

# Appendix: Study of the notion of pushforward measure in the context of dynamic optimal transport

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In this appendix, we study in detail the notion of *pushforward measure*, more precisely the properties useful to our study which follows in the framework of the optimal transport and the interpolation of McCann. The aims of this appendix is to give a proof for Propositions 2.1 and 2.2, which ones are respectively state in the associate main paper [2] under the numbers 5.1 and 5.2.

Before beginning our study, we shall begin by recalling the statements of some classical theorems of the Measure Theory (section 1). The main part of this document, where are stated and proved Propositions 2.1 and 2.2, follows in the section 2.

## 1 Elements of Measure Theory

### 1.1 Some useful properties of sub-gradients of Lipschitzian functions

**Lemma 1.1.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  a  $L$ -Lipschitz function on an open set  $\omega$ . Then*

$$\sup_{x \in \omega} |\partial f(x)| \left( = \sup_{x \in \omega} \sup_{y \in \partial f(x)} |y| \right) \leq L.$$

We recall that if  $f$  is convex, and differentiable at a point  $x$ , we have  $\partial f(x) = \{\nabla f(x)\}$ .

**Lemma 1.2.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a proper lower semicontinuous convex function, and let  $(x_k)_k$  and  $(\xi_k)_k$  be two sequences of  $\mathbb{R}^d$  which converge in  $\mathbb{R}^d$  respectively to  $x$  and  $\xi$ , and such that*

$$\forall k \in \mathbb{N}, \quad \xi_k \in \partial f(x_k).$$

*Then  $\partial f(x)$  is non-empty and  $\xi \in \partial f(x)$ .*

### 1.2 Elements of Measure Theory

**Lemma 1.3** ([1]). *Let  $f$  be a Lipschitz function  $\mathbb{R}^d$ . Then, for all  $A \subset \mathbb{R}^d$ ,*

$$\mathcal{L}^n(A) = 0 \Rightarrow \mathcal{L}(f(A)) = 0$$

*where  $\mathcal{L}$  is the measure of Lebesgue on  $\mathbb{R}$ .*

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For all subset  $A \subset \mathbb{R}^d$ , we define, for all  $x \in \mathbb{R}^d$ , the function  $\mathbb{1}_A$  by  $\mathbb{1}_A(x) = 1$  if  $x \in A$ , and  $\mathbb{1}_A(x) = 0$  otherwise.

**Theorem 1.1** (Urysohn's theorem, restricted to  $\mathbb{R}^d$ , [3] and [4]). 1. Let  $V$  be an open set of  $\mathbb{R}^d$ , and  $K$  be a compact set of  $\mathbb{R}^d$  such that  $K \subset V$ . Then, there exists  $f \in C_c^\infty(\mathbb{R}^d)$  such that:

$$\mathbb{1}_K \leq f \leq \mathbb{1}_V.$$

2. Let  $V$  be an open set  $\mathbb{R}^d$ , and  $K$  be a compact set of  $\mathbb{R}^d$  such that  $K \subset V$ . Then there exists an open set  $U$  with compact closure, such that

$$K \subset U \subset \bar{U} \subset V.$$

We will write that a measure  $\mu$  (positive or not) verifies the properties (I) if it satisfies the following properties:

**Properties (I).** Let  $\Omega$  be an open set of  $\mathbb{R}^d$ , and  $\mu$  be a measure on the tribe of Lebesgue of  $\Omega$ .

1.  $\mu(K) < +\infty$  for all compact set  $K \subset \Omega$ .

2. For all Lebesgue-measurable set  $E$  of  $\Omega$ , we have

$$\mu(E) = \inf\{\mu(V), E \subset V, V \text{ open}\}.$$

3. For all open set  $E$  and for all Lebesgue-measurable  $E$  of  $\Omega$  such that  $\mu(E) < +\infty$ ,

$$\mu(E) = \sup\{\mu(K), K \subset E, K \text{ compact}\}.$$

**Theorem 1.2** (Riesz Representation Theorem [3]). Let  $\Omega$  an open set of  $\mathbb{R}^d$ , and let  $\Lambda$  a positive linear form on  $C_c(\Omega)$ . Then we can build a positive measure  $\mu_1$  on the tribe of Lebesgue of  $\Omega$  which represents  $\Lambda$ , i.e.

$$\forall f \in C_c(\Omega), \quad \Lambda(f) = \int_{\Omega} f d\mu. \quad (1-1)$$

This measure is defined as follows:

1. For all  $V$  open set of  $\Omega$ ,  $\mu(V) = \sup\{\Lambda(f), f \in C_c(\Omega), f \leq \mathbb{1}_V\}$ .

2. For all Lebesgue-measurable set  $E$  of  $\Omega$ ,  $\mu(E) = \inf\{\mu(V), E \subset V, V \text{ ouvert}\}$ .

This measure  $\mu$  satisfies the properties (I).

**Theorem 1.3** ([3]). Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  two measurable spaces (i.e.  $\mathcal{A}$  and  $\mathcal{B}$  are respectively some  $\sigma$ -algebras on  $X$  and  $Y$ ). We associate  $X \times Y$  with the  $\sigma$ -algebra  $\mathcal{A} \times \mathcal{B}$  (i.e. the smallest  $\sigma$ -algebra containing all the sets of the form  $A \times B$ , with  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ ).

For all  $E \in \mathcal{A} \times \mathcal{B}$ , we define for all  $x \in X$  and  $y \in Y$  the sets

$$E_x = \{y \in Y, (x, y) \in E\} \quad \text{et} \quad E^y = \{x \in X, (x, y) \in E\}.$$

Then for all  $x \in X$  and  $y \in Y$ ,  $E_x \in \mathcal{B}$  and  $E^y \in \mathcal{A}$ .

**Theorem 1.4** ([3]). Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  two  $\sigma$ -finished measured spaces (countable coun-  
cils of finite measure subsets). Let  $Q \in \mathcal{A} \times \mathcal{B}$  (defined in Theorem 1.3). We define for all  $x \in X$   
and any  $y \in Y$ ,

$$\Phi(x) = \nu(Q_x), \quad \text{and} \quad \Psi(y) = \mu(Q^y).$$

Then the function  $\Phi$  is measurable relative to  $\mathcal{A}$  and the function  $\Psi$  is measurable relative to  $\mathcal{B}$ .  
Furthermore

$$\int_X \Phi d\mu = \int_Y \Psi d\nu. \quad (1-2)$$

This last Theorem is a special case of the Fubini Theorem which follows (but in fact goes into  
its proof). We will very often use the Fubini Theorem (most of the time implicitly) in the case  
of a classical and unambiguous inversion of integrals. On the other hand, we will explicitly use  
this Theorem in the more complex cases where classical notions of measurement theory can lead  
to ambiguities.

**Theorem 1.5** (Fubini's Theorem [3]). Let  $(X, \mathcal{A}, \mu)$  and  $(Y, \mathcal{B}, \nu)$  two  $\sigma$ -finished measured spaces.  
Let  $f$  a function with real values defined on  $X \times Y$ , and measurable relative to  $\mathcal{A} \times \mathcal{B}$ .

- For all  $x \in X$ , we define on  $Y$  the function  $y \mapsto f_x(y) = f(x, y)$ .
- For all  $y \in Y$ , we define on  $X$  the function  $x \mapsto f^y(x) = f(x, y)$ .

The following assertions are satisfied:

1. For all  $x \in X$ ,  $f_x$  is measurable relative to  $\mathcal{B}$  and, for all  $x \in X$ ,  $f^y$  is measurable relative  
to  $\mathcal{A}$ .
2. If  $0 \leq f \leq +\infty$ , then, by defining for all  $x \in X$  and  $y \in Y$ ,

$$\Phi(x) = \int_Y f_x d\nu, \quad \text{and} \quad \Psi(y) = \int_X f^y d\mu, \quad (1-3)$$

the function  $\Phi$  is measurable relative to  $\mathcal{A}$  and the function  $\Psi$  is measurable relative to  $\mathcal{B}$ .  
Moreover,

$$\int_X \Phi d\mu = \int_{X \times Y} f (d\mu \otimes d\nu) = \int_Y \Psi d\nu. \quad (1-4)$$

3. If  $f$  is a function with complex values and if we have

$$\int_X \Phi^* d\mu < +\infty \quad \text{with} \quad \Phi^*(x) = \int_Y |f_x| d\nu \quad \text{for all } x \in X,$$

or else

$$\int_Y \Psi^* d\nu < +\infty \quad \text{with} \quad \Psi^*(y) = \int_X |f^y| d\mu \quad \text{for all } y \in Y,$$

then  $f \in L^1(\mu \otimes \nu)$ .

4. If  $f \in L^1(\mu \otimes \nu)$ , then  $f_x \in L^1(\nu)$  for almost all  $x \in X$  and  $f^y \in L^1(\mu)$  for almost all  $y \in Y$ .  
Moreover, if we define  $\Phi$  and  $\Psi$  as in relations (1-3) (defined here only for, respectively,  
almost all  $x \in X$  and almost all  $y \in Y$ ), then  $\Phi$  and  $\Psi$  belong respectively to  $L^1(\mu)$  and  
 $L^1(\nu)$ . Moreover, the relation (1-4) is always satisfied by  $\Phi$  and  $\Psi$ .

**Theorem 1.6** (Radon-Nikodym Theorem [3]). *Let  $\Omega$  be an open set of  $\mathbb{R}^d$  and  $\mathcal{A}$  be a  $\sigma$ -algebra on  $\Omega$ . Let  $\mu$  and  $\lambda$  be two measures on  $\mathcal{A}$ , such that  $\mu$  is a bounded positive measure and  $\lambda$  any measure (positive or complex). If  $\lambda$  is absolutely continuous with respect to  $\mu$  (denoted  $\lambda \ll \mu$ ) then there exists a unique  $h \in L^1(\mu)$  such that  $\lambda = h \cdot \mu$  on  $\mathcal{A}$ , i.e.*

$$\forall E \in \mathcal{A}, \quad \lambda(E) = \int_E h d\mu.$$

**Theorem 1.7** (Radon-Nikodym Theorem, specific variant, [3] and [1]). *Let  $\lambda$  be a positive measure on the tribe of Lebesgue, finite on every compact subset of  $\mathbb{R}^d$ , and absolutely continuous with respect to the Lebesgue measure  $\mathcal{L}^d$ . Then we introduce the function  $\overline{D}\lambda$  of  $\mathbb{R}^d$  in  $\mathbb{R}$ , denoted "upper derivative" of  $\lambda$  with respect to  $\mathcal{L}^d$ , and defined by*

$$\forall x \in \mathbb{R}^d, \quad (\overline{D}\lambda)(x) = \limsup_{r \rightarrow 0} f_r(x), \quad \text{with} \quad f_r(x) = \frac{1}{\mathcal{V}_d(B_1)r^d} \lambda(B(x, r)), \quad r > 0,$$

where  $B(x, r)$  denotes the open ball with center  $x$  and radius  $r$ , and  $\mathcal{V}_d(B_1)$  the unit ball volume in dimensions  $d$ . We have the following properties:

1. For  $\mathcal{L}^d$ -almost all  $x \in \mathbb{R}^d$ , the family  $(f_r(x))_{r>0}$  converges when  $r$  tends to 0 (and we have  $\lim_{r \rightarrow 0, r>0} f_r(x) = (\overline{D}\lambda)(x)$ ). We then write that  $\lambda$  is differentiable in  $x$  with respect to  $\mathcal{L}^d$ .
2.  $\overline{D}\lambda$  is Lebesgue-measurable on  $\mathbb{R}^d$  and  $\overline{D}\lambda \in L^1(\mathbb{R}^d)$ .
3. For all Lebesgue-measurable set  $E$ ,

$$\lambda(E) = \int_E (\overline{D}\lambda)(x) d\mathcal{L}^d(x).$$

**Proposition 1.1** ([3]). *Let  $\Omega$  be an open set  $\mathbb{R}^d$ , and let  $\mu$  be a measure on a  $\sigma$ -algebra  $\mathcal{A}$  of  $\Omega$ . Then there exists a measurable function  $f$  such that  $|f(x)| = 1$  for all  $x \in \Omega$ , and such that*

$$d\mu = h d|\mu|, \tag{1-5}$$

where  $|\mu|$  refers to the total variation of the measure  $\mu$ .

**Theorem 1.8** (Generalized Riesz Representation Theorem [3]). *Let  $\Omega$  be an open set of  $\mathbb{R}^d$ , and let  $T$  be a continuous linear form on  $C(\Omega)$ . Then, there exists a unique measure defined on the tribe of Lebesgue (not necessarily positive), satisfying the properties (I), and which represents  $T$ , i.e.*

$$\forall f \in C(\Omega), \quad T(f) = \int_{\Omega} f d\mu, \tag{1-6}$$

with  $\|T\| = |\mu|(\Omega)$ .

**Theorem 1.9** (Lusin Theorem [3]). *Let  $\Omega$  be an open set of  $\mathbb{R}^d$ , and let  $\mu$  be a positive measure satisfying the properties (I).*

*Let  $A \subset \Omega$  be a measurable set such that  $\mu(A) < +\infty$ , and let  $f$  be a measurable function on  $\Omega$  (for the tribe of Lebesgue) which vanishes outside of  $A$ . So, for all  $\epsilon > 0$ , there exists  $g \in C_c(\Omega)$  such that*

$$\mu(\{x, f(x) \neq g(x)\}) < \epsilon, \quad \text{and} \quad \sup_{x \in \Omega} |g(x)| \leq \sup_{x \in \Omega} |f(x)|. \tag{1-7}$$

**Corollary 1.1** ([3]). *With the assumptions of Theorem 1.9, if in addition there exists a constant  $M > 0$  such that  $|f| \leq M$ , then there exists a sequence  $(g_n)_n \in C_c(\Omega)^\mathbb{N}$ , with  $|g_n| \leq M$  for all  $n \in \mathbb{N}$ , such that*

$$f(x) = \lim_{n \rightarrow +\infty} g_n(x), \quad \mu - \text{a.e.}$$

**Corollary 1.2.** *With the assumptions of Theorem 1.9, if in addition  $f \in L^1_{loc}(\Omega, \mu)$ , then*

$$\forall g \in C_c(\Omega), \quad \int_{\Omega} f g d\mu = 0 \implies f = 0, \quad \mu - \text{a.e.}$$

## 2 The pushforward measure

Let us begin by defining more precisely the notion of measurement push (Proposition 2.1 below). For this we will need the following two Lemmas 2.1 and 2.2.

**Lemma 2.1.** *Let  $V$  be an open set  $\mathbb{R}^d$ . There exists a sequence  $(K_n)_{n \in \mathbb{N}}$  of open sets in  $\mathbb{R}^d$  such that for all  $n \in \mathbb{N}$ ,  $K_n \subset K_{n+1}$ , and*

$$V = \bigcup_{n \in \mathbb{N}} K_n.$$

*Proof:* Let us first assume that  $V$  is bounded. Its adherence  $\bar{V}$  is therefore a compact subset of  $\mathbb{R}^d$ . Thus, according to the Borel-Lebesgue property, for every  $n \in \mathbb{N}$ , there exists  $k_n \in \mathbb{N}$  and  $x_0^n, \dots, x_{k_n}^n \in \bar{V}$  such that

$$\bar{V} \subset \bigcup_{i=0..k_n} \bar{B}\left(x_i^n, \frac{1}{n+1}\right), \quad (2-8)$$

where  $\bar{B}(x, r)$  denotes the closed ball with center  $x$  and radius  $r$ . For all  $n \in \mathbb{N}$ , we define

$$A_n = \left\{ i \in \llbracket 0, k_n \rrbracket, \bar{B}\left(x_i^n, \frac{1}{n+1}\right) \subset V \right\} \quad \text{and} \quad K_n = \bigcup_{i \in A_n} \bar{B}\left(x_i^n, \frac{1}{n+1}\right) \subset V,$$

$K_n$  being a compact set because  $A_n$  is finite. Let us show that  $V$  is the union of  $K_n$ . First of all, it is clear that

$$\bigcup_{n \in \mathbb{N}} K_n \subset V.$$

Let  $x \in V$  and  $r > 0$  such that  $B(x, r) \subset V$  ( $B(x, r)$  is an open ball with center  $x$  and radius  $r$ ), and let  $n \in \mathbb{N}$  be such that  $1/(n+1) < r/2$ . According to (2-8), there exists  $x_i^n$  such that  $x \in \bar{B}(x_i^n, 1/(n+1))$ . Therefore, for all  $y \in \bar{B}(x_i^n, 1/(n+1))$ , we have

$$|x - y| \leq |x - x_i^n| + |x_i^n - y| \leq \frac{1}{n+1} + \frac{1}{n+1} < r,$$

then

$$\bar{B}\left(x_i^n, \frac{1}{n+1}\right) \subset B(x, r) \subset V,$$

so that  $i \in A_n$  and  $\bar{B}(x_i^n, 1/(n+1)) \subset K_n$ . It follows that  $x \in K_n$ , and we can thus conclude that

$$V \subset \bigcup_{n \in \mathbb{N}} K_n.$$

Finally, if we no longer suppose that  $V$  is bounded, we can always use the fact that  $V$  can be written as a countable union of bounded open sets (for example, the union of  $B(0, n) \cap V$ ).

Therefore, as a countable union of compact countable union,  $V$  is the union of a countable family  $(K_n)_{n \in \mathbb{N}}$  of compact sets of  $\mathbb{R}^d$ .

With the possible need to define for all  $n \in \mathbb{N}$ ,

$$K'_n = \bigcup_{j=0}^n K_j,$$

we can consider that this family is increasing in the sense of inclusion.  $\square$

**Lemma 2.2.** *Let  $K$  be a compact set of  $\mathbb{R}^d$ . There exists a sequence  $(V_n)_{n \in \mathbb{N}}$  of open sets in  $\mathbb{R}^d$  such that for all  $n \in \mathbb{N}$ ,  $V_{n+1} \subset V_n$ , and*

$$K = \bigcap_{n \in \mathbb{N}} V_n.$$

Moreover, there exists a bounded open set  $V'$  of  $\mathbb{R}^d$  such that for all  $n \in \mathbb{N}$ ,  $V_n \subset V'$ .

*Proof:* By the Borel-Lebesgue property, for all  $n \in \mathbb{N}$ , there exists  $k_n \in \mathbb{N}$  et  $x_0^n, \dots, x_{k_n}^n \in K$  such that

$$K \subset \bigcup_{i=0..k_n} B\left(x_i^n, \frac{1}{n+1}\right), \quad (2-9)$$

where  $B(x, r)$  denotes the open ball with center  $x$  and radius  $r$ . We thus define, for all  $n \in \mathbb{N}$

$$V_n = \bigcup_{i=0..k_n} B\left(x_i^n, \frac{1}{n+1}\right).$$

It is clear that

$$K \subset \bigcap_{n \in \mathbb{N}} V_n.$$

Let  $x \in \bigcap_{n \in \mathbb{N}} V_n$ . For all  $n \in \mathbb{N}$ , there exists  $i_n \in \llbracket 0, k_n \rrbracket$  such that  $x \in B\left(x_{i_n}^n, \frac{1}{n+1}\right)$ .

The sequence  $(x_{i_n}^n)_{n \in \mathbb{N}}$  (elements of  $K$ ) then converges towards  $x$ , and therefore  $x \in K$ . Thus

$$\bigcap_{n \in \mathbb{N}} V_n \subset K.$$

With the possible need to define for all  $n \in \mathbb{N}$

$$V'_n = \bigcap_{j=0}^n V_j,$$

we can consider that the family  $(V_n)_{n \in \mathbb{N}}$  is decreasing in the sense of inclusion.

It will further be noted that for any  $n \in \mathbb{N}$ , we have  $V_n \subset V'$ ,  $V'$  being the open bounded set defined by

$$V' = \bigcup_{x \in K} B(x, 1).$$

$\square$

**Proposition 2.1.** *We associate  $\mathbb{R}^d$  with the tribe of Lebesgue. Let  $\mu$  be a positive measure  $\sigma$ -finite on the tribe of Lebesgue, and let  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  measurable. It is assumed that one of the following two hypotheses is satisfied:*

1. The measure  $\mu$  is finite,
2. The application  $T$  is coercive with respect to  $\mu$ , i.e.  $T^{-1}(A)$  is bounded  $\mu$ -almost everywhere for any bounded Lebesgue-measurable set  $A$  in  $\mathbb{R}^d$ .

Then there exists a positive measure on the Lebesgue tribe  $\nu$  on  $\mathbb{R}^d$ , satisfying the properties (I), such that

$$\forall f \in C_c^0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f d\nu = \int_{\mathbb{R}^d} f(Tx) d\mu. \quad (2-10)$$

Moreover, for every Lebesgue-measurable set  $A \subset \mathbb{R}^d$ , we have  $\nu(A) = \mu(T^{-1}(A))$ .

We then say that  $\nu$  is **the pushforward of  $\mu$  by the operator  $T$** , denoted as

$$\nu = T\#\mu.$$

*Proof:* The existence of a positive measure on the tribe of Lebesgue  $\nu$  on  $\Omega$ , satisfying the properties (I), and relation (2-10) comes from the application of Theorem 1.2 to the positive linear form on  $C_c^0(\mathbb{R}^d)$  defined by

$$f \mapsto \int_{\mathbb{R}^d} f d\nu = \int_{\mathbb{R}^d} f(Tx) d\mu.$$

This linear form is well defined if one of the two hypotheses of the statement is verified ( $\mu$  is finite or  $T$  is coercive).

The less trivial part of this proof will therefore concern the last point of the statement. We shall first show the property for the bounded open sets and the compact sets of  $\mathbb{R}^d$ , and then generalize it to all Lebesgue-measurable sets.

1. Let  $V$  an bounded open set of  $\mathbb{R}^d$ . According to Lemma 2.1, there exists a sequence  $(K_n)_{n \in \mathbb{N}}$  of compact sets, increasing in the meaning of inclusion, such that  $V$  is the union of the  $K_n$ . For all  $n \in \mathbb{N}$ , Theorem 1.1 gives us the existence of  $f_n \in C_c^0(\Omega)$  such that  $\mathbf{1}_{K_n} \leq f_n \leq \mathbf{1}_V$ . The sequences of indicator functions  $(\mathbf{1}_{K_n})_n$  simply converges towards  $\mathbf{1}_V$ , the same holds for the sequence  $(f_n)_n$ , and then for the sequence  $(f_n \circ T)_n$  towards  $\mathbf{1}_V \circ T$ .

According to the first points of the properties (I), we have  $\nu(V) \leq \nu(\overline{V}) < +\infty$ . Let  $V'$  be an open bounded set  $\mathbb{R}^d$  such that  $\overline{V} \subset V'$ . According to Theorem 1.1, there exists a function  $g \in C_c^0(\Omega)$  such that  $\mathbf{1}_{\overline{V}} \leq g \leq \mathbf{1}_{V'}$ . Hence we have

$$f_n \circ T \leq \mathbf{1}_V \circ T \leq \mathbf{1}_{\overline{V}} \circ T \leq g \circ T$$

for all  $n \in \mathbb{N}$ , and

$$\int_{\mathbb{R}^d} g(Tx) d\mu = \int_{\mathbb{R}^d} g d\nu \leq \nu(\overline{V'}) < +\infty.$$

Thus by dominated convergence, we have

$$\begin{aligned} \nu(V) &= \int_{\mathbb{R}^d} \mathbf{1}_V d\nu = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f_n d\nu = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} f_n(Tx) d\mu \\ &= \int_{\mathbb{R}^d} \mathbf{1}_V(Tx) d\mu = \mu(T^{-1}(V)). \end{aligned} \quad (2-11)$$

2. Let  $K$  be a compact set of  $\mathbb{R}^d$ . We argue like above: Lemma 2.2 gives us the existence of a sequence  $(V_n)_{n \in \mathbb{N}}$  of openings, decreasing in the sense of inclusion, such that  $K$  is the intersection of the  $V_n$  and of a bounded open set  $V'$  such that for every  $n \in \mathbb{N}$ , we have  $V_n \subset V'$ . Moreover, for any  $n \in \mathbb{N}$ , Theorem 1.1 gives us the existence of a  $f_n \in C_c^0(\Omega)$  such that  $\mathbb{1}_K \leq f_n \leq \mathbb{1}_{V_n} \leq \mathbb{1}_{V'}$ . The sequence of  $(\mathbb{1}_{V_n})_n$  converges simply towards  $\mathbb{1}_K$ , and then the sequence  $(f_n)_n$  converges to the same limit. Therefore, the sequence  $(f_n \circ T)_n$  converges simply towards  $\mathbb{1}_K \circ T$ , with the upper bound  $f_n \circ T \leq \mathbb{1}_{V'} \circ T$ , with according to the previous point

$$\int_{\mathbb{R}^d} \mathbb{1}_{V'}(Tx) d\mu = \int_{\mathbb{R}^d} \mathbb{1}_{V'} d\nu \leq \nu(V') \leq \nu(\overline{V'}) < +\infty.$$

We can then deduce by dominated convergence that  $\nu(K) = \mu(T^{-1}(K))$ .

3. Let  $A$  be a bounded Lebesgue-measurable set of  $\mathbb{R}^d$ . The measure  $\nu$  satisfies the properties (I), thus especially by the points 2 and 3:

$$\nu(A) = \inf\{\nu(V), A \subset V, V \text{ bounded open set}\}$$

and

$$\nu(A) = \sup\{\nu(K), K \subset A, K \text{ compact set}\}. \quad (2-12)$$

Notice that we can consider that we minimize on bounded open sets since  $A$  is assumed to be bounded. For all compact set  $K$  and all open set  $V$  such that  $K \subset A \subset V$ , we have

$$\nu(K) = \mu(T^{-1}(K)) \leq \mu(T^{-1}(A)) \leq \mu(T^{-1}(V)) = \nu(V),$$

so that

$$\begin{aligned} \nu(A) &= \sup\{\nu(K), K \subset A, K \text{ compact set}\} \\ &= \sup\{\mu(T^{-1}(K)), K \subset A, K \text{ compact set}\} \leq \mu(T^{-1}(A)) \\ &\leq \inf\{\mu(T^{-1}(V)), A \subset V, V \text{ bounded open set}\} \\ &= \inf\{\nu(V), A \subset V, V \text{ bounded open set}\} = \nu(A) \end{aligned} \quad (2-13)$$

and  $\nu(A) = \mu(T^{-1}(A))$ . Finally, considering that  $A$  is any Lebesgue-measurable set, it can be written as a countable union  $(A_n)_n$  of bounded Lebesgue-measurable sets (for example the union of the  $A_n = B(0, n) \cap A$ ). Thus

$$\begin{aligned} \nu(A) &= \lim_{n \rightarrow +\infty} \nu\left(\bigcup_{k=0}^n A_k\right) = \lim_{n \rightarrow +\infty} \mu\left(T^{-1}\left(\bigcup_{k=0}^n A_k\right)\right) = \lim_{n \rightarrow +\infty} \mu\left(\bigcup_{k=0}^n T^{-1}(A_k)\right) \\ &= \mu\left(\bigcup_{k \in \mathbb{N}} T^{-1}(A_k)\right) = \mu\left(T^{-1}\left(\bigcup_{k \in \mathbb{N}} A_k\right)\right) = \mu(T^{-1}(A)). \end{aligned} \quad (2-14)$$

□

Let us now turn to McCann's interpolation: Proposition 2.2 below. We recall that the McCann's interpolations measures  $\mu_t$ , between a measure  $\mu$  and the measure  $T\#\mu$ , are defined for all  $t \in [0, 1]$  by  $\mu_t = [(1-t)\text{id} + tT]\#\mu$ .

As we have shown in our main paper [2], we can consider that  $\phi$  satisfy the property  $(\Gamma_1)$ , that is to say  $\phi$  and  $\phi^*$  are convex, continuous and achieve a minimum on  $\mathbb{R}^d$ .

As we will often treat about the interpolated functions of the transport potential  $\phi$ , we first need to state the following small Lemma:



**Lemma 2.3.** *Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the property  $(\Gamma_1)$ . Then, for all  $t \in [0, 1]$ , the interpolated function  $t\phi + (1-t)|\cdot|^2/2$  also satisfies the property  $(\Gamma_1)$ .*

*Proof:* This is a direct consequence of the fact that  $(t\phi + (1-t)|\cdot|^2/2)^* = {}^{1-t}(t\phi)$  for all  $t \in [0, 1]$ .  $\square$

We recall the definition and the properties of the operator  $\mathbf{p}$  (see [2]):

**Definition 2.1** (Operator  $\mathbf{p}$ ). *Let  $\phi : \mathbb{R}^d \mapsto \mathbb{R}$  satisfying the property  $(\Gamma_1)$  (especially  $\phi$  is convex and continuous at every point of  $\mathbb{R}^d$ , and admitting in each of these point a non-empty and compact sub-differential). The operator  $\mathbf{p}$  is defined as*

$$\begin{aligned} \mathbf{p}_\phi : [0, 1) \times \mathbb{R}^d &\rightarrow \mathbb{R}^d \\ (t, x) &\mapsto \text{Prox}_{\frac{t}{1-t}\phi} \left( \frac{x}{1-t} \right) \end{aligned}$$

$\mathbf{p}_\phi$  satisfies the following properties:

1. for all  $t \in [0, 1)$ ,  $\mathbf{p}_\phi(t, \cdot)$  is  $1/(1-t)$ -Lipschitz,
2. if  $t \in (0, 1)$ , by setting  $\phi_t = (1-t)|\cdot|^2/2 + t\phi$ , then  $(\phi_t)^*$  is of class  $C^1$  on  $\mathbb{R}^d$  and we have:

$$\mathbf{p}_\phi(t, \cdot) = \nabla_x(\phi_t)^*, \quad (2-15)$$

3. for all  $t \in [0, 1)$ ,  $\mathbf{p}_\phi(t, \cdot)$  is surjective on  $\mathbb{R}^d$  and for all  $x, y \in \mathbb{R}^d$ ,

$$y = \mathbf{p}_\phi(t, x) \Leftrightarrow x \in (1-t)y + t\partial\phi(y). \quad (2-16)$$

4. for all  $t \in (0, 1)$  and  $x \in \mathbb{R}^d$ , the velocity  $v$  introduced in [2] can be defined from  $\mathbf{p}_\phi$  by:

$$v_\phi(t, x) = \frac{x - \mathbf{p}_\phi(t, x)}{t}. \quad (2-17)$$

Before coming to Proposition 2.2, let us state the necessary following Lemma. For  $f$  and  $g$  two measurable functions, we will often denote  $f = T\#g$  instead of  $f\mathcal{L}^d = T\#(g\mathcal{L}^d)$ .

**Lemma 2.4.** *Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the property  $(\Gamma_1)$ , and  $\mathbf{p}$  defined per  $\phi$  according to Definition 2.1, that is to say:*

$$\mathbf{p}(t, \cdot) = \mathbf{p}_\phi(t, \cdot) = \text{Prox}_{\frac{t}{1-t}\phi} \left( \frac{x}{1-t} \right) = \nabla_x(\phi_t)^*$$

where  $(\phi_t)^*$  refers to the Legendre transform of  $\phi_t = (1-t)|\cdot|^2/2 + t\phi$ .

For all  $r > 0$ , the function

$$\mathcal{G} : (t, x, y) \mapsto \mathbf{1}_{\mathbf{p}(t, \overline{B(x, r)})}(y)$$

is Lebesgue-measurable on  $[0, 1) \times \mathbb{R}^d \times \mathbb{R}^d$ .

*Proof:* Let  $r > 0$ . Let us first note that  $\mathcal{G} = \mathbf{1}_F$  with

$$F = \left\{ (t, x, y) \in [0, 1) \times \mathbb{R}^d \times \mathbb{R}^d, y \in \mathbf{p} \left( t, \overline{B(x, r)} \right) \right\}.$$

Thus, to prove that  $\mathcal{G}$  is Lebesgue-measurable, it is sufficient to show that  $F$  is an element of the Lebesgue tribe. We will prove more precisely that  $F$  is a closed set into  $[0, 1) \times \mathbb{R}^d \times \mathbb{R}^d$ .

Let  $(t_n, x_n, y_n)_n$  a sequence of elements of  $F$  converging to  $(t^*, x^*, y^*) \in [0, 1) \times \mathbb{R}^d \times \mathbb{R}^d$ . Let us show that  $(t^*, x^*, y^*) \in F$ . For all  $n \in \mathbb{N}$ ,

$$\begin{aligned}
(t_n, x_n, y_n) \in F &\Leftrightarrow y_n \in \mathbf{p}\left(t_n, \overline{B(x_n, r)}\right) \\
&\Rightarrow \exists z_n \in \overline{B(x_n, r)}, y_n = \mathbf{p}(t_n, z_n) \\
&\Leftrightarrow \exists z_n \in \overline{B(x_n, r)}, z_n \in t_n \partial\phi(y_n) + (1 - t_n)y_n \\
&\Leftrightarrow \exists z_n \in \overline{B(x_n, r)}, \exists \xi_n \in \partial\phi(y_n), z_n = t_n \xi_n + (1 - t_n)y_n.
\end{aligned} \tag{2-18}$$

We therefore consider the sequences  $(z_n)_n$  and  $(\xi_n)_n$  as defined in the relation (2-18). The sequence  $(y_n)_n$  is convergent, therefore bounded. Let  $\omega$  be a bounded open containing all the  $y_n$  for  $n \in \mathbb{N}$ . Potential  $\phi$  satisfies the property  $(\Gamma_1)$ , so in particular  $\phi$  is locally Lipschitz on  $\mathbb{R}^d$ , so it is Lipschitz on  $\omega$ . According to Lemma 1.1,  $\partial\phi(\omega)$  is bounded in  $\mathbb{R}^d$ : the sequence  $(\xi_n)_n$  is therefore bounded in  $\mathbb{R}^d$ , so it is the same for the sequence  $(z_n)_n$ . With the possible need to have to extract a subsequence, we can then consider that  $(\xi_n)_n$  and  $(z_n)_n$  converge in  $\mathbb{R}^d$ . We write  $\xi^*$  and  $z^*$  their respective limits. According to Lemma 1.2, we have  $\xi^* \in \partial\phi(y^*)$ , and then

$$z^* = t^* \xi^* + (1 - t^*)y^* \in t^* \partial\phi(y^*) + (1 - t^*)y^* \Rightarrow y^* = \mathbf{p}(t^*, z^*).$$

Moreover, for all  $n \in \mathbb{N}$ , we have  $\|x_n - z_n\| \leq r$ . At the limit, we then have  $\|x^* - z^*\| \leq r$ , i.e.  $z^* \in \overline{B(x^*, r)}$ . Thus,  $y^* \in \mathbf{p}\left(t^*, \overline{B(x^*, r)}\right)$ , i.e.  $(t^*, x^*, y^*) \in F$ .  $\square$

**Proposition 2.2.** *Let  $\rho_0 \in L^1(\mathbb{R}^d, d\mathcal{L}^d)$  with compact support, such that  $\rho_0 \geq 0$  and  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  verify the Property  $(\Gamma_1)$ . Let  $\rho_0 \in L^1(\mathbb{R}^d)$  be a compact support, such that  $\rho_0 \geq 0$ , and  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the property  $(\Gamma_1)$ . Then, for all  $t \in [0, 1)$  there exists a positive measure  $\nu_t$  on the Lebesgue of  $\mathbb{R}^d$ , with bounded support in  $(t\nabla\phi + (1 - t)\text{id})(\text{supp}(\rho_0))$ , satisfying the properties (I), and such that  $\nu_t = (t\nabla\phi + (1 - t)\text{id})\#(\rho_0\mathcal{L}^d)$ , i.e.*

$$\forall f \in C_c^0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f d\nu_t = \int_{\mathbb{R}^d} f(t\nabla\phi(x) + (1 - t)x)\rho_0(x) d\mathcal{L}^d(x). \tag{2-19}$$

Moreover, there exists  $\rho_t \in L^1(\mathbb{R}^d)$  such that  $\nu_t = \rho_t\mathcal{L}^d$  ( $\nu_t \ll \mathcal{L}^d$ ). It is also possible, for any  $t \in [0, 1)$ , to choose a representative of  $\rho_t$  in such a way that  $(t, x) \mapsto \rho_t(x)$  is measurable on  $[0, 1) \times \mathbb{R}^d$ .

Finally, the following properties are satisfied:

1. For all  $h \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$  and  $t \in [0, 1)$ , we have

- $h \circ (t\nabla\phi + (1 - t)\text{id}) \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$ ,
- $\int_{\mathbb{R}^d} h d\nu_t = \int_{\mathbb{R}^d} h(x)\rho_t(x) d\mathcal{L}^d(x) = \int_{\mathbb{R}^d} h(t\nabla\phi(x) + (1 - t)x)\rho_0(x) d\mathcal{L}^d(x)$ .

2. For all  $h \in L_{loc}^\infty([0, 1) \times \mathbb{R}^d, \mathcal{L}^{d+1})$ , we have

- $(t, x) \mapsto h(t, t\nabla\phi(x) + (1 - t)x) \in L_{loc}^\infty([0, 1) \times \mathbb{R}^d)$ ,
- $\int_0^1 \int_{\mathbb{R}^d} h(t, x)\rho_t(x) d\mathcal{L}^d(x) d\mathcal{L}(t) = \int_0^1 \int_{\mathbb{R}^d} h(t, t\nabla\phi(x) + (1 - t)x)\rho_0(x) d\mathcal{L}^d(x) d\mathcal{L}(t)$ .

3. For all  $h \in C_c^0([0, 1) \times \mathbb{R}^d)$ ,  $t \mapsto \int_{\mathbb{R}^d} h(t, \cdot) d\nu_t = \int_{\mathbb{R}^d} h \rho_t d\mathcal{L}^d$  is continuous on  $[0, 1)$ , in other words  $t \mapsto \nu_t$  is continuous from  $[0, 1)$  to  $\mathcal{D}'(\mathbb{R}^d)$ .

*Proof:* For all  $t \in [0, 1)$ , we apply Proposition 2.1 for  $\mu = \rho_0 \mathcal{L}^d$  and  $T = t\nabla\phi + (1-t)\text{id}$ . Here, since  $\rho_0 \in L^1(\mathbb{R}^d, d\mathcal{L}^d)$ ,  $\rho_0 \mathcal{L}^d$  is a finite measure. Proposition 2.1 therefore gives the existence for  $t \in [0, 1)$  of a positive measure  $\nu_t$  on the tribe of Lebesgue of  $\mathbb{R}^d$  as in Theorem 1.2, satisfying the properties (I), and the relation (2-19):

$$\forall f \in C_c^0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f d\nu_t = \int_{\mathbb{R}^d} f(t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x).$$

Thus, for all  $t \in [0, 1)$  and for all  $f \in C_c^0(\mathbb{R}^d)$  such that:

$$\text{supp}(f) \subset \mathbb{R}^d \setminus (t\nabla\phi + (1-t)\text{id})(\text{supp}(\rho_0)),$$

we have

$$\int_{\mathbb{R}^d} f d\nu_t = \int_{\mathbb{R}^d} f(t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x) = 0.$$

Then  $\text{supp}(\nu_t) \subset (t\nabla\phi + (1-t)\text{id})(\text{supp}(\rho_0))$  and,  $\text{supp}(\rho_0)$  being bounded, it is the same for  $(t\nabla\phi + (1-t)\text{id})(\text{supp}(\rho_0))$  according to Lemma 1.1.

Moreover, for all Lebesgue-measurable set  $A$  from  $\mathbb{R}^d$ , we have

$$\nu_t(A) = (\rho_0 \mathcal{L}^d) \left( (t\nabla\phi + (1-t)\text{id})^{-1}(A) \right) = (\rho_0 \mathcal{L}^d) (\mathbf{p}(t, A)) = \int_{\mathbf{p}(t, A)} \rho_0 d\mathcal{L}^d(x) \quad (2-20)$$

Indeed, recall that for every  $x \in \mathbb{R}^d$  where  $\phi$  is differentiable, we have from Definition 2.1,

$$x = \mathbf{p}(t, y) \Leftrightarrow y = (1-t)x + t\nabla\phi(x).$$

For all  $t \in [0, 1)$ ,  $\mathbf{p}(t, \cdot)$  is Lipschitz (according to Definition 2.1), therefore, according to Lemma 1.3, if  $\mathcal{L}^d(A) = 0$ , then  $\mathcal{L}^d(\mathbf{p}(t, A)) = 0$  and  $\nu_t(A) = \int_{\mathbf{p}(t, A)} \rho_0 d\mathcal{L}^d(x) = 0$ . The measure  $\nu_t$  is therefore absolutely continuous with respect to the Lebesgue measure  $\mathcal{L}^d$ .

For any  $t \in [0, 1)$  and all  $x \in \mathbb{R}^d$ , we define

$$\rho_t(x) = (\overline{D}\nu_t)(x) = \limsup_{r \rightarrow 0} \frac{1}{\mathcal{V}_d(B_1)r^d} \nu_t(B(x, r)), \quad r > 0, \quad (2-21)$$

as described in Theorem 1.7. Here, according to (2-20), for all  $r > 0$  and  $(t, x) \in [0, 1) \times \mathbb{R}^d$ ,

$$\nu_t(B(x, r)) = \int_{\mathbf{p}(t, B(x, r))} \rho_0 d\mathcal{L}^d = \int_{\mathbb{R}^d} \mathbf{1}_{\mathbf{p}(t, B(x, r))}(y) \rho_0(y) d\mathcal{L}^d(y).$$

Note that for all  $(t, x) \in [0, 1) \times \mathbb{R}^d$ ,

$$\mathcal{L}^d(\mathbf{p}(t, B(x, r))) \leq \mathcal{L}^d\left(\mathbf{p}\left(t, \overline{B(x, r)}\right)\right) \leq \mathcal{L}^d(\mathbf{p}(t, B(x, r))) + \mathcal{L}^d(\mathbf{p}(t, S(x, r))),$$

with  $S(x, r) = \overline{B(x, r)} \setminus B(x, r)$ . We have  $\mathcal{L}^d(S(x, r)) = 0$ , and  $\mathbf{p}(t, \cdot)$  is Lipschitz on  $\mathbb{R}^d$ , then according to Lemma 1.3,  $\mathcal{L}^d(\mathbf{p}(t, S(x, r))) = 0$ . Thus

$$\mathcal{L}^d(\mathbf{p}(t, B(x, r))) = \mathcal{L}^d\left(\mathbf{p}\left(t, \overline{B(x, r)}\right)\right), \quad \text{and then} \quad \mathbf{1}_{\mathbf{p}(t, B(x, r))}(y) = \mathbf{1}_{\mathbf{p}(t, \overline{B(x, r)})}(y)$$

for  $\mathcal{L}^d$ -almost all  $y \in \mathbb{R}^d$ . Hence we have

$$\nu_t(B(x, r)) = \int_{\mathbb{R}^d} \mathbf{1}_{\mathbf{p}(t, B(x, r))}(y) \rho_0(y) d\mathcal{L}^d(y) = \int_{\mathbb{R}^d} \mathbf{1}_{\mathbf{p}(t, \overline{B(x, r)})}(y) \rho_0(y) d\mathcal{L}^d(y).$$

According to Lemma 2.4, the function  $(t, x, y) \mapsto \mathbf{1}_{\mathbf{p}(t, \overline{B(x, r)})}(y)$  is Lebesgue-measurable on  $[0, 1) \times \mathbb{R}^d \times \mathbb{R}^d$ . According to Theorem 1.5, the function

$$(t, x) \mapsto \nu_t(B(x, r)) = \int_{\mathbb{R}^d} \mathbf{1}_{\mathbf{p}(t, \overline{B(x, r)})}(y) \rho_0(y) d\mathcal{L}^d(y)$$

is then measurable on  $[0, 1) \times \mathbb{R}^d$ . Therefore, the function  $(t, x) \mapsto \rho_t(x)$  is **measurable** on  $[0, 1) \times \mathbb{R}^d$ : it is a simple limit-sup of a countable sequence of measurable functions. Finally, according to Fubini Theorem 1.5, for all  $t \in [0, 1)$  the function  $x \mapsto \rho_t(x)$  is measurable on  $\mathbb{R}^d$  and, according to Theorem 1.7,  $\nu_t = \rho_t \mathcal{L}^d$ , i.e.

$$\forall t \in [0, 1), \nu_t(E) = \int_E \rho_t(x) d\mathcal{L}^d(x),$$

for all Lebesgue-measurable  $E$  subset of  $\mathbb{R}^d$ . Furthermore, we have

$$\int_{[0, 1) \times \mathbb{R}^d} \rho_t(x) (dx \otimes dt) = \int_0^1 \int_{\mathbb{R}^d} \rho_t(x) dx dt = \int_0^1 \nu_t(\mathbb{R}^d) dt = \int_{\mathbb{R}^d} \rho_0 d\mathcal{L}^d < +\infty.$$

Indeed, according to Definition 2.1, for all  $t \in [0, 1)$ ,  $\mathbf{p}(t, \cdot)$  is surjective from  $\mathbb{R}^d$  onto  $\mathbb{R}^d$ . Then from (2-20):

$$\nu_t(\mathbb{R}^d) = \int_{\mathbf{p}(t, \mathbb{R}^d)} \rho_0 d\mathcal{L}^d = \int_{\mathbb{R}^d} \rho_0 d\mathcal{L}^d.$$

Thus  $(t, x) \mapsto \rho_t(x) \in L^1([0, 1) \times \mathbb{R}^d)$ . The uniqueness of such a function  $(t, x) \mapsto \rho_t(x)$  comes from Corollary 1.2.

To conclude the proof of Proposition 2.2, we finally have to verify the three properties defined in the statement.

1. Let  $h \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$ ,

- Let  $V$  be a bounded open set of  $\mathbb{R}^d$  and  $t \in [0, 1)$ . The image set  $(t\nabla\phi + (1-t)\text{id})(V)$  is always bounded according to Lemma 1.1, since  $\phi$  verifies the property  $(\Gamma_1)$ . There exists therefore a constant  $M > 0$  such that, by defining

$$A = \{y \in (t\nabla\phi + (1-t)\text{id})(V), |h(y)| > M\},$$

we have  $\mathcal{L}^d(A) = 0$ . For all  $x \in V$  such that  $\phi$  is differentiable in  $x$  (remind that the set of points for which it is not the case is a zero measure set),

$$|h(t\nabla\phi(x) + (1-t)x)| > M \Leftrightarrow t\nabla\phi(x) + (1-t)x \in A \Leftrightarrow x \in \mathbf{p}(t, A).$$

Here  $\mathcal{L}^d(\mathbf{p}(t, A)) = 0$  (according to Lemma 1.3, since  $\mathbf{p}(t, \cdot)$  is Lipschitz on  $\mathbb{R}^d$ ). Then, for any  $\mathcal{L}^d$ -almost all  $(t, x) \in V$ , whence  $h \circ (t\nabla\phi + (1-t)\text{id}) \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$ .

- Let  $V$  be a bounded open set into  $\mathbb{R}^d$  and let  $t \in [0, 1)$ . The density  $\rho_0$  is assumed to have a compact support, therefore, if we additionally assume that  $\text{supp}(\rho_0) \subset V$ , according to Corollary 1.1, there exists a sequence  $(h_n)_n \in C_c(\mathbb{R}^d)^\mathbb{N}$  with  $|h_n| \leq M$  for all  $n \in \mathbb{N}$ , converging  $\mathcal{L}^d$ -almost everywhere to  $h \cdot \mathbb{1}_{(t\nabla\phi + (1-t)\text{id})(V)}$ .

The set  $B$  of the points of  $(t\nabla\phi + (1-t)\text{id})(V)$  such that the sequence  $(h_n(x))_n$  does not converge to  $h(x)$  or which is not bounded in Euclidean norm by  $M$  is therefore a zero measure set for  $\mathcal{L}^d$ .

We now consider the set of points  $x$  of  $V$  where  $\phi$  is differentiable, and such that the sequence  $(h_n(t\nabla\phi(x) + (1-t)x))_n$  does not converge to  $h(t\nabla\phi(x) + (1-t)x)$  or is not bounded in Euclidean norm by  $M$ . By reasoning as in the previous point, we can show that this last set is included in  $\mathbf{p}(t, B)$ , which is a zero measure set for  $\mathcal{L}^d$ , according to Lemma 1.3: indeed,  $\mathbf{p}(t, \cdot)$  is Lipschitz on  $\mathbb{R}^d$ . From the relation (2-19), we have for all  $n \in \mathbb{N}$ ,

$$\int_{\mathbb{R}^d} h_n d\nu_t = \int_{\mathbb{R}^d} h_n(x)\rho_t(x) d\mathcal{L}^d(x) = \int_{\mathbb{R}^d} h_n(t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x).$$

By dominated convergence, we obtain

$$\int_{\mathbb{R}^d} h d\nu_t = \int_{\mathbb{R}^d} h(x)\rho_t(x) d\mathcal{L}^d(x) = \int_{\mathbb{R}^d} h(t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x),$$

which holds for all  $h \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$  et  $t \in (0, 1)$ .

2. Let  $h \in L_{loc}^\infty([0, 1) \times \mathbb{R}^d, \mathcal{L}^{d+1})$ ,

- We assume that is  $V$  a bounded open set in  $[0, 1) \times \mathbb{R}^d$ .

As before, we can deduce that  $(t \mapsto (t, t\nabla\phi + (1-t)\text{id})(V))$  is bounded. There exists therefore a constant  $M > 0$  such that, by defining

$$A = \{(t, y) \in (t \mapsto (t, (t\nabla\phi + (1-t)\text{id})(V)), |h(t, y)| > M\},$$

we have  $\mathcal{L}^{d+1}(A) = 0$ . For any  $t \in [0, 1]$ , we define the set  $A_t$  by

$$A_t = \{x \in \mathbb{R}^d, (t, x) \in A\}.$$

It is a Lebesgue-measurable set of  $\mathbb{R}^d$  according to Theorem 1.3 (note that we have  $\mathbb{1}_A(t, x) = \mathbb{1}_{A_t}(x)$  for all  $(t, x) \in [0, 1] \times \mathbb{R}^d$ ). For all  $(t, x) \in V$  such that  $\phi$  is differentiable in  $x$  (let us recall that the set of points for which this is not the case is a zero measure set),

$$\begin{aligned} |h(t, t\nabla\phi(x) + (1-t)x)| > M &\Leftrightarrow (t, t\nabla\phi(x) + (1-t)x) \in A \\ &\Leftrightarrow x \in \mathbf{p}(t, A_t) \\ &\Leftrightarrow (t, x) \in B = \{(t', x') \in V, x' \in \mathbf{p}(t', A_{t'})\}. \end{aligned}$$

With the same definition, we can also define  $B_t = \{x \in \mathbb{R}^d, (t, x) \in B\} = \mathbf{p}(t, A_t)$  for all  $t \in [0, 1)$ , which is also a Lebesgue-measurable set. Finally, according to Fubini Theorem,  $t \mapsto \mathcal{L}^d(A_t)$  and  $t \mapsto \mathcal{L}^d(B_t)$  are measurable.

If  $\mathcal{L}^{d+1}(A) = 0$ , then

$$\begin{aligned} 0 = \mathcal{L}^{d+1}(A) &= \int_0^1 \int_{\mathbb{R}^d} \mathbb{1}_A(t, x) d\mathcal{L}^d(x) dt = \int_0^1 \int_{\mathbb{R}^d} \mathbb{1}_{A_t}(x) d\mathcal{L}^d(x) dt \\ &= \int_0^1 \mathcal{L}^d(A_t) dt. \end{aligned} \tag{2-22}$$

Thus  $\mathcal{L}^d(A_t) = 0$  for almost all  $t \in [0, 1]$ . For all  $t \in [0, 1]$ ,  $\mathbf{p}(t, \cdot)$  is Lipschitz: therefore, from Lemma 1.3, if  $\mathcal{L}^d(A_t) = 0$ , we have  $\mathcal{L}^d(\mathbf{p}(t, A_t)) = 0$ .

Thus,  $\mathcal{L}^d(B_t) = \mathcal{L}^d(\mathbf{p}(t, A_t)) = 0$  for almost all  $t \in [0, 1]$ . Consequently, according to Fubini Theorem,

$$\mathcal{L}^{d+1}(B) = \int_0^1 \mathcal{L}^d(B_t) dt = 0, \quad (2-23)$$

so that  $|h(t, t\nabla\phi(x) + (1-t)x)| \leq M$  for  $\mathcal{L}^{d+1}$ -almost all  $(t, x) \in V$ . We can therefore conclude that  $(t, x) \mapsto h(t, t\nabla\phi(x) + (1-t)x) \in L_{loc}^\infty([0, 1] \times \mathbb{R}^d, \mathcal{L}^{d+1})$ .

- For the second point of the property 2, we reason as in the second point of the property 1, via Lusin Theorem (the property being valid for  $h \in C_c^0([0, 1] \times \mathbb{R}^d)$ ).
3. By continuity under integral sign, it is clear that for all  $h \in C_c^0([0, 1] \times \mathbb{R}^d)$ , the function

$$t \mapsto \int_{\mathbb{R}^d} h(t, \cdot) d\nu_t = \int_{\mathbb{R}^d} h(t, t\nabla\phi(x) + (1-t)x) \rho_0(x) d\mathcal{L}^d(x)$$

is continuous on  $[0, 1]$ .

□

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