( \ln \frac{1}{\varepsilon} \right)^{\frac{1}{\gamma-1}} \right)^{\gamma-1} \right) d\varepsilon \\ &\leq C + \frac{1}{\varepsilon_{max}} \int_0^{\varepsilon_{max}} \varepsilon^{-\delta T_{r,\dot{N}}^{\frac{1}{\gamma-1}}} d\varepsilon \\ &< \infty, \end{aligned}$$

where  $C$  corresponds to the integral between  $\varepsilon_{max}$  and 1, bounded by the value of  $T_{A,F}$  for  $\varepsilon = \varepsilon_{max}$ . □

**Remark 3.1.** • *If no lines are forbidden, that is if  $F = \emptyset$ , then  $\alpha$  can be taken as big as we wish, so that the lines used all hit as small a widening of  $A$  as we want.*

- *We may slightly generalize the theorem: instead of using a forbidden area  $F$ , we could use different conditions for which lines to accept. The important property is that we must have enough relevant lines hitting pairs of balls in a  $r_{n+1}$ -net.*
- *There are a few optimisations that could be used to gain slightly in the constants. For example, we have written the proof with one subnet inside each ball. If we had used a single  $r_n$ -net of a correctly widened  $A$ , we would have only about  $\dot{N} 2^d \alpha^{d(n+1)}$  pairs of ball, allowing a bigger  $\varepsilon_{max}$ . The result stays essentially the same, however, as proven by the following proposition.*

**Proposition 3.2.** *For any two points  $x$  and  $y$ , their  $\Pi$ -distance  $T_{xy}$  does not have a moment with higher exponent on the time  $T_{xy}$  than in Formula (4). That is, for any  $\delta > 0$ , for any  $\eta > \gamma - 1$ , we have:*

$$\mathbb{E} \left[ \exp \left( \delta T_{xy}^\eta \right) \right] = \infty.$$

PROOF. Say that  $|x - y| = r$ . A path from  $x$  to  $y$  has to go from  $x$  to the border of  $B(x, r)$ . Hence it must use lines hitting  $B(x, r)$  for a length at least  $r$ . So that  $T_{xy}$  is controlled by the fastest line hitting  $B(x, r)$ . Now,

$$\begin{aligned} \mathbb{P} [v_{max}(B(x, r)) \leq v_\varepsilon] &= \exp \left( -c(r)v_\varepsilon^{1-\gamma} \right) \\ &= \varepsilon \qquad \qquad \qquad \text{with} \\ v_\varepsilon &= \left( \frac{1}{c(r)} \ln \frac{1}{\varepsilon} \right)^{\frac{1}{1-\gamma}}, \end{aligned}$$

where  $c(r)$  depends only on  $r$ .

So that on an event of probability at least  $\varepsilon$ , we have the bound  $T_{xy} \geq r/v_\varepsilon = c_2(r)(\ln(1/\varepsilon))^{1/(\gamma-1)}$ . Hence, for some positive constant  $c_3$  depending on  $\delta, \eta$  and  $r$ :

$$\begin{aligned} \mathbb{E} [\exp(\delta T_{xy}^\eta)] &\geq \int_0^1 \exp\left(-c_3 \left(\ln \frac{1}{\varepsilon}\right)^{\eta/(\gamma-1)}\right) d\varepsilon \\ &= \infty, \end{aligned}$$

since  $\eta/(\gamma-1) > 1$ . □

**4. Almost sure uniqueness of  $\Pi$ -geodesics.** In this section, we prove that the network  $\mathcal{N}$  satisfies the property 1 of a SIRSN, that is that between two specified points  $x$  and  $y$  in  $\mathbb{R}^d$ , there is almost surely only a unique route. Since routes are the  $\Pi$ -geodesics, this is equivalent to almost sure unicity of the geodesic  $g_{xy}$ .

The case in dimension 2 has already been established in Kendall's (2014) paper. The following proof, on the other hand covers all dimensions more than 2, as stated in Theorem 4.8.

The strategy is the following:

- We introduce a concept of *many directions*, with the following property. If a  $\Pi$ -path has many directions near a point  $x$ , any finite  $\Pi$ -path supported by the same lines contains  $x$ .
- We show that all geodesics have many directions near all the ends of the segments they are made of, except their own extremal points.
- We show that for specified  $x$  and  $y$  in  $\mathbb{R}^d$ , almost surely all geodesics from  $x$  to  $y$  are supported exactly on the same lines.
- So that they all contain the same segment ends, and this will prove they are the same.

The author thinks the proof is very technical for something that looks clear enough, but could not find an easier way. Hopefully, the concept of many directions can be useful elsewhere.

**4.1. Many directions.** Having many directions near a point  $x$  intuitively means that the lines used near the point have so many different unit vectors that the only way to touch all those lines (a *tour*) with a finite path is by touching most of them near  $x$ .

**Definition 4.1.** For a set of lines  $\mathcal{L} = \{l_j\}_{j \in J}$  and a subset of the space  $X \subset \mathbb{R}^d$ , a  $\mathcal{L}$ -tour in  $X$  is a sequence  $(x_i)_{i \in \mathbb{N}}$  such that  $x_i \in X$  for all  $i$ , and



for all  $j \in J$ , there is an  $i$  such that  $x_i \in l_j$ . Define the length of the tour as  $\sum_i |x_{i+1} - x_i|$ .

Recalling the notation  $\mathcal{L}_\xi$  for the support of  $\xi$ :

**Definition 4.2.** *A finite  $\Pi$ -path  $\xi$  has many directions near a point  $x$  if, for any  $\varepsilon > 0$ , all  $\mathcal{L}_\xi$ -tours in  $\mathbb{R}^d \setminus B(x, \varepsilon)$  have infinite length.*

As a remark, this concept is only interesting in dimension at least three: in dimension two, a circle is usually a tour, and has finite length.

**Proposition 4.3.** *Let two finite  $\Pi$ -paths  $\xi$  and  $\eta$ . If  $\xi$  has many directions near a point  $x \in \mathbb{R}^d$ , and its support is included in that of  $\eta$ , that is  $\mathcal{L}_\xi \subset \mathcal{L}_\eta$ , then  $x \in \eta$ .*

PROOF. A finite  $\Pi$ -path is closed in  $\mathbb{R}^d$ . It is then sufficient to show that for any positive  $\varepsilon$ ,  $B(x, \varepsilon) \cap \eta \neq \emptyset$ .

Now, if that was not the case for a given  $\varepsilon$ , the path  $\eta$  would have to go through a  $\mathcal{L}_\xi$ -tour on  $\mathbb{R}^d \setminus B(x, \varepsilon)$ . Since  $\xi$  has many directions near  $x$ , the length of  $\eta$  would be infinite. By Theorem 2.4, it would have infinite time length. Contradiction.  $\square$

**Lemma 4.4.** *Let  $d \geq 3$ .*

*Let  $l$  be a random line (not seen as a line of  $\Pi$ ). Almost surely, for all  $x \in l$ , for any  $y \notin l$ , all geodesics  $g_{xy}$  have many directions near  $x$ .*

PROOF. In the course of the proof, we will need the following technical lemmas, whose proof we delay to Section 4.4.

The first lemma yields a control on the proportion of balls in a nested set that are hit by lines faster than a threshold appropriately scaling with their size.

**Lemma 4.5.** *Let  $d \geq 2$ . Let  $\alpha > 1$  be a scale factor. We write  $p = \alpha^{1-d} < 1$ . Let  $B_i = B(x_i, r_i)$  for  $1 \leq i \leq n$  be a set of nested balls with*

$r_i = r_0/\alpha^i$  for some  $r_0$ . For some  $v_0$ , define:

$$\begin{aligned} v_i &= v_0 p^{\frac{i}{\gamma-1}} \\ \mathcal{V}_0 &= \{(l, v) \in \Pi : v \geq v_0\} \\ \mathcal{V}_i &= \{(l, v) \in \Pi : v_{i-1} \geq v \geq v_i\} \\ H_i &= \#\{k > 0 : \mathcal{V}_i \cap [B_{i+k}] \neq \emptyset\} \\ I_n &= \#\{i \leq n : \exists k > 0 : [B_i] \cap \mathcal{V}_{i-k} \neq \emptyset\} \leq \sum_{i=0}^{n-1} H_i. \end{aligned}$$

Notice that  $I_n$  is exactly the number of balls  $B_i$  that are hit by lines faster than  $v_{i-1}$ . Then,  $\lambda \triangleq \mathbb{E}[\#\{\mathcal{V}_i \cap [B_i]\}] = (1-p)\omega_{d-1}r_0^{d-1}/(2v_0^{\gamma-1})$  independently on  $i \geq 1$ . Then, for any  $\delta > 0$ :

$$(8) \quad \mathbb{P}[I_n > \delta n] \leq \left( (1 + \lambda\delta) \left( \frac{1 + \delta}{\delta} \right)^\delta p^\delta \right)^n.$$

We have more precise but more cumbersome bounds in Equations (10) and (11).

The second lemma gives a guarantee that except on exceedingly rare events, no significant fraction of uniformly randomly oriented lines are clumped together in a small number of directions.

**Lemma 4.6.** *Let  $\alpha > \beta > \delta > 0$ . Consider at most  $\alpha n$  lines with uniformly random direction in the projective space  $P\mathbb{R}^{d-1}$ . Recall that we may think of the projective space as the sphere  $\mathbb{S}^{d-1}$  with opposite points identified. It then inherits a metric from the natural metric on the sphere. Let  $\eta > \frac{1}{d-1}$ . Consider a  $n^{-\eta}$ -net of  $P\mathbb{R}^{d-1}$ . Then, there is no subset of  $\beta n$  lines that are all contained in at most  $\delta n$  balls of the net, except on an event of subexponential probability, at most  $(Cn^{(\beta-\delta)(\eta(d-1)-1)})^{-n}$  with  $C$  depending on  $\alpha, \beta$  and  $\delta$ .*

Back to the proof of Lemma 4.4.

It is sufficient to show that the claim holds for all  $x \in l \cap B(0, R)$ , for all  $R > 0$ . It is also sufficient to work with a fixed  $\varepsilon$ , and prove that for any  $y \notin l^\varepsilon$ , for any geodesic  $g_{xy}$ , all  $\mathcal{L}_{g_{xy}}$ -tours on  $\mathbb{R}^d \setminus B(x, \varepsilon)$  have infinite length.

Let  $r_n = r_0/\alpha^n$  for  $r_0 = \varepsilon$  and an  $\alpha > 1$  big enough to be determined later. We consider internal  $\frac{r_n}{4}$ -nets  $N_n$  of  $B(0, R) \cap l$ .

In particular, there are at most  $\frac{4R}{r_n} + 1$  points in  $N_n$ .

Now, for any point  $x_n$  in  $N_n$ , we may build a set of nested balls  $\{B_i\}_{i \leq n}$  with the following properties:

- The ball  $B_n$  is centred on  $x_n$ .
- All balls  $B_i = B(x_i, r_i)$  are centred on a point  $x_i$  in the net  $N_i$ .
- The centres are deep inside the previous ball, that is  $x_i \in B(x_{i-1}, \frac{r_{i-1}}{4})$ .  
This stems from the fact that  $N_{i-1}$  is an  $\frac{r_{i-1}}{4}$ -net.

We also use the notations of Lemma 4.5 for  $v_i$  and  $\mathcal{V}_i$ . We have to choose a good speed  $v_0$ .

We then show that the nested balls have the following properties, with a  $T$  to be determined later, except on an event of probability  $o(r_n^{-1})$ :

1. There are at least  $\frac{2n}{5}$  balls  $B_i$  that no line faster than  $v_{i-1}$  hits.
2. There are at least  $\frac{5n}{6}$  indices  $i$  such that the time diameter of  $B(x_i, \frac{7r_i}{8}) \setminus B(x_i, \frac{r_i}{2})$  on  $\Pi_{[B_i] \setminus [B_{i+1}]}$  is less than  $T\alpha^{i\frac{\gamma-d}{\gamma-1}}$ .
3. There are at least  $\frac{5n}{6}$  balls  $B_i$  that are hit by at most  $\tau$  lines of speed between  $v_i$  and  $v_{i-1}$ , for some fixed  $\tau$ . Moreover, all those lines have independent, uniformly random directions.

The  $T$  will yield the  $v_0$  we need to continue the proof, that is:

$$(9) \quad v_0 = \frac{3r_0}{8T}.$$

Let us first consider Property 3.

Since the sets of lines  $\mathcal{V}_i$  are disjoint, the events that more than  $\tau$  lines in  $\mathcal{V}_i$  hit  $B_i$  are mutually independent (in  $i$ ), as are their directions. Moreover, since the  $B_i$  are balls, the direction of the lines of the isotropic Poisson line process that hit it are uniformly random.

So that the number  $X_n$  of such events is a binomial random variable  $B(n, q)$  where  $q$  is the probability of a single event. The Chernoff bound for a binomial is, for  $\delta > q$ :

$$\mathbb{P}[X_n > \delta n] \leq \left( \left( \frac{q}{\delta} \right)^\delta \left( \frac{1-q}{1-\delta} \right)^{1-\delta} \right)^n.$$

Here we take  $\delta = \frac{1}{6}$ . For  $\alpha$  big enough, if we take  $q \leq \alpha^{-7}$ , the above bound is negligible with respect to  $\alpha^{-n}$ .

Now the probability  $q$  of a single event is the probability that a Poisson random variable with parameter not depending on  $i$  is bigger than  $\tau$ . We then just choose  $\tau$  so that  $q \leq \alpha^{-7}$ .

Now for Property 2.

Since the sets of lines  $[B_i] \setminus [B_{i+1}]$  are disjoint, each of the events that the time diameter of  $B(x_i, \frac{7r_i}{8}) \setminus B(x_i, \frac{r_i}{2})$  on  $\Pi_{[B_i] \setminus [B_{i+1}]}$  is less than  $T\alpha^{i\gamma-d\gamma-1}$  are independent. So that, like for Property 3, we merely have to choose  $T$  such that the probability  $q$  of a single event is less than  $\alpha^{-7}$ .

Now, by scaling, the sets  $A_i = B(x_i, \frac{7r_i}{8}) \setminus B(x_i, \frac{r_i}{2})$  may all be covered by the same number  $\dot{N}$  of balls with radius  $\frac{r_i}{8}$ , and centres in  $A_i$ . Moreover  $A_i$  is connected. Now the  $\frac{r_i}{8}$ -widening of  $A_i$  is included in  $B_i$ , and since  $x_{i+1} \in B(x_i, \frac{r_i}{4})$ , we know that  $x_{i+1}$  is at distance at least  $\frac{1}{4}$  of this widening. So that the maximum solid angle of  $B_{i+1}$  viewed from any point of the widening is a decreasing function of  $\alpha$ , hitting zero when  $\alpha$  goes to infinity. So that for  $\alpha$  big enough, we may apply Lemma 3.1 with  $\Omega < \Omega_{max}$ . Replacing  $\varepsilon$  in bound (3) with  $1/(\varepsilon_{max}\alpha^7)$  to ensure  $q \leq \alpha^{-7}$ , we get that Property 2 is ensured if:

$$\begin{aligned} T &= \dot{N}T_1 \left(\frac{r_0}{8}\right)^{\frac{\gamma-d}{\gamma-1}} (\ln(\alpha^7/\varepsilon_{max}))^{\frac{1}{\gamma-1}} \\ &= O\left(\ln^{\frac{1}{\gamma-1}} \alpha\right). \end{aligned}$$

So that  $v_0 = O\left(\ln^{-\frac{1}{\gamma-1}} \alpha\right)$ .

As for Property 1, we apply Lemma 4.5.

Replacing with  $\delta = \frac{3}{5}$  and recalling  $d-1 \geq 2$  in Equation (8), and recalling  $p = \alpha^{1-d} \leq \alpha^{-2}$  and  $\lambda = O(v_0^{1-\gamma}) = O(\ln \alpha)$ , we get:

$$\begin{aligned} &\mathbb{P}\left[\text{There are at least } \frac{2n}{5} \text{ balls } B_i \text{ that no line faster than } v_{i-1} \text{ hits}\right] \\ &\leq (c \ln(\alpha) \alpha^{-\frac{6}{5}})^n \\ &= o(\alpha^{-n}). \end{aligned}$$

Let us now consider a geodesic from a point outside  $B_0$  to a point inside  $B_n$ . Since the balls are nested, they are all crossed by the geodesic. Even the sets  $A_i = B(x_i, \frac{7r_i}{8}) \setminus B(x_i, \frac{r_i}{2})$  introduced in the proof of Property 2 are all crossed, that is we go from the hull  $S(x_i, \frac{7r_i}{8})$  to the smaller hull  $S(x_i, \frac{r_i}{2})$ .

If the three properties are satisfied, then there are at least  $\frac{n}{15}$  indices  $i$  for which the conditions are simultaneously satisfied. For such an  $i$ , by Property 2 the geodesic will go from  $S(x_i, \frac{7r_i}{8})$  to  $S(x_i, \frac{r_i}{2})$  in time at most  $T\alpha^{i\frac{\gamma-d}{\gamma-1}}$ . Since the two hulls are  $\frac{3r_i}{8}$  apart, the geodesic must use a line with speed

more than  $\frac{3r_i}{8T\alpha^{i(\gamma-d)/(\gamma-1)}} = v_0\alpha^{i\frac{d-1}{\gamma-1}} = v_i$  within  $A_i$ . Now,  $A_i \subset B_i$  and, by Property 1, there is no line faster than  $v_{i-1}$  in  $B_i$ . So that the geodesic must use a line in  $\mathcal{V}_i \cap [B_i]$ , whose cardinal is less than  $\tau$  by Property 3. Since the  $v_i$  are disjoint, we have proved that the geodesic must use at least  $\frac{n}{15}$  distinct lines among a set of size at most  $\tau n$  lines with uniformly random direction.

We may then apply Lemma 4.6.

We take  $1 > \eta > 1/(d-1)$  and  $\delta = 1/20$ . With probability  $1 - O(n^{-cn})$ , all geodesics from a point outside  $B_0$  to a point inside  $B_n$  must use at least  $n/20$  lines (depending on the geodesic), each with a direction in a different ball of a  $n^{-\eta}$ -net of the projective space. Since there are at most a constant number  $c_1$  of points in a  $n^{-\eta}$ -net at distance less than  $3n^{-\eta}$  from a given point, we may choose  $n/(20c_1)$  of those lines, so that each pair of them makes an angle at least  $n^{-\eta}$ .

Let us consider a fixed  $\varepsilon$ . Since  $r_i = \varepsilon\alpha^{-i}$ , up to removing  $\ln n$  of our  $n/(20c_1)$  lines, we know all those lines hit a ball  $B(x, \varepsilon n^{-\eta}/3)$ . So that, by elementary geometry illustrated in Figure 3, for any two lines  $l_1$  and  $l_2$  in our collection, no point of  $l_1 \setminus B(x, \varepsilon)$  is closer to a point of  $l_2 \setminus B(x, \varepsilon)$  than  $\varepsilon n^{-\eta}/4$ . Hence any tour on the support of the geodesic in  $\mathbb{R}^d \setminus B(x, \varepsilon)$  must have length at least  $c_2 n^{1-\eta}$ , going to infinity with  $n$ .

All those properties were obtained except on a set of probability  $o(\alpha^{-n})$ . Since the net  $N_n$  has cardinal of order  $\alpha^n$ , this is true simultaneously for all sets of nested balls built on all the points  $x_n$  in  $N_n$ , except on an event of probability  $\varepsilon_n$  going to zero.

Letting  $n$  go to infinity yields the lemma. □

#### 4.2. Geodesics use the same lines.

**Theorem 4.7.** *For all  $x$  and  $y$  in  $\mathbb{R}^d$ , almost surely all geodesics  $g_{xy}$  are supported on exactly the same lines.*

PROOF. A similar result is used during the proof of Theorem 4.4 in Kendall's (2014) article. The idea of the proof is to "slow down" the lines not used by a specific geodesic. Then all geodesics that use one of those lines become slower and are no longer geodesics. The set of slowed speeds has infinite measure relative to the initial speeds.

Almost surely, there exists a finite  $R$  such that all geodesics from  $x$  to  $y$  are included in  $B(0, R)$ , by Theorem 2.6 of Kendall. Almost surely, for any  $R$ , there are countably many lines intersecting  $B(0, R)$ , all with different speeds  $v_1 > v_2 > \dots$  (the information here is that there is an order-preserving

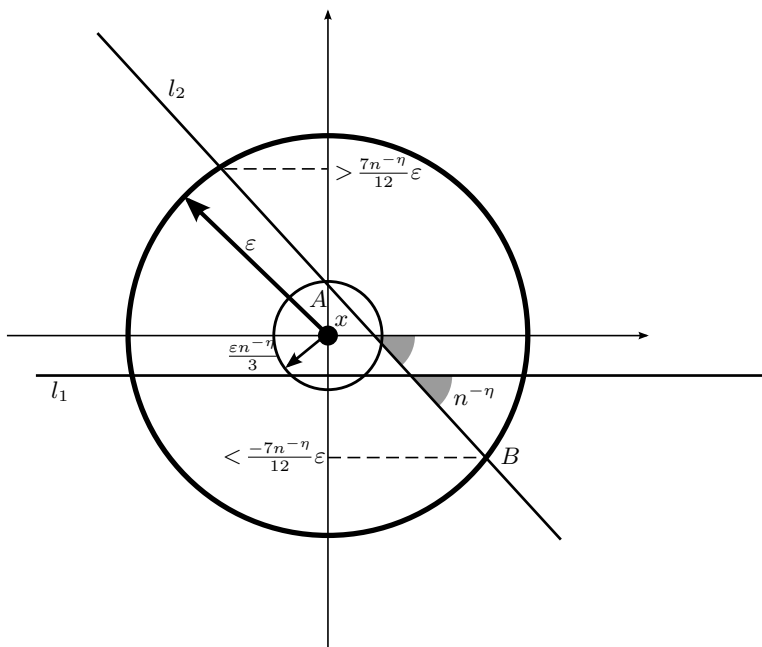


Fig 3: We project on the vector space spanned by  $l_1$  and  $l_2$ , with  $l_1$  parallel to the first coordinate and  $x$  as the origin. All points of  $l_1$  then have second coordinate at most  $n^{-\eta}\epsilon/3$  in absolute value. On the other hand, let  $A$  be a point in  $l_2 \cap B(x, n^{-\eta}\epsilon/3)$  and  $B$  a point in  $l_2 \setminus B(x, \epsilon)$ . Then  $\overrightarrow{AB}$  has absolute second coordinate at least  $(\epsilon(1 - n^{-\eta}\epsilon/3) \sin(n^{-\eta}) > \frac{11}{12}\epsilon n^{-\eta}$ , so that  $B$  has absolute second coordinate at least  $\frac{7}{12}\epsilon n^{-\eta}$ .

bijection from  $\mathbb{N}$  to the inverse line speeds). We then write  $l_i$  for the line with speed  $v_i$ .

Let  $i \in \mathbb{N}$  and let us fix all the other speeds  $v_j$  for  $j \neq i$ . Let us suppose there is a speed  $v_i^-$  between  $v_{i-1}$  and  $v_{i+1}$  such that if  $v_i = v_i^-$ , then there are at least two geodesics  $g_{xy}^i$  and  $g_{xy}^{-i}$  from  $x$  to  $y$ , one of which is supported by  $l_i$ , and the other not. That is  $l_i \sqsubset g_{xy}^i$  and  $l_i \not\sqsubset g_{xy}^{-i}$ .

Now, the time length of a path not supported by  $l_i$  does not depend on  $v_i$ . So that  $g_{xy}^{-i}$  is the fastest of these paths for all  $v_i$ , with constant time. On the other hand, by decomposition (2), the time length of  $g_{xy}^i$  is decreasing in  $v_i$ . Hence if  $v_i > v_i^-$ , no geodesic from  $x$  to  $y$  is supported by  $l_i$ . Conversely, if  $v_i < v_i^-$ , no path supported by  $l_i$  is a geodesic from  $x$  to  $y$ . Indeed, such a path would be as fast as  $g_{xy}^{-i}$  at speed  $v_i$ , hence faster at speed  $v_i^-$ .

So that, when the other speeds are fixed, there is at most one value  $v_i^-$  of  $v_i$  such that  $l_i$  is in the support of some geodesic from  $x$  to  $y$  and not in the support of another such geodesic.

Now, we may disintegrate the measures on the line speeds. For almost all fixed line speeds for all lines except  $l_i$ , the measure  $\mu_i$  for  $v_i$  has a density (namely, it is proportional to  $v_i^{-\gamma} \mathbf{1}_{v_{i-1} \leq v_i \leq v_{i+1}}$ ). Hence  $\mu_i(\{v_i^-\}) = 0$ . So that almost surely, either  $l_i \sqsubset g$  for all geodesics  $g$  from  $x$  to  $y$ , or  $l_i \not\sqsubset g$  for any such geodesic. This is true for all lines  $l_i$ , ending the proof.  $\square$

4.3. *Almost sure unicity.* We may now state this section's main result:

**Theorem 4.8.** *In dimension  $d \geq 2$ , for all  $x$  and  $y$  in  $\mathbb{R}^d$ , almost surely there is a unique geodesic  $g_{xy}$  from  $x$  to  $y$ .*

PROOF. Kendall (2014) proved the dimension 2 case in Theorem 4.4. We then assume from here on that  $d \geq 3$ .

Consider any line  $l \in \Pi$ . Intuitively, since  $\Pi$  is a Poisson process,  $\Pi \setminus \{l\}$  still has the same distribution as  $\Pi$ . So that almost surely, by Lemma 4.4, all geodesics on  $\Pi \setminus \{l\}$  with an endpoint  $z \in l$  have many directions near  $z$ . Since the lines are countable, this is true for all lines simultaneously.

The rigorous way of writing the former paragraph is through Slivnyak-Mecke formula (originally proved by Slivnyak (1962), see for example Møller and Waagepetersen's (2003) book, Theorem 3.2., for a modern treatment). A Poisson point process  $X$  with intensity  $\mu$  takes value in the set of locally finite point configurations  $N_{lf}$ . For any nonnegative measurable function  $h$  on  $\mathbb{R}^d \times N_{lf}$ , we have:

$$\mathbb{E} \left[ \sum_{y \in X} h(y, X \setminus \{x\}) \right] = \int_{\mathbb{R}^d} \mathbb{E} [h(y, X)] d\mu(x).$$

We apply the formula with the underlying point process for  $\Pi$ , so that  $y = (l, v)$  are marked lines. The function  $h$  is the indicator function:

$$h((l, v), \Pi) = [\exists x \in l, y \notin l, g_{xy} \text{ } \Pi\text{-geodesic : } \\ g_{xy} \text{ does not have many directions near } x],$$

with Iverson bracket notation.

Lemma 4.4 then ensures that the expectation in the integrand in the right-hand side is uniformly zero, so that, almost surely, for all  $l \in \Pi$ , all geodesics on  $\Pi \setminus \{l\}$  with an endpoint  $z \in l$  have many directions near  $z$ .

Let  $g_{xy}$  be a geodesic from  $x$  to  $y$ . Let  $\{z_i\}$  be the set of endpoints of all the segments of the geodesic  $g_{xy}$  except  $x$  and  $y$ . For any  $z_i$ , since it's a segment endpoint (on line  $l$ ), there is a  $u$  such that  $g_{uz_i}$  or  $g_{z_i u}$  is included in  $g_{xy}$ , and such that  $l$  is not in the support of this sub-geodesic. Hence this sub-geodesic is also a geodesic in  $\Pi \setminus \{l\}$ , and has many directions near  $z_i$ . *A fortiori*,  $g_{xy}$  has many directions near  $z_i$ . Moreover, by Lemma 4.7 almost surely all other geodesics from  $x$  to  $y$  have the same support. Hence all these geodesics include all the segment endpoints of all the geodesics from  $x$  to  $y$ .

Now two geodesics from  $x$  to  $y$  must pass through the same points in the same order: indeed if  $g_{xy}^2 = g_{xa}^2 g_{ab}^2 g_{by}^2$  and  $g_{xy}^2 = g_{xb}^2 g_{ba}^2 g_{ay}^2$ , then  $g_{xa}^1 g_{ay}^2$  or  $g_{xb}^2 g_{by}^1$  would be shorter than both.

So that all geodesics from  $x$  to  $y$  pass through their segment endpoints in the same order, so they are the same. □

A byproduct of the proof is the following remark:

**Corollary 4.9.** *Let  $d \geq 3$ . Almost surely, for any point  $x$  not on a line of  $\Pi$ , that is  $x \notin \mathcal{S}$ , all geodesics containing  $x$  have many directions near  $x$ .*

PROOF. We use the step in the former proof, that almost surely, for all  $l \in \Pi$ , all geodesics on  $\Pi \setminus \{l\}$  with an endpoint  $z \in l$  have many directions near  $z$ .

For any  $\varepsilon$ , since  $x \notin \mathcal{S}$ , the geodesic to  $x$  will leave a line of  $\Pi$  at a point  $z \in B(x, \varepsilon/2)$ . Now the restriction of the geodesic to  $g_{zx}$  has many directions near  $z$ . Since  $B(z, \varepsilon/2) \subset B(x, \varepsilon)$ ,  $g$  has also many directions near  $x$ . □

#### 4.4. Proof of technical lemmas.



PROOF OF LEMMA 4.5. We have:

$$\begin{aligned}\mathbb{E}[\#\{\mathcal{V}_i \cap [B]_{i+k}\}] &= \mathbb{E}[\#\{\mathcal{V}_i \cap [B]_i\}] p^k \\ &= \lambda p^k\end{aligned}\quad \text{for all } i \geq 1 \text{ and } k \geq 0,$$

$$\begin{aligned}\mathbb{E}[\#\{\mathcal{V}_0 \cap [B]_k\}] &= \lambda p^k \sum_{m=0}^{\infty} p^m \\ &= \frac{1}{1-p} \lambda p^k\end{aligned}\quad \text{for all } k \geq 0.$$

So that, using that  $[B_{i+1}] \subset [B_i]$  on line 3:

$$\begin{aligned}\mathbb{P}[H_i = 0] &\leq 1 && \text{for all } i \geq 0, \\ \mathbb{P}[H_i = k > 0] &= \mathbb{P}[\mathcal{V}_i \cap [B]_{i+k} \neq \emptyset \text{ and } \mathcal{V}_i \cap [B]_{i+k+1} = \emptyset] \\ &= \exp(-\lambda p^{k+1}) - \exp(-\lambda p^k) \\ &\leq 1 - \exp(-\lambda(p^k - p^{k+1})) \\ &\leq \lambda p^k (1-p) && \text{for } i \geq 1. \\ \mathbb{P}[H_0 = k > 0] &= \exp\left(-\frac{\lambda p^{k+1}}{1-p}\right) - \exp\left(-\frac{\lambda p^k}{1-p}\right) \\ &\leq \lambda p^k.\end{aligned}$$

Hence, for all  $0 < a < 1/p$ :

$$\begin{aligned}\mathbb{E}[a^{H_i}] &= 1 + \sum_{k=1}^{\infty} \lambda p^k (1-p) a^k \\ &= 1 + \lambda(1-p) \frac{ap}{1-ap}\end{aligned}\quad \text{for all } i \geq 1,$$

$$\begin{aligned}\mathbb{E}[a^{H_0}] &= 1 + \sum_{k=1}^{\infty} \lambda p^k a^k \\ &= 1 + \lambda \frac{ap}{1-ap},\end{aligned}$$

$$\begin{aligned}\mathbb{E}[a^{I_n}] &\leq \mathbb{E}\left[a^{\sum_{i=0}^{n-1} H_i}\right] \\ &\leq \left(1 + \lambda(1-p) \frac{ap}{1-ap}\right)^n \frac{1 + ap(\lambda - 1)}{1 + ap(\lambda(1-p) - 1)} \\ &\leq \left(1 + \lambda \frac{ap}{1-ap}\right)^n.\end{aligned}$$

We may now use Markov inequality:

$$(10) \quad \mathbb{P}[I_n > \delta n] \leq \frac{\mathbb{E}[a^{I_n}]}{a^{\delta n}},$$

and optimize upon  $a$ . Exact optimization requires solving a degree-two equation and yields a cumbersome solution, so we shall only use here the solution for the second approximation when  $\lambda = 1$ , that is  $a = \delta/((1 + \delta)p)$ , so that  $ap/(1 - ap) = \delta$ . We get:

$$(11) \quad \mathbb{P}[I_n > \delta n] \leq \frac{1 + \lambda\delta}{1 + \lambda(1 - p)\delta} \left( (1 + \lambda(1 - p)\delta) \left( \frac{1 + \delta}{\delta} \right)^\delta p^\delta \right)^n$$

$$(12) \quad \leq \left( (1 + \lambda\delta) \left( \frac{1 + \delta}{\delta} \right)^\delta p^\delta \right)^n.$$

□

PROOF OF LEMMA 4.6. Recall that  $\mu_{d-1}^{(o)}$  is the natural probability measure on  $P\mathbb{R}^{d-1}$ . Each ball in the  $n^{-\eta}$ -net has a  $\mu_{d-1}^{(o)}$ -measure that is between  $c_1 n^{-\eta(d-1)}$  and  $c_2 n^{-\eta(d-1)}$ . Since all the points of the net are in disjoint balls of radius  $n^{-\eta}/2$ , we also know that there are at most  $c_3 n^{\eta(d-1)}$  points in the net. Now, with  $c_4, c_5, c_6$  depending on  $\alpha, \beta$  and  $\delta$ :

$$\begin{aligned} & \mathbb{P}[\text{There are } \beta n \text{ lines all contained in } \delta n \text{ balls of the net}] \\ & \leq \# \{ \text{subsets of } \beta n \text{ lines} \} \# \{ \text{subsets of } \delta n \text{ balls} \} \\ & \quad \cdot \mathbb{P}[\beta n \text{ specific lines are all contained in } \delta n \text{ specific balls}] \\ & \leq \binom{\alpha n}{\beta n} \binom{c_3 n^{\eta(d-1)}}{\delta n} (\delta n c_2 n^{-\eta(d-1)})^{\beta n} \\ & \leq 2^{\alpha n} (c_4 n^{\eta(d-1)-1})^{\delta n} (c_5 n^{1-\eta(d-1)})^{\beta n} \\ & = (c_6 n^{(\beta-\delta)(\eta(d-1)-1)})^n. \end{aligned}$$

□

**5. Geodesic length has finite expectation.** This is property 3 of a SIRSN.

**Theorem 5.1.** *Let  $x, y \in \mathbb{R}^d$ . Then the Euclidean length of the  $\Pi$ -geodesic between  $x$  and  $y$  has finite expectation:*

$$\mathbb{E}[L_{xy}] < \infty$$

PROOF. Let us first suppose that the time length  $T_{xy}$  is known.

For any  $r$ , any  $\Pi$ -path of Euclidean length more than  $r$  has to have an intersection at least  $r$  with  $B(x, r)$ . Hence it must spend time at least  $r/v_{max}(B(x, r))$  in that sphere. This yields the constraint

$$(13) \quad v_{max}(B(x, r)) \geq r/T_{xy}.$$

By Theorem 3.1, there is a  $T$  such that with probability at least  $2^{-(n+1)}$ :

$$T_{xy} \leq T(n+1)^{\frac{1}{\gamma-1}}.$$

Let  $\kappa < \gamma - 1$ . We consider:

$$r_0 = \left( \frac{2^{\gamma-1}c}{2^{\gamma-1-\kappa} - 1} \right)^{\frac{1}{\gamma-d}} \left( T(n+1)^{\frac{1}{\gamma-1}} \right)^{\frac{\gamma-1}{\gamma-d}},$$

$$r_l = 2^l r_0,$$

with  $c = \mu_d([B(0, 1)])$ , and  $l$  ranging from 0 to  $m = \lfloor (n+1)/\kappa \rfloor$ .

We must satisfy constraint (13) for each  $r_l$ . Starting with  $r_0$ , this is the case if and only if there is an increasing sequence of  $(1+k)$  integers  $0 = l_0, l_1, \dots, l_k$  between 0 and  $m$  such that:

$$2^{l_{i+1}-l_i} r_{l_i} / T_{xy} \geq v_{max}(B(x, r_{l_i})) \geq 2^{l_{i+1}-l_i-1} r_{l_i} / T_{xy} \quad \text{if } i < k,$$

$$v_{max}(B(x, r_{l_k})) \geq 2^{m-l_k} r_{l_k} / T_{xy}.$$

Starting from  $r_0$ , each of these events is conditioned on the former, so that it is conditioned on information in  $B(x, r_{l_{i-1}})$  only, in particular that  $v_{max}(B(x, r_{l_{i-1}})) \leq r_{l_{i-1}}/T_{xy}$ . So that its conditional probability is no more than the unconditioned:

$$\begin{aligned} \mathbb{P} \left[ v_{max}(B(x, r_{l_i})) \geq 2^{l_{i+1}-l_i-1} r_{l_i} / T_{xy} \right] &= 1 - \exp \left( -c \frac{r_{l_i}^{d-1}}{(2^{l_{i+1}-l_i-1} r_{l_i} / T_{xy})^{\gamma-1}} \right) \\ &\leq c \frac{r_{l_i}^{d-1}}{(2^{l_{i+1}-l_i-1} r_{l_i} / T_{xy})^{\gamma-1}} \\ &\leq 2^{-(l_{i+1}-l_i)(\gamma-1)} (2^{\gamma-1-\kappa} - 1), \end{aligned}$$

where we used  $r_0 \leq r_{l_i}$  and our bound on  $T_{xy}$ , and implicitly  $l_{k+1} = m + 1$ .

Regardless of which  $k$  indices were the  $l_i$ , the product of the latter bound on all  $i$  yields  $2^{-(m+1)(\gamma-1)} (2^{\gamma-1-\kappa} - 1)^{k+1}$ .

So that:

$$\begin{aligned}
(14) \quad & \mathbb{P}[\forall 0 \leq l \leq m : v_{max}(B(x, r_l)) \geq r_l/T_{xy}] \\
& \leq \sum_{k=0}^m \binom{m}{k} 2^{-(m+1)(\gamma-1)} (2^{\gamma-1-\kappa} - 1)^{k+1} \\
& = 2^{-(m+1)(\gamma-1)} (2^{\gamma-1-\kappa} - 1 + 1)^m (2^{\gamma-1-\kappa} - 1) \\
& \leq 2^{-\kappa(m+1)} \\
& \leq 2^{-(n+1)}.
\end{aligned}$$

We have thus proved that with probability at least  $1 - 2^{-n}$ , the Euclidean length is bounded by:

$$L_{xy} \leq 2^{(n+1)/\kappa} (n+1)^{\frac{1}{\gamma-d}} C,$$

where  $C$  is a positive constant depending on  $T$  and  $\kappa$ , but not on  $n$ . So that  $L_{xy}$  has a  $\delta$ -moment for all  $\delta < \kappa$ . Since  $\kappa$  was anything less than  $\gamma - 1$ , we have a  $\delta$ -moment for all  $\delta < \gamma - 1$ . Since  $\gamma > d \geq 2$ , this ends the proof.  $\square$

**Remark 5.1.** • *The main part of the proof is really just saying that the Euclidean diameter of a  $\Pi$ -ball has finite mean. Together with Theorem 3.1, this implies that the random metric space generated by the  $\Pi$ -length is almost surely homeomorphic to  $\mathbb{R}^d$ .*

- *Notice that for  $\kappa$  close to  $\gamma - 1$ , the main term in the sum (14) is the one for  $k = 0$ . In other words, it is easier to have one single extremely fast line close to  $x$  than to have many successively faster lines, if we want to be abnormally fast at each distance of  $x$ .*
- *A first way to improve on the moments of  $L_{xy}$  starts with noticing that if we use a line to go far away very fast, we need to use another line to come back, since a geodesic never crosses itself.*
- *More precisely, the author conjectures that the structure yielding long geodesics with highest probability is the following: a line with speed at least  $2r/T_{xy}$  hits  $B(x, \varepsilon)$ , another hits  $B(y, \varepsilon)$  and they hit a common  $\varepsilon$ -ball at distance  $r$ . Since the two first events have probability in  $r^{-(\gamma-1)}$  and the other in  $r^{-(d-1)}$ , we would conclude that  $\mathbb{E} \left[ L_{xy}^{2\gamma+d-3} \right] = \infty$ , but  $\mathbb{E} \left[ L_{xy}^\delta \right]$  for all  $\delta < 2\gamma + d - 3$ .*

**6. Finite intensity of long-distance network.** We now turn to property 4 of a SIRS, that is their key property.

Intuitively, it means that there are “highways”. If we look at all the geodesics simultaneously, except a ball around their endpoints, they total to only a finite length in each compact set: all the geodesics follow the same segments. Contrast with the Euclidean case where the whole space is used.

In our context, we have to prove that the intensity  $p(1)$  of the following long-distance network  $\mathcal{F}$  is finite: let  $\{\Xi_n, n \in \mathbb{N}^*\}$  be a collection of Poisson processes on  $\mathbb{R}^d$  with intensity  $n$  times Lebesgue, all independent from  $\Pi$ , and coupled so that  $\Xi_n \subset \Xi_{n+1}$ . Write  $\Xi = \bigcup_{n \in \mathbb{N}^*} \Xi_n$ . Then:

$$\mathcal{F} = \bigcup_{x,y \in \Xi} (g_{xy} \setminus (B(x,1) \cup B(y,1))).$$

Notice that almost surely  $g_{xy}$  is unique for all  $x$  and  $y$  since they are countable.

By translational invariance, it is enough to prove that the intersection of  $\mathcal{F}$  with a given ball has finite mean Hausdorff measure of dimension 1. Indeed if  $\ell = m_1(\mathcal{F} \cap B(x,r))$ , then  $\mathbb{E}[\ell] = \frac{\omega_{d-1} r^{d-1}}{2} p(1)$ . Notice that scale equivariance yields similar results if the long-distance network  $\mathcal{F}$  is defined by removing balls of any fixed radius instead of radius 1.

The main argument will be pigeon-hole principle: a geodesic must use one of few fast lines to come near a point, and to leave it. And by unicity of geodesics, two geodesics with two common points are the same between those points.

**Theorem 6.1.** *Let  $\gamma > d \geq 2$ . With the notations above, let  $\ell = m_1(\mathcal{F} \cap B(0, \frac{1}{3}))$  be the length of the long-distance network in  $B(0, \frac{1}{3})$ . For  $\varepsilon < \varepsilon_{max}$ , with probability at least  $1 - \varepsilon$ , this length is less than  $C(\ln(C_1/\varepsilon))^2$ , for constants  $\varepsilon_{max}$ ,  $C$  and  $C_1$  depending only on  $\gamma$  and  $d$ . Consequently we have this moment, for any  $\delta < \sqrt{C}$ :*

$$(15) \quad \mathbb{E} \left[ \exp \left( \delta \sqrt{\ell} \right) \right] < \infty.$$

*In particular,  $\ell$  has finite mean.*

PROOF. Since  $\Xi$  is countable, by Theorem 4.8, almost surely all geodesics between its points are unique. In the proof, we only consider such geodesics and subgeodesics, so that we assume unicity.

Consider the balls  $B(0, \frac{1}{3})$  and  $B(0, \frac{2}{3})$ . We call their set difference  $B(0, \frac{2}{3}) \setminus B(0, \frac{1}{3})$  the *border*.

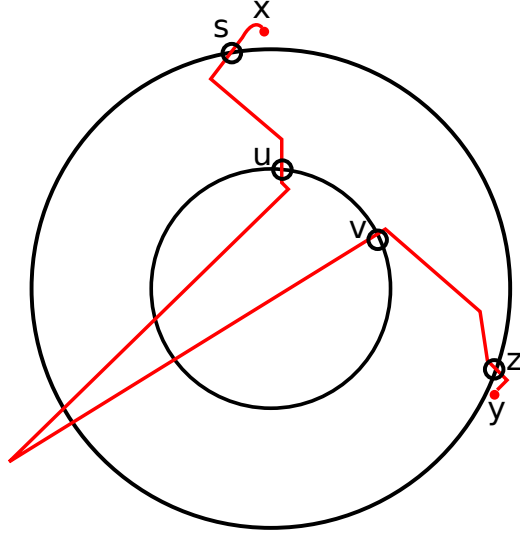


Fig 4: The geodesic from  $x$  to  $y$  hits  $B(0, \frac{2}{3})$  for the first time at  $s$  and the last time at  $z$ . It hits  $B(0, \frac{1}{3})$  for the first time at  $u$  and the last time at  $v$ .

Now if a point  $x \in B(0, \frac{2}{3})$ , then  $B(0, \frac{1}{3}) \subset B(x, 1)$ . Hence geodesics with an endpoint in  $B(0, \frac{2}{3})$  make no contribution to  $\ell$ . We have:

$$\ell \leq m_1 \left( \bigcup_{x,y \in \Xi \setminus B(0, \frac{2}{3})} (g_{xy} \cap B(0, \frac{1}{3})) \right).$$

Hence geodesics  $g_{xy}$  making contributions to  $\ell$  are structured in the following way, illustrated in Figure 4:

- They hit  $B(0, \frac{2}{3})$  for the first time at point  $s$  on the corresponding sphere.
- Then they hit  $B(0, \frac{1}{3})$  for the first time at point  $u$  on the corresponding sphere.
- Then they hit  $B(0, \frac{1}{3})$  for the last time at point  $v$  on the corresponding sphere.
- Then they hit  $B(0, \frac{2}{3})$  for the last time at point  $z$  on the corresponding sphere.

In particular, the contribution to  $\ell$  is included in the subgeodesic  $g_{uv}$ .

Now, uniformly on  $x$  and  $y$ , the time  $T_{sz}$  between  $s$  and  $z$  is bounded by the time diameter of  $B(0, \frac{2}{3})$ . So that, by Theorem 3.1, with probability

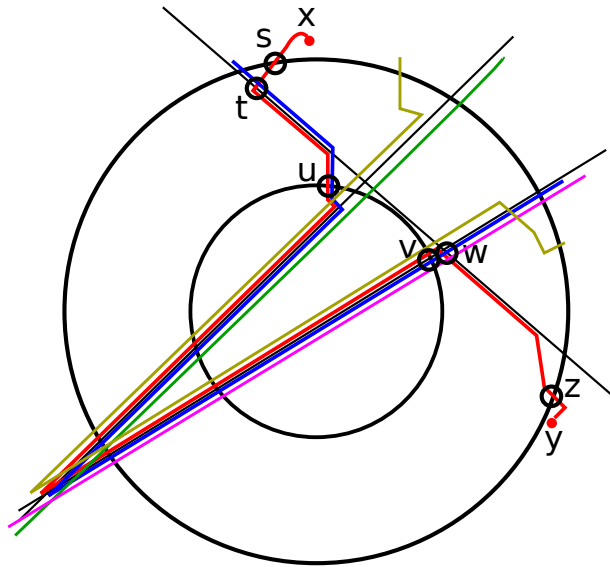


Fig 5: The red geodesic  $g_{xy}$  has a common point  $t$  with the blue geodesic on a fast (black) line when first crossing the border, and another  $w$  when last crossing the border. Any other geodesic contributing to  $\ell$  would meet one of the geodesics shown in the same way.

$1 - \varepsilon/2$ , we have:

$$(16) \quad \begin{aligned} T_{sz} &\leq T_\varepsilon \\ &\hat{=} T_{2/3,1} \left( \ln \frac{2}{\varepsilon \cdot \varepsilon_{max}} \right). \end{aligned}$$

We call *fast lines* the lines faster than  $v_\varepsilon = 1/6T_\varepsilon$ . Conversely, lines slower than  $v_\varepsilon$  are *slow lines*. We write  $\mathcal{V}_\varepsilon = \mathcal{S}_{v_\varepsilon} \cap B(0, \frac{2}{3})$  for the intersection of all those fast lines with the ball  $B(0, \frac{2}{3})$ .

The number of fast lines hitting  $B(0, \frac{2}{3})$  is a Poisson variable with parameter  $\lambda_\varepsilon = \frac{\omega_{d-1}(2/3)^{d-1}}{2v_\varepsilon^{\gamma-1}} = C_2 \ln(C_1/\varepsilon)$  for explicit constants  $C_1$  and  $C_2$ . In particular, for  $X$  with law  $\text{Poisson}(\lambda_\varepsilon)$ :

$$\begin{aligned} \mathbb{P}[X \geq x] &\leq e^{-\lambda_\varepsilon} \left( \frac{e\lambda_\varepsilon}{x} \right)^x, \\ \mathbb{P}[X \geq C_3\lambda_\varepsilon] &\leq \left( \frac{\varepsilon}{C_1} \right)^{C_2 \ln(C_3^{C_3} e^{1-C_3})} \\ &\leq \frac{\varepsilon}{2}, \end{aligned}$$

for  $C_3$  an explicit, big enough constant. So that with probability  $1 - \frac{\varepsilon}{2}$ , there are at most  $C_3\lambda_\varepsilon$  fast lines hitting  $B(0, \frac{2}{3})$ .

Since  $T_{sz} \leq T_\varepsilon$ , the intersection of  $g_{sz}$  with slow lines has length at most  $v_\varepsilon/T_\varepsilon = \frac{1}{6}$ .

So that, since  $s$  and  $u$  (*resp.*  $v$  and  $z$ ) are at least  $\frac{1}{3}$  apart, the geodesic must have fast segments for length at least  $\frac{1}{6}$  between  $s$  and  $u$  (*resp.*  $v$  and  $z$ ), that is:

$$\begin{aligned} m_1(\mathcal{V}_\varepsilon \cap g_{su}) &\geq \frac{1}{6}, \\ m_1(\mathcal{V}_\varepsilon \cap g_{vz}) &\geq \frac{1}{6}. \end{aligned}$$

Now

$$\begin{aligned} m_1(\mathcal{V}_\varepsilon) &\leq \# \{\text{fast lines}\} \cdot \text{Diam} \left( B(0, \frac{2}{3}) \right) \\ &\leq \frac{4C_3\lambda_\varepsilon}{3} \\ &= C_4 \ln \left( \frac{C_1}{\varepsilon} \right). \end{aligned}$$



So that

$$\frac{m_1 \otimes m_1(\mathcal{V}_\varepsilon \otimes \mathcal{V}_\varepsilon)}{m_1 \otimes m_1(g_{su} \cap \mathcal{V}_\varepsilon \otimes g_{vz} \cap \mathcal{V}_\varepsilon)} \leq C_5 \left( \ln \frac{C_1}{\varepsilon} \right)^2.$$

Hence, by pigeon-hole property illustrated in Figure 5, we may find a maximal family  $\{g^i\}$  of at most  $(C_5 \left( \ln \frac{C_1}{\varepsilon} \right)^2 + 1)$  geodesics such that:

- $g^i$  is a geodesic between  $x^i$  and  $y^i$  in  $\Xi \setminus B(0, \frac{2}{3})$ , passing through  $s^i$ ,  $u^i$ ,  $v^i$  and  $z^i$  defined like  $s, u, v$  and  $z$  above.
- Any geodesic  $g_{xy}$  contributing to  $\ell$  crosses one of the  $g^i$  when first and last crossing the border, that is there are points  $t$  and  $w$  such that  $t \in g_{s^i u^i} \cap g_{su}$  and  $w \in g_{v^i z^i} \cap g_{vz}$ .

By unicity of geodesics,  $g^i$  and  $g_{xy}$  coincide on  $g_{tw}$ . In particular, they coincide on  $g_{uv}$ . Hence the intersection  $\mathcal{F} \cap B(0, \frac{1}{3})$  is included in the finite number of geodesics  $g_{u^i v^i}$ . We may then conclude by separating contributions from fast and slow lines:

$$\begin{aligned} \ell &\leq \sum_i m_1(g_{u^i v^i}) \\ &\leq m_1(\mathcal{V}_\varepsilon) + \sum_i m_1(g_{u^i v^i} \cap (\Pi \setminus \mathcal{V}_\varepsilon)) \\ &\leq C_4 \ln \left( \frac{C_1}{\varepsilon} \right) + \left( C_5 \left( \ln \frac{C_1}{\varepsilon} \right)^2 + 1 \right) \cdot \frac{1}{6} \\ &\leq C \left( \ln \frac{C_1}{\varepsilon} \right)^2. \end{aligned}$$

□

**Theorem 6.2.** *The network  $\mathcal{N}$  made of the time geodesics is a SIRS $\mathcal{N}$ .*

PROOF. Property 1 of a SIRS $\mathcal{N}$  is a consequence of almost sure unicity of geodesics between two points, that is Theorem 4 in dimension at least 3, and Theorem 4.4. in Kendall's (2014) article in dimension 2.

Property 2 of a SIRS $\mathcal{N}$  is because the underlying Poisson line process is invariant by translation and rotation. As for change of scale, the underlying Poisson line process is invariant by a transformation where scale is multiplied by  $\alpha$  and speed by  $\alpha^{(d-1)/(\gamma-1)}$ . Hence all paths have their time length multiplied by the same  $\alpha^{(\gamma-d)/(\gamma-1)}$ , so that the geodesics are the same and  $\mathcal{N}$  is invariant.

Property 3 of a SIRS $\mathcal{N}$  is Theorem 5.1.

Property 4 of a SIRS $\mathcal{N}$  is Theorem 6.1. □

**7. Conclusion.** We have established that the improper Poisson line process with adequate speed limits yield a SIRSN.

Along the way, a few questions have been raised. Is there an easier, more natural way to prove unicity of geodesics ? What are the tightest moments of the Euclidean length of a geodesic ?

On a more general note, we may wonder which property of our network  $\mathcal{N}$  translate to general SIRSNs, or to SIRSNs made of geodesics of a metric space. For example, it should be easy to show that not all SIRSNs have the equivalent of Lemma 4.4 or Corollary 4.9: it is certainly not true for the binary hierarchy model by Aldous.

Finally, another somewhat tangential direction of research would be to study more closely the properties of the random metric space. For example, being a SIRSN entails coalescence of geodesics, a very hyperbolic-like property. We may also draw comparisons with a well-known random metric space such as the Brownian map (see for example Le Gall, 2014).

The Brownian map is a random metric space homeomorphic to the sphere  $\mathbb{S}^2$ . It has Hausdorff dimension 4. All its geodesics minus their endpoints is of Hausdorff dimension 1. The cut-locus of its geodesics starting from a given point has Hausdorff dimension 2, and the topology of an open continuous tree.

On the other hand, we have shown that our metric space is homeomorphic to  $\mathbb{R}^d$ . It should be easy, by scaling arguments, to show that its Hausdorff dimension is  $(d\gamma - d)/(\gamma - d)$ , which is bigger than  $d$ . In particular, with  $d = 2$  and  $\gamma = 3$ , we have the same dimensions as the Brownian map. If any geodesic can be appropriately approximated by geodesics between points of Poisson point processes, it should also be easy that the geodesics minus their endpoints is of dimension 1. However, the cut-locus might have a very different behaviour.

**Acknowledgements.** I would like to thank Wilfrid Kendall for introducing us to the problem during a talk, and helpful discussions since then.

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LABORATOIRE PAUL PAINLEVÉ (UMR 8524)  
UNIVERSITÉ DE LILLE 1, CNRS  
CITÉ SCIENTIFIQUE—BÂT. M2  
59655 VILLENEUVE D'ASCQ CEDEX  
FRANCE  
E-MAIL: jonas.kahn@math.univ-lille1.fr