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► **To cite this version:**

Françoise Dal'Bo-Milonet, Marc Peigné, Jean-Claude Picaud, Andrea Sambusetti. Volume growth and rigidity of negatively curved manifolds of finite volume. 2015. hal-01130605

HAL Id: hal-01130605

<https://hal.science/hal-01130605>

Preprint submitted on 12 Mar 2015

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VOLUME GROWTH AND RIGIDITY OF NEGATIVELY CURVED MANIFOLDS OF FINITE VOLUME

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March 12, 2015

Abstract

We study the asymptotic behaviour of the volume growth function of simply connected, Riemannian manifolds X of strictly negative curvature admitting a non-uniform lattice Γ . If X is asymptotically 1/4-pinched, we prove that Γ is divergent, with finite Bowen-Margulis measure, and that the volume growth of balls $B(x, R)$ in X is asymptotically equivalent to a purely exponential function $c(x)e^{\omega(X)R}$, where $\omega(X)$ is the volume entropy of X . This generalizes Margulis' celebrated theorem for negatively curved spaces with compact quotients. A crucial step for this is a finite-volume version of the entropy-rigidity characterization of constant curvature spaces: any finite volume n -manifold with sectional curvature $-b^2 \leq k(X) \leq -1$ and volume entropy equal to $(n-1)$ is hyperbolic. In contrast, we show that for spaces admitting lattices which are not 1/4-pinched, depending on the critical exponent of the parabolic subgroups and on the finiteness of the Bowen-Margulis measure, the growth function can be exponential, lower-exponential or even upper-exponential.

1 Introduction

Let X be a complete, simply connected manifold with strictly negative curvature. In the sixties, G. Margulis [22], using measure theory on the foliations of the Anosov system defined by the geodesic flow, showed that if Γ is a uniform lattice of X (i.e. a torsionless, discrete group of isometries such that $\bar{X} = \Gamma \backslash X$ is compact), then the *orbital function* of Γ is asymptotically equivalent to a purely exponential function:

$$v_\Gamma(x, y, R) = \#\{\gamma \in \Gamma \mid d(x, \gamma y) < R\} \sim c_\Gamma(x, y)e^{\delta(\Gamma)R}$$

where $\delta(\Gamma) = \lim_{R \rightarrow \infty} R^{-1} \ln v_\Gamma(x, x, R)$ is the *critical exponent* of Γ , and \sim means that the quotient tends to 1 when $R \rightarrow \infty$. By integration over fundamental domains, one then obtains an asymptotic equivalence for the *volume growth function* of X :

$$v_X(x, R) = \text{vol}B(x, R) \sim m(x)e^{\delta(\Gamma)R}.$$

It is well-known that the exponent $\delta(\Gamma)$ equals, for uniform lattices, the *volume entropy* $\omega(X) = \limsup \frac{1}{R} \ln v_X(x, R)$ of the manifold X ; the function $m(x)$, depending on the center of the ball, is the *Margulis function* of X .

Since then, this result has been generalized in different directions. Notably, G. Knieper showed in [21] that the volume growth function of a Hadamard space X (a complete, simply connected manifolds with nonpositive curvature) admitting uniform lattices is *purely exponential*, provided that X has rank one, that is:

$$v_X(x, R) \asymp e^{\omega(X)R}$$

where $f \asymp g$ means that $1/A < f(R)/g(R) < A$ for some positive A , when $R \gg 0$. In general, he showed that $v_X(x, R) \asymp R^{\frac{d-1}{2}} e^{\omega(X)R}$ for rank d manifolds; however, as far as the authors are aware, it is still unknown whether there exists a Margulis function for Hadamard manifolds of rank 1 with uniform lattices, i.e. a function $m(x)$ such that $v_X(x, R) \sim m(x)e^{\omega(X)R}$, even in the case of surfaces. Another remarkable case is that of *asymptotically harmonic manifolds* of strictly negative curvature, where the strong asymptotic homogeneity implies the existence of a Margulis function, even without compact quotients, cp. [9].

In another direction, it seems natural to ask what happens for a Hadamard space X of negative curvature admitting *nonuniform lattices* Γ (i.e. $\text{vol}(\Gamma \backslash X) < \infty$): *is v_X purely exponential and, more precisely, does X admit a Margulis function?* Let us emphasize that if X also admits a uniform lattice then X is a symmetric space of rank one (by [15], Corollary 9.2.2); therefore, we are interested in spaces which do not have uniform lattices, i.e. the universal covering of finite volume, negatively curved manifolds which are not locally symmetric.

It is worth to stress here that the orbital function of Γ is closely related to the volume growth function of X , but it generally has, even for lattices, a different asymptotic behaviour than $v_X(x, R)$. The weak equivalence $v_\Gamma(x, R) \asymp e^{\delta(\Gamma)R}$ is known for convex-cocompact discrete subgroups of isometries of \mathbb{H}^n since [29], [24], by Patterson-Sullivan theory. A precise asymptotic equivalence for v_Γ was proved by T. Roblin [26] in a very general setting. Namely, he proved that for any nonelementary group of isometries Γ of a CAT(-1) space X with non-arithmetic length spectrum¹ and $\bar{X} = \Gamma \backslash X$, one has:

- (a) $v_\Gamma(x, y, R) \sim c_\Gamma(x, y)e^{\delta(\Gamma)R}$ if the Bowen-Margulis measure of $U\bar{X}$ is finite;
- (b) $v_\Gamma(x, y, R) = o(R)e^{\delta(\Gamma)R}$, where $o(R)$ is infinitesimal, otherwise.

Thus, the behaviour of $v_\Gamma(x, R)$ strongly depends on the finiteness of the Bowen-Margulis measure μ_{BM} ; also, the asymptotic constant can be expressed in terms of μ_{BM} and of the family of Patterson-Sullivan measures (μ_x) of Γ , as $c_\Gamma(x, y) = \frac{\|\mu_x\| \|\mu_y\|}{\|\mu_{BM}\|}$. A useful criterion ensuring that $\mu_{BM}(U\bar{X}) < \infty$, hence a precise asymptotics for $v_\Gamma(x, R)$, is the following (see [10])

FINITENESS CRITERION. *Let Γ be a divergent, geometrically finite group, $\bar{X} = \Gamma \backslash X$. We have $\mu_{BM}(U\bar{X}) < \infty$ if and only if for every maximal parabolic subgroup P of Γ*

$$\sum_{p \in P} d(x, px) e^{-\delta(\Gamma)d(x, px)} < +\infty. \quad (1)$$

¹This means that the additive subgroup of \mathbb{R} generated by the length of closed geodesics in $\bar{X} = \Gamma \backslash X$ is dense in \mathbb{R} ; it is the case, for instance, if $\dim(X) = 2$, or when Γ is a lattice.

On the other hand, any convergent group Γ exhibits a behaviour as in (b), since it certainly has infinite Bowen-Margulis measure (by Poincaré recurrence, $\mu_{BM}(U\bar{X}) < \infty$ implies that the geodesic flow is totally conservative, and this is equivalent to divergence, by Hopf-Tsuji-Sullivan’s theorem). Notice that, whereas uniform lattices always are divergent and with finite Bowen-Margulis measure, for nonuniform lattices Γ divergence and condition (1) in general may fail. Namely, this can happen only in case Γ has a “very large” parabolic subgroup P , that is such that $\delta(P) = \delta(\Gamma)$: we will call *exotic* such a lattice Γ , and we will say that such a P is a *dominant* parabolic subgroup. Convergent, exotic lattices are constructed by the authors in [14]; also, one can find in [14] some original counting results for the orbital function of Γ in infinite Bowen-Margulis measure, more precise than (b).

However, as we shall see, the volume growth function v_X has a wilder behaviour than v_Γ . In [12] we proved that for nonuniform lattices in pinched, negatively curved spaces X , the functions v_Γ and v_X can have different exponential growth rates, i.e. $\omega(X) \neq \delta(\Gamma)$. In the Example 6.2 we will see that the function v_X might as well have different superior and inferior exponential growth rates $\omega^\pm(X)$ (notice, in contrast, that $\delta(\Gamma)$ always is a true limit). Nevertheless, $\delta(\Gamma)$ still encodes a lot of information on the manifold X even if Γ is non-uniform. The first result we prove in this paper is a generalization of a volume-entropy characterization of constant curvature spaces, due to G. Knieper when the quotient $\Gamma \backslash X$ is compact (cp. [20]; see also [3] for an analogue in case of convex-cocompact lattices):

Theorem 1.1 *Let X be a Hadamard manifold with curvature $-b^2 \leq K_X \leq -a^2 < 0$ and Γ a nonuniform lattice of X . If $\delta(\Gamma) = (n-1)a$, i.e. if it equals the volume entropy of the space \mathbb{H}_a^n with constant curvature $-a^2$, then X has constant curvature $-a^2$.*

The volume-entropy characterization of constant curvature (and locally symmetric) metrics has a long history and has been declined in many different ways so far, for uniform lattices or convex-cocompact representations (beyond [20] and [3], see also [16], [6], [19], [1], [2], [8]). To prove Theorem 1.1, we use the barycenter method initiated by Besson-Courtois-Gallot in [1]-[2]. There exist finite-volume versions of [1]& [2], given by Boland-Connell-Souto [5] and Storm [28]: these two works, together, imply that if a Hadamard manifold X with curvature $K_X \leq -1$ has a quotient of finite volume $\bar{X} = \Gamma \backslash X$ and $\omega(X) = n-1$, then it is hyperbolic, *provided that one knows that \bar{X} is homotopically equivalent to a finite-volume, hyperbolic manifold \bar{X}_0* . In contrast, notice that in Theorem 1.1 no supplementary topological assumption on the quotient manifold \bar{X} is made. Also, notice that if we drop the assumption $K_X \geq -b^2$, the manifold \bar{X} might as well be of infinite type (i.e. with infinitely generated fundamental group, or even without any cusp, see examples in [23]), hence not even homotopically equivalent to a finite-volume, hyperbolic manifold.

The second result of the paper concerns the Bowen-Margulis measure and an asymptote for the volume growth function of $\frac{1}{4}$ -pinched spaces with lattices. This strongly relies on the above characterization and on a Counting Formula (Proposition 3.1), which enables us to reduce the computation of v_X to the *analytic profile* of the cusps of \bar{X} and v_Γ (so, in the last instance, to T.Roblin’s asymptotics $(a)\mathcal{E}(b)$):

Theorem 1.2 *Let X be a Hadamard space with curvature $-b^2 \leq K_X \leq -a^2$, and let Γ be a nonuniform lattice of X . If $\bar{X} = \Gamma \backslash X$ has asymptotically $1/4$ -pinched curvature (that is, for any $\epsilon > 0$, the metric satisfies $-k_+^2 \leq K_X \leq -k_-^2$ with $k_+^2 \leq 4k_-^2 + \epsilon$ on each cuspidal end outside some compact set $\bar{C}_\epsilon \subset \bar{X}$), then:*

- (i) Γ is divergent;
- (ii) the Bowen-Margulis measure μ_{BM} of $U\bar{X}$ is finite;
- (iii) $\omega^+(X) = \omega^-(X) = \delta(\Gamma)$;
- (iv) there exists a function $\bar{m}(x) \in L^1(\bar{X})$ such that $v_X(x, R) \sim m(x)e^{\delta(\Gamma)R}$, where $m(x)$ is the lift of \bar{m} to X .

From the divergence of Γ , it then follows that the geodesic flow of any asymptotically $\frac{1}{4}$ -pinched, negatively curved manifold is ergodic and totally conservative w.r. to μ_{BM} , by the celebrated Hopf-Tsuji-Sullivan Theorem (see [29], [26]). Condition (iv) also implies that volume equidistributes on large spheres, i.e. the volume $v_X^\Delta(x, R)$ of annuli in X of thickness Δ satisfies the asymptotics $v_X^\Delta(x, R) \sim 2m(x) \sinh(\Delta\delta(\Gamma))e^{\omega(X)R}$. Notice that the above theorem also covers the classical case of noncompact symmetric spaces of rank one (where the proof of the divergence and the asymptotics is direct).

One may wonder about the meaning (and necessity) of the $\frac{1}{4}$ -pinching condition. This turns out to be an asymptotic, geometrical condition on the influence and wildness of maximal parabolic subgroups of Γ associated to the cusps of $\bar{X} = \Gamma \backslash X$. Parabolic groups, being elementary, do not necessarily have a critical exponent which can be interpreted as a true limit; rather, for a parabolic group of isometries P of X , one can consider the limits

$$\delta^+(P) = \limsup_{R \rightarrow \infty} \frac{1}{R} \ln v_P(x, R) \quad \delta^-(P) = \liminf_{R \rightarrow \infty} \frac{1}{R} \ln v_P(x, R)$$

and the critical exponent $\delta(P)$ of the Poincaré series of P coincides with $\delta^+(P)$. Accordingly, we say that a lattice Γ is *sparse* if it has a maximal parabolic subgroup P such that $\delta^+(P) > 2\delta^-(P)$ (conversely, we will say that Γ is *parabolically $\frac{1}{2}$ -pinched* if it is not sparse). Such parabolic groups in Γ are precisely associated to cusps whose growth can wildly change and this can globally influence the growth function of X . Namely, we can prove:

Theorem 1.3 *Let X be a Hadamard manifold with pinched, negative curvature $-b^2 \leq K_X \leq -a^2 < 0$. If X has a nonuniform lattice Γ which is neither exotic nor sparse, then Γ is divergent and with finite Bowen-Margulis measure; moreover, $v_X \asymp v_\Gamma$ and X has a Margulis function $m(x)$, whose projection is L^1 on $\bar{X} = \Gamma \backslash X$.*

Theorem 1.2 is a particular case of Theorem 1.3, as (using the volume-entropy characterization 1.1 of constant curvature spaces) we can show that any lattice in a negatively curved, $\frac{1}{4}$ -pinched space is neither exotic nor sparse.

The last part of the paper is devoted to studying sparse and exotic lattices, and the following result shows that Theorem 1.3 is the best that we can expect for Hadamard spaces with quotients of finite volume:

Theorem 1.4 *Let X be a Hadamard manifold with pinched negative curvature $-b^2 \leq K_X \leq -a^2 < 0$ admitting a nonuniform lattice Γ .*

(i) *If Γ is exotic and the dominant subgroups P satisfy $\delta(\Gamma) = \delta^+(P) < 2\delta^-(P)$, then both v_X and v_Γ are purely exponential or lower-exponential, with the same exponential growth rate $\omega(X) = \delta(\Gamma)$. Namely*

- *either $\mu_{BM} < \infty$, and then v_X is purely exponential and X has a Margulis function;*
- *or $\mu_{BM} = \infty$, and in this case v_X is lower-exponential.*

The two cases can actually occur, cp. Examples 6.3(a) & (b).

(ii) *If Γ is exotic and a dominant subgroup P satisfies $\delta(\Gamma) = \delta^+(P) = 2\delta^-(P)$, then $\omega(X) = \delta(\Gamma)$ but in general $v_X \neq v_\Gamma$, and X does not admit a Margulis function. Namely, there exist cases (Examples 6.4(a) & (b)) where:*

- *$\mu_{BM} < \infty$, with v_Γ purely exponential and v_X upper-exponential;*
- *$\mu_{BM} = \infty$, with v_Γ lower-exponential and v_X upper-exponential.*

By lower- (respectively, upper-) exponential we mean a function f with exponential growth rate $\omega = \limsup_{R \rightarrow \infty} \frac{1}{R} \ln f(R)$, but such that $\liminf_{R \rightarrow \infty} f(R)/e^{\omega R} = 0$ (resp. $\limsup_{R \rightarrow \infty} f(R)/e^{\omega R} = +\infty$).

We shall see that all these examples can be obtained as lattices in $(\frac{1}{4} - \epsilon)$ -pinched spaces, for arbitrary $\epsilon > 0$, which shows the optimality of the $\frac{1}{4}$ -pinching condition.

On the other hand, if Γ is sparse, one can even have $\omega^+(X) > \omega^-(X) > \delta(\Gamma)$, and the Example 6.2 shows that virtually any asymptotic behaviour for v_X can occur. Thus, the case of exotic lattices with a parabolic subgroup such that $\delta^+(P) = 2\delta^-(P)$ can be seen as the critical threshold where a transition happens, from functions v_Γ, v_X with same asymptotic behaviour to functions with even different exponential growth rate.

NOTATIONS.

Given two functions $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we will systematically write $f \stackrel{C}{\prec} g$ for $R > R_0$ (or $g \stackrel{C}{\succ} f$) if there exists $C > 0$ such that $f(R) \leq Cg(R)$ for these values of R . We say that f and g are *weakly asymptotically equivalent* and write $f \stackrel{C}{\asymp} g$ when $g \stackrel{C}{\prec} f \stackrel{C}{\prec} g$ for $R \gg 0$; we will simply write $f \asymp g$ and $f \prec g$ when the constants C and R_0 are unessential. We say that f and g are *asymptotically equivalent* and write $f \sim g$ when $\lim_{R \rightarrow +\infty} f(R)/g(R) = 1$.

We define the upper and lower exponential growth rates of a function f respectively as:

$$\omega^+(f) = \limsup_{R \rightarrow +\infty} R^{-1} \ln f(R) \quad \text{and} \quad \omega^-(f) = \omega(f) = \liminf_{R \rightarrow +\infty} R^{-1} \ln f(R)$$

and we simply write $\omega(f)$ when the two limits coincide. Also, we will say that f is *purely exponential* if $f \asymp e^{\omega(f)R}$, and that f is *lower-exponential* (resp. *upper-exponential*) when $\liminf_{R \rightarrow +\infty} \frac{f(R)}{e^{\omega(f)R}} = 0$ (resp. $\limsup_{R \rightarrow +\infty} \frac{f(R)}{e^{\omega(f)R}} = +\infty$).

Finally, if f and g are two real functions, we will use the notation $f *_\Delta g$ for the discrete convolution of f and g with gauge Δ , defined by $(f *_\Delta g)(R) = \sum_{h+k=\lfloor R/\Delta \rfloor} f(h\Delta)g(k\Delta)$. We notice here that, for nondecreasing functions f and g , this is weakly equivalent to the usual convolution, namely

$$\Delta \cdot (f *_\Delta g)(R - \Delta) \leq (f * g)(R) = \int_0^R f(t)g(R-t)dt \leq 2\Delta \cdot (f *_\Delta g)(R + 2\Delta).$$

2 Growth of parabolic subgroups and of lattices modulo parabolic subgroups

Throughout all the paper, unless otherwise stated, X will be a Hadamard space of dimension n , with pinched negative sectional curvature $-b^2 \leq K_X \leq -a^2 < 0$.

For $x, y \in X$ and $\xi \in X(\infty)$, we will denote $[x, y]$ (resp. $[x, \xi]$) the geodesic segment from x to y (resp. the ray from x to ξ). We will repeatedly make use of the following, classical result in strictly negative curvature: there exists $\epsilon(a, \vartheta) = \frac{1}{|a|} \log\left(\frac{2}{1-\cos\vartheta}\right)$ such that any geodesic triangle xyz in X making angle $\vartheta = \angle_z(x, y)$ at z satisfies:

$$d(x, y) \geq d(x, z) + d(z, y) - \epsilon(a, \vartheta). \quad (2)$$

Let $b_\xi(x, y) = \lim_{z \rightarrow \xi} d(x, z) - d(z, y)$ be the Busemann function centered at ξ . The level set $\partial H_\xi(x) = \{y \mid b_\xi(x, y) = 0\}$ (resp. the suplevel set $H_\xi(x) = \{y \mid b_\xi(x, y) \geq 0\}$) is the horosphere (resp. the horoball) with center ξ and passing through x . From (2) we easily deduce the following:

Lemma 2.1 *For any $d > 0$, there exists $\epsilon_1 = \epsilon_1(a, d) \geq \epsilon(a, \frac{\pi}{2})$ with the following property: given two disjoint horoballs H_1, H_2 at distance $d = d(H_1, H_2) = d(z_1, z_2)$ with $z_i \in \partial H_i$, then for any $x \in H_1$ and $y \in H_2$ we have*

$$d(x, z_1) + d(z_1, z_2) + d(z_2, y) - \epsilon_1(a, d) \leq d(x, y) \leq d(x, z_1) + d(z_1, z_2) + d(z_2, y).$$

Proof. As $K_X \leq -a^2$ and horoballs are convex, for any $y \in H_2$ the angle $\vartheta(y) = \angle_{z_1} z_2, y$ satisfies $\tan \vartheta(y) \leq \frac{1}{\sinh(d/|a|)}$ (cp. for instance [27], Prop.8). Then, we have $\angle_{z_1} x, y \geq \frac{\pi}{2} - \vartheta(y) \geq \vartheta(d)$ with $\vartheta(d) > 0$ for $d \neq 0$, hence, by (2),

$$d(x, y) \geq d(x, z_1) + d(z_1, y) - \epsilon(a, \vartheta(d)) \geq d(x, z_1) + d(z_1, z_2) + d(z_2, y) - \epsilon_1(a, d)$$

for $\epsilon_1(a, d) = \epsilon(a, \vartheta(d)) + \epsilon(a, \frac{\pi}{2})$. \square

Let d_ξ denote the horospherical distance between two points on a same horosphere centered at ξ . If $\psi_{\xi, t} : X \rightarrow X$ denotes the radial flow in the direction of ξ , we define:

$$t_\xi(x, y) = \begin{cases} \inf\{t > 0 \mid d_\xi(\psi_{\xi, t+\Delta}(x), \psi_{\xi, t}(y)) < 1\} & \text{if } b_\xi(x, y) = \Delta \geq 0; \\ \inf\{t > 0 \mid d_\xi(\psi_{\xi, t}(x), \psi_{\xi, t-\Delta}(y)) < 1\} & \text{if } b_\xi(x, y) = \Delta < 0. \end{cases} \quad (3)$$

If y is closer to ξ than x , let $x_\Delta = [x, \xi] \cap \partial H_\xi(y)$: then, $t_\xi(x, y)$ represents the minimal time we need to apply the radial flow $\psi_{\xi, t}$ to the points x_Δ and y until they are at horospherical distance less than 1. Using (2) and the lower curvature bound $K_X \geq -b^2$, we obtain in [12] the following estimate, which is also crucial in our computations:

Approximation Lemma 2.2

There exists $\epsilon_0 = \epsilon_0(a, b) \geq \epsilon(a, \frac{\pi}{2})$ such that for all $x, y \in X$ and $\xi \in X(\infty)$ we have:

$$2t_\xi(x, y) + |b_\xi(x, y)| - \epsilon_0 \leq d(x, y) \leq 2t_\xi(x, y) + |b_\xi(x, y)| + \epsilon_0$$

In this section we give estimates for the growth of annuli in a parabolic subgroup and in quotients of a lattice by a parabolic subgroup, which will be used later. So, let us fix some notations. We let $A^\Delta(x, R) = B(x, R + \frac{\Delta}{2}) \setminus B(x, R - \frac{\Delta}{2})$ be the annulus of radius R and thickness Δ around x . For G acting on X , we will consider the orbital functions

$$v_G(x, y, R) = \#(B(x, R) \cap Gy) \quad v_G^\Delta(x, y, R) = \#(A^\Delta(x, R) \cap Gy)$$

and we set $v_G(x, R) = v_G(x, x, R)$, $v_G^\Delta(x, R) = v_G^\Delta(x, x, R)$ and $v_G^\Delta(x, R) = \emptyset$ for $\Delta < 0$. We will also need to consider the growth function of coset spaces, endowed with the natural quotient metric: if $H < G$, we define $d_x(g_1H, g_2H) := d(g_1Hx, g_2Hx)$ and

$$v_{G/H}(x, R) := \#\{gH \mid |gH|_x = d_x(H, gH) < R\}$$

$$v_{G/H}^\Delta(x, R) = v_{G/H}\left(x, R + \frac{\Delta}{2}\right) - v_{G/H}\left(x, R - \frac{\Delta}{2}\right).$$

We will use analogous notations for the growth functions of balls and annuli in the spaces of left and double cosets $H \setminus G$, $H \setminus G/H$ with the metrics

$$d_x(Hg_1, Hg_2) := d(Hg_1x, Hg_2x) = |g_1^{-1}Hg_2|_x$$

$$d_x(Hg_1H, Hg_2H) := d(Hg_1Hx, Hg_2Hx) = |g_1^{-1}Hg_2H|_x.$$

The growth of the orbital function of a bounded parabolic group P is best expressed by introducing the horospherical area function. Let us recall the necessary definitions:

Definitions 2.3 Let P be a *bounded* parabolic group of X fixing $\xi \in X(\infty)$: that is, acting cocompactly on $X(\infty) - \{\xi\}$ (as well as on every horosphere ∂H centered at ξ). Given $x \in X$, let $\mathcal{D}(P, x)$ be a *Dirichlet domain* centered at x for the action of P on X ; that is, a convex fundamental domain contained in the closed subset

$$\overline{\mathcal{D}}(P, x) = \{y \in X \mid d(x, y) \leq d(px, y) \text{ for all } p \in P\}$$

We set $\mathcal{S}_x = \mathcal{D}(P, x) \cap \partial H_\xi(x)$ and $\mathcal{C}_x = \mathcal{D}(P, x) \cap H_\xi(x)$, and denote by $\mathcal{S}_x(\infty)$ the trace at infinity of $\mathcal{D}(P, x)$, minus ξ ; these are, respectively, fundamental domains for the actions of P on $\partial H_\xi(x)$, $H(x)$ and $X(\infty) - \{\xi\}$.

The *horospherical area function* of P is the function

$$\mathcal{A}_P(x, R) = \text{vol}[P \setminus \psi_{\xi, R}(\partial H_\xi(x))] = \text{vol}[\psi_{\xi, R}(\mathcal{S}_x)]$$

where the vol is the Riemannian measure of horospheres. We also define the *cuspidal function* of P , which is the function

$$\mathcal{F}_P(x, R) = \text{vol}[B(x, R) \cap H_\xi(x)]$$

that is, the volume of the intersection of a ball centered at x and the horoball centered at ξ and passing through x . Notice that the functions $\mathcal{A}_P(x, R)$, $\mathcal{F}_P(x, R)$ only depend on the choice of the initial horosphere $\partial H_\xi(x)$.

REMARK 2.4 Well-known estimates of the differential of the radial flow (cp. [18]) yield, when $-b^2 \leq K_X \leq -a^2 < 0$,

$$e^{-bt} \|v\| \leq \|d\psi_{\xi,t}(v)\| \leq e^{-at} \|v\| \quad (4)$$

Therefore we deduce that, for any $\Delta > 0$,

$$e^{-(n-1)b\Delta} \leq \frac{\mathcal{A}_P(x, R + \Delta)}{\mathcal{A}_P(x, R)} \leq e^{-(n-1)a\Delta} \quad (5)$$

The following Propositions show how the horospherical area \mathcal{A}_P and the cuspidal function \mathcal{F}_P are related to the orbital function of P ; they refine and precise some estimates given in [12] for $v_P(x, R)$.

Proposition 2.5 *Let P be a bounded parabolic group of X fixing ξ , with $\text{diam}(\mathcal{S}_x) \leq d$. There exist $C = C(n, a, b, d)$ and $C' = C'(n, a, b, d; \Delta)$ such that:*

$$v_P(x, y, R) \stackrel{C}{\asymp} \mathcal{A}_P^{-1}\left(x, \frac{R + b_\xi(x, y)}{2}\right) \quad \forall R \geq b_\xi(x, y) + R_0 \quad (6)$$

$$v_P^\Delta(x, y, R) \stackrel{C'}{\asymp} \mathcal{A}_P^{-1}\left(x, \frac{R + b_\xi(x, y)}{2}\right) \quad \forall R \geq b_\xi(x, y) + R_0 \quad \text{and} \quad \forall \Delta > \Delta_0 \quad (7)$$

for explicit constants R_0 and Δ_0 only depending on n, a, b, d .

Proposition 2.6 *Same assumptions as in Proposition 2.5. We have:*

$$\mathcal{F}_P(x, R) \stackrel{C}{\asymp} \int_0^R \frac{\mathcal{A}_P(x, t)}{\mathcal{A}_P\left(x, \frac{R+t}{2}\right)} dt \quad \forall R \geq R_0 \quad (8)$$

REMARK 2.7 More precisely, we will prove (and use later) that:

- (i) $v_P(x, y, R) \stackrel{C}{\asymp} \mathcal{A}_P^{-1}\left(x, \frac{R + b_\xi(x, y)}{2}\right)$ for all $R > 0$;
- (ii) $v_P^\Delta(x, y, R) \stackrel{C'}{\asymp} \mathcal{A}_P^{-1}\left(x, \frac{R + b_\xi(x, y)}{2}\right)$ for all $\Delta, R > 0$;
- (iv) $\mathcal{F}_P(x, R) \stackrel{C}{\asymp} \int_0^R \frac{\mathcal{A}_P(x, t)}{\mathcal{A}_P\left(x, \frac{R+t}{2}\right)} dt$ for all $R > 0$.

As a direct consequence of (8) and (6) we have (see also Corollary 3.5 in [12]):

Corollary 2.8 *Let P be a bounded parabolic group of X . Then:*

$$\delta^-(P) \leq \omega^-(\mathcal{F}_P) \leq \omega^+(\mathcal{F}_P) \leq \max\{\delta^+(P), 2(\delta^+(P) - \delta^-(P))\} \quad (9)$$

Proof of Proposition 2.5.

Since $v_P(x, y, R) = v_P(y, x, R)$ and $\mathcal{A}_P(x, R) = \mathcal{A}_P(y, R - b_\xi(x, y))$, we can assume that $t = b_\xi(x, y) \geq 0$. If $z \in \partial H_\xi(y)$ and $d(x, z) = R$, we know by Lemma 2.2 that

$$2t_\xi(x, z) + t - \epsilon_0 \leq d(x, z) \leq 2t_\xi(x, z) + t + \epsilon_0$$

so $|t_\xi(x, z) - \frac{R-t}{2}| \leq \epsilon_0/2$. We deduce that $d_\xi\left(\psi_{\xi, \frac{R+t+\epsilon_0}{2}}(x), \psi_{\xi, \frac{R-t+\epsilon_0}{2}}(z)\right) \leq 1$, so the set $\psi_{\xi, \frac{R-t+\epsilon_0}{2}}(B(x, R) \cap \partial H_\xi(y))$ is contained in the unitary ball B^+ of the horosphere $\partial H_\xi(x^+)$, centered at $x^+ = \psi_{\xi, \frac{R+t+\epsilon_0}{2}}(x)$. Similarly, if $R > t + \epsilon_0$ then $t_\xi(x, z) > 0$, so $d_\xi\left(\psi_{\xi, \frac{R+t-\epsilon_0}{2}}(x), \psi_{\xi, \frac{R-t-\epsilon_0}{2}}(z)\right) \geq 1$, and the set $\psi_{\xi, \frac{R-t-\epsilon_0}{2}}(B(x, R) \cap \partial H_\xi(y))$ contains the unitary ball B^- of $\partial H_\xi(x^-)$, centered at the point $x^- = \psi_{\xi, \frac{R+t-\epsilon_0}{2}}(x)$. We know that, by Gauss' equation, the sectional curvature of horospheres of \tilde{X} is between $a^2 - b^2$ and $2b(b - a)$ (see, for instance, [4], §1.4); therefore, there exist positive constants $v^- = v^-(a, b)$ and $v^+ = v^+(a, b)$ such that $\text{vol}(B^+) < v^+$ and $\text{vol}(B^-) > v^-$. Now, let $\mathcal{S}_y = \psi_{\xi, t}(\mathcal{S}_x)$ be the fundamental domain for the action of P on $\partial H_\xi(y)$ deduced from \mathcal{S}_x . There are at least $v_P(x, y, R - d)$ distinct fundamental domains $p\mathcal{S}_y$ included in $B(x, R) \cap \partial H_\xi(y)$; since the radial flow $\psi_{\xi, t}$ is equivariant with respect to the action of P on the horospheres centered at ξ , there are also at least $v_P(x, y, R - d)$ distinct fundamental domains $\psi_{\xi, \frac{R-t+\epsilon_0}{2}}(p\mathcal{S}_y)$ included in $\psi_{\xi, \frac{R-t+\epsilon_0}{2}}(B(x, R) \cap \partial H_\xi(y))$.

We deduce that $v_P(x, y, R - d) \cdot \mathcal{A}_P(x, \frac{R+t+\epsilon_0}{2}) < v^+$ and, by (5), this gives $v_P(x, y, R) \prec \mathcal{A}_P^{-1}(x, \frac{R+t}{2})$ for all $R \geq 0$. On the other hand, if $R > t + \epsilon_0$, we can cover the set $B(x, R) \cap \partial H_\xi(y)$ with $v_P(x, y, R + d)$ distinct fundamental domains $p\mathcal{S}_y$; then, again, $\psi_{\xi, \frac{R-t-\epsilon_0}{2}}(B(x, R) \cap \partial H_\xi(y))$ can be covered by $v_P(x, y, R + d)$ fundamental domains $\psi_{\xi, \frac{R-t-\epsilon_0}{2}}(p\mathcal{S}_y)$ as well, hence we deduce that $v_P(x, y, R + d) \cdot \mathcal{A}_P(x, \frac{R+t-\epsilon_0}{2}) \geq v^-$.

This implies that $v_P(x, y, R) \prec \mathcal{A}_P^{-1}(x, \frac{R+t}{2})$ for all $R > t + R_0$, for $R_0 = \epsilon_0 + d$ and a constant $C = C(n, a, b, d)$.

To prove the weak equivalence (7), we just write, for $R + \frac{\Delta}{2} > t + R_0$:

$$\begin{aligned} v_P^\Delta(x, y, R) &= v_P^\Delta(x, y, R + \Delta/2) - v_P^\Delta(x, y, R - \Delta/2) \geq \frac{C^{-1}}{\mathcal{A}_P\left(\frac{R+t+\Delta/2}{2}\right)} - \frac{C}{\mathcal{A}_P\left(\frac{R+t-\Delta/2}{2}\right)} \\ &\geq \frac{C^{-1}e^{(n-1)a\frac{\Delta}{4}} - Ce^{-(n-1)a\frac{\Delta}{4}}}{\mathcal{A}_P\left(x, \frac{R+t}{2}\right)} = 2 \sinh\left[\frac{1}{4}(n-1)a\Delta - \ln C\right] \cdot \mathcal{A}_P^{-1}\left(\frac{R+t}{2}\right) \end{aligned}$$

again by (5), if $\Delta > \Delta_0 = \frac{4 \ln C}{(n-1)a}$. Reciprocally, we have for all $R, \Delta > 0$:

$$v_P^\Delta(x, y, R) \leq v_P(x, y, R + \frac{\Delta}{2}) \leq \frac{C}{\mathcal{A}_P\left(x, \frac{R+t+\Delta/2}{2}\right)} \leq \frac{C'(n, a, b, d; \Delta)}{\mathcal{A}_P\left(x, \frac{R+t}{2}\right)} \quad \square$$

Proof of Proposition 2.6.

We just integrate (6) over a fundamental domain \mathcal{C}_x for the action of P on $H_\xi(x)$:

$$\mathcal{F}_P(x, R) = \sum_{p \in P} \text{vol}[B(x, R) \cap p\mathcal{C}_x] = \int_{\mathcal{C}_x} \sum_{p \in P} 1_{B(x, R)}(pz) dz = \int_{\mathcal{C}_x} v_P(x, y, R) dy$$

so, integrating over each slice $\psi_{\xi, t}(\mathcal{S}_x)$ by the coarea formula, we obtain

$$\int_0^{R-R_0} \int_{\psi_{\xi, t}(\mathcal{S}_x)} \mathcal{A}_P^{-1}\left(x, \frac{R+t}{2}\right) dt \stackrel{C}{\prec} \mathcal{F}_P(x, R) \stackrel{C}{\prec} \int_0^R \int_{\psi_{\xi, t}(\mathcal{S}_x)} \mathcal{A}_P^{-1}\left(x, \frac{R+t}{2}\right) dt$$

(the left inequality holding for $R > R_0$). By (5), both sides are weakly equivalent to

the integral $\int_0^R \frac{\mathcal{A}_P(x, t)}{\mathcal{A}_P(x, \frac{R+t}{2})} dt$, up to a multiplicative constant $c = c(n, a, b, d)$. \square

REMARK 2.9 Thus, we see that the curvature bounds imply that $v_P^\Delta(x, R) \asymp v_P(x, R)$ for Δ and R large enough. This also holds in general for *non-elementary* groups Γ with finite Bowen-Margulis measure, as in this case $v_\Gamma^\Delta(x, R) \sim \frac{2\|\mu_x\|^2}{\|\mu_{BM}\|} \sinh[\frac{\Delta}{2}\delta(\Gamma)]$ by Roblin's asymptotics. On the other hand, it is unclear whether the weak equivalence $v_\Gamma^\Delta \asymp v_\Gamma$ holds for non-elementary lattices Γ , when $\|\mu_{BM}\| = \infty$.

In the next section we will also need estimates for the growth of annuli in the spaces of left and right cosets of a lattice Γ of X , modulo a bounded parabolic subgroup P . Notice that, if P fixes $\xi \in X(\infty)$, the function $v_{P \backslash \Gamma}(x, R)$ counts the number of points $\gamma x \in \Gamma x$ falling in the Dirichlet domain $\mathcal{D}(P, x)$ of P with $d(x, \gamma x) < R$; on the other hand, the function $v_{\Gamma/P}(x, R)$ counts the number of horoballs $\gamma H_\xi(x)$ at distance (almost) less than R from x . It is remarkable that, even if these functions count geometrically distinct objects, they are weakly asymptotically equivalent, as the following Proposition will show. Actually, let H_ξ be a horoball centered at the parabolic fixed point ξ of $P < \Gamma$; we call $\text{depth}(H_\xi)$ the minimal distance $\min_{\Gamma \setminus \{e\}} d(H_\xi, \gamma H_\xi)$. Then, for \mathcal{S}_x defined as in Definition 2.3 we have:

Proposition 2.10 *Let Γ be a torsionless, non-elementary, discrete group of isometries of X , let P a bounded parabolic subgroup of Γ , and let $x \in X$ be fixed. Assume that $\max\{\text{diam}(\mathcal{S}_x), 1/\text{depth}(H_\xi(x))\} \leq d$, and let ℓ be the minimal displacement $d(x, \gamma x)$ of the elements $\gamma \in \Gamma$ whose domains of attraction $\mathcal{U}^\pm(\gamma, x) = \{y \mid d(\gamma^{\pm 1}x, y) \leq d(x, y)\}$ are included in the Dirichlet domain $\mathcal{D}(P, x)$.*

There exists a constant $\delta_0 = \delta_0(a, d)$ such that, for all $\Delta, R > 0$:

- (i) $v_{P \backslash \Gamma}^{\Delta - \delta_0}(x, R) \leq v_{\Gamma/P}^\Delta(x, R) \leq v_{P \backslash \Gamma}^{\Delta + \delta_0}(x, R)$;
- (ii) $\frac{1}{2}v_\Gamma^{\Delta - 2\ell}(x, R) \leq v_{P \backslash \Gamma}^\Delta(x, R) \leq v_\Gamma^\Delta(x, R)$;
- (iii) $\frac{1}{2}v_\Gamma^{\Delta - \delta_0 - 2\ell}(x, R) \leq v_{\Gamma/P}^\Delta(x, R) \leq v_\Gamma^{\Delta + \delta_0}(x, R)$;
- (iv) $\frac{1}{4}v_\Gamma^{\Delta - \delta_0 - 4\ell}(x, R) \leq v_{P \backslash \Gamma/P}^\Delta(x, R) \leq v_\Gamma^\Delta(x, R)$.

Notice that (iv) strenghtens a result of S. Hersensky and F. Paulin on the number of rational lines with depth smaller than R (cp. [17] Theorem 1.2, where the authors furthermore assume the condition $\delta_P < \delta_\Gamma$). Actually, let H_ξ be the largest horosphere centered at ξ non intersecting any other γH_ξ for $\gamma \neq e$, and recall that the *depth of a geodesic* $c = (\xi, \gamma\xi)$ is defined as the length of the maximal subsegment $\hat{c} \subset c$ outside ΓH_ξ . The double coset space $P \backslash (\Gamma - P) / P$ can be identified with the set of oriented geodesics $(\xi, \gamma\xi)$ of X with $\gamma \in \Gamma - P$. Then, if $x \in \partial H_\xi$, the counting function $v_{P \backslash (\Gamma - P) / P}(x, R)$ corresponds to the number of geodesics of $\bar{X} = \Gamma \backslash X$ which travel a time R outside the cusp $\bar{\mathcal{C}} = P \backslash H_\xi$, before entering and definitely staying (in the future and in the past) in $\bar{\mathcal{C}}$.

Proof. The right-hand inequalities in (ii), (iii), (iv) are trivial.

Let us prove (i). We first define two sections of the projections $P \backslash \Gamma \leftarrow \Gamma \rightarrow \Gamma / P$. Consider the fundamental domain $\mathcal{S}_x(\infty)$ for the action of P on $X(\infty) - \{\xi\}$ defined in 2.3, and choose for each $\gamma \in \Gamma$, a representative $\hat{\gamma}$ of γP which minimizes the distance to x .

Then, we set

$$\widehat{\Gamma} = \{\widehat{\gamma} \mid \gamma P \in \Gamma/P\}$$

$$\Gamma_0 = \{\gamma_0 \mid \gamma_0 \in \Gamma, \gamma_0 \xi \in \mathcal{S}_x(\infty)\} \cup \{e\}.$$

We have bijections $\widehat{\Gamma} \cong \Gamma/P$ and $\Gamma_0 \cong P \backslash \Gamma$, as $\mathcal{S}_x(\infty)$ is a fundamental domain. Moreover, every $\gamma_0 \in \Gamma_0$ *almost* minimizes the distance to x in its right coset $P\gamma_0$. Actually, for all $\gamma \in \Gamma$ set $z(\gamma) = (\xi, \gamma\xi) \cap \partial H_\xi(x)$ and $z'(\gamma) = (\xi, \gamma\xi) \cap \gamma \partial H_\xi(x)$; then, for all $p \in P$ we have, by Lemma 2.1

$$d(x, p\gamma_0 x) \geq d(x, pz(\gamma)) + d(pz(\gamma), pz'(\gamma)) + d(pz'(\gamma), p\gamma_0 x) - \epsilon_1(a, d) \geq d(x, \gamma_0 x) - c \quad (10)$$

as $d(H_\xi(x), p\gamma_0 H_\xi(x)) = d(pz(\gamma), pz'(\gamma))$, for $c = 2d + \epsilon_1(a, d)$.

We will now define a bijection between pointed metric spaces $i : (P \backslash \Gamma, x_0) \rightarrow (\Gamma/P, x_0)$ which almost-preserves the distance to their base point $x_0 = P$ (with respect to their quotient distances $|\cdot|_x = d_x(P, \cdot)$ as seen at the beginning of the section), as follows. For every $\gamma \in \Gamma$ we can write $\gamma = \widehat{\gamma}p_\gamma$, for uniquely determined $\widehat{\gamma} \in \widehat{\Gamma}$ and $p_\gamma \in P$; given a right coset $P\gamma$, we take $\gamma_0 \in \Gamma_0$ representing $P\gamma$ and then set $i(P\gamma) := p_{\gamma_0}\widehat{\gamma}_0 P$. The map i is surjective. Actually, given γP , we take $p \in P$ such that $p\gamma\xi \in \mathcal{S}_x(\infty)$, so that $P\gamma = P\gamma_0$, for $\gamma_0 = p\gamma \in \Gamma_0$; then, we write $\gamma_0 = \widehat{\gamma}_0 p_{\gamma_0}$, and we deduce that $i(P\gamma) = i(P\gamma_0) = i(P\widehat{\gamma}_0 p_{\gamma_0}^{-1}) = p_{\gamma_0}^{-1}\widehat{\gamma}_0 P = p_{\gamma_0}^{-1}\gamma_0 p_{\gamma_0} P = \gamma P$.

We now check that i is injective. Given $\gamma_0 = \widehat{\gamma}_0 p_{\gamma_0}$ and $\gamma'_0 = \widehat{\gamma}'_0 p_{\gamma'_0}$ in Γ_0 representing two right cosets $P\gamma$ and $P\gamma'$, assume that $p_{\gamma_0}\widehat{\gamma}_0 P = p_{\gamma'_0}\widehat{\gamma}'_0 P$. Then, $\widehat{\gamma}_0 \xi = p_{\gamma'_0} \widehat{\gamma}'_0 \xi$ for $p = p_{\gamma_0}^{-1} p_{\gamma'_0} \in P$, which yields $p_{\gamma_0} = p_{\gamma'_0}$ as $\widehat{\gamma}_0 \xi, \widehat{\gamma}'_0 \xi \in \mathcal{S}_x(\infty)$ and $\mathcal{S}_x(\infty)$ is a fundamental domain for the left action of P ; so, $\widehat{\gamma}_0 P = \widehat{\gamma}'_0 P$, which implies that $\widehat{\gamma}_0 = \widehat{\gamma}'_0$ too (as $\widehat{\Gamma}$ is a section of Γ/P). Therefore, $P\gamma = P\gamma_0 = P\widehat{\gamma}_0 p_{\gamma_0} = P\widehat{\gamma}'_0 p_{\gamma'_0} = P\gamma'_0 = P\gamma'$. To show that i almost preserves $|\cdot|_x$, we notice that, given a class $P\gamma$ and writing its representative in Γ_0 as $\gamma_0 = \widehat{\gamma}_0 p_{\gamma_0}$, we have

$$|P\gamma|_x \leq |\gamma_0|_x \leq d(x, \widehat{\gamma}_0 x) + d(\widehat{\gamma}_0 x, \widehat{\gamma}_0 p_{\gamma_0} x) = |\widehat{\gamma}_0|_x + |p_{\gamma_0}|_x$$

while, by (10) and by Lemma 2.1

$$|P\gamma|_x \geq |\gamma_0|_x - c \geq d(x, z'(\gamma_0)) + d(z'(\gamma_0), \widehat{\gamma}_0 p_{\gamma_0} x) - \epsilon_1(a, d) - c \geq |\widehat{\gamma}_0|_x + |p_{\gamma_0}|_x - 2c$$

as $d(z'(\gamma_0), \widehat{\gamma}_0 x) < d$. On the other hand

$$|i(P\gamma)|_x = |p_{\gamma_0}\widehat{\gamma}_0 P|_x \leq d(x, p_{\gamma_0} x) + d(p_{\gamma_0} x, p_{\gamma_0}\widehat{\gamma}_0 P x) = |p_{\gamma_0}|_x + |\widehat{\gamma}_0|_x$$

while, as $z(p_{\gamma_0}\widehat{\gamma}_0) = p_{\gamma_0} z(\widehat{\gamma}_0)$ and $z'(p_{\gamma_0}\widehat{\gamma}_0) = p_{\gamma_0} z'(\widehat{\gamma}_0)$, we get by Lemma 2.1

$$|i(P\gamma)|_x \geq d(x, p_{\gamma_0} z(\widehat{\gamma}_0)) + d(p_{\gamma_0} z(\widehat{\gamma}_0), p_{\gamma_0}\widehat{\gamma}_0 P x) - \epsilon_1(a, d) \geq |p_{\gamma_0}|_x + |\widehat{\gamma}_0|_x - c.$$

This shows that $|P\gamma|_x - c \leq |i(P\gamma)|_x \leq |P\gamma|_x + 2c$. We then immediately deduce that $v_{P \backslash \Gamma}(x, R - 2c) \leq v_{\Gamma/P}(x, R) \leq v_{P \backslash \Gamma}(x, R + c)$, as well as (i) for $\delta_0 = 4c$.

The proof of the left-hand inequality in (ii) is a variation for annuli of a trick due to Roblin, cp. [26]. Actually, as $LP \subsetneq L\Gamma$, we can choose a $\bar{\gamma} \in \Gamma$ with $d(x, \bar{\gamma}x) = \ell$ and such that the domains of attraction $\mathcal{U}^\pm(\bar{\gamma}, x)$ are included in the domain $\mathcal{D}(P, x)$. Let $v_{\mathcal{D}(P, x)}(x, R)$ be the number of points of the orbit Γx falling in $\mathcal{D}(P, x) \cap B(x, R)$.

We have:

$$v_{\widehat{\Gamma}}^{\Delta}(x, R) \leq v_{\mathcal{D}(P, x)}^{\Delta}(x, R) + v_{\mathcal{D}(P, x)}^{\Delta+2\ell}(x, R) \leq 2v_{\mathcal{D}(P, x)}^{\Delta+2\ell}(x, R)$$

since, for $\gamma x \in A^{\Delta}(x, R)$, either $\gamma x \in \mathcal{D}(P, x)$, or $\bar{\gamma}\gamma x \in \mathcal{D}(P, x)$ and $\bar{\gamma}\gamma x \in A^{\Delta+2\ell}(x, R)$. As the points of P falling in $\mathcal{D}(P, x)$ minimize the distance to x modulo the left action of P , we also have $v_{\mathcal{D}(P, x)}^{\Delta+2\ell}(x, R) = v_{P \setminus \Gamma}^{\Delta+2\ell}(x, R)$, which proves (ii).

Assertion (iii) follows directly from (i) and (ii). To show (iv), we need to estimate the number of classes γP modulo the left action of P , that is the elements of $\widehat{\Gamma}$ such that $\widehat{\gamma}x$ belongs to the fundamental domain $\mathcal{D}(P, x)$. We choose an element $\bar{\gamma} \in \Gamma$ with $\mathcal{U}^{\pm}(\bar{\gamma}, x) \subset \mathcal{D}(P, x)$ as before, and apply again Roblin's trick to the classes γP . The set $\widehat{\Gamma}x$ can be parted in two disjoint subsets: the subset $\widehat{\Gamma}_1 := \widehat{\Gamma} \cap \mathcal{D}(P, x)$, and the subset $\widehat{\Gamma}_2 := \widehat{\Gamma} \cap \mathcal{D}(P, x)^c$, whose elements $\widehat{\gamma}$ then satisfy $\bar{\gamma}\widehat{\gamma} \in \mathcal{D}(P, x)$ and $|\bar{\gamma}\widehat{\gamma}|_x \leq |\widehat{\gamma}|_x + \ell$. Then $v_{\widehat{\Gamma}/P}^{\Delta}(x, R) = v_{\widehat{\Gamma}_1}^{\Delta}(x, R) + v_{\widehat{\Gamma}_2}^{\Delta}(x, R) \leq 2v_{P \setminus \Gamma/P}^{\Delta+2\ell}(x, R)$ and we conclude by (iii). \square

3 Orbit-counting estimates for lattices

In this section we give estimates of the orbital function $v_{\Gamma}(x, y, R)$ and of $v_X(R)$ in terms of the orbital function of the parabolic subgroups P_i and the associated cuspidal functions \mathcal{F}_{P_i} of Γ . These estimates will be used in §4 and §6; they stem from an accurate dissection of large balls in compact and horospherical parts, assuming that ambient space X admits a nonuniform lattice action.

Let Γ be a lattice of X . The quotient manifold $\bar{X} = \Gamma \backslash X$ is geometrically finite, and we have the following classical results due to B. Bowditch [7] concerning the structure of the limit set Γ and of \bar{X} :

(a) $L(\Gamma) = X(\infty)$ and it is the disjoint union of the radial limit set $L_{rad}(\Gamma)$ with finitely many orbits $L_{bp}\Gamma = \Gamma\xi_1 \cup \dots \cup \Gamma\xi_m$ of *bounded* parabolic fixed points; this means that each $\xi_i \in L_{bp}G$ is the fixed point of some maximal bounded parabolic subgroup P_i of Γ ;

(b) (*Margulis' lemma*) there exist closed horoballs $H_{\xi_1}, \dots, H_{\xi_m}$ centered respectively at ξ_1, \dots, ξ_m , such that $gH_{\xi_i} \cap H_{\xi_j} = \emptyset$ for all $1 \leq i, j \leq m$ and all $\gamma \in \Gamma - P_i$;

(c) \bar{X} can be decomposed into a disjoint union of a compact set $\bar{\mathcal{K}}$ and finitely many "cusps" $\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_m$: each $\bar{\mathcal{C}}_i$ is isometric to the quotient of H_{ξ_i} by the maximal bounded parabolic group $P_i \subset \Gamma$. We refer to $\bar{\mathcal{K}}$ and to $\bar{\mathcal{C}} = \cup_i \bar{\mathcal{C}}_i$ as to the *compact core* and the *cuspidal part* of \bar{X} .

Throughout this section, we fix $x \in X$ and we consider a Dirichlet domain $\mathcal{D}(\Gamma, x)$ centered at x ; this is a convex fundamental subset, and we may assume that \mathcal{D} contains the geodesic rays $[x, \xi_i[$. Accordingly, setting $\mathcal{S}_i = \mathcal{D} \cap \partial H_{\xi_i}$ and $\mathcal{C}_i = \mathcal{D} \cap H_{\xi_i} \simeq \mathcal{S}_i \times \mathbb{R}_+$, the fundamental domain \mathcal{D} can be decomposed into a disjoint union

$$\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$$

where \mathcal{K} is a convex, relatively compact set containing x in its interior (projecting to a subset $\bar{\mathcal{K}}$ in \bar{X}), while \mathcal{C}_i and \mathcal{S}_i are, respectively, connected fundamental domains for the action of P_i on H_{ξ_i} and ∂H_{ξ_i} (projecting respectively to subsets $\bar{\mathcal{C}}_i, \bar{\mathcal{S}}_i$ of \bar{X}).

Finally, as $LP_i = \{\xi_i\}$, for every $1 \leq i \leq m$ we can find an element $\gamma_i \in \Gamma$, with $\ell_i = d(x, \gamma_i x)$, which is *in Schottky position with P_i relatively to x* , i.e. such that the domains of attraction $\mathcal{U}^\pm(\gamma_i) = \{y \mid d(\gamma_i^{\pm 1} x, y) \leq d(x, y)\}$ are included in the Dirichlet domain $\mathcal{D}(P_i, x)$, as in Proposition 2.10.

For the following, we will then set $d = \max\{\text{diam}(\mathcal{K}), \text{diam}(\mathcal{S}_i), 1/\text{depth}(H_{\xi_i}), \ell_i\} \geq \epsilon_0$.

Proposition 3.1 (Counting Formula)

Assume that $x, y \in X$ project respectively to the compact core $\bar{\mathcal{K}}$ and to a cuspidal end $\bar{\mathcal{C}}_i$ of $\bar{X} = \Gamma \backslash X$. There exists $C'' = C(n, a, b, d)$ such that:

$$[v_\Gamma(x, \cdot) * v_{P_i}(x, y, \cdot)](R - D_0) \stackrel{C''}{\prec} v_\Gamma(x, y, R) \stackrel{C''}{\prec} [v_\Gamma(x, \cdot) * v_{P_i}(x, y, \cdot)](R + D_0) \quad \forall R \geq 0$$

for a constant D_0 only depending on n, a, b, d .

Proof. We will write, as usual, $|\gamma|_x = d(x, \gamma x)$ and $|\gamma P|_x = d(x, \gamma P x)$, and choose a constant $\Delta > \max\{R_0, \Delta_0, 2\delta_0 + 4d\}$, where R_0, Δ_0, δ_0 are the constants of Propositions 2.5 and 2.10. We first show that

$$B(x, R) \cap \Gamma y \subset \bigcup_{k=1}^N \bigcup_{\bar{\gamma} \in \Gamma, |\bar{\gamma}| \leq k\Delta} B(\bar{\gamma} x, (N - k)\Delta) \cap (\bar{\gamma} P_i) y \quad (11)$$

for $N = \lfloor \frac{R}{\Delta} \rfloor + 2$. Actually, let $\gamma y \in B(x, R) \cap \gamma H_{\xi_i}$ and set $\bar{y}_i = [x, \gamma \xi] \cap \gamma \partial H_{\xi_i}$. By using the action of the group $\gamma P_i \gamma^{-1}$ on γH_{ξ_i} , we can find $\bar{\gamma} = \gamma p$, with $p \in P_i$, such that $\bar{y}_i \in \bar{\gamma} \mathcal{C}_i$. Since the angle $\angle_{\bar{y}_i}(x, \gamma y)$ at \bar{y}_i is greater than $\frac{\pi}{2}$, we have:

$$d(x, \gamma y) \leq d(x, \bar{y}_i) + d(\bar{y}_i, \gamma y) \leq d(x, \gamma y) + \epsilon_0 < R + \epsilon_0$$

with $|\bar{\gamma}| \leq d(x, \bar{y}_i) + d < R + d + \epsilon_0 \leq N\Delta$. Then, if $k\Delta \leq |\bar{\gamma}| < (k + 1)\Delta$, we deduce

$$d(\bar{\gamma} x, \gamma y) \leq d(\bar{y}_i, \gamma y) + d \leq R + \epsilon_0 - d(x, \bar{\gamma} x) + 2d < (N - k)\Delta$$

which shows that $\gamma y = \bar{\gamma} p^{-1} y \in B(\bar{\gamma} x, (N - k)\Delta) \cap (\bar{\gamma} P_i) y = \bar{\gamma} [B(x, (N - k)\Delta) \cap P_i y]$. Thus, we obtain:

$$v_\Gamma(x, y, R) \leq \sum_{k=1}^N v_\Gamma(x, k\Delta) \cdot v_{P_i}(x, y, (N - k)\Delta) \prec v_\Gamma * v_{P_i}(R + 2\Delta)$$

This proves the right hand side of our inequality. The left hand is more delicate, as we need to dissect the ball $B(x, R)$ in disjoint annuli. So, consider the set $\hat{\Gamma}_i$ of minimal representatives of Γ/P_i as in the proof of Proposition 2.10. We have:

$$A^{4\Delta}(x, R) \cap \Gamma y \supset \bigsqcup_{k=0}^N \bigsqcup_{\substack{\hat{\gamma} \in \hat{\Gamma}_i \\ k\Delta - \frac{\Delta}{2} \leq |\hat{\gamma}| < k\Delta + \frac{\Delta}{2}}} A^\Delta(\hat{\gamma} x, (N - k)\Delta) \cap (\hat{\gamma} P_i) y \quad (12)$$

for $N = \lfloor \frac{R}{\Delta} \rfloor + 1$. In fact, given $\gamma y = \hat{\gamma} p_i y \in A^\Delta(\hat{\gamma} x, (N - k)\Delta)$ with $\hat{\gamma} x \in A^\Delta(x, k\Delta)$ we have again

$$N\Delta - 2\Delta \leq |\hat{\gamma}| + d(\hat{\gamma} x, \gamma y) - 2d - \epsilon_0 \leq d(x, \gamma y) \leq |\hat{\gamma}| + d(\hat{\gamma} x, \gamma y) < N\Delta + \Delta$$

as $\Delta > 2d + \epsilon_0$, hence $\gamma y \in A^{4\Delta}(x, R)$. Notice that (12) is a disjoint union, as the annuli with the same center do not intersect by definition, while for $\hat{\gamma} \neq \hat{\gamma}'$ the orbits $\hat{\gamma}P_i y$ and $\hat{\gamma}'P_i y$ lie on different horospheres $\hat{\gamma}H_i \neq \hat{\gamma}'H_i$, which are disjoint by Margulis' Lemma. From (12) and by Proposition 2.10 we deduce that for all $R > 0$ it holds:

$$v_\Gamma^{4\Delta}(x, y, R) \geq \frac{1}{2} \sum_{k=0}^N v_\Gamma^{\Delta/2}(x, k\Delta) \cdot v_{P_i}^\Delta(x, y, (N-k)\Delta) \quad (13)$$

as $\Delta > 2\ell_i$. Now, we set $h_i = b_{\xi_i}(x, y)$ and we sum (13) over annuli of radii $R_n = n\Delta$, and we get:

$$\begin{aligned} v_\Gamma(x, y, R) &\geq \frac{1}{4} \sum_{n=0}^{\lfloor \frac{R}{\Delta} \rfloor - 2} v_\Gamma^{4\Delta}(x, y, n\Delta) \succ \sum_{k=0}^{\lfloor \frac{R}{\Delta} \rfloor - 1} \left[\sum_{n \geq k}^{\lfloor \frac{R}{\Delta} \rfloor - 1} v_\Gamma^{\Delta/2}(x, (n-k)\Delta) \right] \cdot v_{P_i}^\Delta(x, y, k\Delta) \geq \\ &\geq \sum_{k \geq \frac{h_i}{\Delta} + 1}^{\lfloor \frac{R}{\Delta} \rfloor - 1} v_\Gamma(x, R - (k+2)\Delta) \cdot v_{P_i}^\Delta(x, y, k\Delta) \succ \sum_{k = \frac{h_i}{\Delta} + 1}^{\lfloor \frac{R}{\Delta} \rfloor - 1} \frac{v_\Gamma(x, R - (k+2)\Delta)}{\mathcal{A}_{P_i}(x, \frac{k\Delta + h_i}{2})} \end{aligned} \quad (14)$$

as $v_{P_i}^\Delta(x, y, k\Delta) \succ \mathcal{A}_{P_i}^{-1}\left(x, \frac{k\Delta + h_i}{2}\right)$ if $k\Delta \geq h_i + \Delta > h_i + R_0$ by Proposition 2.5.

Using again Proposition 2.5 and (5), it is easily verified that the expression in (14) is greater than the continuous convolution $v_\Gamma(x, \cdot) * v_{P_i}(x, y, \cdot)(R + 4\Delta)$, up to a multiplicative constant $CC'\Delta$. This ends the proof, by taking $D_0 = 4\Delta$. \square

The Counting Formula enables us to reduce the estimate of the growth function v_X to a group-theoretical calculus, that is to the estimate of a the convolution of v_Γ with the cuspidal functions \mathcal{F}_{P_i} of maximal parabolic subgroups P_i of Γ :

Proposition 3.2 (Volume Formula)

There exists a constant $C''' = C'''(n, a, b, d, \text{vol}(\mathcal{K}))$, such that:

$$\left[v_\Gamma(x, \cdot) * \sum_i \mathcal{F}_{P_i}(x, \cdot) \right] (R - 2D_0) \stackrel{C'''}{\prec} v_X(x, R) \stackrel{C'''}{\prec} \left[v_\Gamma(x, \cdot) * \sum_i \mathcal{F}_{P_i}(x, \cdot) \right] (R + 2D_0) \quad \forall R \geq 0 \quad (15)$$

for $D_0 = D_0(n, a, b, d)$ as in Proposition 3.1.

Proof. Let $h_i = d(x, H_{\xi_i})$; we may assume that the constants R_0, D_0 of Propositions 2.5 and 3.1 satisfy $D_0 \gg d \geq \text{diam}(\mathcal{K}) \geq h_i \gg R_0$. Now call $\mathcal{S}_i(h) = \psi_{\xi_i, h}[\mathcal{S}_i]$; integrating $v_\Gamma(x, y, R)$ over the fundamental domain \mathcal{D} yields, by Proposition 3.1:

$$\begin{aligned} v_X(x, R + 2D_0) &= \int_{\mathcal{D}} v_\Gamma(x, y, R + 2D_0) dy = \int_{\mathcal{K}} v_\Gamma(x, y, R + 2D_0) dy + \sum_{i=1}^m \int_{\mathcal{C}_i} v_\Gamma(x, y, R + 2D_0) dy \\ &\stackrel{C''}{\succ} \sum_{i=1}^m \int_{2h_i}^{R+D_0} v_\Gamma(x, R + 2D_0 - t) \left[\int_{h_i}^{t-h_i} \int_{\mathcal{S}_i(h)} v_{P_i}(x, y, t) dy dh \right] dt \end{aligned}$$

which then gives by Propositions 2.5 and 2.6, as $h = b_{\xi_i}(x, y) \leq t - h_i < t - R_0$,

$$\begin{aligned} v_X(x, R + 2D_0) &\stackrel{C''}{\succ} \int_{2h_i}^{R+D_0} v_\Gamma(x, R + 2D_0 - t) \left[\sum_{i=1}^m \int_{h_i}^{t-h_i} \frac{\mathcal{A}_{P_i}(x, h)}{\mathcal{A}_{P_i}(x, \frac{t+h}{2})} dh \right] dt \\ &\geq \int_0^{R+D_0-2h_i} v_\Gamma(x, R - t) \left[\sum_{i=1}^m \int_0^t \frac{\mathcal{A}_{P_i}(x, s + h_i)}{\mathcal{A}_{P_i}(x, \frac{t+s+3h_i}{2})} ds \right] dt \succ \int_0^R v_\Gamma(x, R - t) \sum_{i=1}^m \mathcal{F}_{P_i}(x, t) dt. \end{aligned}$$

Reciprocally, we have $v_\Gamma(x, R - D_0) \leq v_\Gamma(x, y, R) \leq v_\Gamma(x, R + D_0)$ so again by Proposition 3.1 and Remarks 2.7 we obtain

$$\begin{aligned} v_X(x, R - 2D_0) &\stackrel{C''}{\prec} \text{vol}(\mathcal{K}) \cdot v_\Gamma(x, R - D_0) + \sum_{i=1}^m \int_{\mathcal{C}_i} \left[\int_0^{R-2D_0} v_\Gamma(x, t) v_{P_i}(x, y, R - t) dt \right] dy \\ &\stackrel{C'''}{\prec} v_\Gamma(x, R - D_0) + \int_0^{R-2D_0} v_\Gamma(x, t) \left[\sum_{i=1}^m \int_0^{R-t} \frac{\mathcal{A}_{P_i}(x, h)}{\mathcal{A}_{P_i}\left(x, \frac{R-t+h}{2}\right)} dh \right] dt \end{aligned}$$

as $v_{P_i}(x, y, R - t) = 0$ for $R - t < b_{\xi_i}(x, y) = h$. This proves the converse inequality, since $v_\Gamma(x, R - D_0) \prec v_\Gamma(x, R - D_0) \mathcal{F}_{P_i}(R_0) \leq \frac{1}{D_0 - R_0} \int_{R-D_0}^{R-R_0} v_\Gamma(x, t) \mathcal{F}_{P_i}(x, R - t) dt$. \square

As a consequence of the Volume Formula and of Corollary 2.8, we deduce²:

Corollary 3.3 *If \mathcal{F}_{P_i} are the cuspidal functions of the parabolic subgroups of Γ :*

- (i) $\omega^+(X) = \max\{\delta(\Gamma), \omega^+(\mathcal{F}_{P_1}), \dots, \omega^+(\mathcal{F}_{P_m})\}$.
- (ii) $\omega^+(X) = \omega^-(X) = \delta(\Gamma)$ if Γ is $\frac{1}{2}$ -parabolically pinched.

4 Margulis function for regular lattices

In this section we assume that Γ is a lattice which is neither sparse nor exotic. We need to recall a general criterion for the divergence of the Poincaré series of Γ , which can be found in [10], [13]:

DIVERGENCE CRITERION. *Let Γ be a geometrically finite group: if $\delta^+(P) < \delta(\Gamma)$ for every parabolic subgroup P of Γ , then Γ is divergent.*

Proof of Theorem 1.3.

Let Γ be a nonuniform lattice of X which is neither sparse nor exotic. As Γ is not exotic, it satisfies the gap property $\delta(P) < \delta(\Gamma)$ for all parabolic subgroups; by the Divergence and Finiteness Criterion we deduce that the group is divergent and that $\mu_{BM}(U\bar{X}) < \infty$. Therefore $v_\Gamma(x, R) \stackrel{c_\Gamma(x)}{\asymp} e^{\delta(\Gamma)R}$ is purely exponential (for some $c_\Gamma(x)$ depending on Γ, x). We will now show that X has a Margulis function.

Let \mathcal{D} be the fundamental domain for Γ and P_i the maximal parabolic subgroups fixing ξ_i as at the beginning of §3: we call $w(x, y, R) = v_X(x, R)e^{-\delta(\Gamma)R}$, so that have

$$\frac{v_X(x, R)}{e^{\delta(\Gamma)R}} = \int_{\mathcal{D}} \frac{v_\Gamma(x, y, R)}{e^{\delta(\Gamma)R}} dy = \int_{\mathcal{K}} w(x, y, R) dy + \sum_{i=1}^m \int_{\mathcal{C}_i} w(x, y, R) dy \quad (16)$$

We know that $v_\Gamma(x, y, R) \leq v_\Gamma(x, R + d) \leq c_\Gamma(x)e^{\delta(\Gamma)R}$ for $y \in \mathcal{K}$, so we can pass to the limit for $R \rightarrow \infty$ under the integral sign in the first term. For the integrals over the cusps, we have:

$$w(x, y, R) \stackrel{C''}{\prec} \frac{[v_\Gamma(x, \cdot) * v_{P_i}(x, y, \cdot)](R + D_0)}{e^{\delta(\Gamma)R}} \stackrel{c_\Gamma(x)}{\prec} \int_{b_{\xi_i}(x, y)}^\infty \frac{e^{-\delta(\Gamma)t}}{\mathcal{A}_{P_i}\left(x, \frac{b_{\xi_i}(x, y) + t}{2}\right)} dt = w(x, y)$$

²Part (i) of this corollary already appears in [12], where an *upper* estimate for v_X is proved. Notice that in [12] we erroneously stated that also $\omega^-(X) = \max\{\delta(\Gamma), \omega^-(\mathcal{F}_{P_1}), \dots, \omega^-(\mathcal{F}_{P_m})\}$; an explicit counterexample to this is given in Example 6.2.

Notice that the dominating function $w(x, y)$ is finite as $\delta^+(P_i) < \delta(\Gamma)$.

We will now show that $w(x, y) \in L^1(\mathcal{C}_i)$. With the same notations $h_i = d(x, H_{\xi_i})$ and $\mathcal{S}_i(h) = \psi_{\xi_i, h}(\mathcal{S}_i)$ as before, we have for all i :

$$\begin{aligned} \int_{\mathcal{C}_i} w(x, y) dy &= \int_{h_i}^{\infty} \int_{y \in \mathcal{S}_i(h)} \left[\int_{b_{\xi_i}(x, y)}^{\infty} \frac{e^{-\delta(\Gamma)t}}{\mathcal{A}_{P_i}\left(x, \frac{b_{\xi_i}(x, y)+t}{2}\right)} dt \right] dy dh = \int_{h_i}^{\infty} \int_h^{\infty} \frac{e^{-\delta(\Gamma)t} \mathcal{A}_{P_i}(h)}{\mathcal{A}_{P_i}\left(x, \frac{h+t}{2}\right)} dt dh \\ &= \int_{h_i}^{\infty} e^{-\delta(\Gamma)t} \left[\int_{h_i}^t \frac{\mathcal{A}_{P_i}(h)}{\mathcal{A}_{P_i}\left(x, \frac{h+t}{2}\right)} dh \right] dt \stackrel{C}{\asymp} \int_{h_i}^{\infty} e^{-\delta(\Gamma)t} \mathcal{F}_{P_i}(t) dt \end{aligned} \quad (17)$$

which converges, as Γ is not sparse and so $\omega^+(\mathcal{F}_{P_i}) \leq \delta^+(P_i) < \delta(\Gamma)$, by Corollary 2.8. We therefore obtain from (16), by dominated convergence, using Roblin's asymptotics

$$\lim_{R \rightarrow +\infty} \frac{v_X(x, R)}{e^{\delta(\Gamma)R}} = \frac{\|\mu_x\|}{\|\mu_{BM}\|} \int_{\mathcal{D}} \|\mu_y\| dy =: m(x) < +\infty.$$

Notice that $m(x)$ defines an L^1 -function on $\bar{X} = \Gamma \backslash X$, as its integral over \mathcal{D} is finite. \square

Proof of Theorem 1.4(i).

We assume now that X has a nonuniform lattice Γ which is exotic, with the dominant parabolic subgroups P_i , for $i = 1, \dots, d$, satisfying $\delta := \delta(\Gamma) = \delta^+(P_i) \leq 2(\delta^-(P_i) - \epsilon)$, for some $\epsilon > 0$. When $\mu_{BM}(U\bar{X}) < \infty$, the same lines of the above proof apply: $v_{\Gamma}(x, R) \asymp c_{\Gamma}(x)e^{\delta R}$ is purely exponential, and for the same functions $w(x, y, R)$, $w(x, y)$ we again obtain (17); but we need some more work to deduce that, for the dominant cusps P_i , the integral of $e^{-\delta t} \mathcal{F}_{P_i}(t) dt$ converges. So, for every dominant subgroup P_i , we write $v_{P_i}(x, t) = o_i(t)e^{\delta t}$, for some subexponential functions $o_i(t)$; so, $\mathcal{A}_{P_i}(x, t) \asymp e^{-2\delta t}/o_i(2t)$ for $t \geq R_0$. As Γ is exotic, the dominant parabolic subgroups P_i are convergent: actually, for any divergent subgroup $\Gamma_0 < \Gamma$ with limit set $L\Gamma_0 \subsetneq L\Gamma$ one has $\delta(\Gamma_0) < \delta(\Gamma)$ (see [11]). Therefore, the Poincaré series of P_i gives, for $\Delta > \Delta_0 \gg 0$

$$\infty > \sum_{p \in P_i} e^{-\delta d(x, px)} \asymp \sum_{k \geq 1} \frac{v_{P_i}^{\Delta}(x, k\Delta)}{e^{\delta k}} \asymp \int_{\Delta}^{\infty} o_i(t) dt$$

by Proposition 2.5, so the functions $o_i(t)$ are integrable. This shows that

$$w(x, y) = \int_{b_{\xi_i}(x, y)}^{\infty} \frac{e^{-\delta t}}{\mathcal{A}_{P_i}\left(x, \frac{b_{\xi_i}(x, y)+t}{2}\right)} dt = e^{\delta b_{\xi_i}(x, y)} \int_{b_{\xi_i}(x, y)}^{\infty} o_i(h+t) dt < \infty$$

Moreover, as every dominant P_i is strictly $\frac{1}{2}$ -pinched, we have $v_{P_i}(x, t) \asymp e^{\frac{1}{2}(\delta+\epsilon)t}$ for some $\epsilon > 0$, that is $\mathcal{A}_{P_i}(x, t) \prec e^{-(\delta+\epsilon)t}$ for all $t > 0$. Then Proposition 2.6 yields

$$\mathcal{F}_{P_i}(R) \asymp \int_0^R \frac{\mathcal{A}_{P_i}(s)}{\mathcal{A}_{P_i}\left(\frac{s+R}{2}\right)} ds \prec e^{\delta R} \int_0^R e^{-\epsilon s} o_i(s+R) ds \quad \text{for } R \gg 0 \quad (18)$$

hence (17) gives in this case:

$$\int_{\mathcal{C}_i} w(x, y) dy \stackrel{C}{\asymp} \int_{h_i}^{\infty} e^{-\delta(\Gamma)t} \mathcal{F}_{P_i}(t) dt \asymp \int_{h_i}^{\infty} \left[\int_0^t e^{-\epsilon s} o_i(s+t) ds \right] dt \leq \int_0^{\infty} e^{-\epsilon s} \left[\int_s^{\infty} o_i(s+t) dt \right] ds$$

which converges, since o_i is integrable. We can therefore pass to the limit for $R \rightarrow \infty$ under the integral in (16), obtaining the asymptotics for $v_X(x, R)$ as before.

On the other hand, if $\mu_{BM}(U\bar{X}) = \infty$, then $v_\Gamma(x, R) = o_\Gamma(R)e^{\delta R}$ is lower-exponential, and by (18) we have $\mathcal{F}_{P_i}(x, R) = f_i(R)e^{\delta R}$ with $f_i(R) = \int_0^R e^{-\epsilon s} o_i(s+R) ds$ for the dominant cusps, and $f_i(R) \prec e^{-\epsilon R}$, with $\epsilon > 0$, for the others; in both cases, $f_i \in L^1$, since the functions $o_i(t)$ are subexponential. Proposition 3.2 then gives, for any arbitrarily small $\varepsilon' > 0$

$$\begin{aligned} \frac{v_X(x, R)}{e^{\delta R}} &\prec \frac{1}{e^{\delta R}} \int_0^R v_\Gamma(x, t) \sum_i \mathcal{F}_{P_i}(R-t) dt \prec \int_0^R o_\Gamma(t) \sum_i f_i(R-t) dt \\ &\leq \sum_i \|f_i\|_1 \cdot \sup_{t > \frac{R}{2}} o_\Gamma(t) + \|o_\Gamma\|_\infty \cdot \sum_i \int_{R/2}^R f_i(t) dt \leq \varepsilon' \cdot \left(\sum_i \|f_i\|_1 + \|o_\Gamma\|_\infty \right) \end{aligned}$$

provided that $R \gg 0$, since $o_\Gamma(t)$ is infinitesimal and the f_i are integrable. This shows that $v_X(x, R)$ is lower-exponential too. \square

REMARK 4.1 We have seen that, if $\mu_{BM}(U\bar{X}) = \infty$, then $v_\Gamma(x, R) = o_\Gamma(R)e^{\delta R}$ and $v_X(x, R) = o_X(R)e^{\delta R}$, with o_Γ, o_X infinitesimal, and $\mathcal{F}_{P_i}(x, R) = f_i(R)e^{\delta R}$ with $f_i \in L^1$; so,

$$\|o_\Gamma\|_1 \prec \|o_X\|_1 \leq \int_0^\infty \frac{v_X(x, R)}{e^{\delta R}} dR \prec \int_0^\infty \int_0^R o_\Gamma(t) \sum_i f_i(R-t) dt dR \leq \|o_\Gamma\|_1 \cdot \sum_i \|f_i\|_1$$

and we can say that o_Γ is L^1 if and only if o_X is.

5 Entropy rigidity and $\frac{1}{4}$ -pinched manifolds

This section is devoted to the proof of the rigidity Theorem 1.1 and Theorem 1.2. We prove it for $a = 1$, as the general case follows from this by applying an homothety. The proof is through the method of barycenter, initiated by Besson-Courtois-Gallot [1], [2], and follows the lines of [8] (Theorem 1.6, holding for compact quotients). The main difficulty in the finite volume case is to show that the map produced by the barycenter method is proper: we recall the main steps of the construction, referring the reader to [8] for estimates which are now well established, and we focus on the new estimates necessary to prove properness.

Let $\bar{X} = \Gamma \backslash X$, fix a point $x_0 \in X$ and call for short $b_\xi(x) = b_\xi(x, x_0)$. The function b_ξ is strictly convex if $K_X \leq -1 < 0$, since for every point y we have

$$\text{Hess}_y b_\xi \geq g_y - (db_\xi)_y \otimes (db_\xi)_y \quad (19)$$

where g denotes the metric tensor of X ; moreover, it is folklore that if the equality holds in (19) at every point y and for every direction ξ , then the sectional curvature is constant, and X is isometric to the hyperbolic space \mathbb{H}^n . The idea of the proof is to show that the condition $\delta(\Gamma) = n - 1$ forces the equality in (19).

Recall that, for every measure μ on $X(\infty)$ whose support is not reduced to one point, we can consider its *barycenter*, denoted $\text{bar}[\mu]$, that is the unique point of minimum of the function $y \mapsto \mathcal{B}_\mu(y) = \int_{X(\infty)} e^{b_\xi(y)} d\mu(\xi)$ (which is C^2 and strictly convex function, as $b_\xi(y)$ is). If $\text{supp}(\mu)$ is not a single point, it is easy to see that $\lim_{y \rightarrow X(\infty)} \mathcal{B}_\mu(y) = +\infty$, cp. [8]. Consider now the map $F : X \rightarrow X$ defined by

$$F(x) = \text{bar} \left[e^{-b_\xi(x)} \mu_x \right] = \text{argmin} \left[y \mapsto \int_{X(\infty)} e^{b_\xi(y,x)} d\mu_x(\xi) \right]$$

where (μ_x) is the family of Patterson-Sullivan measures associated with the lattice Γ .

We briefly recall the main properties of the family (μ_x) , cp. [29], [24]:

- (i) they are absolutely continuous w.r. to each other, and $\frac{d\mu_{x'}}{d\mu_x}(\xi) = e^{-\delta(\Gamma)b_\xi(x',x)}$;
- (ii) $\mu_x(\gamma^{-1}A) = \mu_{\gamma x}(A)$ for every isometry γ of X and every Borel set $A \subset X(\infty)$;
- (iii) if Γ is a lattice, then the support of μ_x is the whole boundary $X(\infty)$.

In [8] it is proved that the map F satisfies the following properties:

- a.** F is equivariant with respect to the action of Γ , i.e. $F(\gamma x) = \gamma F(x)$;
- b.** F is C^2 , with Jacobian

$$|\text{Jac}_x F| \leq \left(\frac{\delta(\Gamma) + 1}{n} \right)^n \cdot \det^{-1}(k_x) \quad (20)$$

where $k_x(u, v)$ is the bilinear form on $T_x X$ defined as

$$k_x(u, v) = \frac{\int_{X(\infty)} e^{b_\xi(F(x),x)} \left[(db_\xi)_{F(x)}^2 + \text{Hess}_{F(x)} b_\xi \right] (u, v) d\mu_x(\xi)}{\int_{X(\infty)} e^{b_\xi(F(x),x)} d\mu_x(\xi)} \quad (21)$$

Notice that the eigenvalues of k_x are all greater or equal than 1, by (19).

Property **(a)** stems from the equivariance (i) of the family of Patterson-Sullivan measures with respect to the action of Γ , and from the cocycle formula for the Busemann function: $b_\xi(x_0, x) + b_\xi(x, y) = b_\xi(x_0, y)$.

Property **(b)** comes from the fact that the Busemann function is C^2 on Hadamard manifolds, and is proved by direct computation, which does not use cocompactness. By equivariance, the map F defines a quotient map $\bar{F} : \bar{X} \rightarrow \bar{X}$, which is homotopic to the identity through the homotopy

$$\bar{F}_t(x) = \text{bar} \left[e^{-b_\xi(x)} (t\mu_x + (1-t)\lambda_x) \right] \text{ mod } \Gamma$$

where λ_x is the visual measure from x . Actually, the map $F_t = \text{bar} \left[e^{-b_\xi(x)} (t\mu_x + (1-t)\lambda_x) \right]$ passes to the quotient since it is still Γ -equivariant; moreover, we have $\text{bar} \left[e^{-b_\xi(x)} \lambda_x \right] = x$ as for all $v \in T_x X$:

$$\left(dB_{e^{-b_\xi(x)} \lambda_x} \right)_x (v) = \int_{X(\infty)} (db_\xi)_x(v) e^{b_\xi(x)} e^{-b_\xi(x)} d\lambda_x(\xi) = \int_{U_x X} g_x(u, v) du = 0.$$

We now prove that:

Proposition 5.1 *The homotopy map \bar{F}_t is proper.*

Assuming for a moment Proposition 5.1, the proof of Theorem 1.1 follows by the degree formula: since \bar{F} is properly homotopic to the identity, it has degree one, so

$$\text{vol}(\bar{X}) = \int_{\bar{X}} \bar{F}^* dv_g \leq \int_{\bar{X}} |\text{Jac}_x \bar{F}| dv_g \leq \left(\frac{\delta(\Gamma) + 1}{n} \right)^n \int_{\bar{X}} \det^{-1}(k_x) dv_g \leq \left(\frac{\delta(\Gamma) + 1}{\delta(\mathbb{H}^n) + 1} \right)^n \cdot \text{vol}(\bar{X})$$

as $\det(k_x) \geq 1$ everywhere. So, if $\delta(\Gamma) = \delta(\mathbb{H}^n) = n - 1$, we deduce that $\det(k_x) = 1$ everywhere and $k = g$, hence the equality in the equation (19) holds for every $y = F(x)$ and ξ . Since F is surjective, this shows that X has constant curvature -1 . \square

To show that the map \bar{F}_t is proper, we need some precise estimates on the Patterson-Sullivan measures of a lattice. For $x \in X$ and $\zeta \in X(\infty)$, let $x\zeta(t)$ be the geodesic ray from x to ζ ; we define the ‘‘spherical cap’’ $V_\zeta(x, R) \subset X(\infty)$ as the set of points at infinity ξ whose projection on $x\zeta$ falls between $x\zeta(R)$ and ζ . The following estimates are proved in [25]:

Lemma 5.2 *Let Γ be a nonuniform lattice of X , with curvature $k_X \leq -a^2$. Let $x_0 \in X$, and let $\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$ a decomposition of the Dirichlet domain of Γ centered at x_0 as in Sect. §3, corresponding to maximal, bounded parabolic subgroups P_1, \dots, P_m with fixed points ξ_1, \dots, ξ_m . There exists a constant c such that for every $\zeta \in X(\infty)$, if $x_R = x_0\zeta(R) \in \gamma\mathcal{C}_i$ and $r = b_{\xi_i}(x_0, \gamma^{-1}x_R)$ we have:*

$$(a) \quad \mu_{x_0}(V_\zeta(x_0, R)) \stackrel{c}{\asymp} e^{-\delta(\Gamma)(R-r)} \cdot \sum_{\substack{p \in P_i \\ d(x_0, px_0) > 2r + c'}} e^{-\delta(\Gamma)d(x_0, px_0)}, \quad \text{if } \gamma\xi_i \in V_\zeta(x_0, R)$$

$$(b) \quad \mu_{x_0}(V_\zeta(x_0, R)) \stackrel{c}{\asymp} e^{-\delta(\Gamma)(R+r)} \cdot v_{P_i}(2r - c') \quad \text{otherwise.}$$

Proof of Proposition 5.1. We denote by \bar{z} the projection of points $z \in X$ to \bar{X} . Call $\delta = \delta(\Gamma)$ and $\mu_x^t = e^{-b_\xi(x)}(t\mu_x + (1-t)\lambda_x)$; we need to show that if $t_k \rightarrow t_0$ and if $\bar{x}_k \rightarrow \infty$ in \bar{X} , then $\bar{y}_k = \bar{F}_{t_k}(\bar{x}_k) = \text{bar}[\mu_{\bar{x}_k}^{t_k}]$ goes to infinity too. Assume by contradiction that the points \bar{y}_k stay in a compact subset of \bar{X} : so (up to a subsequence) \bar{x}_k, \bar{y}_k lift to points x_k, y_k such that $y_k \rightarrow y_0 \in X$ and $d(y_0, x_k) = d(\bar{y}_0, \bar{x}_k) = R_k \rightarrow \infty$. By the cocycle relation $b_\xi(y, x) = b_\xi(y, y_0) + b_\xi(y_0, x)$ and by the density formula for the Patterson-Sullivan measures $\frac{d\mu_x}{d\mu_{y_0}}(\xi) = e^{-b_\xi(x, y_0)}$, we have

$$(d\mathcal{B}_{\mu_x^t})_y(v) = \int_{X(\infty)} e^{b_\xi(y, x)} (db_\xi)_y(v) d\mu_x^t$$

$$= t \int_{X(\infty)} e^{b_\xi(y, y_0)} e^{(\delta+1)b_\xi(y_0, x)} (db_\xi)_y(v) d\mu_{y_0} + (1-t) \int_{X(\infty)} e^{b_\xi(y, x)} (db_\xi)_y(v) d\lambda_x \quad (22)$$

We will now estimate the two terms in the above formula, and show that $(d\mathcal{B}_{\mu_{x_k}^{t_k}})_{y_k}$ does not vanish for $R_k \gg 0$, a contradiction. So, let ζ_k be the endpoints of the geodesic rays $y_0x_k(t)$, and let $v_k = (\nabla b_{\zeta_k})_{y_k}$. Also, consider the caps $V_{\zeta_k}(y_0, R_k)$ and $V_{\zeta_k}(y_0, R_k/2)$. Let us first consider the contributions of the integrals in (22) over $X \setminus V_{\zeta_k}(y_0, R_k/2)$. If $\xi \in X(\infty) \setminus V_{\zeta_k}(y_0, R_k/2)$, the projection P of ξ over $y_0\zeta_k$ falls closer to y_0 than to x_k , hence $b_\xi(y_0, x_k) \leq 2\epsilon_0$ by (2); moreover, $b_\xi(y_k, y_0) \leq d(y_k, y_0) \rightarrow 0$, so the first integral on $X \setminus V_{\zeta_k}(y_0, R_k/2)$ for $x = x_k, y = y_k$ and $v = v_k$ gives:

$$\left| \int_{X \setminus V_{\zeta_k}(y_0, \frac{R_k}{2})} e^{b_\xi(y_k, y_0)} e^{(\delta+1)b_\xi(y_0, x_k)} (db_\xi)_{y_k}(v_k) d\mu_{y_0} \right| < 2e^{2\epsilon_0(\delta+1)} \|\mu_{y_0}\|$$

for $k \gg 0$. Analogously, the second integral on $X \setminus V_{\zeta_k}(y_0, R_k/2)$ yields

$$\left| \int_{X \setminus V_{\zeta_k}(y_0, \frac{R_k}{2})} e^{b_\xi(y_k, x_k)} (db_\xi)_{y_k}(v_k) d\lambda_{x_k} \right| < 2e^{2\epsilon_0} \text{vol}(S^{n-1})$$

for $k \gg 0$, since $|b_\xi(y_k, x_k) - b_\xi(y_0, x_k)| < d(y_k, y_0)$. So, these contributions are bounded. We now compute the contributions of the integrals over $V_{\zeta_k}(y_0, R_k/2) \setminus V_{\zeta_k}(y_0, R_k)$. For all $\xi \in V_{\zeta_k}(y_0, R_k/2)$ we have that $(\nabla b_\xi)_{y_0} \cdot (\nabla b_{\zeta_k})_{y_0}$ is close to 1, for $R_k \gg 0$; moreover, as $|(\nabla b_\xi)_{y_k} \cdot v_k - (\nabla b_\xi)_{y_0} \cdot (\nabla b_{\zeta_k})_{y_0}| \xrightarrow{n \rightarrow 0} 0$, we deduce that $(db_\xi)_{y_k}(v_k) > \frac{1}{2}$ on $V_{\zeta_k}(y_0, R_k/2)$ for $n \gg 0$, hence these contributions are positive.

Finally, let us compute the contributions of these integrals on the caps $V_{\zeta_k}(y_0, R_k)$. For $\xi \in V_{\zeta_k}(y_0, R_k)$, consider the ray $y_0\xi(t)$ from y_0 to ξ , and the projection $P(t)$ of $y_0\xi(t)$ on the geodesic $y_0\zeta_k$. We have, again by Lemma (2)

$$b_\xi(y_0, x_k) \geq \lim_{t \rightarrow \infty} [d(y_0, P(t)) + d(P(t), \xi(t))] - [d(\xi(t), P(t)) + d(P(t), x_k)] - 2\epsilon_0 \geq R_k - 2\epsilon_0$$

therefore we deduce that, for $k \gg 0$

$$\int_{V_{\zeta_k}(y_0, R_k)} e^{b_\xi(y_k, y_0)} e^{(\delta+1)b_\xi(y_0, x_k)} (db_\xi)_{y_k}(v_k) d\mu_{y_0} \geq \frac{1}{2} t_k e^{(\delta+1)(R_k - 2\epsilon_0)} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) \quad (23)$$

$$\int_{V_{\zeta_k}(y_0, R_k)} e^{b_\xi(y_k, x_k)} (db_\xi)_{y_k}(v_k) d\lambda_{x_k} \geq \frac{1}{4} (1 - t_k) e^{(R_k - d(y_k, y_0) - 2\epsilon_0)} \text{vol}(S^{n-1}) \quad (24)$$

It is clear that this last integral goes to infinity when $R_k \gg 0$; we will now prove that the right-hand side of (23) also diverges for $R_k \rightarrow \infty$. This will conclude the proof, as it will show that $d\mathcal{B}_{\mu_{x_k}^{t_k}}(v_k)$, being a convex combination of two positive diverging terms, does not vanish for $k \gg 0$.

So, let $\mathcal{D} = \mathcal{K} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$ be a decomposition of the Dirichlet domain of Γ centered at y_0 as in Sect. §3, corresponding to maximal, bounded parabolic subgroups P_1, \dots, P_m with fixed points ξ_1, \dots, ξ_m . We know that \bar{x}_k belongs to some cusp of \bar{X} , so $x_k \in \gamma\mathcal{C}_i$, for some γ , so let $r_k = b_{\xi_i}(y_0, \gamma^{-1}x_k) \leq R_k$.

If $\gamma\xi_i$ falls in $V_{\zeta_k}(y_0, R_k)$ and $\Delta \gg 0$, as $K_X \geq -b^2$ we can use Lemma 5.2 and Proposition 2.5 and deduce that there exist constants c, C' such that

$$\begin{aligned} e^{(\delta+1)R_k} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) &\stackrel{c}{\asymp} e^{(\delta+1)R_k} e^{-\delta(R_k - r_k)} \cdot \sum v_{P_i}^\Delta((h+1)\Delta) e^{-\delta h \Delta} \\ &\stackrel{C'}{\asymp} e^{R_k + \delta r_k} \cdot \sum_{h > \frac{2r_k}{\Delta}} e^{(\delta^-(P_i) - \epsilon - \delta)k\Delta} \succ e^{R_k} \cdot e^{(2\delta^-(P_i) - \delta - 2\epsilon)r_k} \end{aligned}$$

for arbitrarily small $\epsilon > 0$. Since $K_X \leq -1$, we know that $\mathcal{A}_{P_i}(y_0, R) \prec e^{-(n-1)R}$, so $\delta^-(P_i) \geq \frac{n-1}{2}$ by Proposition 2.5; as $R_k \geq r_k$ the integral in (23) diverges in this case. If, on the other hand, $\gamma\xi_i \notin V_{\zeta_k}(y_0, R_k)$, always by 5.2 and 2.5, we have:

$$e^{(\delta+1)R_k} \mu_{y_0}(V_{\zeta_k}(y_0, R_k)) \stackrel{c}{\asymp} e^{R_k - \delta r_k} \cdot v_P(y_0, 2r_k - c) \succ e^{R_k + (n-1-\delta)r_k}$$

which also diverges as $\delta \leq n-1$. This concludes the proof that the map \bar{F}_t is proper. \square

We are now ready to prove Theorem 1.2 :

Proof of Theorem 1.2.

Assume that Γ is a nonuniform lattice in a $\frac{1}{4}$ -pinched negatively curved manifold X , i.e. $-b^2 \leq K_X \leq -a^2$ with $b^2 \leq 4a^2$. If $X = \mathbb{H}_a^n$, then clearly $v_X(x, R) \asymp v_\Gamma(x, R)$ is purely exponential, X has a Margulis function, and Γ is divergent. Otherwise, let P_i be the maximal parabolic subgroups of Γ , up to conjugacy. By the formulas (5), we know that for all $x \in X$ $e^{-(n-1)bR} \prec \mathcal{A}_{P_i}(x, R) \prec e^{-(n-1)aR}$, so by Proposition 2.5 we have

$$\frac{a(n-1)}{2} \leq \delta^-(P_i) \leq \delta^+(P_i) \leq \frac{b(n-1)}{2}$$

for all P_i . Thus, Γ is parabolically $\frac{1}{2}$ -pinched. It follows from Corollary 3.3 that $\omega^+(X) = \omega^-(X) = \delta(\Gamma)$. Moreover, for all P_i we have

$$\delta^+(P_i) \leq \frac{b(n-1)}{2} \leq a(n-1) < \omega(X) = \delta(\Gamma)$$

the strict inequality following by the rigidity Theorem 1.1, since $X \neq \mathbb{H}_a^n$. The same argument applies when \bar{X} is only asymptotically $\frac{1}{4}$ -pinched, by replacing $-a^2, -b^2$ with the bounds $-k_+^2 - \epsilon \leq K_X \leq -k_-^2 + \epsilon$ on the cusps $\bar{\mathcal{C}}_i$. Then, Γ is also non-exotic, and we can conclude by Theorem 1.3 that Γ is divergent, with finite Bowen-Margulis measure, $v_X \asymp v_\Gamma$ and X has a L^1 Margulis function $m(x)$. \square

6 Examples

In this section we show that all the cases presented in Theorem 1.4 do occur, by providing examples of spaces X with exotic or sparse lattices Γ which do not admit a Margulis function, and with functions v_Γ, v_X having different behaviour.

If $\bar{\mathcal{C}} = P \backslash H_\xi(o)$ is a cusp of $\bar{X} = \Gamma \backslash X$, we write the metric of X in horospherical coordinates on $H_\xi(o) \cong \partial H_\xi(o) \times \mathbb{R}^+$ as $g = T(x, t)^2 dy^2 + dt^2$, for $x \in \partial H_\xi(o)$ and $t = b_\xi(o, \cdot)$. We call the function $T(x, t)$ the *analytic profile of the cusp* $\bar{\mathcal{C}}$. The horospherical area $\mathcal{A}_P(x, t)$ is then obtained by integrating $T^{n-1}(x, t)$ over a compact fundamental domain S for the action of P on $\partial H_\xi(o)$; thus, we have

$$\mathcal{A}_P(x, t) \stackrel{c}{\asymp} T^{n-1}(x, t) \quad \text{for all } x \in \bar{\mathcal{C}}$$

(for a constant c depending on X and o). Also, notice that, in the particular case where $T(y, t) = T(t)$, for points x, y belonging to a same horosphere H_ξ we have by the Approximation Lemma 2.2

$$d(x, y) \sim 2T^{-1} \left(\frac{T(0)}{d_\xi(x, y)} \right) \quad \text{for } R = d(x, y) \rightarrow \infty. \quad (25)$$

We will repeatedly make use of the following lemma, which is a easy modification of one proved in [12]:

Lemma 6.1 *Let $b > a > 0$, $\beta > \alpha > 0$ and $\epsilon > 0$ be given.*

There exist $D = D(a, b, \alpha, \beta, \epsilon) > 1$ and $D' = D'(a, b, \alpha, \beta) > 0$ such that if $[p, q]$, $[r, s]$ are disjoint intervals satisfying $r \geq Dq$ and $p \geq D'$, then there exist C^2 , convex and decreasing functions $\phi_\epsilon, \psi_\epsilon$ on $[p, s]$ satisfying:

$$\left\{ \begin{array}{l} \forall t \in [p, q], \quad \phi_\epsilon(t) = t^\beta e^{-bt} \\ \forall t \in [r, s], \quad \phi_\epsilon(t) = t^\alpha e^{-at} \\ \forall t \in [p, s], \quad t^\beta e^{-bt} \leq \phi_\epsilon(t) \leq t^\alpha e^{-at} \\ \forall t \in [p, s], \quad a^2 - \epsilon \leq \frac{\phi_\epsilon''(t)}{\phi_\epsilon(t)} \leq b^2 + \epsilon \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \forall t \in [p, q], \quad \psi_\epsilon(t) = t^\alpha e^{-at} \\ \forall t \in [r, s], \quad \psi_\epsilon(t) = t^\beta e^{-bt} \\ \forall t \in [p, s], \quad t^\beta e^{-bt} \leq \psi_\epsilon(t) \leq t^\alpha e^{-at} \\ \forall t \in [p, s], \quad a^2 - \epsilon \leq \frac{\psi_\epsilon''(t)}{\psi_\epsilon(t)} \leq b^2 + \epsilon \end{array} \right.$$

Example 6.2 *Sparse lattices.*

Sparse lattices satisfying $\omega^+(X) > \delta(\Gamma)$ were constructed by the authors in [12]. Here, we modify that construction to show that, for spaces X admitting sparse lattices, one can have $\omega^+(X) > \omega^-(X) > \delta(\Gamma)$ (in contrast, notice that $\delta(\Gamma)$ always is a true limit); this shows in particular that sparse lattices generally do not have a Margulis function. We start from a hyperbolic surface $\bar{X}_0 = X_0 \setminus \Gamma$ of finite volume, homeomorphic to a 3-punctured sphere, and, for any arbitrary small $\epsilon > 0$, we perturb the hyperbolic metric g_0 on one cusp $\bar{C} = P \setminus H_\xi(x)$ into a metric g_ϵ by choosing an analytic profile T_ϵ oscillating, on infinitely many horospherical bands, from e^{-t} to e^{-bt} .

Namely, choose $a = 1, b > 2$ and $\epsilon > 0$ arbitrarily small, and let D, D' be the constants given by Lemma 6.1. For $M \gg 1$, we define a sequence of disjoint subintervals of $[M^{4n}, M^{4n+1}]$:

$$[p_n, q_n] := [M^{4n}, 2M^{4n}], \quad [r_n, s_n] := \left[\frac{p_n + M^{4n+1}}{2}, \frac{q_n + M^{4n+1}}{2} \right]$$

such that $r_n \geq Dq_n, p_{n+1} \geq Ds_n, p_1 \geq D'$ (we can choose any $M \geq \max\{4D - 1, \sqrt[3]{D}\}$ in order that these conditions are satisfied). Notice that $\frac{t + M^{4n+1}}{2} \in [r_n, s_n]$ for all $t \in [p_n, q_n]$. Then, by Lemma 6.1, we consider a C^2 , decreasing function $T_\epsilon(t)$ satisfying:

- (i) $T_\epsilon(t) = e^{-t}$ for $t \in [M^{4n-2}, M^{4n}] \cup [p_n, q_n]$, and $T_\epsilon(t) = e^{-bt}$ for $t \in [r_n, s_n]$;
- (ii) $e^{-bt} \leq T_\epsilon(t) \leq e^{-t}$ and $-b^2 - \epsilon \leq T_\epsilon''(t)/T_\epsilon(t) \leq -1 + \epsilon$.

Thus, the new analytic profile $T_\epsilon(t)$ of \bar{C} coincides with the profile of a usual hyperbolic cusp on $[M^{4n-2}, 2M^{4n}]$, and with the profile of a cusp in curvature $-b^2$ on the bands $[r_n, s_n] \subset [M^{4n}, M^{4n+1}]$. We have, with respect to the metric g_ϵ :

a. $\delta^+(P) = \frac{b}{2}$ and $\delta^-(P) = \frac{1}{2}$, by (i) and (ii), because of Proposition 2.5;

b. $\omega^+(\mathcal{F}_P) \geq \frac{b}{2} + \delta$ for $\delta = \frac{1}{M}(\frac{b}{2} - 1) > 0$, because for $R = M^{4n+1}$

$$\mathcal{F}_P(x, R) \succ \int_0^R \frac{\mathcal{A}_\epsilon(x, t)}{\mathcal{A}_\epsilon(x, \frac{t+R}{2})} dt \geq \int_{p_n}^{q_n} \frac{e^{-t}}{e^{-b(\frac{t+R}{2})}} dt \succ e^{\frac{b}{2}R} \cdot M^{4n} e^{(\frac{b}{2}-1)p_n} \geq e^{\frac{b}{2}R} \cdot e^{\frac{1}{M}(\frac{b}{2}-1)R} \quad (26)$$

as $p_n/R = \frac{1}{M}$;

c. $\omega^-(\mathcal{F}_P) \leq \frac{1}{2}$ if $M > 2$, as for $R \in [M^{4n+3}, M^{4n+4}]$ we obtain:

$$\mathcal{F}_P(x, R) \prec \int_0^R \frac{e^{-t}}{e^{-(\frac{t+R}{2})}} dt \prec e^{\frac{R}{2}} \quad (27)$$

since $M^{4n+4} \geq \frac{t+R}{2} \geq \frac{M^{4n+3}}{2} \geq M^{4n+2}$;

d. $\delta(\Gamma)$ is arbitrarily close to $\delta^+(P)$, let's say $\delta(\Gamma) \leq \frac{b}{2} + \frac{\delta}{2}$, if we perturb the hyperbolic metric sufficiently far in the cusp \bar{C} , i.e. if $r_1 \gg 0$ (this is Proposition 5.1 in [12]).

It follows that $\omega^-(X) > \delta(\Gamma)$. Actually, assume that $v_\Gamma(x, R) \succ e^{(\delta(\Gamma)-\eta)R}$, for arbitrarily small η . By Proposition 3.2 and (26), for any $R \gg 0$, if $M^{4n+1} \leq R < M^{4n+5}$

$$v_X(x, R + 2D_0) \geq v_\Gamma(x, \cdot) * \mathcal{F}_P(x, \cdot)(x, R) \succ e^{(\delta(\Gamma)-\eta)(R-M^{4n+1})} \cdot e^{(\frac{b}{2}+\delta)M^{4n+1}}$$

by taking just the term $v_\Gamma(x, R-t)\mathcal{F}_P(x, t)$ of the convolution with t closest to M^{4n+1} , where $\mathcal{F}_P(t) \succ e^{(\frac{b}{2}+\delta)t}$; as $M^{4n+1} \geq R/M^4$ we get $v_X(x, R + 2\Delta) \succ e^{(\delta(\Gamma)-\eta+\frac{\delta/2+\eta}{M^4})R}$ which gives $\omega^-(X) \geq \delta(\Gamma) + \frac{\delta}{2M^4}$, η being arbitrary.

Finally, we show that $\omega^+(X) > \omega^-(X)$. In fact, the cusps different from $\bar{\mathcal{C}}$ being hyperbolic, we have, always by Proposition 3.2, that $\omega^+(X) = \omega^+(\mathcal{F}_P) \geq \frac{b}{2} + \delta$.

On the other hand, we know that $\omega^+(\mathcal{F}_P) \leq \max\{\delta^+(P), 2(\delta^+(P) - \delta^-(P))\} = b - 1$, by Corollary 2.8; thus, assuming $\mathcal{F}_P(x, t) \prec e^{(b-1+\eta)t}$, for arbitrarily small η , equation (27) yields for $R = M_{4n+4}$

$$\begin{aligned} v_X(x, R - 2D_0) &\leq \int_0^{M^{4n+3}} v_\Gamma(x, R-t) \cdot \mathcal{F}_P(x, t) dt + \int_{M^{4n+3}}^R v_\Gamma(x, R-t) \cdot \mathcal{F}_P(x, t) dt \\ &\prec \int_0^{M^{4n+3}} e^{\delta(\Gamma)(R-t)} \cdot e^{(b-1+\eta)t} dt + \int_{M^{4n+3}}^R e^{\delta(\Gamma)(R-t)} \cdot e^{\frac{1}{2}t} dt \\ &\prec e^{\delta(\Gamma)R} \cdot e^{(b-1+\eta-\delta(\Gamma))M^{4n+3}} \leq e^{(\frac{b}{2}+\frac{\delta}{2}+\frac{(b/2+\eta-1)}{M})R} \end{aligned}$$

being $\frac{b}{2} \leq \delta(\Gamma) \leq \frac{b}{2} + \frac{\delta}{2}$ and $M^{4n+3} = \frac{R}{M}$. Hence $\omega^-(X) < \frac{b}{2} + \delta \leq \omega^+(X)$, if $M \gg 0$ and η small enough.

Examples 6.3 *Exotic, strictly $\frac{1}{2}$ -parabolically pinched lattices.*

We say that a lattice Γ is *strictly $\frac{1}{2}$ -parabolically pinched* when every parabolic subgroup $P < \Gamma$ satisfies the strict inequality $\delta^+(P) < 2\delta^-(P)$. Let $\bar{X} = \Gamma \backslash X$ as before; we show here that, for Γ exotic and strictly $\frac{1}{2}$ -parabolically pinched, the following cases which appear in Theorem 1.4 do occur:

- (a) $\mu_{BM}(U\bar{X}) = \infty$ and v_X is lower-exponential;
- (b) $\mu_{BM}(U\bar{X}) < \infty$ and v_X is purely exponential.

We start by an example of lattice satisfying (a).

In [14] the authors show how to construct *convergent lattices*, in pinched negative curvature and any dimension n ; we will take $n = 2$ here by the sake of simplicity. In those examples, the metric is hyperbolic everywhere but one cusp \mathcal{C} , which has analytic profile $T(t) = t^\beta e^{bt}$ for $t \geq t_0 \gg 0$, with $\beta > 1$ and $b > 2$. Therefore, there is just one dominant maximal parabolic subgroup P , with $\mathcal{A}_P(x, t) \asymp T(t) \asymp e^{bt}$, and $\delta^+(P) = \delta^-(P) = \frac{b}{2}$; moreover, the subgroup P is convergent as

$$\sum_{p \in P} e^{-\frac{b}{2}d(x, px)} \leq \sum_{k \geq 0} v_P(x, k) e^{-\frac{b}{2}k} \asymp \int_1^\infty \frac{e^{-\frac{b}{2}t}}{\mathcal{A}_P(x, \frac{t}{2})} dt \asymp \int_1^\infty \frac{e^{-\frac{b}{2}t}}{(t)^\beta \cdot e^{-b\frac{t}{2}}} dt \asymp \int_1^\infty t^{-\beta} dt < \infty.$$

By decomposing the elements of Γ in geodesic segments which, alternatively, either go very deep in the cusp or stay in the hyperbolic part of X , we show in [14] that Γ is convergent too, provided that $t_0 \gg 0$. Then, Γ is exotic with infinite Bowen-Margulis measure, and $v_\Gamma(x, R)$ is lower-exponential by Roblin's asymptotics. By Theorem 1.4(i), the function v_X is lower-exponential behaviour as well, with the same exponential growth rate.

We now give an example for **(b)**.

This is more subtle, as we need to take a *divergent, exotic lattice* Γ : the existence of such lattices is established, in dimension 2, in [14]. Again, the simplest example is homeomorphic to a three-punctured sphere, with three cusps, and hyperbolic metric outside one cusp \bar{C} , which has analytic profile

$$T(t) = \begin{cases} e^{-t} & \text{for } t \leq A \\ e^{-bt} & \text{for } t \in [A, A+B] + D \\ t^3 \cdot e^{-bt} & \text{for } t \gg D + A + B \end{cases}$$

with $b > 2$ and $A, B, D \gg 0$. As before, we have one dominant and convergent maximal parabolic subgroup P , with $\delta^+(P) = \delta^-(P) = \frac{b}{2}$. In [14] it is proved that, according to the values of A and B , the behaviour of the group Γ is very different: it is convergent with critical exponent $\delta(\Gamma) = \delta^+(P)$, for $A \gg 0$ and $B = 0$, while it is divergent with $\delta(\Gamma) > \delta^+(P)$ if $B \gg A$. By perturbation theory of transfer operators, it is then proved that there exists a value of B for which Γ is divergent with $\delta(\Gamma) = \delta^+(P)$ precisely. Thus, for this particular value of B , the lattice Γ we is exotic, and has finite Bowen-Margulis measure by the Finiteness Criterion, as

$$\sum_{p \in P} d(x, px) e^{-\delta(\Gamma)d(x, px)} \prec \int_1^\infty \frac{te^{-\frac{b}{2}t}}{\mathcal{A}_P(x, \frac{t}{2})} dt \prec \int_1^\infty \frac{te^{-\frac{b}{2}t}}{t^3 \cdot e^{-b\frac{t}{2}}} dt \asymp \int_1^\infty t^{-2} dt < \infty \quad (28)$$

It follows that $v_X \asymp v_\Gamma$ is purely exponential, by Theorem 1.4(i).

Examples 6.4 *Exotic, exactly $\frac{1}{2}$ -parabolically pinched lattices.*

We say that a lattice Γ is *exactly $\frac{1}{2}$ -parabolically pinched* when it is $\frac{1}{2}$ -parabolically pinched and has a parabolic subgroup $P < \Gamma$ satisfying the quality $\delta^+(P) = 2\delta^-(P)$. We show here that for an exotic and exactly $\frac{1}{2}$ -parabolically pinched lattice Γ , the following cases can occur:

- (a) $\mu_{BM}(U\bar{X}) < \infty$, with v_Γ purely exponential and v_X upper-exponential;
- (b) $\mu_{BM}(U\bar{X}) = \infty$, with v_Γ lower-exponential and v_X upper-exponential.

We start by **(a)**. Consider a surface with three cusps as in the Examples 6.3, now perturbing the hyperbolic metric on the cusp \bar{C} to an analytic profile defined as follows. First, choose a sequence of disjoint subintervals of $[M^{2n}, M^{2n+1}]$

$$[p_n, q_n] := [M^{2n}, \mu M^{2n+1}], \quad [r_n, s_n] := \left[\frac{p_n + M^{2n+1}/2}{2}, \frac{q_n + M^{2n+1}}{2} \right] \quad (29)$$

and then define, for $b > 1$ and $0 < \gamma < 1$

$$T(t) = \begin{cases} e^{-t} & \text{for } t \leq A \\ e^{-bt} & \text{for } t \in [A, A+B] + D \\ t \cdot e^{-\frac{b}{2}t} & \text{for } t \in [p_n, q_n] \\ t^{2+\gamma} \cdot e^{-bt} & \text{for } t \in [r_n, s_n] \end{cases}$$

with $t^{2+\gamma}e^{-bt} \leq T(t) \leq t \cdot e^{-\frac{b}{2}t}$ for all $t \geq t_0 \gg 0$ (in order that the conditions of Lemma 6.1 are satisfied, it is enough to choose any $0 < \mu < \frac{1}{4D}$ and $M > D$).

As before, the profile T gives a divergent, exotic lattice Γ for a suitable value of B and $A \gg 0$, with dominant parabolic subgroup P having $\delta^+(P) = \frac{b}{2} = \delta(\Gamma)$, and $\delta^-(P) = \frac{b}{4}$. The Bowen-Margulis measure of Γ is finite, as (28) also holds in this case; thus, v_Γ is purely exponential. Let us now show that v_X is upper exponential: for every $R = M^{2n+1}$ we have, by Proposition 3.2,

$$\begin{aligned} v_X(x, R + 2D_0) &\succ [v_X(x, \cdot) * \mathcal{F}_P(x, \cdot)](R) \asymp \int_0^R v_\Gamma(x, R-t) \left[\int_0^t \frac{\mathcal{A}_P(x, s)}{\mathcal{A}_P(x, \frac{s+t}{2})} ds \right] dt \\ &= \int_0^R \mathcal{A}_P(x, s) \left[\int_s^R \frac{v_\Gamma(x, R-t)}{\mathcal{A}_P(x, \frac{s+t}{2})} dt \right] ds \geq \int_{p_n}^{q_n} \mathcal{A}_P(x, s) \left[\int_{\frac{R}{2}}^R \frac{v_\Gamma(x, R-t)}{\mathcal{A}_P(x, \frac{s+t}{2})} dt \right] ds \end{aligned}$$

since $q_n < \frac{R}{2}$. As $\frac{s+t}{2} \in [r_n, s_n]$ if $s \in [p_n, q_n]$ and $t \in [\frac{R}{2}, R]$, by the definition of $T(t) \asymp \mathcal{A}_P(x, t)$ on $[r_n, s_n]$, this yields

$$v_X(x, R) \succ \int_{p_n}^{q_n} s e^{-\frac{b}{2}s} \left[\int_{\frac{R}{2}}^R \frac{e^{\frac{b}{2}(R-t)}}{e^{-b(\frac{s+t}{2})(s+t)^{2+\gamma}}} dt \right] ds \succ e^{\frac{b}{2}R} \int_{p_n}^{q_n} \frac{Rs}{(s+R)^{2+\gamma}} ds$$

with $\int_{p_n}^{q_n} \frac{Rs}{(s+R)^{2+\gamma}} ds \geq \int_{\frac{1}{M}}^{\mu} \frac{u}{(1+u)^{2+\gamma}} du \asymp R^{1-\gamma}$, so v_X is upper-exponential.

Producing examples for case **(b)** is more difficult; for this, we will need an exotic lattice Γ whose orbital function satisfies $v_\Gamma(o, R) \asymp \frac{1}{R^\gamma} e^{\delta(\Gamma)R}$. Lattices with lower-exponential growth and infinite Bowen-Margulis measure are investigated in [14], where a refined counting result is proved, according to the behaviour of the profile functions of the cusps (the examples in [14] are, as far as we know, the only precise estimates of the orbital function for groups with infinite Bowen-Margulis measure). Here we only give the necessary analytic profiles of the cusps in order to have a function v_X which is exponential or upper-exponential, referring to [14] for the precise estimate of v_Γ .

We again start from a hyperbolic surface $\bar{X}_0 = X_0 \setminus \Gamma$ with three cusps as in 6.3, and perturb now the metric on *two* cusps. We choose $b > 2$ and $1 + \gamma < \beta < 2 + \gamma$, and define the profiles for \bar{C}_1 and \bar{C}_2 as

$$T_1(t) = \begin{cases} e^{-t} & \text{for } t \leq A \\ e^{-bt} & \text{for } t \in [A, A+B] + D \\ t \cdot e^{-\frac{b}{2}t} & \text{for } t \in [p_n, q_n] \\ t^\beta \cdot e^{-bt} & \text{for } t \in [r_n, s_n] \end{cases} \quad \text{and} \quad T_2(t) = \begin{cases} e^{-t} & \text{for } t \leq A \\ t^{1+\gamma} e^{-bt} & \text{for } t \gg A \end{cases}$$

for the same sequence of intervals $[p_n, q_n]$, $[r_n, s_n]$ as in (29).

If P_1, P_2 are the associated maximal parabolic subgroups, we have $\delta^-(P_1) = \frac{b}{4}$ and $\delta^+(P_1) = \frac{b}{2}$, while $\delta^+(P_2) = \delta^-(P_2) = \frac{b}{2}$ by construction. It is easily verified that these parabolic subgroups are convergent as $\gamma > 0$. Again, pushing the perturbation far in the cusps (i.e. choosing $A \gg 0$) and for a suitable value of B , the lattice Γ becomes exotic and divergent; it has two dominant cusps, it is exactly $\frac{1}{2}$ -parabolically pinched, and has infinite Bowen-Margulis measure, because (as $\gamma < 1$)

$$\sum_{p \in P_2} d(x, px) e^{-\delta(\Gamma)d(x, px)} \prec \int_1^\infty \frac{t e^{-\frac{b}{2}t}}{t^{1+\gamma} \cdot e^{-b\frac{t}{2}}} dt \asymp \int_1^\infty t^{-\gamma} dt = \infty.$$

Accordingly, v_Γ is lower-exponential. In [14] it is proved that the *least convergent* dominant parabolic subgroup determines the asymptotics of v_Γ ; in this case, the parabolic subgroup P_1 converges faster than P_2 , and the chosen profile for \tilde{C}_2 then gives $v_\Gamma(o, R) \asymp \frac{1}{R^{1-\gamma}} e^{\delta(\Gamma)R}$, provided that $\gamma \in (\frac{1}{2}, 1)$, cp. [14].

Let us now estimate $v_X(x, R)$, for $R = M^{2n+1}$. Writing $T_1(t) = \tau^+(t)e^{-bt} = \tau^-(t)e^{-\frac{b}{2}t}$ so that $\tau^+(t) = t^\beta$ on $[r_n, s_n]$ and $\tau^-(t) = t$ on $[p_n, q_n]$, we compute as in case **(a)**:

$$\begin{aligned} v_X(x, R + 2D_0) &\asymp (v_\Gamma(x, \cdot) * \mathcal{F}_{P_1}(x, \cdot))(R) = \int_0^R \int_0^t \frac{\mathcal{A}_{P_1}(x, s)}{\mathcal{A}_{P_1}(x, \frac{t+s}{2})} v_\Gamma(x, R-t) dt ds \\ &\asymp \int_0^R \int_0^t \frac{\tau^-(s) \cdot e^{-\frac{b}{2}s} \cdot e^{\frac{b}{2}(R-t)}}{\tau^+(\frac{t+s}{2}) \cdot (R-t)^{1-\gamma} \cdot e^{-b(\frac{t+s}{2})}} dt ds = e^{\frac{b}{2}R} \int_0^R \tau^-(s) \left[\int_s^R \frac{dt}{\tau^+(\frac{t+s}{2})(R-t)^{1-\gamma}} \right] ds \\ &\asymp e^{\frac{b}{2}R} \int_{p_n=\frac{R}{M}}^{q_n=\mu R} s \left[\int_{\frac{R}{2}}^R \frac{dt}{R^\beta (R-t)^{1-\gamma}} \right] ds \asymp \left(\mu - \frac{1}{M} \right) R^{2+\gamma-\beta} e^{\frac{b}{2}R} \end{aligned}$$

which is upper-exponential as $\beta < 2 + \gamma$.

REMARK 6.5 Notice that in all these examples b can be chosen arbitrarily close to $2a = 2$. Thus, by the last condition in Lemma 6.1, the analytic profiles give metrics with curvature $-4a^2 - \epsilon \leq K_X \leq -a^2$, for arbitrarily small $\epsilon > 0$.

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