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# PROVING IN NUMBER THEORY AT THE TRANSITION FROM SECONDARY TO TERTIARY LEVEL: BETWEEN ORGANIZING AND OPERATIVE DIMENSIONS

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*At the beginning of French University, the failure rate in proving in mathematics is important. In this paper, we explore this phenomenon in the case of number theory. In the proving process, we distinguish two complementary dimensions, namely the organizing one and the operative one. This distinction permits to situate the autonomy devolved to learners in proving tasks. We focus on the transfer of this autonomy in the secondary-tertiary transition from examples out of examinations given in 2006 in Grade 12 and in first year at University concerning congruences. According to us, this captures one of the sources of difficulties to prove in number theory faced by students arriving at University.*

## INTRODUCTION

In our researches in didactic of number theory, we are especially interested in proving in the secondary-tertiary transition<sup>1</sup>. Within didactic researches related to this transition (Guedet, 2008), we propose to study some of the ruptures at stake in terms of autonomy devolved to Grade 12-pupils and students. To characterize this autonomy in the process of proving in number theory, we exploit the distinction in the reasoning between the organizing dimension and the operative dimension (Battie, 2007).

We distinguish two complementary dimensions. The organizing dimension concerns the mathematician's « aim » (i.e. his or her « program », explicit or not). For example, besides usual figures of mathematical reasoning, especially *reductio ad absurdum*, we identify in organizing dimension induction (and other forms of exploitation in reasoning of the well-ordering  $\leq$  of the natural numbers), reduction to the study of finite number of cases, and factorial ring's method. The operative dimension relates to those treatments operated on objects and developed for implementing the different steps of the program. For instance, we identify forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring  $Z$ ) and standard order  $\leq$  (the well-ordered set  $N$ ). Among the numerous didactic researches on mathematical reasoning and proving (International Newsletter on the Teaching and Learning of Mathematical Proof and, especially for Number theory, see (Zazkis & Campbell, 2002 & 2006)), we can put into perspective our distinction between organizing dimension and

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<sup>1</sup> In secondary-tertiary transition, number theory is primarily concerned with structures and properties of the integers (i.e. Elementary number theory). For a detailed consideration of various facets falling under the rubric of number theory, see Campbell and Zazkis, 2002.

operative dimension (in the reasoning in number theory) with the “structuring mathematical proofs” of Leron (1983). As we showed (Battie, 2007), an analogy is *a priori* possible, but only on certain types of proofs. According to us, the theoretical approach of Leron is primarily a hierarchical organization of mathematical sub-results necessary to demonstrate the main result, independently of the specificity of mathematical domains at stake. As far as we know, Leron’s point of view does not permit access that gives our analysis in terms of organizing and operative dimensions, namely the different nature of mathematical work according to whether a dimension or another and, so essential, interactions that take place between these two dimensions.

First, we’ll present characteristics of number theory teaching in Grade 12 and in the first year of University. We’ll study two examples out of examinations given in 2006 in Grade 12 and in first year at University concerning congruences (for the study of University’s examination, we have an interview with the author and the papers of 73 students<sup>2</sup>). Then, in a second part, we’ll illustrate the influence of teaching culture on the proving practice of Grade 12-pupils and students.

## **NUMBER THEORY TEACHING IN GRADE 12 AND IN THE FIRST YEAR OF UNIVERSITY**

In Grade 12, number theory curriculum as an option comprises: divisibility, Euclidian division, Euclid’s algorithm, congruence, integers relatively prime, prime numbers, existence and uniqueness of prime factorization, least common multiple (LCM), Bézout’s identity and Gauss’ theorem<sup>3</sup>. In the first year of University, there are two distinct populations: only part of students have attended number theory course in Grade 12. In the first year of University, number theory curriculum is the same than in Grade 12 but algebraic structures are clarified.

In *baccalauréat*’s exercises (Battie, 2003a), autonomy devolved to pupils to solve number theory problems is mainly located at the operative dimension. Autonomy is devolved at the organizing dimension only for routine tasks as resolution of Diophantine equations  $ax+by=c$  ( $\gcd(a,b)$  divide  $c$ ), and when the organizing dimensions involved are considered as non-problematic by the institution, such as the treatment of logical equivalences. An example will be given afterwards. We make the hypothesis that *baccalauréat*’s exercises have an important influence on the choice of teachers and then, we can suppose that proposed tasks to pupils in class are mainly located at the operative dimension. This hypothesis is supported by our analysis of a training test for the *baccalauréat* (Battie, 2003b).

In the first year of University, the situation is different. This is an example of number theory examinations (December 2006):

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<sup>2</sup> The corpus comes from the master dissertation of Florence Buisson a student of us.

<sup>3</sup> If an integer divides the product of two other integers, and the first and second integers are coprime, then the first integer divides the third integer.

The purpose of this exercise is to solve in  $Z$  the following system of congruences in unknown  $n$ : (E) 
$$\begin{cases} n \equiv 3 \pmod{7} \\ n \equiv 0 \pmod{16} \end{cases}$$

- 1) Prove that 7 and 16 are coprime and write a corresponding Bézout's identity.
- 2) What is the set of integers both multiple of 7 and 16?
- 3) By using Bézout's identity, find an integer  $n_0$  multiple of 16 and checking  $n_0 \equiv 3 \pmod{7}$ . Deduce a solution of the system (E).
- 4) Deduce from two previous questions the set of solutions of (E).

This problem is a particular case of Chinese remainder theorem. To prove this theorem, the main organizing dimension refers to an equivalence that can be interpreted in terms of existence and uniqueness of a solution of the system or in terms of surjective and injective function which is, in this case, a ring's isomorphism (let  $m_1, m_2$  be coprime integers, for all  $x$ , element of  $Z$ , the application at stake, from  $Z/m_1m_2$  to  $Z/m_1 \times Z/m_2$ , associates to each element  $x \pmod{m_1m_2}$  the sequence of  $x \pmod{m_1}$  and  $x \pmod{m_2}$ ). For the operative dimension, the key to prove the existence of a solution is Bézout's identity ( $m_1$  and  $m_2$  are relatively prime); this is precisely the subject of Question 1. To prove the uniqueness of such a solution, the essential operative element is the result stating that if an integer is divisible by  $m_1$  and  $m_2$  then it is divisible by the product  $m_1m_2$  and this can be achieved as a consequence of Gauss' theorem or via the concept of LCM ( $\text{LCM}(m_1m_2) = m_1m_2$ ); this is the subject of Question 2. Thus, the author of the exam highlights the two operative keys in the first two questions, but it doesn't mark out the main organizing dimension. The only evidence of this kind appears through cutting issues 3 (existence of a solution) and 4 (obtaining all solutions from the solution *i.e.* uniqueness of the solution modulo  $7 \times 16$ ). In this way, the author of the exam wants to bring out the linearity relation as he told in the interview by F.Buisson:

“[...] we will use something which is general and that they [students] have already seen, it's linearity phenomena which is omnipresent in mathematics. And then, in this case, the goal of this development<sup>4</sup> of the problem is a parallel with the linear differential equation, resolution of linear equation, linear system in any field, why not in  $R$ ”.

For each of the last two questions, the operative key is mentioned in response to questions 1 and 2 (“With Bézout's identity [...]”, “Deduce from two previous questions [...]”). In conclusion, autonomy devolved to students is mainly located at the organizing dimension: keys of operative dimension are explicitly given to students while they have to build almost the entire organizing dimension (Question 4 illustrates particularly well this characteristic). And, for the author of

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<sup>4</sup> The French term used is “acheminement”.

the examination, skills related to the organizing dimension are important criteria to evaluate students:

“[...] me, what I want to see it is: has he [the student] got the idea to go to question 4 from questions 2 and 3. Agree? So it is the higher level of the exercise to see a little, well, a little the best ones, the “ happy few ”, because then there are not many [students] who remain here.”

We find also the problem of solving a system of congruences in the 2006 national *baccalauréat*:

The purpose is to solve in  $Z$  the system (S)  $\begin{cases} n \equiv 13 \pmod{19} \\ n \equiv 6 \pmod{12} \end{cases}$

1) Prove that exists an ordered pair of integers  $(u, v)$  such that  $19u + 12v = 1$  (in this question it's not required to give an example of such an ordered pair). Check that for such an ordered pair  $N = 13 \times 12v + 6 \times 9u$  is a solution of (S).

2) a) Let  $n_0$  be a solution of (S). Check that the system (S) is equivalent to  $\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$   
 b) Prove that the system  $\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$  is equivalent to  $n \equiv n_0 \pmod{12 \times 19}$ .

3) a) Find a ordered pair  $(u, v)$  solution of the equation  $19u + 12v = 1$  and calculate the corresponding value of  $N$ .

b) Determine the set of solutions of (S) (it's possible using question 2)b).

In this *baccalauréat*'s exercise, there are more indications related to organizing dimension than in the University's examination and those indications are more detailed while, for the operative dimension, only Bézout's identity is mentioned (without it appellation appears). This comparison permits to enlighten that the University's examination offers several possibilities in operative dimension: instead of using Gauss theorem, essential to treat the *baccalauréat*'s exercise (question 2b), it's possible to use the concept of LCM and work in  $Z/7Z$  (questions 2 and 3), that the author had not anticipated.

In the secondary-tertiary transition, specifically in the Grade 12-University transition, there is therefore a transfer of the autonomy devolved in problems wording of number theory. At the entrance of University, students are much more responsible than in Grade 12 for the development of the organizing dimension of proofs.

## **INFLUENCE OF TEACHING CULTURE ON MATHEMATICAL PRACTICE OF GRADE 12-PUPILS AND STUDENTS**

An analysis in terms of organizing and operative dimensions has permitted to examine the proving process of Grade 12-pupils from transcripts of pupils' research (Battie, 2007 & 2008). In 2001, we proposed to a class of Grade 12-pupils to produce number theory proofs of the irrationality of  $\sqrt{2}$  and  $\sqrt{3}$ . In 2007, we proposed to another such class to prove the fundamental theorem of (elementary) number theory<sup>5</sup> from historical texts. It appears that elements of the

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<sup>5</sup> Existence and uniqueness of the factorization of a natural integer into a product of primes.

operative dimension (especially using emblematic theorems of the teaching culture such as Bézout's identity and Gauss' theorem) mostly guide pupils in their research. The few explicit tracks for the organizing dimension generally come from a vague recollection of what has been done in the classroom. Pupils mention organizing dimensions encountered at school as induction, *reductio ad absurdum* and separating even and odd cases, but they awkwardly introduce them and, mostly, they don't use them afterwards in their research. There is an imbalance in the pupils' proving process in terms of control of both organizing and operative dimensions. Some operative automatisms, coming from class practice, are entrenched so that they can be a source of genuine creativity in pupils' mathematical researches. Some embryos of original proofs emerge but the fragile evolution of pupils in the organizing dimension prevents that those proofs could be built without the intervention of the teacher. Here, we identify one of the differences between pupils and experts in the number theory proving process: the balance in terms of control between what is "occurring" in each of two dimensions. In particular, experts, unlike pupils, have the opportunity to catch up with a failure to a given dimension by controlling the other one.

In the first year of University, papers coming from the exam presented previously have been analyzed (73 students including 30 who attended optional number theory course in Grade 12). We focus on the last question that is emblematic of the gap between secondary and higher education as we have highlighted. This question was addressed by half of the students and we find proving elements in 25 papers. Only 4 students (all of them have attended optional number theory course in Grade 12) manage to exploit the indications and solve the system of congruences. Breaking with the project of the author, this system can be rewritten in the form of the Diophantine equation  $16x - 7y = 3$ ; with this change of objects in operative working, the problem becomes the routine task mentioned for Grade 12, also encountered in the first year of University ; all (operative) work on congruences is bypassed. Among the 21 remaining papers, 8 students (3 of them have attended optional number theory course in Grade 12) follow this process and none will be able to solve the system. We have already identified this breaking phenomenon during a training test for the *baccalauréat* (Battie, 2003b): even with a wording apparently fixed on the organizing dimension, Grade 12-pupils are able to gain independence from the organizing dimension behind the wording to build another one of their own. Our hypothesis is that this is not a bifurcation consciously constructed by pupils but a process induced by a choice made in the operative work left to the autonomy of pupils, the trigger being institutional origin. According to us, this realizes the dialectical process that could exist, in pupils' research, between organizing and operative dimensions.

## CONCLUSION

An analysis in terms of organizing and operative dimensions permits to situate the autonomy devolved to pupils and students in the wording of number theory problems, and more generally in exams. This type of analysis also offers the

opportunity to observe the effectiveness of this autonomy for Grade 12-pupils and students from their written productions or, still better, through observing and transcribing their researches. In Grade 12-University transition, we notice a transfer of what is under the responsibility of learners: breaking with the culture of Grade 12-teaching, the skills related to organizing dimension become important at the University. According to us, this transfer is one of the sources of difficulties encountered by students arriving at University to prove in number theory: except for routine tasks, their control of organizing level is very too low. In students' papers that we have analyzed, we have confirmation of this weakness contrasted with the success rate to operative questions (questions 1 and 2 in particular with a higher success rate for students who have attended number theory course in Grade 12). To continue this investigation we aim to focus on the years after the University entrance where we suspect that the transfer of autonomy to the organizing dimension is even more crucial. Another perspective is to test our hypothesis (Battie, 2008) that, in Grade 12 textbooks, too little attention is paid to existing springs to work at organizing dimension.

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