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# Quasi-Hankel low-rank matrix completion: a convex relaxation

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**Abstract**—The completion of matrices with missing values under the rank constraint is a non-convex optimization problem. A popular convex relaxation is based on minimization of the nuclear norm (sum of singular values) of the matrix. For this relaxation, an important question is whether the two optimization problems lead to the same solution. This question was addressed in the literature mostly in the case of random positions of missing elements and random known elements. In this contribution, we analyze the case of structured matrices with fixed pattern of missing values, in particular, the case of Hankel and quasi-Hankel matrix completion, which appears as a subproblem in the computation of symmetric tensor canonical polyadic decomposition. We extend existing results on completion of rank-one real Hankel matrices to completion of rank- $r$  complex Hankel and quasi-Hankel matrices.

**Index Terms**—Hankel; quasi-Hankel; matrix completion; nuclear norm; tensor CP decomposition

## I. INTRODUCTION

The problem of completing matrices with missing entries can be traced back to the works of Prony in 1795, and has been addressed since in various fields including: compressed sensing [1], [2], [3], system identification and control [4], [5], [6], graph theory [7], collaborative filtering [8], information theory [9], chemometrics [10], seismics [11], [12], estimation problems and sensor networks [8], to cite a few. It also appears as a subproblem in the computation of symmetric tensor Canonical Polyadic (CP) decompositions [13].

### A. Structured matrix completion

We are interested in structured matrices of the form

$$\mathcal{S}(\mathbf{p}) = \mathbf{S}_0 + \sum_{k=1}^N p_k \mathbf{S}_k, \quad (1)$$

where  $\mathbf{S}_k$ ,  $k \in \{0, \dots, N\}$  are known linearly independent  $n \times n$  matrices, and  $\mathbf{p} = [p_1, \dots, p_N]^\top$ . Without loss of generality, we assume that the matrices  $\mathbf{S}_k \in \mathbb{C}^{n \times n}$  are symmetric (a nonsymmetric matrix completion problem can be always symmetrized). Thus,  $\mathcal{S}$  is an injective map  $\mathbb{C}^N \rightarrow \mathbb{C}^{n \times n}$ , called

affine matrix structure. The Structured Low-Rank Matrix Completion (SLRMC) for complex affine matrix structures is then stated formally as

$$\tilde{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{C}^N} \text{rank}\{\mathcal{S}(\mathbf{p})\}. \quad (2)$$

In the literature on matrix completion, the problem (2) is often formulated in dual form, *i.e.*, as rank minimization subject to linear constraints, and is called affine rank minimization [14]. In this paper we prefer the formulation (2), which seems to be more suitable for completion of structured (*e.g.*, Hankel-like) matrices.

There are special cases when the solution to SLRMC can be found in polynomial time: partially known block-triangular matrices [15], Hankel/Toeplitz matrices [16], [17], block-Hankel matrices [18] and some cases of quasi-Hankel matrices [19]. In the general case, the low-rank matrix completion is NP-hard (see, *e.g.*, [20]).

SLRMC can be also considered as an extreme case of Structured Low-rank matrix Approximation (SLRA) problem with missing data [21]. The latter problem is more relevant in practice, where we may have noise in addition to missing data.

### B. A convex relaxation

A popular approach in machine learning [20], [14] is to build a relaxation of LRMC, by replacing the rank with the nuclear norm (*i.e.* the sum of singular values):

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{C}^N} \|\mathcal{S}(\mathbf{p})\|_*. \quad (3)$$

Now (3) is a convex optimization problem, and a variety of convex optimization methods can be used to find its global minimum. A central question is when do the solutions of (3) and (2) coincide (*i.e.*, the conditions on low-rank matrix recovery with the nuclear norm).

A special case of SLRMC is when each matrix  $\mathbf{S}_k$  contains only one nonzero element. This is the most common unstructured matrix completion:  $\mathbf{S}_0$  is the known part of the matrix and the positions of nonzero elements in  $\mathbf{S}_k$  indicate the values to be completed. There is a vast literature on the subject on low-rank matrix recovery, where mainly the case of unstructured matrix completion is treated. Most

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of the results are proved in a random setting [2], [14], [3]: the positions of missing entries (non-zero elements of  $\mathcal{S}_k$ ,  $k \in \{1, \dots, N\}$ ) are assumed to be drawn randomly; often the known entries are also assumed to be random. There are also results on the general affine rank minimization (which is equivalent to SLRMC), but the linear constraints are also assumed to be drawn randomly according to a certain probability distribution.

Up to the authors' knowledge, there exists only one result [22], which treats the case of fixed structure, in a very simple case.

**Theorem 1** ([22, Thm. 1]). *Let  $\mathcal{S}$  be a Hankel matrix*

$$\mathcal{S}(\mathbf{p}) = \begin{bmatrix} 1 & \lambda & \cdots & \lambda^n \\ \lambda & \lambda^2 & \cdots & p_1 \\ \vdots & \cdots & \cdots & \vdots \\ \lambda^n & p_1 & \cdots & p_{n-1} \end{bmatrix},$$

where  $\lambda \in (-1; 1)$ . Then the solution of (3), with  $\mathbf{p} \in \mathbb{R}^{n-1}$ , is unique and coincides with the minimal rank (rank-1) completion, which is given by  $p_k = \lambda^{n+k}$ .

In this paper we extend the results of [22] in two directions: (i) to Hankel matrices with arbitrary rank, and (ii) to quasi-Hankel matrices, which are particularly interesting in the context of symmetric tensor CP decomposition [13]. As in [22], we consider cases when the solution to SLRMC is known, and establish conditions on  $\mathcal{S}$  when the solution of (2) and (3) coincide.

Our results are more general (for example, applicable both in the real and complex cases), and do not depend on the results of [22]. However, we believe that our exposition may also help to understand the complicated logic of the proof in [22].

In Section II we introduce the main notation, in particular quasi-Hankel matrices, which are generalizations of Hankel matrices. In Section III we formally state the considered quasi-Hankel matrix completion problem, and summarize some known results (this summary is needed for stating the main results of the paper in Section VI). In Section IV we recall optimality conditions for convex optimization problems, and specialize them to nuclear norm minimization for structured matrices.

Section V contains the main lemmas needed for proofs of the results of the paper. The lemmas imply that the optimality conditions are satisfied if the column space of completed matrices is not too far from a certain simply structured subspace. Finally, in Section VI, we prove the main results of the paper, and illustrate them with numerical experiments in Section VII.

## II. BACKGROUND AND MAIN NOTATION

### A. Sets of multi-indices

We denote by  $\mathbb{N}$  the set of nonnegative integers. First, for a multi-index  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{N}^m$ , the monomial  $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$  will be denoted as  $\mathbf{x}^{\boldsymbol{\alpha}}$ , and its total degree is  $|\boldsymbol{\alpha}| \stackrel{\text{def}}{=} \sum_{\ell} \alpha_{\ell}$ .

For ordering the multi-indices, we use in this paper an ordering denoted by  $\prec$ . It is defined recursively on the size of vectors  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  as:

$$\boldsymbol{\alpha} \prec \boldsymbol{\beta} \Leftrightarrow \begin{cases} |\boldsymbol{\alpha}| < |\boldsymbol{\beta}| \\ \text{or} \\ |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| \text{ and } (\alpha_2, \dots, \alpha_m) \prec (\beta_2, \dots, \beta_m) \end{cases}$$

Next, we shall denote by  $\blacktriangle^{(m,d)}$  the set of multi-indices  $\{\boldsymbol{\alpha} \in \mathbb{N}^m : |\boldsymbol{\alpha}| \leq d\}$ . By  $\triangle^{(m,d)}$  we denote the set  $\{\boldsymbol{\alpha} \in \mathbb{N}^m : |\boldsymbol{\alpha}| = d\}$ . It is easy to see that

$$\blacktriangle^{(m,d)} = \triangle^{(m,0)} \cup \triangle^{(m,1)} \cup \dots \cup \triangle^{(m,d)}.$$

**Example 1.** Take  $m = 2$  and  $d = 3$ . Then

$$\begin{aligned} \blacktriangle^{(2,3)} &= \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \\ &\quad (3,0), (2,1), (1,2), (0,3)\}, \\ \triangle^{(2,0)} &= \{(0,0)\}, \quad \triangle^{(2,1)} = \{(1,0), (0,1)\}, \\ \triangle^{(2,2)} &= \{(2,0), (1,1), (0,2)\}, \\ \triangle^{(2,3)} &= \{(3,0), (2,1), (1,2), (0,3)\}. \end{aligned}$$

The multi-indices are ordered according to  $\prec$  in the sets defined above. In fact, when the total degrees are the same, terms are sorted only by increasing degrees in the second variable.

For sets  $\mathcal{A}, \mathcal{B} \subset \mathbb{N}^m$ , we define their Minkowski sum as  $\mathcal{A} + \mathcal{B} \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} + \boldsymbol{\beta} : \boldsymbol{\alpha} \in \mathcal{A}, \boldsymbol{\beta} \in \mathcal{B}\}$ , with a shorthand notation  $2\mathcal{A} \stackrel{\text{def}}{=} \mathcal{A} + \mathcal{A}$ . It is easy to see that  $\triangle^{(m,d_1)} + \triangle^{(m,d_2)} = \triangle^{(m,d_1+d_2)}$  and  $\blacktriangle^{(m,d_1)} + \blacktriangle^{(m,d_2)} = \blacktriangle^{(m,d_1+d_2)}$ .

For  $m = 1$ , we have that  $\blacktriangle^{(1,d)} = \{0, \dots, d\}$  and  $\{0, \dots, d_1\} + \{0, \dots, d_2\} = \{0, \dots, d_1+d_2\}$ . For  $m = 2$ , an example is shown in Fig. 1 (the multi-indices are depicted as black dots).

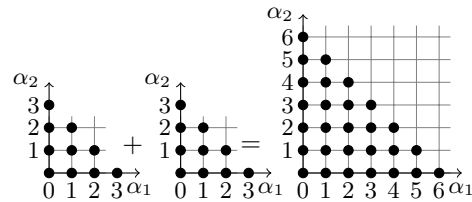


Fig. 1. Minkowski sum:  $\blacktriangle^{(2,3)} + \blacktriangle^{(2,3)} = \blacktriangle^{(2,6)}$ .

Finally, for  $\mathcal{A} \in \mathbb{N}^m$ , we define its extension<sup>1</sup> as

$$\mathcal{A}^+ \stackrel{\text{def}}{=} \mathcal{A} \cup (\mathcal{A} + \{\mathbf{e}_1\}) \cup \dots \cup (\mathcal{A} + \{\mathbf{e}_m\}),$$

where  $\mathbf{e}_k$ ,  $1 \leq k \leq m$ , denotes the vector of  $\mathbb{N}^m$  having a 1 in the  $k$ -th entry and zeros elsewhere. The exterior boundary of  $\mathcal{A}$  is defined as

$$\delta\mathcal{A} = \mathcal{A}^+ \setminus \mathcal{A}.$$

For example, for  $\mathcal{A} = \blacktriangle^{(m,d)}$  we have that

$$\mathcal{A}^+ = \blacktriangle^{(m,d+1)}, \quad \delta\mathcal{A} = \triangle^{(m,d+1)}.$$

### B. Quasi-Hankel and quasi-Vandermonde matrices

Let  $\mathcal{A}$  be a multi-index set ordered according to  $\prec$ :

$$\mathcal{A} = \{\alpha_1, \dots, \alpha_M\}, \quad \alpha_k \in \mathbb{N}^m \quad (4)$$

and  $\mathbf{h} = \{h_\alpha\}_{\alpha \in \mathcal{D}} \subset \mathbb{C}$  be an array of numbers, where  $2\mathcal{A} \subset \mathcal{D} \subset \mathbb{N}^m$ . Then the symmetric quasi-Hankel [23] matrix is defined as

$$\mathcal{H}_{\mathcal{A}}(\mathbf{h}) \stackrel{\text{def}}{=} [h_{\alpha_i + \alpha_j}]_{i,j=1}^{M,M}$$

**Example 2.** For  $\mathcal{A} = \blacktriangle^{(1,d)} = \{0, \dots, d\}$ ,  $M = d+1$ , the quasi-Hankel matrix is the ordinary Hankel matrix

$$\mathcal{H}_{\mathcal{A}}(\mathbf{h}) = [h_{k+l}]_{k,l=0}^{d,d} = \begin{bmatrix} h_0 & h_1 & \dots & h_d \\ h_1 & h_2 & \dots & h_{d+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_d & h_{d+1} & \dots & h_{2d} \end{bmatrix}. \quad (5)$$

**Example 3.** In case  $m > 1$ , for example  $\mathcal{A} = \blacktriangle^{(2,3)}$  (as in Fig. 1) the matrix  $\mathcal{H}_{\mathcal{A}}(\mathbf{h})$  has the form

$$\begin{bmatrix} h_{0,0} & h_{1,0} & h_{0,1} & h_{2,0} & h_{1,1} & h_{0,2} & h_{3,0} & h_{2,1} & h_{1,2} & h_{0,3} \\ h_{1,0} & h_{2,0} & h_{1,1} & h_{3,0} & h_{2,1} & h_{1,2} & h_{4,0} & h_{3,1} & h_{2,2} & h_{1,3} \\ h_{0,1} & h_{1,1} & h_{0,2} & h_{2,1} & h_{1,2} & h_{0,3} & h_{3,1} & h_{2,2} & h_{1,3} & h_{0,4} \\ h_{2,0} & h_{3,0} & h_{2,1} & h_{4,0} & h_{3,1} & h_{2,2} & h_{5,0} & h_{4,1} & h_{3,2} & h_{2,3} \\ h_{1,1} & h_{2,1} & h_{1,2} & h_{3,1} & h_{2,2} & h_{1,3} & h_{4,0} & h_{3,2} & h_{2,3} & h_{1,4} \\ h_{0,2} & h_{1,2} & h_{0,3} & h_{2,2} & h_{1,3} & h_{0,4} & h_{3,2} & h_{2,3} & h_{1,4} & h_{0,5} \\ h_{3,0} & h_{4,0} & h_{3,1} & h_{5,0} & h_{4,1} & h_{3,2} & h_{6,0} & h_{5,1} & h_{4,2} & h_{3,3} \\ h_{2,1} & h_{3,1} & h_{2,2} & h_{4,1} & h_{3,2} & h_{2,3} & h_{5,1} & h_{4,2} & h_{3,3} & h_{2,4} \\ h_{1,2} & h_{2,2} & h_{1,3} & h_{3,2} & h_{2,3} & h_{1,4} & h_{4,2} & h_{3,3} & h_{2,4} & h_{1,5} \\ h_{0,3} & h_{1,3} & h_{0,4} & h_{2,3} & h_{1,4} & h_{0,5} & h_{3,3} & h_{2,4} & h_{1,5} & h_{0,6} \end{bmatrix}, \quad (6)$$

where we omitted parentheses in subscripts for conciseness. Note that in (6), each block corresponds to a subset  $\triangle^{(2,k)}$ ,  $0 \leq k \leq 6$ .

Let  $\mathcal{A} \subset \mathbb{N}^m$  be the ordered multi-index set defined in (4), and  $\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{C}^m$  be a set of points. Then the quasi-Vandermonde matrix is defined as

$$\mathcal{V}_{\mathcal{A}}(\mathbf{z}_1, \dots, \mathbf{z}_r) \stackrel{\text{def}}{=} [(\mathbf{z}_j)^{\alpha_i}]_{i,j=1}^{N,r}.$$

In particular, for  $m = 1$ ,  $\mathcal{A} = \blacktriangle^{(1,d)} = \{0, \dots, d\}$ , it is just an ordinary rectangular Vandermonde matrix.

<sup>1</sup>In the papers [13], [19] and others this notation was used for the set of monomials.

**Example 4.** For  $\mathcal{A} = \blacktriangle^{(2,3)}$ ,  $r = 3$  and  $\mathbf{z}_k = \begin{bmatrix} \lambda_k \\ \mu_k \end{bmatrix}$ ,  $k \in \{1, 2, 3\}$ , we have that  $\mathcal{V}_{\mathcal{A}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) =$

$$\begin{bmatrix} 1 & \lambda_1 & \mu_1 & \lambda_1^2 & \lambda_1 \mu_1 & \mu_1^2 & \lambda_1^3 & \lambda_1^2 \mu_1 & \lambda_1 \mu_1^2 & \mu_1^3 \\ 1 & \lambda_2 & \mu_2 & \lambda_2^2 & \lambda_2 \mu_2 & \mu_2^2 & \lambda_2^3 & \lambda_2^2 \mu_2 & \lambda_2 \mu_2^2 & \mu_2^3 \\ 1 & \lambda_3 & \mu_3 & \lambda_3^2 & \lambda_3 \mu_3 & \mu_3^2 & \lambda_3^3 & \lambda_3^2 \mu_3 & \lambda_3 \mu_3^2 & \mu_3^3 \end{bmatrix}^T. \quad (7)$$

Finally, we will use the following definition.

**Definition 2.** The points  $\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{C}^m$  are called  $\mathcal{A}$ -independent if

$$\text{rank } \mathcal{V}_{\mathcal{A}}(\mathbf{z}_1, \dots, \mathbf{z}_r) = r.$$

The notion of  $\mathcal{A}$ -independence is equivalent to the fact the monomials  $\{\mathbf{x}^\alpha\}_{\alpha \in \mathcal{A}}$  taken on the grid of points  $\{\mathbf{z}_1, \dots, \mathbf{z}_r\}$  form a set of  $\#\mathcal{A}$  vectors spanning a linear space of dimension  $r$ . Hence, these monomials can interpolate any function on this grid.

If  $m = 1$ , the points are always  $\mathcal{A}$ -independent for any  $\mathcal{A}$  with  $r \leq \#\mathcal{A}$ . Although for  $m > 1$  this is no longer the case, the following remark holds true.

**Remark 3.**  $\mathcal{A}$ -independence is a generic property, i.e., points  $\{\mathbf{z}_j\}_{j=1}^r$  randomly drawn according to an absolutely continuous distribution are  $\mathcal{A}$ -independent if  $r \leq \#\mathcal{A}$ . This follows from the fact that the set of  $\mathcal{A}$ -independent points is open in the Zariski topology.

### III. (QUASI-)HANKEL MATRIX COMPLETION

In this paper, we consider the following problem.

**Problem 1.** Given  $\mathcal{A} = \blacktriangle^{(m,d)}$ , and  $\{h_\alpha\}_{\alpha \in \mathcal{A}}$ ,

$$\underset{h_\alpha, \alpha \in 2\mathcal{A} \setminus \mathcal{A}}{\text{minimize}} \text{rank } \mathcal{H}_{\mathcal{A}}(\mathbf{h}), \quad (8)$$

where  $\mathbf{h} = \{h_\alpha\}_{\alpha \in 2\mathcal{A}}$ .

If  $m = 1$ , we have  $\mathcal{A} = \{0, \dots, d\}$  (the matrix  $\mathcal{H}_{\mathcal{A}}(\mathbf{h})$  shown in (5)); in this case, only the values  $h_0, \dots, h_d$  are known (shown in gray in (5)) and  $h_{d+1}, \dots, h_{2d}$  are to be completed. The matrix completion for Hankel matrices is entirely solved [17], and its solution is related to the Sylvester's algorithm for CP decomposition of  $2 \times \dots \times 2$  tensors [24]. We recall the Hankel case in Section III-A.

In the general case ( $m > 1$ ), the upper block-triangular part of the quasi-Hankel matrix is known (e.g., in (6) it is shown in gray). As it was shown in [13], symmetric CP decomposition can be reduced to solving Problem 1. The general quasi-Hankel matrix completion problem (8) can be solved in some cases when the rank is sufficiently small (this is exactly the case when the CP tensor decomposition can be solved without considering incomplete quasi-Hankel matrices, e.g., using the Catalecticant algorithm [24], [25]). We recall these cases in Section III-B.

### A. Solution of Hankel low-rank completion

Probably the first complete solution of Problem 1 for  $m = 1$  is contained in [16, Ch. II], where the problem of matrix completion is called ‘‘singular extension’’ of Hankel matrices. In this section, we will use the theory from [17], and we will provide a summary of results in a simplified form.

For a finite sequence  $\mathbf{h}_d = [h_0 \ \dots \ h_d]^\top \in \mathbb{C}^{d+1}$ , we denote by  $\text{hrank}(\mathbf{h}_d)$  the smallest number  $r$  such that there exists a non-zero vector  $\mathbf{q} = [q_0 \ q_1 \ \dots \ q_r]^\top \neq 0$  for which

$$q_0 h_k + \dots + q_r h_{k+r} = 0, \quad \forall k \in \{0, \dots, d-r\}. \quad (9)$$

The value  $\text{hrank}(\mathbf{h}_d)$  is equal to the maximal rank of the submatrices of the known triangular part of the Hankel matrix (5) (or, equivalently to the maximal rank of a Hankel matrix that can be constructed only from the elements of the vector  $\mathbf{h}_d$ ). The number  $\text{hrank}(\mathbf{h}_d)$  is called *the first characteristic degree* of  $\mathbf{h}_d$  [17, Def. 5.3, page 81]. We will also call the vector  $\mathbf{q}$  as *the characteristic vector*<sup>2</sup> of  $\mathbf{h}_d$ . Next, we recall the following basic properties of  $\mathbf{h}_d$  and  $\mathbf{q}$ .

- For any  $\mathbf{h}_d \in \mathbb{C}^{d+1}$ ,  $\text{hrank}(\mathbf{h}_d) \leq \frac{d+2}{2}$  (see [17, Prop. 5.4]).
- If, in addition,  $\text{hrank}(\mathbf{h}_d) < \frac{d+2}{2}$ , then the characteristic vector  $\mathbf{q}$  is unique up to scaling by a constant (see [17, page 84]).

In what follows, we assume that  $q_r \neq 0$  (this is a generic case). The treatment of the case  $q_r = 0$  can be found in [17, page 84]). Then an explicit representation of  $\mathbf{h}_d$  is determined by the *characteristic polynomial*

$$q(z) = \sum_{j=0}^r q_j z^j = q_r \sum_{k=1}^s (z - \lambda_k)^{\nu_k}, \quad (10)$$

where  $\lambda_k \in \mathbb{C}$  are distinct,  $\nu_k$  are the multiplicities of the roots, and  $r = \nu_1 + \dots + \nu_s$ .

**Proposition 4** (A special case of [17, Thm 8.1]). *If  $q_r \neq 0$  and all the roots of  $q(z)$  are simple ( $\nu_k = 1$ ,  $s = r$ ), then the sequence  $\mathbf{h}_d$  admits a representation*

$$h_k = \sum_{j=1}^r c_j \lambda_j^k. \quad (11)$$

**Example 5.** For  $\mathbf{h}_d = [1 \ \lambda \ \dots \ \lambda^d]^\top$  (see Theorem 1), we have  $\text{hrank}(\mathbf{h}_d) = 1$  and the characteristic vector can be chosen as  $\mathbf{q} = [-\lambda \ 1]^\top$ .

The representation (11) is called a *canonical representation* of  $\mathbf{h}_d$  (or the unique canonical representation in case  $\text{hrank}(\mathbf{h}_d) < \frac{d+2}{2}$ ). In the general case (10),

<sup>2</sup>In [17],  $\mathbf{q}$  does not have a specific name and is denoted by  $p$ .

there are multiple roots and the canonical representation has a more complicated form, as stated by the proposition below.

**Proposition 5** (A special case of [17, Thm 8.1]). *Let  $q_r \neq 0$ ,  $\lambda_1 = 0$  and  $\lambda_k \neq 0$ , and the multiplicities  $\nu_k$  of the roots  $\lambda_k$  in (10) are such that  $\nu_1 \geq 0$  and  $\nu_k > 1$  for  $k \geq 2$ . Then the sequence  $\mathbf{h}_d$  has the form*

$$h_k = \sum_{j=2}^s c_j(k) \lambda_j^k + \sum_{l=0}^{\nu_1-1} c_{1,l} \delta(k, l), \quad (12)$$

where  $c_j(k)$  are polynomials of degree  $(\nu_k - 1)$  and  $\delta(k, l)$  is the Kronecker symbol.

**Remark 6.** *In the statement of Proposition 5, if  $\nu_1 = 0$  then the second term in (12) is absent.*

**Example 6.** *In the extreme case  $q(z) = q_r z^r$  (i.e.,  $r = \nu_1$ ), the canonical representation (12) becomes*

$$\mathbf{h}_d = [c_{1,0} \ \dots \ c_{1,r-1} \ 0 \ \dots \ 0]^\top, \quad (13)$$

Apart from the explicit form of the sequence  $\mathbf{h}_d$ , the characteristic polynomial gives a solution to the rank minimization problem (8) (for  $m = 1$ ).

**Proposition 7** (A corollary of [17, Thm. 5.14]). *Let  $\mathbf{q} \in \mathbb{C}^{r+1}$  be a characteristic vector of  $\mathbf{h}_d$  with  $q_r \neq 0$ ,  $r = \text{hrank}(\mathbf{h}_d)$ . Then it holds that (i) The rank of the minimal rank completion of  $\mathcal{H}_A(\mathbf{h})$  in (5) is  $r$ . (ii) A minimum rank completion  $h_{d+1}, \dots, h_{2d}$  is given by the recursive continuation, for  $k > d - r$ :*

$$h_{k+r} = -\frac{1}{q_r} (q_0 h_k + \dots + q_{r-1} h_{k+r-1}) \quad (14)$$

*In addition, the values of  $h_{k+r}$  for  $k > d - r$  can be obtained by using the corresponding formula of canonical representation, namely (11) or (12). (iii) If  $r < \frac{d+2}{2}$ , then the minimum rank completion is unique.*

### B. Solution of quasi-Hankel low-rank completion

Unlike Hankel matrices, for  $m > 1$  the quasi-Hankel completion problem does not have a closed form solution. The simplest case when the solution can be easily found is given by a flat extension theorem [19, Thm. 1.4].

**Proposition 8** (Corollary of [19, Thm. 1.4]). *Let  $\mathcal{A} = \blacktriangle^{(m,d)}$ , and the values  $\{h_\alpha\}_{\alpha \in \mathcal{A}}$  be given. Moreover, let  $d' \stackrel{\text{def}}{=} \lfloor \frac{d}{2} \rfloor$ ,  $\mathcal{B} \stackrel{\text{def}}{=} \blacktriangle^{(m,d'-1)}$ , and it holds that*

$$\text{rank}\{\mathcal{H}_{\mathcal{B}}(\mathbf{h})\} = \text{rank}\{\mathcal{H}_{\mathcal{B}+}(\mathbf{h})\} = r.$$

*Then the unique solution of Problem 1 has rank  $r$ .*

Proposition 8 induces bounds on rank that are too restrictive. For example, for  $\mathcal{A} = \blacktriangle^{(2,3)}$ ,  $d' = 1$  and only rank-one cases can be treated. In what follows,

we consider arrays of special form, for which Proposition 8 can be extended.

Let  $\mathcal{C} \subset \mathbb{N}^m$  be arbitrary,  $\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{C}^m$ ,  $c_1, \dots, c_r \in \mathbb{C}$  be some coefficients, and the array of coefficients  $\mathbf{h} = \{h_\alpha\}_{\alpha \in 2\mathcal{C}}$  be defined as

$$h_\alpha = \sum_{k=1}^r c_k \mathbf{z}_k^\alpha. \quad (15)$$

It is easy to see that the following lemma holds true

**Lemma 9.** *For  $\mathbf{h}$  defined as in (15), the quasi-Hankel matrix admits the factorization  $\mathcal{H}_{\mathcal{C}}(\mathbf{h}) =$*

$$\mathcal{V}_{\mathcal{C}}(\mathbf{z}_1, \dots, \mathbf{z}_r) \text{diag}\{c_1, \dots, c_r\} (\mathcal{V}_{\mathcal{C}}(\mathbf{z}_1, \dots, \mathbf{z}_r))^\top \quad (16)$$

and  $\text{rank}\{\mathcal{H}_{\mathcal{C}}(\mathbf{h})\} \leq r$ . If, in addition,  $c_k \neq 0$  for all  $k$ , and  $\mathbf{z}_1, \dots, \mathbf{z}_r$  are  $\mathcal{C}$ -independent, then

$$\text{rank}\{\mathcal{H}_{\mathcal{C}}(\mathbf{h})\} = r.$$

Next, we describe an easy improvement of Proposition 8 for arrays of form (15).

**Proposition 10.** *Let  $\mathcal{A} = \mathbf{\Delta}^{(m,d)}$ ,  $d' \stackrel{\text{def}}{=} \lfloor \frac{d}{2} \rfloor$ ,  $\mathcal{B} \stackrel{\text{def}}{=} \mathbf{\Delta}^{(m,d')}$  (it is easy to see that  $2\mathcal{B} \subset \mathcal{A}$ ). Assume that the coefficients  $\{h_\alpha\}_{\alpha \in \mathcal{A}}$  have the form (15), where the points  $\mathbf{z}_1, \dots, \mathbf{z}_r$  are  $\mathcal{B}$ -independent and  $c_1, \dots, c_r$  are nonzero. Then, the following hold true (i) The rank of the minimal completion in Problem 1 is  $r$ . A minimal rank completion is obtained by defining  $\{h_\alpha\}_{\alpha \in 2\mathcal{A}}$  using the same formula (15) (called canonical completion). (ii) If  $d$  is odd, the canonical completion is unique. (iii) If  $d$  is even, and  $\mathbf{z}_1, \dots, \mathbf{z}_r$  are  $\mathbf{\Delta}^{(m,d'-1)}$ -independent, then the completion is unique (the canonical completion of Eq. (15) is the only possible).*

*Proof.* See the extended version of the paper [26].  $\square$

For example, for the completion of structure shown in (6) ( $m = 2$  and  $d = 3$ ), Proposition 10 covers cases up to rank 3. Indeed, let  $\mathbf{z}_k = \begin{bmatrix} \lambda_k \\ \mu_k \end{bmatrix}$ ,  $k \in \{1, 2, 3\}$ , we have that if the Vandermonde matrix has full rank (7), then the matrix completion is unique and is given by the canonical completion.

**Remark 11.** *If the points  $\mathbf{z}_1, \dots, \mathbf{z}_r$  are generic, due to Remark 3, we can replace the  $\mathcal{B}$ -independence and  $\mathbf{\Delta}^{(m,d'-1)}$ -independence by the bounds  $r \leq \binom{d'+m}{m}$  and  $r \leq \binom{d'+m-1}{m}$  respectively.*

Finally, we note that for generic cases, the rank bound in Remark 11 can be further improved by using results of [25, Theorem 2.4] (since uniqueness of matrix completion is related to uniqueness of tensor decomposition). For our purposes, however, Proposition 10 will be sufficient.

## IV. OPTIMALITY CONDITIONS OF NUCLEAR NORM MINIMIZATION

### A. Optimality conditions in convex optimization

We recall some definitions from the field of convex optimization [27, Chapter D]. For a convex (possibly non-differentiable) function  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ , the subdifferential [27, Def. 1.2.1] of  $f$  is defined as the set  $\partial f(\mathbf{x}) \subset \mathbb{R}^N$

$$\partial f(\mathbf{x}) \stackrel{\text{def}}{=} \{\mathbf{d} : f(\mathbf{y}) - f(\mathbf{x}) \geq \langle \mathbf{d}, \mathbf{y} - \mathbf{x} \rangle \quad \forall \mathbf{y} \in \mathbb{R}^N\}.$$

In particular, if  $f$  is differentiable at a point  $\mathbf{x}$ , then the subgradient has only one element: the usual gradient, i.e.,  $\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}$ .

For the unconstrained convex optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^N} f(\mathbf{x}), \quad (17)$$

we recall the first-order optimality condition, which is necessary and sufficient in this case.

**Lemma 12** (First-order optimality, [27, Thm. 2.2.1]). *A point  $\mathbf{x}$  is a minimum point of (17) if and only if*

$$\mathbf{0} \in \partial f(\mathbf{x}). \quad (18)$$

In particular, for differentiable functions, the condition of Lemma 12 reduces to  $\nabla f(\mathbf{x}) = \mathbf{0}$ . Next, we use a simple uniqueness condition for the minimizer.

**Lemma 13** (Sufficient condition of uniqueness). *A point  $\mathbf{x}$  is the unique minimizer of (17) if*

$$\mathbf{0} \in \text{int}(\partial f(\mathbf{x})), \quad (19)$$

where  $\text{int}(\cdot)$  denotes the interior of a set.

*Proof.* In this case for any  $\mathbf{y} \neq \mathbf{x}$  there exists  $\delta > 0$  such that  $\delta \cdot (\mathbf{y} - \mathbf{x}) \in \partial f(\mathbf{x})$ . By definition of  $\partial f(\mathbf{x})$ , we have that  $f(\mathbf{y}) - f(\mathbf{x}) \geq \delta \|\mathbf{y} - \mathbf{x}\|_2^2 > 0$ .  $\square$

### B. Subdifferential of the nuclear norm

First, we recall the form of the subdifferential of the nuclear norm of a matrix. Let  $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$  be a matrix of rank  $r$ , and let  $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$  be an SVD of  $\mathbf{X}$ , where  $\mathbf{U} \in \mathbb{R}^{n_1 \times r}$ ,  $\mathbf{V} \in \mathbb{R}^{n_2 \times r}$  and  $\mathbf{\Sigma} \in \mathbb{R}^{r \times r}$  is a diagonal matrix of nonzero singular values. Next, let  $\mathbf{U}_\perp \in \mathbb{R}^{n_1 \times (n_1 - r)}$ ,  $\mathbf{V}_\perp \in \mathbb{R}^{n_2 \times (n_2 - r)}$  be the bases of the left and right nullspaces of  $\mathbf{X}$ , respectively. Then, according to [28, p. 41], the subdifferential of the nuclear norm at  $\mathbf{X}$  is equal to  $\partial \|\mathbf{X}\|_* =$

$$\{\mathbf{B} + \mathbf{U}_\perp \mathbf{W} \mathbf{V}_\perp^\top : \mathbf{W} \in \mathbb{R}^{(n_1 - r) \times (n_2 - r)}, \|\mathbf{W}\|_2 \leq 1\},$$

where  $\mathbf{B}$  is defined as

$$\mathbf{B} \stackrel{\text{def}}{=} \mathbf{U} \mathbf{V}^\top, \quad (20)$$

and  $\|\cdot\|_2$  is the spectral norm (largest singular value).

**Remark 14.** When the SVD of  $\mathbf{X}$  is not unique, then the matrix  $\mathbf{B}$  from (20) is still defined uniquely. In fact, if  $\mathbf{X}$  has multiple singular values, its matrix of left (resp. right) singular vectors is of the form  $\mathbf{U}\Theta$  (resp.  $\mathbf{V}\Theta$ ), where  $\Theta$  is an orthonormal matrix which commutes with  $\Sigma$ .

Next we consider a real nuclear norm minimization problem

$$\hat{\mathbf{p}} = \arg \min_{\mathbf{p} \in \mathbb{R}^N} f(\mathbf{p}), \quad f(\mathbf{p}) \stackrel{\text{def}}{=} \|\mathcal{S}(\mathbf{p})\|_*, \quad (21)$$

where  $\mathcal{S}$  is defined in (1). By an analogue of chain rule [27, Thm. 4.2.1], we immediately have that

$$\partial f(\mathbf{p}) = \left\{ \left[ \langle \mathbf{S}_1, \mathbf{H} \rangle_F \cdots \langle \mathbf{S}_N, \mathbf{H} \rangle_F \right]^\top : \mathbf{H} \in \partial \|\mathbf{X}\|_*, \mathbf{X} = \mathcal{S}(\mathbf{p}) \right\}, \quad (22)$$

where  $\langle \cdot, \cdot \rangle_F$  denotes the Frobenius inner product. In the following section, we will give a more compact form or subdifferentials and optimality conditions.

### C. Real nuclear norm minimization

First, we assume that  $\mathbf{X}$  is symmetric and  $n_1 = n_2 = n$ , then we have that  $\mathbf{U}_\perp \mathbf{U}_\perp^\top = \mathbf{V}_\perp \mathbf{V}_\perp^\top = \mathbf{Q}$ , where  $\mathbf{Q}$  is the projector on the nullspace of  $\mathbf{X}$ . Then the subdifferential can be rewritten as

$$\partial \|\mathbf{X}\|_* = \{ \mathbf{B} + \mathbf{Q}\mathbf{M}\mathbf{Q} : \|\mathbf{M}\|_2 \leq 1 \}, \quad (23)$$

where  $\mathbf{B}$  is defined as in (20). We note that  $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ , where  $\mathbf{P}$  is the orthogonal projector on the column space of  $\mathbf{X}$ , which can be obtained as

$$\mathbf{P} \stackrel{\text{def}}{=} \mathbf{U}\mathbf{U}^\top = \mathbf{B}\mathbf{B}^\top. \quad (24)$$

Next, we define the matrix

$$\mathbf{S} \stackrel{\text{def}}{=} [\text{vec}(\mathbf{S}_1) \cdots \text{vec}(\mathbf{S}_N)], \quad (25)$$

and for a matrix  $\mathbf{P} \in \mathbb{R}^{n \times n}$  we define

$$\mathcal{A}(\mathbf{P}) \stackrel{\text{def}}{=} \mathbf{S}^\top ((\mathbf{I} - \mathbf{P}) \otimes (\mathbf{I} - \mathbf{P})) \in \mathbb{R}^{N \times n^2}, \quad (26)$$

where  $\otimes$  denotes the Kronecker product. Then, by (22), the subgradient of  $f$  can be rewritten as  $\partial f(\mathbf{p}) =$

$$\{ \mathbf{S}^\top \text{vec}(\mathbf{B}) + \mathcal{A}(\mathbf{P}) \text{vec}(\mathbf{M}) : \|\mathbf{M}\|_2 \leq 1 \}. \quad (27)$$

with  $\mathbf{M} \in \mathbb{R}^{n \times n}$ . Using (27), we can rewrite Lemma 12 and Lemma 13 as follows.

**Lemma 15** (First-order optimality conditions for real nuclear norm minimization). *A point  $\mathbf{p}^*$  is a minimizer of (21) if and only if there exists a matrix  $\mathbf{M} \in \mathbb{R}^{n \times n}$  with  $\|\mathbf{M}\|_2 \leq 1$  such that*

$$\mathcal{A}(\mathbf{P}) \text{vec}(\mathbf{M}) = -\mathbf{S}^\top \text{vec}(\mathbf{B}). \quad (28)$$

where  $\mathbf{B}$  is as in (20),  $\mathbf{P}$  is as in (24), and  $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{P}$ .

*Proof.* Follows from (27) and Lemma 12.  $\square$

**Lemma 16** (Sufficient condition of uniqueness). *A point  $\mathbf{p}$  is the unique minimizer of (3) if there exists  $\mathbf{M} \in \mathbb{R}^{n \times n}$  with  $\|\mathbf{M}\|_2 < 1$  such that (28) holds and the matrix  $\mathcal{A}(\mathbf{P})$  is full row rank (i.e., is of rank  $N$ ).*

*Proof.* Follows from (27) and Lemma 13.  $\square$

### D. Complex nuclear norm minimization

Now let  $\mathcal{S}(\mathbf{p})$  be the matrix structure given in (1), but now  $\mathbf{S}_0 \in \mathbb{C}^{n \times n}$ ,  $\mathbf{S}_k \in \mathbb{R}^{n \times n}$  for  $k \geq 1$ , and the parameter is complex  $\mathbf{p}_\mathbb{C} \in \mathbb{C}^N$ , i.e., (1) defines a complex-valued map  $\mathcal{S} : \mathbb{C}^N \rightarrow \mathbb{C}^{n \times n}$ .

In order to derive the optimality conditions for the complex-valued case, we construct an extended real-valued structure  $\mathcal{S}_{ext} : \mathbb{R}^{2N} \rightarrow \mathbb{R}^{2n \times 2n}$  as follows.

For  $\mathbf{p}_{ext} = \begin{bmatrix} \mathbf{p}_\mathcal{R} \\ \mathbf{p}_\mathcal{I} \end{bmatrix}$ , where  $\mathbf{p}_\mathcal{R}, \mathbf{p}_\mathcal{I} \in \mathbb{R}^N$ , we define an equivalent (non-symmetric) matrix structure:

$$\mathcal{S}_{ext}(\mathbf{p}_{ext}) \stackrel{\text{def}}{=} \begin{bmatrix} \mathcal{S}(\mathbf{p}_\mathcal{R}) & -\mathcal{S}(\mathbf{p}_\mathcal{I}) \\ \mathcal{S}(\mathbf{p}_\mathcal{I}) & \mathcal{S}(\mathbf{p}_\mathcal{R}) \end{bmatrix} \quad (29)$$

From [29], the singular values of  $\mathcal{S}_{ext}(\mathbf{p}_{ext})$  contain two copies of singular values of  $\mathcal{S}(\mathbf{p}_\mathbb{C})$ . Hence,

$$\text{rank}\{\mathcal{S}_{ext}(\mathbf{p}_{ext})\} = 2 \text{rank}\{\mathcal{S}(\mathbf{p}_\mathbb{C})\},$$

and the complex rank minimization problem (2) is equivalent to real rank minimization

$$\tilde{\mathbf{p}}_{ext} = \arg \min_{\mathbf{p}_{ext} \in \mathbb{R}^{2N}} \text{rank} \mathcal{S}_{ext}(\mathbf{p}_{ext}). \quad (30)$$

Moreover, the problem (3), is equivalent to the real nuclear norm minimization problem

$$\hat{\mathbf{p}}_{ext} = \arg \min_{\mathbf{p}_{ext} \in \mathbb{R}^{2N}} \|\mathcal{S}_{ext}(\mathbf{p}_{ext})\|_*. \quad (31)$$

The following proposition shows that we can rewrite the optimality conditions for the problem (31) in a convenient form, as in Lemmae 15 and 16.

**Proposition 17** (Complex optimality conditions). *Let  $\mathbf{p}_\mathbb{C} = \mathbf{p}_\mathcal{R} + i\mathbf{p}_\mathcal{I} \in \mathbb{C}^N$ ,  $\mathcal{S}(\mathbf{p}_\mathbb{C}) = \mathbf{U}_\mathbb{C}\Sigma\mathbf{V}_\mathbb{C}^\mathbf{H}$  be an SVD of the symmetric matrix  $\mathcal{S}(\mathbf{p}_\mathbb{C})$ , and define  $\mathbf{B}_\mathbb{C} := \mathbf{U}_\mathbb{C}\mathbf{V}_\mathbb{C}^\mathbf{H}$ ,  $\mathbf{P}_\mathbb{C} = \mathbf{U}_\mathbb{C}\mathbf{U}_\mathbb{C}^\mathbf{H}$ , and  $\mathbf{Q}_\mathbb{C} = \mathbf{I}_n - \mathbf{P}_\mathbb{C}$ . Then it holds that:*

(i) *The point  $\mathbf{p}_\mathbb{C}$  is a minimizer of (31) if and only if there exists  $\mathbf{M} \in \mathbb{C}^{n \times n}$  with  $\|\mathbf{M}\|_2 \leq 1$  that satisfies (28).*

(ii) *The point  $\mathbf{p}_{ext}$  is the unique minimizer of (31) if there exists  $\mathbf{M} \in \mathbb{C}^{n \times n}$  with  $\|\mathbf{M}\|_2 < 1$  that satisfies (28), and  $\mathcal{A}(\mathbf{P})$  is full row rank.*

*Proof.* See the extended version of the paper [26].  $\square$

**Remark 18.** *Lemmae 15, 16, and Proposition 17 are related to optimality conditions used in [22]. Indeed,*

the rank condition on  $\text{rank}\{\mathcal{A}(\mathbf{P})\} = N$  is equivalent to  $\mathbf{Q}\mathbf{H}\mathbf{Q}^\top \neq \mathbf{0}$  for all  $\mathbf{H}$  of the form

$$\forall \mathbf{H} = \sum_{k=1}^N \Delta p_k \mathbf{S}_k, \quad \Delta \mathbf{p} \in \mathbb{C}^N \setminus \{\mathbf{0}\},$$

which corresponds to [22, Proposition 2]. Next, the condition (28) can be rewritten as

$$\langle \mathbf{S}_k, \mathbf{B} + \mathbf{Q}\mathbf{M}\mathbf{Q}^\top \rangle_F = \mathbf{0}, \quad \forall k \in \{1, \dots, N\},$$

which corresponds to the condition in [22, Lemma 2]. The authors of [22], however, do not make explicit connection with general optimality conditions for convex optimization problems, presented in Section IV-A.

## V. MAIN LEMMAS: OPTIMALITY CONDITIONS AND SIMPLE PROJECTORS

### A. Quasi-Hankel completion: basis matrices

First, we put the rank minimization problem (8) in the (standard) form (2). In order to do this, we explicitly write down the matrices  $\mathbf{S}_k$  in (1). The constant part has the form

$$\mathbf{S}_0 \stackrel{\text{def}}{=} \mathcal{H}_{\mathcal{A}}(\mathbf{c}),$$

where  $\mathbf{c} = \{c_\alpha\}_{\alpha \in 2\mathcal{A}}$  is defined as

$$c_\alpha \stackrel{\text{def}}{=} \begin{cases} h_\alpha, & \alpha \in \mathcal{A}, \\ 0, & \alpha \in 2\mathcal{A} \setminus \mathcal{A}. \end{cases}$$

**Example 7.** In the case of  $\mathcal{A} = \blacktriangle^{(2,3)}$  (depicted in (6)), we have that  $\mathbf{S}_0 = \mathcal{H}_{\mathcal{A}}(\mathbf{c}) =$

$$\begin{bmatrix} h_{0,0} & h_{1,0} & h_{0,1} & h_{2,0} & h_{1,1} & h_{0,2} & h_{3,0} & h_{2,1} & h_{1,2} & h_{0,3} \\ h_{1,0} & h_{2,0} & h_{1,1} & h_{3,0} & h_{2,1} & h_{1,2} & 0 & 0 & 0 & 0 \\ h_{0,1} & h_{1,1} & h_{0,2} & h_{2,1} & h_{1,2} & h_{0,3} & 0 & 0 & 0 & 0 \\ \hline h_{2,0} & h_{3,0} & h_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_{1,1} & h_{2,1} & h_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_{0,2} & h_{1,2} & h_{0,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline h_{3,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_{0,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the case of Hankel matrices ( $\mathcal{A} = \blacktriangle^{(1,d)}$ , see (5)), we have that  $n = d + 1$ , and  $\mathbf{S}_0$  is given by

$$\mathbf{S}_0 = \begin{bmatrix} h_0 & h_1 & \cdots & h_{d-1} & h_d \\ h_1 & h_2 & \cdots & h_d & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{d-1} & h_d & \cdots & 0 & 0 \\ h_d & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (32)$$

The number of free parameters in (1) is  $N = \#(2\mathcal{A}) - \#(\mathcal{A})$  (i.e., the number of unknowns in (8)). The corresponding set of multi-indices can be represented as

$$2\mathcal{A} \setminus \mathcal{A} = \{\beta_1, \dots, \beta_N\}, \quad \beta_1 \prec \cdots \prec \beta_N. \quad (33)$$

Next, we define basis arrays  $\mathbf{e}^\beta \stackrel{\text{def}}{=} \{e^\beta\}_{\alpha \in \mathbb{N}^m}$  as

$$e^\beta_\alpha \stackrel{\text{def}}{=} \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

and  $\mathbf{S}_k$  as

$$\mathbf{S}_k := \mathcal{H}_{\mathcal{A}}(\mathbf{e}^{\beta_k}). \quad (34)$$

Finally, it is easy to see that

$$\mathcal{H}_{\mathcal{A}}(\mathbf{h}) = \mathcal{S}(\mathbf{p}),$$

if  $\mathbf{p}$  is defined as  $p_k = \mathbf{h}_{\beta_k}$ .

**Example 8.** For  $\mathcal{A} = \blacktriangle^{(2,3)}$ , shown in (6), we have

$$\mathbf{S}_1 = \begin{bmatrix} 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|1|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ \hline 0|0|0|1|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ \hline 0|1|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|1|0|0 \\ 0|0|0|0|0|0|0|1|0|0 \\ \hline 0|0|0|0|1|0|0|0|0|0 \\ 0|0|0|1|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ \hline 0|0|1|0|0|0|0|0|0|0 \\ 0|1|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \end{bmatrix},$$

$$\mathbf{S}_3 = \begin{bmatrix} 0|0|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|1|0 \\ 0|0|0|0|0|0|0|1|0|0 \\ \hline 0|0|0|0|0|1|0|0|0|0 \\ 0|0|0|0|1|0|0|0|0|0 \\ 0|0|0|1|0|0|0|0|0|0 \\ \hline 0|0|0|0|0|0|0|0|0|0 \\ 0|0|1|0|0|0|0|0|0|0 \\ 0|1|0|0|0|0|0|0|0|0 \\ 0|0|0|0|0|0|0|0|0|0 \end{bmatrix}, \dots$$

For Hankel structure, ( $\mathcal{A} = \blacktriangle^{(1,d)}$ , shown in (5)), we have that  $N = d$ , and the matrices  $\mathbf{S}_k$ ,  $k \in \{1, \dots, N\}$  are defined as

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}, \quad \dots, \quad \mathbf{S}_{N-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{S}_N = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \quad (35)$$

### B. Simple matrices

Now we show that optimality conditions hold true for simple matrices. We take  $\mathcal{A} = \blacktriangle^{(m,d)}$ , and consider the matrix completion problem (8), where  $\mathbf{S}_k$  are defined in Section V-A.

**Lemma 19.** Let  $d' \stackrel{\text{def}}{=} \lfloor \frac{d}{2} \rfloor$ ,  $\mathbf{P}_0 \in \mathbb{C}^{n \times n}$  be a matrix of the form

$$\mathbf{P}_0 \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{P}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}' \in \mathbb{C}^{s \times s}, \quad (36)$$

where  $s \leq \binom{m+d'}{m}$ . Then it holds that (i) For any  $k \in \{1, \dots, N\}$ ,  $\mathbf{P}_0^\top \mathbf{S}_k \mathbf{P}_0 = \mathbf{0}$ . (ii) If, in addition,  $s \leq n'' := \binom{d'+m}{m}$ , where  $d'' := \lfloor \frac{d-1}{2} \rfloor$ , then  $\text{rank}\{\mathcal{A}(\mathbf{P}_0)\} = N$ .



*Proof.* See the extended version of the paper [26].  $\square$  Finally,

**Example 9.** In the case of Hankel matrices ( $\mathcal{A} = \mathbf{A}^{(1,d)}$ ) the first bound is  $s \leq \lfloor \frac{d+2}{2} \rfloor$ , and the first statement of Lemma 19 is evident from (35). The second bound is  $s \leq \lfloor \frac{d+1}{2} \rfloor$ , and a right lower block of the matrix  $\mathbf{Q}_0 \mathbf{H} \mathbf{Q}_0^\top$  (where  $\mathbf{H}$  is defined in Remark 18) has the form

$$\begin{bmatrix} 0 & \cdots & 0 & \Delta p_1 & \cdots & \Delta p_r \\ \vdots & \ddots & \ddots & \ddots & \ddots & \Delta p_{r+1} \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \Delta p_1 & \ddots & \ddots & \ddots & \ddots & \Delta p_{N-2} \\ \vdots & \ddots & \ddots & \ddots & \Delta p_{N-2} & \Delta p_{N-1} \\ \Delta p_r & \Delta p_{r+1} & \cdots & \Delta p_{N-2} & \Delta p_{N-1} & \Delta p_N \end{bmatrix},$$

and hence all  $\Delta p_k$  are present in this block.

Lemma 19 immediately proves that the SLRMC is solved by nuclear norm minimization in very simple cases.

**Corollary 20.** In the completion problem (8) for the structure (5), let the vector  $\mathbf{h}_d$  be of the form (13), with  $r \leq \frac{d+2}{2}$ . Then we have the following.

(i) A solution of the rank minimization problem is given by  $h_k = 0$  for  $k > d$ , and coincides with the solution of (3). (ii) If, in addition,  $r < \frac{d+2}{2}$ , the solution of (3) is unique.

*Proof.* (i) Let  $\mathbf{P}_0$  be as in (36), with  $s = r$ , and  $\mathbf{P}'_0 = \mathbf{I}_s$ . Then, the matrix  $\mathbf{B}_\mathbb{C}$  in (20) has the form  $\mathbf{B}_\mathbb{C} = \mathbf{P}_0 \mathbf{A} \mathbf{P}_0^\top$ . By Lemma 19, we have that  $\langle \mathbf{S}_k, \mathbf{B}_\mathbb{C} \rangle_F = \langle \mathbf{P}_0^\top \mathbf{S}_k \mathbf{P}_0, \mathbf{A} \rangle_F = 0$ . The rest follows from Remark 18. (ii) Follows from Remark 18.  $\square$

### C. Perturbations of simple projectors

Now we show that the conditions of Proposition 17 hold true for perturbations of simple matrices. First, we need a simple inequality on distance between projectors.

**Lemma 21.** Let  $\mathbf{B} = \mathbf{B}^\top = \mathbf{U}\mathbf{V}^\mathbf{H}$ , where  $\mathbf{U}, \mathbf{V} \in \mathbb{C}^{n \times r}$  such that  $\mathbf{U}^\mathbf{H}\mathbf{U} = \mathbf{V}^\mathbf{H}\mathbf{V} = \mathbf{I}_r$ . Let  $\mathbf{P}_0$  be an orthogonal projector on a subspace of  $\mathbb{C}^n$ , such that  $\text{rank}(\mathbf{P}_0) = r$ . Then we have that

$$\|\mathbf{B} - \mathbf{P}_0 \mathbf{B} \mathbf{P}_0^\top\|_F^2 \leq 2\|(\mathbf{I} - \mathbf{P}_0)\mathbf{U}\|_F^2 = \|\mathbf{P} - \mathbf{P}_0\|_F^2,$$

where  $\mathbf{P} = \mathbf{B}\mathbf{B}^\mathbf{H} = \mathbf{U}\mathbf{U}^\mathbf{H}$ .

*Proof.* Since  $\mathbf{B} - \mathbf{P}_0 \mathbf{B} \mathbf{P}_0^\top = \mathbf{B} - \mathbf{P}_0 \mathbf{B} + \mathbf{P}_0 \mathbf{B} - \mathbf{P}_0 \mathbf{B} \mathbf{P}_0^\top$ ,

$$\begin{aligned} \|\mathbf{B} - \mathbf{P}_0 \mathbf{B} \mathbf{P}_0^\top\|_F^2 &\leq \|(\mathbf{I} - \mathbf{P}_0)\mathbf{B}\|_F^2 + \|\mathbf{P}_0 \mathbf{B}(\mathbf{I} - \mathbf{P}_0^\top)\|_F^2 \\ &\leq 2\|(\mathbf{I} - \mathbf{P}_0)\mathbf{B}\|_F^2 = 2\|(\mathbf{I} - \mathbf{P}_0)\mathbf{U}\|_F^2 \end{aligned}$$

$$\begin{aligned} \|\mathbf{P} - \mathbf{P}_0\|_F^2 &= \text{trace}\{\mathbf{P}^2 + \mathbf{P}_0^2 - \mathbf{P}\mathbf{P}_0 - \mathbf{P}_0\mathbf{P}\} \\ &= 2\text{trace}\{2(\mathbf{I} - \mathbf{P}_0)\mathbf{P}\} = 2\|(\mathbf{I} - \mathbf{P}_0)\mathbf{U}\|_F^2, \end{aligned} \quad (37)$$

which completes the proof.  $\square$

**Remark 22.** For a symmetric matrix  $\mathcal{S}(\mathbf{p})$ , the matrix  $\mathbf{B}_\mathbb{C}$  defined in Proposition 17 is also symmetric.

Next, we prove a perturbation lemma that uses a bound on the distance between projectors.

**Lemma 23.** Let  $\mathbf{S}$  and  $\mathcal{A}(\mathbf{P})$  be defined in (25) and (26). Let  $\mathbf{P}_0 \in \mathbb{C}^{n \times n}$  be a rank- $r$  projector matrix such that  $\text{rank}\{\mathcal{A}(\mathbf{P}_0)\} = N$ ,  $\mathbf{P}_0^\top \mathbf{S}_k \mathbf{P}_0 = 0$  for all  $k \in \{1, \dots, N\}$ , and let  $\mathbf{Q}_0 \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{P}_0$ .

Then there exists a constant  $\varepsilon$  such that for any  $\mathbf{B}$  as in Lemma 21 satisfying

$$\|\mathbf{P} - \mathbf{P}_0\|_F < \varepsilon,$$

where  $\mathbf{P} = \mathbf{B}\mathbf{B}^\mathbf{H}$ , it holds that (i) The matrix  $\mathcal{A}(\mathbf{P})$  is full row rank. (ii) There exists a matrix  $\mathbf{M} \in \mathbb{C}^{n \times n}$ ,  $\|\mathbf{M}\|_2 < 1$ , such that (28) is satisfied.

*Proof.* (i) Since  $\mathcal{A}(\mathbf{P})$  depends polynomially on the entries of  $\mathbf{P}$ , and  $\text{rank}\mathcal{A}(\mathbf{P}_0) = N$ , the rank is preserved in a neighborhood of  $\mathbf{P}_0$ . Thus  $\text{rank}\mathcal{A}(\mathbf{P}) = N$  in a neighbourhood of  $\mathbf{P}_0$ .

(ii) Let  $\varepsilon_1 > 0$  be a number such that for any  $\mathbf{P}$ ,  $\|\mathbf{P} - \mathbf{P}_0\|_F \leq \varepsilon_1$  we have  $\mathcal{A}(\mathbf{P}) = N$  and  $0 < \delta_0 < \sigma_{\min}(\mathcal{A}(\mathbf{P}))$  for some number  $\delta_0$ . (We can choose such  $\varepsilon_1$  by continuity of the smallest singular value.) Then, in the  $\varepsilon_1$ -neighborhood, a solution of (28) always exists. Next, we consider the minimum Frobenius norm solution  $\mathbf{M}_* = \mathbf{M}_*(\mathbf{P})$  of (28), which is given by

$$\begin{aligned} \text{vec}(\mathbf{M}_*(\mathbf{P})) &= -\mathcal{A}(\mathbf{P})^\dagger \mathbf{S}^\top \text{vec}(\mathbf{B}) \\ &= -\mathcal{A}(\mathbf{P})^\dagger \mathbf{S}^\top \text{vec}(\mathbf{B} - \mathbf{P}_0 \mathbf{B} \mathbf{P}_0^\top), \end{aligned}$$

where the matrix  $\mathcal{A}(\mathbf{P})^\dagger$  is the pseudoinverse of  $\mathcal{A}(\mathbf{P})$ , and the last equality holds since  $\mathbf{P}_0^\top \mathbf{S}_k \mathbf{P}_0 = 0$  for all  $k \in \{1, \dots, N\}$ . Then, by Lemma 21,

$$\begin{aligned} \|\mathbf{M}_*\|_2 &\leq \|\text{vec}(\mathbf{M}_*(\mathbf{P}))\|_2 \leq \\ \|\mathcal{A}(\mathbf{P})^\dagger\|_2 \|\mathbf{S}^\top\|_2 \|\mathbf{B} - \mathbf{P}_0 \mathbf{B} \mathbf{P}_0^\top\|_F &< \frac{\|\mathbf{S}\|_2}{\delta_0} \|\mathbf{P} - \mathbf{P}_0\|_F. \end{aligned} \quad (38)$$

Finally, define  $\varepsilon \stackrel{\text{def}}{=} \min(\varepsilon_1, \frac{\delta_0}{\|\mathbf{S}\|_2})$ . For such an  $\varepsilon$ , the right hand side of (38) is less than or equal to 1 if  $\|\mathbf{P} - \mathbf{P}_0\|_F < \varepsilon$ , which completes the proof.  $\square$

## VI. MAIN RESULTS

### A. Hankel case

First, we prove a result that generalizes Theorem 1.

**Theorem 24.** *For any  $d$ ,  $n = d + 1$ , and  $r < \frac{d+2}{2}$  there exists  $\rho = \rho(d, r) > 0$  such that for all  $\mathbf{h}_d$  with  $\text{hrank}(\mathbf{h}_d) = r$  and  $|\lambda_k| < \rho$  in (10), the solution of (3) is unique and coincides with (14).*

*Proof.* See the extended version of the paper [26].  $\square$

**Remark 25.** *Theorem 1 proves a special case of Theorem 24 and tells that  $\rho(d, 1) = 1$  (for  $\mathbf{h}_d \in \mathbb{R}^{d+1}$ ).*

### B. Quasi-Hankel case

We first prove a lemma on the limits of projectors on column spaces of quasi-Hankel matrices of the form (16) (for  $\mathbf{z}_1, \dots, \mathbf{z}_r$  in general position).

**Lemma 26.** *Let  $\mathcal{A} = \mathbf{\blacktriangle}^{(m,d)}$ ,  $r \leq \binom{m+d-1}{m}$ , and  $\{\mathbf{y}_1, \dots, \mathbf{y}_r\} \subset \mathbb{C}^m$  be some points. Furthermore, assume that there exists a set  $\mathcal{D}$ ,  $\mathbf{\blacktriangle}^{(m,d_0-1)} \subset \mathcal{D} \subseteq \mathbf{\blacktriangle}^{(m,d_0)}$  (for some  $d_0$ ,  $0 \leq d_0 \leq d - 1$ ) such that the points  $\mathbf{y}_1, \dots, \mathbf{y}_r$  are  $\mathcal{D}$ -independent.*

*Let  $\mathbf{P}(\rho)$  denote the projector on the column space of  $\mathcal{V}_{\mathcal{A}}(\rho\mathbf{y}_1, \dots, \rho\mathbf{y}_r)$ . Then we have that*

- if  $r = \binom{m+d_0}{m}$  (i.e.,  $\mathcal{D} = \mathbf{\blacktriangle}^{(m,d_0)}$ ), then

$$\lim_{\rho \rightarrow 0} \mathbf{P}(\rho) = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (39)$$

- if  $r < \binom{m+d_0}{m}$  (i.e.,  $\mathcal{D} \subset \mathbf{\blacktriangle}^{(m,d_0)}$ ), then

$$\lim_{\rho \rightarrow 0} \mathbf{P}(\rho) = \begin{bmatrix} \mathbf{I}_K & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (40)$$

where  $\mathbf{P}_2 \in \mathbb{C}^{L \times L}$ , with  $L = \binom{m+d_0}{m} - K$ ,  $\text{rank}\{\mathbf{P}_2\} = r - K$ , and  $K \stackrel{\text{def}}{=} \binom{m+d_0-1}{m}$ .

*Proof.* See the extended version of the paper [26].  $\square$

**Remark 27.** *Generic points  $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$  with  $r \leq \binom{m+d-1}{m}$  satisfy conditions of Lemma 26. Indeed, the condition of  $\mathcal{D}$ -independence is equivalent to*

$$\det \mathcal{V}_{\mathcal{D}}(\mathbf{y}_1, \dots, \mathbf{y}_r) \neq 0,$$

which holds for generic  $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$ .

By combining Lemma 23 and Lemma 26 we obtain the following theorem.

**Theorem 28.** *Let  $d'' := \lfloor \frac{d-1}{2} \rfloor$  and  $r \leq N'' := \binom{m+d''}{m}$ . Furthermore, let  $\{\mathbf{y}_1, \dots, \mathbf{y}_r\} \subset \mathbb{C}^m$  satisfy the conditions of Lemma 26. Then there exist a constant  $\rho_0 = \rho_0(\mathbf{y}_1, \dots, \mathbf{y}_r) > 0$  such that for any  $\rho$ :  $0 < \rho < \rho_0$  and points  $\mathbf{z}_k$  defined as  $\mathbf{z}_k = \rho\mathbf{y}_k$ , the following holds true.*

*For any nonzero coefficients  $c_1, \dots, c_r$  and the initial array defined as (15), the canonical completion (15) (in the problem (8)) is also the unique solution of the nuclear norm minimization problem (3).*

*Proof.* By Lemma 26 and Lemma 19, the matrix  $\mathbf{P}_0 \stackrel{\text{def}}{=} \lim_{\rho \rightarrow 0} \mathbf{P}(\rho)$  satisfies the conditions of Lemma 23. Next, take  $\varepsilon$  as in Lemma 23. Then, by Lemma 26, there exists  $\rho_0$ , such that

$$\|\mathbf{P}(\rho) - \mathbf{P}_0\|_F < \varepsilon,$$

which completes the proof.  $\square$

Note in the case  $m = 1$ , Theorem 28 is a weak version of Theorem 24. However, for  $m > 1$ , in Theorem 24, it is impossible to give a uniform bound on the size of exponents (elements of  $\mathbf{z}_k$ ) due to fundamental issues in multivariate polynomial interpolation.

## VII. NUMERICAL RESULTS

### A. Hankel case

In this case we illustrate on the examples Theorem 24. Theorem 24 only states the existence of such radius  $\rho$ , and a lower bound for  $\rho$  may be obtained along the lines of the proof of the theorem. However, this bound may be too small, and in this section we aim at showing on numerical experiments what would be the largest lower bound.

All the numerical experiments are reproducible and available on request. The MATLAB package CVXOPT [30] with default settings is used for nuclear norm minimization.

The setup for the following experiments will be similar. We take a specific  $\mathbf{h}_d$ , compute the solution of (8), and measure the Frobenius norm between the computed solution and the canonical completion (14).

1) *The rank-one case:* First, we consider the rank-one case. We take  $n = 6$ , and a rank-one exponential sequence  $\mathbf{h}_d$ , i.e.  $h_k = \lambda^k$ , where  $\lambda = a + bi$ ,  $a, b \in (-1, 1)^2$ , and plot the results in Fig. 2.

In Fig. 2, we see that the largest lower bound for  $\rho$  is very close to 1. This coincides with the bound for real  $\lambda$  given by Theorem 1.

2) *The rank-two case:* We consider real sequences  $\mathbf{h}_d$  with  $\text{hrank}(\mathbf{h}_d) = 2$ . In this case the following three situations are possible:

- 1)  $\lambda_1, \lambda_2 \in \mathbb{R}$  (two simple real roots);
- 2)  $\lambda_2 = \bar{\lambda}_1 \notin \mathbb{R}$  (two simple complex conjugate roots);
- 3)  $\lambda_1 \in \mathbb{R}$ ,  $\nu_1 = 2$  (double real root).

The first case is considered in [22, Fig. 1], where it is shown (numerically) that the radius  $\rho$  is less than 1. In this section, we examine the second and the third cases. We generate the corresponding  $\mathbf{h}_d$  and compute

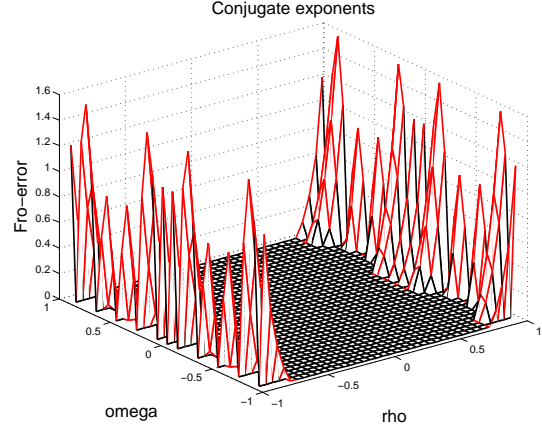
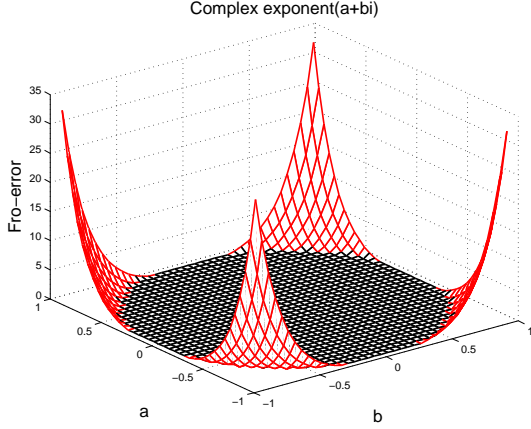
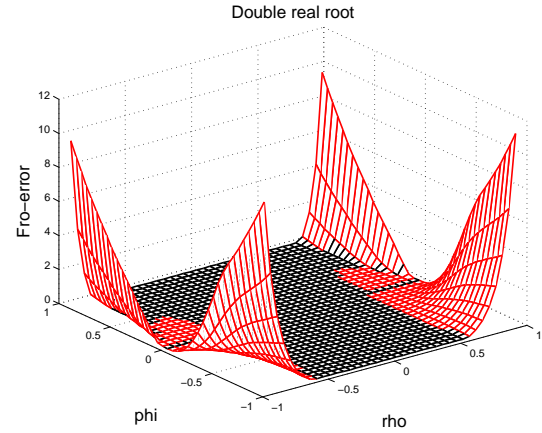


Fig. 2. Nuclear norm reconstruction (Frobenius distance),  $h_k = \lambda^k$ ,  $\lambda = a + bi$ . Black area correspond to the values less than  $10^{-6}$ .

the Frobenius distance between the solutions of (2) and (3), i.e., we compute  $\|\mathcal{S}(\tilde{\mathbf{p}}) - \mathcal{S}(\hat{\mathbf{p}})\|_F$ .

In Fig. 3, we plot the nuclear norm reconstructions for a  $6 \times 6$  matrix and the last two cases. As seen in Fig. 3, the radius is also strictly less than 1. The radius is smaller in the case of a double root and also in the case when two conjugate roots are close to each other.



### B. Multiple exponents

In this experiment, we aim at estimating the radius  $\rho$  based on random realizations of  $\lambda_k$ . We fix  $n = 9$ , and for each  $r \in \{1, \dots, 4\}$  and for  $\rho \in (0, 1)$ , we generate randomly the set  $\{\lambda_1, \dots, \lambda_r\}$ , such that  $|\lambda_1| = \rho$  and  $|\lambda_k| \leq \rho$ .

We consider two situations:

- real roots  $\lambda_k = \rho_k$  (in this case  $\rho_1 = \rho$ , and  $\rho_k$ ,  $k > 1$  are independent and uniformly distributed in  $[\rho; -\rho]$ )
- complex roots  $\lambda_k = \rho_k e^{i\pi\phi_k}$ , where  $\rho_k$  are as in the previous example,  $\phi_k$  are independent, uniformly distributed in  $[0; 1]$ , and independent of  $\rho_k$ .

We repeat the experiment  $M = 100$  times, and select the maximum Frobenius error across all the realizations. The results are plotted in Fig. 4.

The results in Fig. 4 confirm the conclusions of Theorem 24. We see that the nuclear norm heuristic works up to a certain  $\rho$ . Note that the bounds are similar for the real and complex cases.

### C. Quasi-Hankel case

1) *Random exponents*: In this section, we demonstrate Theorem 28. In the experiments, we fix  $m = 2$ ,

Fig. 3. Nuclear norm reconstruction (Frobenius distance). Top: for two complex conjugate roots  $\lambda_{1,2} = \rho \cdot \exp(\pm i\pi\omega)$  (corresponds to the damped cosine sequence  $h_t = \rho^t \cdot \cos(\pi(\omega + 1)t)$ ). Bottom: for a double root  $\rho$  (corresponds to the damped linear function  $h_t = (t \cdot \tan(0.75 \cdot \pi \cdot \varphi) + 1) \cdot \rho^t$ ). Black areas correspond to the values less than  $10^{-6}$ .

$d = 3$  and  $r = 3$  (thus we are dealing with the completion of the structure (6)). We generate random  $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathbb{C}^m$ , such that

$$\mathbf{y}_k = \begin{bmatrix} a_k + ib_k \\ e_k + if_k \end{bmatrix},$$

and  $a_k, b_k, e_k, f_k$  are independent identically distributed on  $[-0.5; 0.5]$ .

For  $\rho \in (0; 1]$ , we define  $\mathbf{z}_k$  as in Theorem 28, and consider the matrix completion with nuclear norm for the array (15), where  $c_k$  are chosen to be  $c_1 = c_2 = c_3 = 1$ . We numerically compute the solution  $\hat{\mathbf{p}}$  of the nuclear norm minimization (3), and compare it with the solution  $\tilde{\mathbf{p}}$  of the rank minimization (2) (which is known from Proposition 10). We repeat the experiment  $M = 10$  times, and plot in Fig. 5 the dependence of  $\|\hat{\mathbf{p}} - \tilde{\mathbf{p}}\|_2$ .

In Fig. 5, we see that for each realization of  $\mathbf{y}_k$ -s, there exists a radius  $\rho_0$  predicted by Theorem 28.

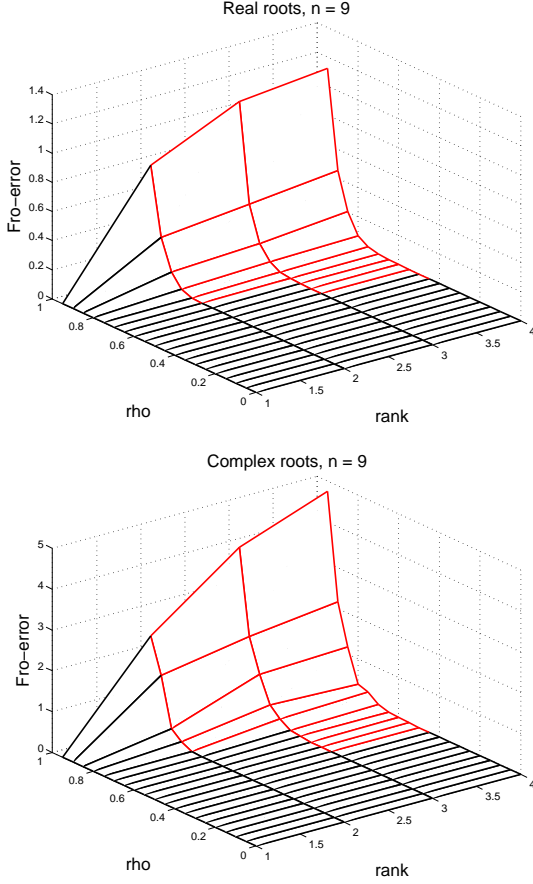


Fig. 4. Nuclear norm reconstruction,  $n = 9$ ,  $M = 400$  (Frobenius distance). Top: real roots, bottom: complex roots. Black area correspond to the values less than  $10^{-5}$ .

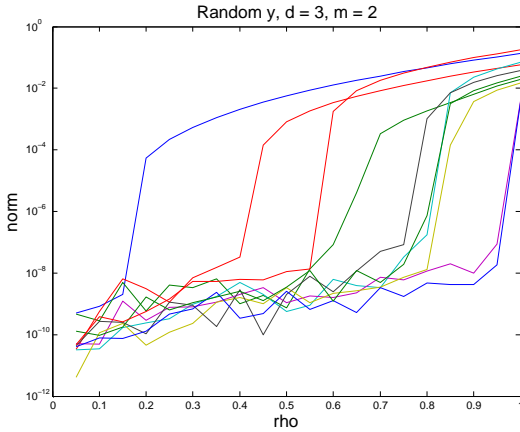


Fig. 5. Nuclear norm reconstruction for quasi-Hankel matrices (parameter vector distance). Each of the curves represents a realization and shows the dependence of  $\|\hat{p} - \tilde{p}\|_2$  on  $\rho$ .

2) *Beyond the theoretical results: nonunique decompositions:* As in the previous section, we consider the

case  $m = 2$ ,  $d = 3$ . But we choose the rank  $r = 4$ , for which the conditions of Theorem 28 no longer hold true. For  $3 \times 3 \times 3$  symmetric tensors, this is exactly the value of generic rank [24].

The construction of the example is based on the correspondence with symmetric tensors. We take  $d$  linearly independent vectors in  $\mathbb{R}^d$ , i.e.,

$$\mathbf{v}_1, \dots, \mathbf{v}_d \in \mathbb{R}^d.$$

Next, we consider the symmetric tensor obtained by symmetrization of a non-symmetric rank-one tensor:

$$\mathcal{T} = \text{Sym}(\mathbf{v}_1 \otimes \dots \otimes \mathbf{v}_d).$$

By [31], the rank of the tensor  $\mathcal{T}$  is equal to  $2^{d-1}$ , and the tensor admits an infinite family of decompositions, parameterized by

$$\mathcal{T} = \frac{1}{2^{d-1}d!} \sum_{\varepsilon_2, \dots, \varepsilon_d=0}^1 (-1)^{\varepsilon_2 + \dots + \varepsilon_d} \otimes^d \mathbf{b}_{\varepsilon_2, \dots, \varepsilon_d},$$

where

$$\mathbf{b}_{\varepsilon_2, \dots, \varepsilon_d} \stackrel{\text{def}}{=} (\gamma_1 \mathbf{v}_1 + \gamma_2 (-1)^{\varepsilon_2} \mathbf{v}_2 + \dots + (-1)^{\varepsilon_d} \gamma_d \mathbf{v}_d),$$

and  $\gamma_1, \dots, \gamma_d \in \mathbb{R}$  satisfy  $\gamma_1 \dots \gamma_d = 1$ .

In order to set up the matrix completion problem, for given  $\gamma_1, \dots, \gamma_d$ , we first construct the array  $\{h_\alpha\}_{\alpha \in \mathbb{N}^m}$  (which is equivalent to the specific tensor decomposition), where  $m \stackrel{\text{def}}{=} d - 1$ , as follows:

$$h_\alpha = \sum_{\varepsilon_2, \dots, \varepsilon_d=0}^1 c_{\varepsilon_2, \dots, \varepsilon_d} \boldsymbol{\lambda}_{\varepsilon_2, \dots, \varepsilon_d}^\alpha, \quad (41)$$

where

$$c_{\varepsilon_2, \dots, \varepsilon_d} \stackrel{\text{def}}{=} \frac{1}{2^{d-1}d!} (-1)^{\varepsilon_2 + \dots + \varepsilon_d} (\mathbf{b}_{\varepsilon_2, \dots, \varepsilon_d, 1})^d,$$

$$\boldsymbol{\lambda}_{\varepsilon_2, \dots, \varepsilon_d} \stackrel{\text{def}}{=} b_{\varepsilon_2, \dots, \varepsilon_d, 1}^{-1} \begin{bmatrix} b_{\varepsilon_2, \dots, \varepsilon_d, 2} \\ \vdots \\ b_{\varepsilon_2, \dots, \varepsilon_d, d} \end{bmatrix},$$

$$\mathbf{b}_{\varepsilon_2, \dots, \varepsilon_d} = \begin{bmatrix} b_{\varepsilon_2, \dots, \varepsilon_d, 1} \\ \vdots \\ b_{\varepsilon_2, \dots, \varepsilon_d, d} \end{bmatrix}.$$

Finally, we solve the nuclear norm minimization problem (3), for the structure defined as in (6).

Assume for instance  $(d, m) = (3, 2)$  and consider

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

For this case, the nuclear norm minimization yields a 4-rank matrix completion, which corresponds to the minimal rank. However, the obtained completion differs from the initial completion (41), which indicates that the nuclear norm minimization resulted in

a different decomposition from the infinite family of decompositions.

We repeat the experiment for random  $v_k$ , generated as

$$[v_1 \quad v_2 \quad v_3] = A + E,$$

where the elements of  $E$  are i.i.d. uniform on  $[-1; 1]$ , and  $A$  is one of the two matrices:

$$A = 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ or } A = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By drawing  $M = 100$  random realizations, we check when the Euclidean norm of the vector of the last singular values  $(\sigma_5, \dots, \sigma_{10})$  is less than  $10^{-4}$  (we say that the nuclear norm heuristics succeeds in this case). We observed that in the first scenario, the nuclear norm heuristic always failed, but for the second scenario in 77 cases the nuclear norm heuristic succeeded.

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