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# On the convergence of augmented Lagrangian method for optimal transport between nonnegative densities

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## Abstract

In the beginning of the 2000 years, J. D. Benamou and Y. Brenier have proposed a dynamical formulation of the optimal transport problem, corresponding to the time-space search of a density and a momentum minimizing a transport energy between two densities. They proposed, in order to solve this problem in practice, to deal with it by looking for a saddle point of some Lagrangian by an augmented Lagrangian algorithm. Using the theory of non-expansive operators, we will study the convergence of this algorithm to a saddle point of the Lagrangian, in the most general conditions, particularly in cases where initial and final densities cancel on some areas of the transportation domain. The principal difficulty of our study will consist of the proof, in these conditions, of the existence of a saddle point, and especially in the uniqueness of the density-momentum component. Indeed, these conditions imply to have to deal with non-regular optimal transportation maps: that is why an important part of our works will have for object a detailed study of the properties of the velocity field associated to an optimal transportation map in quadratic space.

## 1 Introduction

The optimal transport problem is generally formulated as follows: considering two sets of particles or probability measures, the question is to find the assignment between those discrete or continuous objects minimizing a given cost. This is referred to as optimal transport or optimal assignment: even if these two denominations describe the same problem, they reflect however two different approaches. Indeed, while it was initially a problem of optimal displacement, the pioneer Gaspard Monge, acknowledging the fact that the optimal trajectory from one point to another was the straight line, himself reduced this problem to a simple assignment problem. The same holds for the formulation given later by Leonid Kantorovitch (another pioneer): his problem was also reduced to a single assignment (or allocation) problem of the elements of a given resource to be transported. A such, the trajectories are not involved in the transport cost, which only reflects the price to pay to move a mass from one point to another.

The reduction of the optimal transport problem to an assignment problem first makes it easier to tackle theoretically. (see, for instance, the work of Kantorovitch [16]). However, when it comes to deal with the problem practically, that is to describe more precisely the optimal assignment plan, this formulation is less efficient. Some approaches even choose to reintroduce the notion of displacement: this will be the case of the method that we will deal with here.

The first attempt to link the optimal assignment and optimal displacement problems was proposed by R. J. McCann [18]. There was considered the continuous displacements between two measures, that is, to determine a continuum of intermediate measures between two measures

$\mu$  and  $\nu$ , so that the integral sum of local optimal costs (ie the distance of Wasserstein step by step) would be equal to the global optimal allocation cost between  $\mu$  and  $\nu$ . The idea is that for infinitely close measures, there is no difference between optimal assignment and optimal displacement.

Relying on this continuous interpolation idea between the densities  $\rho_0$  and  $\rho_1$  by densities  $\rho_t$  of constant mass, second-order algorithms have been proposed to solve the problem efficiently [4]. This type of approach is nevertheless limited to non-vanishing densities. As we know, the only numerical method handling the general case is based on the work of Y. Brenier and J.-D. Benamou. They introduced a continuous formulation, originating from fluid mechanics, of the optimal transport problem in [3]: this amounts to determine directly the evolution of the density  $\rho = t \mapsto \rho_t$  as well as the velocity field  $v$  which advects this density, corresponding to the optimal map  $\nabla\phi$ . Such a pair  $(\rho, v)$  therefore verifies a continuity equation.

One of the main interest of the dynamic formulation of the optimal transport problem is its expression in terms of fluid mechanics quantities: this makes the model very flexible, and allows easy generalization to other physical constraints [17, 8] which could be relevant in applications. The introduction of these physical constraints (anisotropy of the domain, constraints of free divergence or rigidity on the velocity field) in the dynamic problem was the subject of [15].

But the main interest advanced by the authors of [3] was the ability to introduce an algorithm to exploit this physical formulation of the optimal transport problem, based on a convex reformulation of the latter. For this purpose, they had chosen to introduce the impulsion  $m = \rho v$ , and to reduce the problem to the search for an optimal  $\mu = (\rho^*, m^*)$  rather than  $(\rho^*, v^*)$ . In other words, to find the optimal velocity field  $v^*$  on the support of the density  $\rho^*$  (the velocity field  $v$  can indeed vary outside the support of  $\rho$  without modifying the value of the travel energy  $\mathcal{K}$ ). Such a formulation makes it possible to convexify the transport energy and to linearize the mass conservation equation. The problem then becomes the following: minimizing a convex, proper, lower semicontinuous energy.

The authors of [3] had in this same article proposed to rephrase this convex dynamic problem as the search for a saddle point  $(\psi, q, \mu)$  of a Lagrangian  $\mathbf{L}$ , and then to address this new problem with an augmented Lagrangian algorithm (more precisely ADMM) such as developed in [11]. This algorithm has been used since 1999 and works well in practice, although a number of theoretical questions remain unanswered: the existence of solutions in the context of this new dynamic formulation, the existence of a saddle point for  $\mathbf{L}$ , the convergence (weak or strong) of the algorithm, and the possible uniqueness of a solution, in the general setting of possibly vanishing densities.

Before going further, let us first introduce the dynamic formulation of the optimal transport problem proposed by Benamou and Brenier in [3].

## 2 Formulation of the problem and presentation of the algorithm

### 2.1 The problem of Monge-Kantorovich in $\mathbb{R}^d$ for a quadratic cost

In all the following, we denote by  $|\cdot|$  the Euclidean norm on  $\mathbb{R}^d$  (for all  $d \in \mathbb{N}$ ). Consider two nonnegative densities  $(\rho_0, \rho_1)$  on  $\mathbb{R}^d$  ( $d \in \mathbb{N}^*$ ), with bounded supports and of the same mass. The problem of Monge-Kantorovich is to determine an optimal transport plan  $T$  between  $\rho_0 \mathcal{L}^d$  and  $\rho_1 \mathcal{L}^d$  minimizing

$$\int_{\mathbb{R}^d} d(x, T(x))^2 \rho_0(x) dx, \quad (2-1)$$

where  $d(x, y)$  is a distance on  $\Omega$ . We write  $T\#(\rho_0 \mathcal{L}^d) = \rho_1 \mathcal{L}^d$  the push forward by  $T$  of  $\rho_0 \mathcal{L}^d$  on  $\rho_1 \mathcal{L}^d$ , i.e. such that for any bounded subset  $A$  of  $\mathbb{R}^d$ ,  $\int_A \rho_0 = \int_{T^{-1}(A)} \rho_1$ . The quadratic distance

of Wasserstein  $\mathcal{W}_2(\rho_0, \rho_1)$  is defined by:

$$\mathcal{W}_2(\rho_0, \rho_1)^2 = \inf_{T \# (\rho_0 \mathcal{L}^d) = \rho_1 \mathcal{L}^d} \int_{\Omega} d(x, T(x))^2 \rho_0(x) dx. \quad (2-2)$$

In the Euclidean case (where  $d(x, y) = |x - y|$ ), there exists a unique transport plane  $T$  between  $\rho_0$  and  $\rho_1$  that can be written as the gradient of a lower semi-continuous (l.s.c.) convex function  $\phi$  (Brenier's Theorem [20] p.66) i.e.

$$\mathcal{W}_2(\rho_0, \rho_1)^2 = \int_{\mathbb{R}^d} |\nabla \phi(x) - x|^2 \rho_0(x) dx = \inf_{T \# (\rho_0 \mathcal{L}^d) = \rho_1 \mathcal{L}^d} \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_0(x) dx. \quad (2-3)$$

This problem, in the dynamic formulation of the Monge problem introduced by J. -D. Benamou and Y. Brenier [3], can be rephrased as a minimization problem of a kinetic energy  $\mathcal{K}$ , depending on a mass  $\rho$  and a velocity field  $v$ , such that  $\rho$  is transported from  $\rho_0$  to  $\rho_1$ , by  $v$ .

Let us begin by formulating in details this new optimization problem in a framework that will be convenient for its resolution by the augmented Lagrangian algorithm [3], and which will be the main object of our study.

## 2.2 Convex and augmented lagrangian formulation

We propose to study the following problem: let  $\rho_0, \rho_1 \in L^2(\mathbb{R}^d)$  be two probability densities with bounded supports. The dynamic optimal transport formulation consists in increasing the dimension of the problem by adding a temporal variable  $t \in [0, 1]$ . Formally, we look for a couple  $(\rho, v)$ , where  $\rho$  denotes a nonnegative density, and  $v$  a vector field with values in  $\mathbb{R}^d$ , both defined on  $]0, 1[ \times \Omega$ , where  $\Omega$  is a bounded open convex set of  $\mathbb{R}^d$  containing  $\text{supp}(\rho_0)$  and  $\text{supp}(\rho_1)$ . This couple is required to satisfy the continuity equation,

$$\partial_t \rho + \text{div}(\rho v) = 0 \quad (2-4)$$

with homogeneous Neumann boundary conditions on  $\rho v$ , and with initial and final conditions on  $\rho$ :

$$\rho(0, x) = \rho_0(x), \quad \rho(1, x) = \rho_1(x). \quad (2-5)$$

Among all such couples  $(\rho, v)$  we look for a minimizer of  $\mathcal{K}(\rho, v) = (1/2) \int_0^1 \int_{\Omega} |v|^2 \rho dx dt$ . As  $\mathcal{K}$  is not convex, and the constraint (2-4) is nonlinear, the authors in [3] proposed as a new variable  $m = \rho v$  in place of  $v$ , and consider the transport cost:

$$\tilde{\mathcal{K}}(\rho, m) = \int_0^1 \int_{\Omega} \frac{|m(t, x)|^2}{2\rho(t, x)} dx dt, \quad (2-6)$$

with the corresponding continuity constraint:

$$\partial_t \rho + \text{div}_x m = 0 \quad (2-7)$$

still with homogeneous Neumann boundary conditions on  $m$  and initial / final conditions (2-5). The nonnegativity constraint on  $\rho$  turns to  $\{\rho > 0 \text{ or } (\rho, m) = (0, 0)\}$  through the change of variable  $m = \rho v$ . By introducing a Lagrange multiplier  $\psi$  to handle the linear constraints (2-7) and (2-5), we can write a saddle-point formulation of the problem:

$$\inf_{(\rho, m)} \sup_{\psi} \left[ \int_{]0, 1[ \times \Omega} \frac{|m|^2}{\rho} - \int_{]0, 1[ \times \Omega} (\partial_t \psi \rho - \nabla_x \psi \cdot m) + \int_{\Omega} (\psi(0, \cdot) \rho_0 - \psi(1, \cdot) \rho_1) \right]. \quad (2-8)$$

Another crucial idea of [3], is to encode the above nonnegativity constraint by introducing the Legendre transform of  $(\rho, m) \mapsto |m|^2/(2\rho)$ :

$$F(q) = F(a, b) = \sup_{(\rho, m)} \left( \rho a + \langle m, b \rangle - \frac{|m|^2}{2\rho} \right) \Leftrightarrow F(q) = \begin{cases} 0 & \text{if } q \in \mathcal{P} \\ +\infty & \text{otherwise} \end{cases}$$

with  $q = (a, b) \in \mathbb{R} \times \mathbb{R}^d$  and  $\mathcal{P} = \{(a, b) \in \mathbb{R} \times \mathbb{R}^d, a \leq -|b|^2/2\}$ . Since the transport cost is now convex and l.s.c., it is equal to its biconjugate by the Legendre transform. We therefore have  $|m|^2/(2\rho) = \sup_{(a, b)} (\rho a + m \cdot b - F(a, b))$ . The problem is thus partially linearized with respect to the variables  $(\rho, m)$ : the non-linear part (i.e.  $F$ ) reduces to the indicator function of  $\mathcal{P}$ , which will be implemented as a projection on that convex subset.

By some manipulations as sup-integral or inf-sup inversions, and by setting  $q = (a, b)$  and  $\mu = (\rho, m)$ , the saddle point problem (2-8) is reformulated as  $\inf_{(\psi, q)} \sup_{\mu} L(\psi, p, \mu)$  where

$$\mathbf{L}(\psi, p, \mu) = \tilde{F}(q) + G(\psi) + \langle \mu, \nabla_{t,x} \psi - q \rangle_{L^2} \quad (2-9)$$

with  $G(\psi) = \int_{\Omega} \psi(0, \cdot) \rho_0 dx - \int_{\Omega} \psi(1, \cdot) \rho_1 dx$  and  $F = \chi_{\tilde{\mathcal{P}}}$  (meaning  $F(q) = 0$  if  $q \in \tilde{\mathcal{P}}$  and  $F(q) = +\infty$  otherwise), where  $\tilde{\mathcal{P}} = \{(\tilde{a}, \tilde{b}) \in L^2(Q) \times L^2(Q)^d, \tilde{a} \leq -|\tilde{b}|^2/2\}$ . In the following we will write  $\mathcal{P}$  in place  $\tilde{\mathcal{P}}$ . The augmented Langrangian formulation is finally given, for some parameter  $r > 0$ , by:

$$\mathbf{L}_r(\psi, q, \mu) = F(q) + G(\psi) + \langle \mu, \nabla_{t,x} \psi - q \rangle_{L^2} + \frac{r}{2} \|\nabla_{t,x} \psi - q\|_{L^2}^2. \quad (2-10)$$

### 2.3 Benamou-Brenier algorithm

To solve the saddle point problem associated to (2-10), the authors of [3] have proposed an algorithm based on a Uzawa method: the ALG2 algorithm introduced by M. Fortin and R. Glowinski in [11]. This consists in performing the following steps, starting from  $(\psi^{n-1}, q^{n-1}, \mu^n)$ :

1. Step A: Find the unique  $\psi^n$  such that  $\mathbf{L}_r(\psi^n, q^{n-1}, \mu^n) \leq \mathbf{L}_r(\psi, q^{n-1}, \mu^n), \forall \psi$ .
2. Step B: Find the unique  $q^n = (a^n, b^n)$  such that  $\mathbf{L}_r(\psi^n, q^n, \mu^n) \leq \mathbf{L}_r(\psi^n, q, \mu^n), \forall q$ .
3. Step C: Update  $(\rho^{n+1}, m^{n+1})$ , setting  $\mu^{n+1} = \mu^n + r(\nabla_{t,x} \psi^n - q^n)$ ,

More precisely, the algorithm breaks down as follows: Step A can be interpreted as a projection on fields which are gradients. We look for the unique  $\psi^n \in H^1(Q)/\mathbb{R}$  such that:

$$\forall h \in H^1(Q), G(h) + \langle \mu^n, \nabla_{t,x} h \rangle + r \langle \nabla_{t,x} \psi^n - q^{n-1}, \nabla_{t,x} h \rangle = 0.$$

Formally, this corresponds to find  $\psi^n$  solution of  $-r\Delta_{t,x} \psi^n = \text{div}_{t,x} (\mu^n - r q^{n-1})$ , with as initial and final conditions:

$$r\partial_t \psi^n(0, \cdot) = \rho_0 - \rho^n(0, \cdot) + r a^{n-1}(0, \cdot), \text{ and } r\partial_t \psi^n(1, \cdot) = \rho_1 - \rho^n(1, \cdot) + r a^{n-1}(1, \cdot),$$

and homogeneous Neumann boundary conditions on  $(0, 1) \times \partial\Omega$ . This operation corresponds to a kind of Helmholtz decomposition.

Step B is an orthogonal projection on  $\mathcal{P}$ :  $q^n = P_{\mathcal{P}}((1/r)\mu^n + \nabla_{t,x} \psi^n)$ . This projection is an orthogonal  $L^2$  projection, which corresponds to a point-wise projection.

Step C uses the computed gradient of step A to implement a projection on the affine space of constraints (2-4) and (2-5):  $\mu^{n+1} = \mu^n + r(\nabla_{t,x} \psi^n - q^n)$ .

**Remark.** We chose to take the same parameter  $r > 0$  for the Uzawa step C to ensure the positivity constraint of  $\rho^n$  and cancellation of  $m^n$  when  $\rho^n$  vanishes. Indeed step B can be rewritten as  $\forall q' \in \mathcal{P} \langle \mu^{n+1}, q' - q^n \rangle \leq 0$ , which means that  $\mu^{n+1} = (\rho^{n+1}, m^{n+1})$  is orthogonal to the paraboloid  $\mathcal{P}$  at  $q^n$ . We deduce by the strict convexity of  $\mathcal{P}$ , that for all  $(t, x) \in ]0, 1[ \times \Omega$ ,  $\rho^n(t, x) > 0$  or  $(\rho^n(t, x), m^n(t, x)) = 0$ .

## 2.4 Objectives and related existing works

The main object of this article is to propose a theoretical framework allowing to answer the three following points: existence of saddle points, their unicity properties and convergence of the considered algorithm. The proposed study will also be the opportunity to characterize rigorously some properties of the velocity field associated with an optimal transport plan.

A first study of the Benamou-Brenier algorithm was carried out in 2004 by K. Guittet in [13]. This study considered periodic in space boundary condition:  $\Omega = \mathbb{T}^d$  (where  $\mathbb{T}^d$  denotes the torus in dimension  $d$ , i.e.  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ ). The strongest assumption of this first study was that the density  $\rho^*$ , solving the problem (2-6), had to be larger than a positive constant. This assumption implied in particular a certain regularity of the associated transport plan (discontinuity free).

Indeed, under such conditions, the potential  $\phi$  must be of class  $C^1$  and with Lipschitzian gradient. The regularity of an optimal transport plan on a convex domain (i.e. more precisely here the potential  $\phi$ ) depending on the regularity of the initial and final densities  $\rho_0$  and  $\rho_1$ , additionally assumed to be strictly positive, was studied by Caffarelli in [6] and [7]. A special case mentioned in [13] is the one when we consider  $\rho_0$  and  $\rho_1$  strictly positive on  $\mathbb{T}^d$  and belonging to  $C^{\alpha, l}(\mathbb{T}^d)$ , for some  $l \in ]0, 1[$ , and  $\alpha \in \mathbb{N}^d$ . These conditions imply, according to a study of Cordero-Erausqui in [9], that the optimal transport potential  $\phi$  is of class  $C^{\alpha, l+2}$  and, for any  $t \in [0, 1]$ , the density  $\tilde{\rho}_t$  also has a  $C^{\alpha, l}$  regularity on  $\mathbb{T}^d$  and bounded from below by a strictly positive constant independent of  $t$ .

Under the above assumptions, the author of [13] first shows the existence of a solution  $(\rho^*, m^*)$  for the dynamic formulation of optimal transportation; solution from which is proven the existence of a saddle point  $(\psi^*, q^*, \mu^*)$  for the Lagrangian  $\mathbf{L}$ . However, there is no uniqueness result for the density-momentum couple  $\mu^* = (\rho^*, m^*)$ .

There is also a convergence result of the Benamou-Brenier algorithm: however, this does not explicitly give the strong or weak convergence of the main component of the triplet  $(\psi_n, q_n, \mu_n)$ . Indeed, considering the problem (2-6), the component of interest is the density-momentum component  $\mu_n = (\rho_n, m_n)$ . However in this convergence result, only the strong convergences in  $H^1/\mathbb{R}$  and  $L^2$  of the components  $\psi_n$  and  $q_n$  are proven. Moreover, the proof of this convergence seems incomplete (see section 8).

In this article, we consider a more general framework. The open set  $\Omega$  will here be assumed to be convex and bounded, with homogeneous Neumann boundary conditions on the momentum  $m$ , but more importantly, the density  $\tilde{\rho}$  **will not be assumed to be minored** by a strictly positive regular constant. We will simply assume that the densities  $\rho_0$  and  $\rho_1$  are elements of  $L^2(\Omega)$  (thus potentially non-regular) in the neighborhood of  $\partial\Omega$ .

In this framework, we will show the existence of a saddle point for the Lagrangian  $\mathbf{L}$ , solution of the problem (2-6), as well as the uniqueness among the set of saddle points  $(\psi^*, q^*, \mu^*)$  of the Lagrangian  $\mathbf{L}$  of  $\mu^* = (\rho^*, m^*)$ , which shows that the density corresponds to the McCann interpolation. The uniqueness result established in this article only concerns the component  $\mu$ , since, as we will see at the beginning of the section 6, **there is no uniqueness of the saddle points of  $\mathbf{L}$** : in fact, the components  $\psi$  and  $q$  can vary outside the support of  $\rho$ .

These first two points (results of existence and uniqueness) will be established after a preliminary study on the regularity of the velocity field  $v$  inherent to an optimal transport. In

particular, we would like to underline the new results of this paper:  $\nabla_{t,x}v \in L^p((0,1), L^q_{loc}(\mathbb{R}^d))$  for all  $1 < p \leq +\infty$  and  $1 \leq q < +\infty$  such that  $1/p + 1/q > 1$ , and especially  $v \in W^{1,p}_{loc}([0,1] \times \mathbb{R}^d)$  (i.e.  $\forall \Omega \subset\subset \mathbb{R}^d, v \in W^{1,p}((0,1) \times \Omega)$ ) for all  $1 \leq p < 2$  (see Theorem 4.2 and corollary 4.3).

This study will also lead us to characterize accurately a velocity field inherent to an optimal transport in  $L^2$ . More precisely, we will determine sufficient hypothesis on a velocity field  $v \in L^\infty_{loc}([0,1], \mathbb{R}^d)$  for a density transported by  $v$  to be the McCann interpolation of an optimal (unique) transport Theorem 7.1. These hypothesis will be reduced to the usual characteristics of optimal isotropic transport, in particular straight-line trajectories, at constant speed, and without crossing.

Finally, we will establish the weak convergence of the Benamou-Brenier algorithm, as well as the strong convergence of a relaxed version of it, towards a saddle point of the Lagrangian  $\mathbf{L}$ .

### 3 Existence of a saddle point

The main objective of this first part is to directly build a saddle point of  $\mathbf{L}$  defined in (2-10). Let us therefore define the framework rigorously: let  $\rho_0$  and  $\rho_1$  be two probability densities (i.e. non-negative and integral equal to 1) of  $L^2(\mathbb{R}^d)$  with bounded supports, and  $\Omega$  a sufficiently regular bounded convex open set of  $\mathbb{R}^d$  such that  $\text{supp}(\rho_0) \cup \text{supp}(\rho_1) \subset \Omega$ . In the remainder of this paper, we denote  $Q = (0,1) \times \Omega$ .

For all  $r > 0$ , we write  $\mathbf{L}_r^{ps}(\rho_0, \rho_1, \Omega)$  the set of Lagrangian saddle points  $\mathbf{L}_r$  which are elements of  $Sg = H^1(Q)/\mathbb{R} \times L^2(Q)^{d+1} \times L^2(Q)^{d+1}$ .

Let us define the following three properties for a triplet  $(\psi, q, \mu) \in Sg$  of  $\mathbf{L}_r$ :

**Properties (I).**  $(\psi, q, \mu) \in Sg$  verifies:

$$(P1) \quad \forall q' \in \tilde{\mathcal{P}}, \quad \langle \mu, q' - q \rangle \leq 0,$$

$$(P2) \quad \forall h \in H^1(Q), \quad G(h) + \langle \mu, \nabla_{t,x}h \rangle = 0,$$

$$(P3) \quad \nabla_{t,x}\psi = q.$$

where the paraboloid  $\tilde{\mathcal{P}}$  is defined by

$$\tilde{\mathcal{P}} = \left\{ (a, b) \in L^2(Q) \times L^2(Q)^d, \quad a + \frac{|b|^2}{2} \leq 0 \right\}, \quad (3-1)$$

and the linear operator  $G$  by  $G(h) = \int h(0, \cdot) \rho_0 dx - \int h(1, \cdot) \rho_1 dx$ , for all  $h \in H^1(Q)$ .

**Proposition 3.1.** *A saddle point  $(\psi^*, q^*, \mu^*) \in Sg$  of  $\mathbf{L}_r$  is characterized by the properties (I), for all  $r \geq 0$ .*

*Idea of proof:* We first check that for a triplet  $(\psi^*, q^*, \mu^*) \in Sg$  satisfying the properties (I), we have the relation

$$\mathbf{L}_r(\psi, q, \mu^*) \geq \mathbf{L}_r(\psi^*, q^*, \mu^*) \geq \mathbf{L}_r(\psi^*, q^*, \mu)$$

for all  $(\psi, q, \mu) \in Sg$ , which characterizes a saddle point of  $\mathbf{L}_r$ . Conversely, for a saddle point  $(\psi^*, q^*, \mu^*) \in Sg$  of  $\mathbf{L}$ , one verifies one by one the properties (I), first by fixing  $\psi = \psi^*$  and  $q = q^*$  (one then shows (P3)), then fixing  $q = q^*$  et  $\mu = \mu^*$  (we show (P2)), and finally by fixing  $\mu = \mu^*$  and  $\psi = \psi^*$  (we show (P1)). Since the saddle points of the Lagrangian  $\mathbf{L}_r$  are independent of  $r \geq 0$ , then we will only consider the (not increased) Lagrangian  $\mathbf{L}$ .  $\square$



By setting  $\mu = (\rho, m)$  and  $q = (a, b)$  (with  $a + \frac{|b|^2}{2} \leq 0$ ), we can reinterpret the properties (P1) and (P2). (P1) means that if  $\mu(t, x)$  is nonzero then it is orthogonal to the paraboloid at the point  $q(t, x)$ , i.e. **co-linear** to the vector  $(1, b(t, x))$ . The property (P1) can be translated as follows:  $\rho \geq 0$ ,  $m = \rho b$ ,  $\rho(a + |b|^2/2) = 0$ . Next, (P2) corresponds to the **mass conservation equation** verified by  $\rho$  and  $b$  (i.e.  $\partial_t \rho + \operatorname{div}_x(\rho v) = 0$ , taking  $v = b$ ) for the initial and final densities  $\rho_0$  and  $\rho_1$ .

We now recall that according to Brenier's theorem [20] (p. 66), there exists a convex potential  $\phi$  verifying  $\rho_1 \mathcal{L}^d = \nabla \phi \# (\rho_0 \mathcal{L}^d)$  from which we define the following quantities:

**Definition 3.1.** For all  $t \in ]0, 1[$ , we define:

1. The characteristic displacement at the instant  $t$ ,

$$X(t, \cdot) = (1 - t) \operatorname{id} + t \nabla_x \phi = \nabla_x \phi_t \quad \text{avec } \phi_t = (1 - t) \frac{|\cdot|^2}{2} + t \phi, \quad (3-2)$$

2. The associated velocity field  $v$ ,

$$v(t, \cdot) = \frac{\operatorname{id} - \nabla_x(\phi_t)^*}{t} \quad \text{avec } \phi_t = (1 - t) \frac{|\cdot|^2}{2} + t \phi, \quad (3-3)$$

where  $(\phi_t)^* = (\phi_t)^*$  denotes the Legendre transform of the potential  $\phi_t$  (defined above).

3. The convex density  $\rho$ , composed by the union on  $[0, 1]$  of the McCann interpolation densities  $\rho_t$  between  $\rho_0 \mathcal{L}^d$  and  $\rho_1 \mathcal{L}^d$  [18]:

$$\rho(t, \cdot) \mathcal{L}^d = \rho_t \mathcal{L}^d = X(t, \cdot) \# (\rho_0 \mathcal{L}^d). \quad (3-4)$$

Let us also define the  $(\Gamma_1)$  property on the potential  $\phi$ :

**Hypothesis  $(\Gamma_1)$ .**  $\phi$  and  $\phi^*$  are convex, continuous and admit a minimum on  $\mathbb{R}^d$ .

Here  $\phi^*$  always designates the Legendre transform of  $\phi$ .

**Reminder:** A convex and continuous function on  $\mathbb{R}^d$  is locally Lipschitz.

For the purpose of our study, we complete the Brenier Theorem ([20] p. 66) as follows:

**Proposition 3.2.** Let  $\rho_0$  be a probability density Lebesgue-measurable on  $\mathbb{R}^d$  and  $\mu_1$  a probability measure defined on the tribe of Lebesgue in  $\mathbb{R}^d$ . There exists a potential  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ , satisfying the property  $(\Gamma_1)$ , such as  $\nabla \phi \# (\rho_0 \mathcal{L}^d) = \mu_1$ .

*Idea of proof:* One can first show that the optimal transport potential  $\phi$  given by the Brenier's Theorem (convex, semicontinuous inferior and gradient bounded almost everywhere on  $\operatorname{supp}(\rho_0 \mathcal{L}^d)$ ) is finite and of bounded gradient on an open neighborhood of the support of  $\rho_0 \mathcal{L}^d$ . It is then possible to extend the restriction of  $\phi$  to this neighborhood by a finite  $\bar{\phi}$  convex function on  $\mathbb{R}^d$  continuous, supralinear and sub-quadratic. The supralinearity of  $\bar{\phi}$  implies the existence of a global minimum for the latter, and ensures that its Legendre transform  $\bar{\phi}^*$  will also be finite (and thus continuous) on  $\mathbb{R}^d$ . The sub-quadratic character of  $\bar{\phi}$  ensures the supralinearity of  $\bar{\phi}^*$ , and therefore the existence of a global minimum.  $\square$

We can now define the following set:

**Definition 3.2.** Let  $\rho_0$  be a probability density Lebesgue-measurable on  $\mathbb{R}^d$  and  $\mu_1$  a probability measure defined on the tribe of Lebesgue in  $\mathbb{R}^d$ . We denote by  $\Phi(\rho_0 \mathcal{L}^d, \mu_1)$  the set of functions  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the property  $(\Gamma_1)$  such that  $\nabla \phi \# (\rho_0 \mathcal{L}^d) = \mu_1$ .



### 3.1 Reformulation of the Properties (I)

Let us give an idea of the approach that will be followed: let us set  $\mu = (\rho, \rho v)$  and  $q = (-\frac{1}{2}|v|^2, v)$ . To construct a saddle point, such  $\mu$  and  $q$  should satisfy the properties (I). It will be sufficient to verify that with  $\rho$  and  $v$  defined like in (3-4) and (3-3) satisfy the following properties:

**Properties (I').**

(P1') The density  $\rho$  verifies  $\rho \in L^2(Q)$ .

(P2') The velocity field  $v$  is an element of  $L^\infty(Q)^d$ .

(P3') The velocity field  $v$  satisfies the Burgers equation in the sense of the distributions:

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0$$

(P4') The potential  $(\rho, v)$  satisfies the mass conservation equation in the distributions sense for the initial and final conditions  $\rho_0$  and  $\rho_1$  and the homogeneous Neumann boundary conditions:

$$\int_Q (\partial_t h + v \cdot \nabla h) \rho + \int_\Omega h(0, x) \rho_0(x) dx - \int_\Omega h(1, x) \rho_1(x) dx = 0 \quad (3-5)$$

We will see that  $\rho$  and  $v$  satisfying the properties (I') is sufficient to build a triplet  $(\psi, q, \mu)$  satisfying properties (I). However, the inverse is not true: a triplet  $(\psi, q, \mu)$  satisfying the properties (I) is not enough to build a density-velocity field pair  $(\rho, v)$  satisfying properties (I'), and such that  $\mu = (\rho, \rho v)$  and  $q = (-\frac{1}{2}|v|^2, v)$ . Indeed, the component  $q$  may not belong to the boundary of the paraboloid  $(a, b) \rightarrow a + (1/2)|b|^2 \leq 0$  outside the support of  $\mu$ .

The properties (P1') and (P2'), respectively established in Lemma 5.1 and 4.2, ensure that the saddle point is in the correct space, i.e. in  $Sg$  (defined at the beginning of the section 3). Indeed, we have  $q \in L^2(Q)^{d+1} \Leftrightarrow v \in L^4(Q)^d \subset L^\infty(Q)^d$ ,  $\mu \in L^2(Q)^{d+1} \Leftrightarrow \rho \in L^2(Q)$  et  $\rho v \in L^2(Q)^d$ , and, for the potential  $\psi$ , we have  $W^{1,\infty}(Q) \subset H^1(Q)$ .

The properties (P2') and (P3') involve the property (P3). Indeed, having  $q$  deriving from a space-time potential amounts to verifying, for a dimension  $d \leq 2$ , that  $\text{curl}_{t,x}(q) = 0$  (recalling that  $q = (-\frac{1}{2}|v|^2, v)$ ) in the sense of distributions (see [12] Theorem 2.9 p.31), so that

$$\begin{cases} \partial_t v + \frac{1}{2} \nabla |v|^2 = 0, \\ \text{curl}_x(v) = 0 \Leftrightarrow \exists \psi \in \mathcal{D}'(Q), v = \nabla_x \psi \end{cases}$$

It is clear that the velocity **derives from a potential in space** in the sense of the distributions, namely:

$$v(t, \cdot) = \nabla_x \left( \frac{1}{t} \left( \frac{1}{2} |\cdot|^2 - (\phi_t)^* \right) \right) \quad (3-6)$$

(see the definition of  $v$  in (3-3), and this potential from which  $v$  is derived is an element of  $L^1_{loc}(Q)$ ). According to the Lemma 5.2, this proves the property (P3), provided that the field  $v$  is an element of  $L^\infty(Q)^d$  and verifies the Burgers equation

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0$$

in the sense of distributions.

Even if the notion of rotational is no longer defined for  $d > 2$ , the Lemma 5.2 allows us to state the property (P3) from (P2') and (P3'), whatever the dimension  $d$  is, provided that we

have  $v \in L^\infty(Q)^d$ . Let us note that the rotational can in fact be defined in dimension  $d + 1 > 3$ , but this one will then have an image in a  $d(d - 1)/2$  dimension space.

Finally, note that the property (P4') translates the property (P2) of (I). Indeed, we can easily extend the relation for  $h \in H^1(Q)$  once it is established for  $h \in C^\infty(Q)$ . Finally notice that with the above results, the Property (P1) is verified by setting  $m = \rho v$ .

### 3.2 Main results of existence, uniqueness and regularity

We say that  $v$  satisfies properties (II) if and only if:

#### Properties (II).

1. There exists  $\psi \in W_{loc}^{1,\infty}([0, 1] \times \Omega)$  (i.e.  $\psi \in W^{1,\infty}((0, 1) \times \omega)$ , for all bounded open set  $\omega \subset \Omega$ ) such that  $v = \nabla_x \psi$ .
2. The velocity field  $v$  satisfies *the Burgers equation* in the sense of the distributions, namely the relation

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0. \quad (3-7)$$

According to the Lemma 5.2 that will be stated below, the properties (II) are equivalent to the following ones:

1.  $v \in L_{loc}^\infty([0, 1] \times \Omega)^d$  i.e.  $v \in L^\infty((0, 1) \times \omega)^d$ , for all bounded open set  $\omega \subset \Omega$ ,
2.  $\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0$  (in the sense of distributions),
3. there exists  $\psi \in L_{loc}^1((0, 1) \times \Omega)$ , such that  $v = \nabla_x \psi$ ,

which correspond to the properties (P2') and (P3').

The properties (II) contain the characteristics of an isotropic optimal transport for a quadratic cost: the first point (i.e.  $v = \nabla_x \psi$ ) corresponds to the property of **non crossing trajectories** (recalling that in dimension less than 3 this property is equivalent to a rotational free velocity field  $v$ ); and the second point (the Burgers equation) is in line with the property of **straight-line displacement**.

At the end of section 7, we will give a framework in which we can rigorously characterize an optimal transport-type mass displacement from these properties alone.

We say that  $v$  satisfies properties (B) if and only if:

#### Properties (B).

1.  $v$  is locally Lipschitz on  $]0, 1[ \times \Omega$ ,
2.  $\nabla_{t,x} v \in L^\infty((0, 1), L_{loc}^1(\Omega))$ ,
3.  $\nabla_{t,x} v \in L^p((0, 1), L_{loc}^q(\Omega))$  with  $1/p + 1/q > 1$  (in particular  $v \in W_{loc}^{1,p}([0, 1] \times \Omega)$  with  $1 \leq p < 2$ ).

The principal results of existence and uniqueness we show are the following:

**Theorem 3.1** (Existence of a saddle point). *Let  $\rho_0$  and  $\rho_1$  two probability densities of  $L^2(\mathbb{R}^d)$  with bounded supports, and let  $\Omega$  a sufficiently regular bounded open set of  $\mathbb{R}^d$  such that  $\text{supp}(\rho_0) \cup \text{supp}(\rho_1) \subset \Omega$ . For all  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$  (see relation 3.2), there is a non-negative density  $\rho_\phi \in C^0([0, 1], L^2(\Omega))$ , such that for all  $[0, 1] \times \mathbb{R}^d$ , we have*

$$\rho_\phi(t, \cdot) \mathcal{L}^d = X_\phi(t, \cdot) \# (\rho_0 \mathcal{L}^d), \text{ avec } X_\phi(t, \cdot) = \nabla_x \phi_t = (1 - t) \text{id} + t \nabla_x \phi,$$

and a velocity field  $v_\phi$ , defined with respect to  $\phi$  as in (4-12). Then, setting  $\mu_\phi = (\rho_\phi, \rho_\phi v_\phi)$  and  $q_\phi = (-(1/2)|v_\phi|^2, v_\phi)$ , there exists  $\psi_\phi \in W_{loc}^{1,\infty}([0, 1] \times \Omega)$  such that  $q_\phi = \nabla_{t,x} \psi_\phi$  and such that  $(\psi_\phi, q_\phi, \mu_\phi)$  (or at least its restriction on  $]0, 1[ \times \Omega$ ) is a saddle point Lagrangian  $\mathbf{L}$ . In addition,  $v_\phi$  satisfies the properties (II) and (B).

As we will see in section 7, the fact that  $v$  satisfies the properties (II) is sufficient to characterize an optimal transport in  $L^2$ . On the other hand, the fact that  $v$  verifies the properties (B) is in itself a result apart from this construction: these properties, although interesting in themselves, will not be directly used to characterize an optimal transport velocity field in  $L^2$ . However, very close properties will be considered to show the different statements on the uniqueness of the component  $(\rho, m)$  of the saddle points of  $\mathbf{L}$ , and the results related to the characterization of an optimal transport-type velocity field.

**Theorem 3.2** (Unicity of density and momentum). *If  $(\psi^*, q^*, \mu^*)$  is a saddle point of  $\mathbf{L}$  (the assumptions on  $\rho_0, \rho_1$  and  $\Omega$  being the same as in the Theorem 3.1), then for any potential  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$ , we have  $\mu^* = (\rho^*, m^*) = (\rho_\phi, \rho_\phi v_\phi)$ , with the velocity field  $v_\phi$  defined with respect to  $\phi$  as in (4-12), and*

$$\rho_\phi(t, \cdot) \mathcal{L}^d = X_\phi(t, \cdot) \# (\rho_0 \mathcal{L}^d) \in C([0, 1], L^2(\Omega)), \text{ with } X_\phi(t, \cdot) = \nabla_x \phi_t = (1 - t) \text{id} + t \nabla_x \phi.$$

In general, the set of saddle points  $(\psi, q, \mu)$  of  $\mathbf{L}$  is **not reduced** to a single element: only the component  $\mu = (\rho, m)$  is unique. In other words, the set of points  $(\psi, q, \mu)$  of  $\mathbf{L}$  share the same component  $\mu$ , i.e. there is uniqueness of the density  $\rho$  and the velocity field  $v$  on the support of  $\rho$ . The components  $q$  and  $\psi$  can indeed vary outside the support of  $\rho$ . For more details, see sub-section 4.2.1 of [14].

## 4 Velocity field properties

In this section, we define and characterize a velocity field associated to an optimal transport map using Brenier's Theorem. This constitutes **the basis** of the existence and uniqueness results concerning the saddle points of  $\mathbf{L}$  (see sections 5 and 6), as well as the generalized results of the section 7.

Let us begin by introducing the notion of infimal convolution, or **inf-convolution**:

**Definition 4.1** (inf-convolution ([2] chapter 12)). *Let  $f$  and  $g$  be two functions from  $\mathbb{R}^d$  to  $] - \infty, +\infty]$ . The inf-convolution of  $f$  and  $g$ , denoted by  $f \square g$ , is defined by*

$$f \square g : \mathbb{R}^d \rightarrow ] - \infty, +\infty] : x \mapsto \inf_{y \in \mathbb{R}^d} \{f(x - y) + g(y)\}. \quad (4-1)$$

In the remainder of our problem, we will need the following property, in conjunction with the Legendre transform ([2] chapter 13): for all functions  $f$  and  $g$  from  $\mathbb{R}^d$  to  $] - \infty, +\infty]$ , we have

$$(f \square g)^* = f^* + g^*. \quad (4-2)$$

We also recall the definition of **the proximal operator**:

**Definition 4.2** (The proximal operator ([2] chapter 12)). Let  $f$  be a function of  $\mathbb{R}^n$  ( $n \in \mathbb{N}^*$ ) in  $\mathbb{R}$  proper, l.s.c. and convex, and let  $x \in \mathbb{R}^n$ . The proximal operator of  $f$  in  $x$ , denoted by  $\text{Prox}_f(x)$  is the unique minimizer of  $f + \frac{1}{2}|x - \cdot|^2$  in  $\mathbb{R}^n$ . In other words:

$$\text{Prox}_f(x) = \underset{y \in \mathbb{R}^n}{\text{argmin}} \left( f(y) + \frac{1}{2}|y - x|^2 \right). \quad (4-3)$$

The proximal operator can be characterized by the following relation:

$$y = \text{Prox}_f(x) \Leftrightarrow x - y \in \partial f(y). \quad (4-4)$$

The operators  $\text{Prox}_f$  and  $\text{id} - \text{Prox}_f$  are non-expansive (1-Lipschitz). Let us recall the identity of Moreau (linking the proximal operator of  $f$  with that of its Legendre transform  $f^*$ ):

$$\text{Prox}_{\gamma f^*} = \text{id} - \gamma \text{Prox}_{f/\gamma}(\cdot/\gamma) \quad (4-5)$$

We have here defined the operator on  $\mathbb{R}^n$ , but this can be defined on more general spaces (Hilbert spaces for example), with the same properties.

Let us finally define the Moreau envelope:

**Definition 4.3** (Moreau envelope ([2] chapter 12)). Let  $f : \mathbb{R}^d \rightarrow ]-\infty, +\infty]$  convex, l.s.c. proper and let  $\gamma > 0$ . The Moreau envelope of  $f$  with parameter  $\gamma$  is defined by:

$$\gamma f = f \square \left( \frac{1}{2\gamma} |\cdot|^2 \right). \quad (4-6)$$

By definition of the proximal operator, we can therefore also characterize  $\gamma f$ , for all  $x \in \mathbb{R}^d$  and  $\gamma > 0$ , by:

$$\gamma f(x) = f(\text{Prox}_{\gamma f}(x)) + \frac{1}{2\gamma} |x - \text{Prox}_{\gamma f}(x)|^2 \quad (4-7)$$

where  $\gamma f$  is convex and Fréchet-differentiable on  $\mathbb{R}^d$ . Using (4-5), its gradient reads:

$$\nabla(\gamma f) = \gamma^{-1}(\text{id} - \text{Prox}_{\gamma f}) = \text{Prox}_{f^*/\gamma}(\cdot/\gamma). \quad (4-8)$$

The mapping  $\nabla(\gamma f)$  is therefore  $\gamma^{-1}$ -Lipschitz. Moreover, for every  $x \in \mathbb{R}^d$ , from (4-8) and (4-4), we have

$$\nabla(\gamma f)(x) \in \partial f(\text{Prox}_{\gamma f}(x)). \quad (4-9)$$

#### 4.1 Definition and first properties of the velocity field

With the above definition 4.3, the velocity field (3-3) of an optimal transport can be written as a proximal operator  $\mathbf{p}$ . This will allow us to deal more easily with the problems of "breaks" of the velocity field (which are not necessarily discontinuities). An interesting property of this proximal operator is that it realizes a bijection in the regular areas of the velocity, while being able to close the potential "breaks" of the velocity.

**Definition 4.4** (Operator  $\mathbf{p}$ ). Let  $\phi : \mathbb{R}^d \mapsto \mathbb{R}$  satisfying the property  $(\Gamma_1)$  (especially  $\phi$  is convex and continuous at every point of  $\mathbb{R}^d$ , and admits in each of these point a non-empty and compact sub-differential). The operator  $\mathbf{p}$  is defined as

$$\begin{aligned} \mathbf{p}_\phi : [0, 1[ \times \mathbb{R}^d &\rightarrow \mathbb{R}^d \\ (t, x) &\mapsto \text{Prox}_{\frac{t}{1-t}\phi} \left( \frac{x}{1-t} \right) \end{aligned}$$

$\mathbf{p}_\phi$  satisfies the following properties:

1. for all  $t \in [0, 1[$ ,  $\mathbf{p}_\phi(t, \cdot)$  is  $1/(1-t)$ -Lipschitz,

2. if  $t \in ]0, 1[$ , by setting  $\phi_t = (1-t)|\cdot|^2/2 + t\phi$ , then  $(\phi_t)^*$  is of class  $C^1$  on  $\mathbb{R}^d$  and we have:

$$\mathbf{p}_\phi(t, \cdot) = \nabla_x(\phi_t)^*, \quad (4-10)$$

3. for all  $t \in [0, 1[$ ,  $\mathbf{p}_\phi(t, \cdot)$  is surjective on  $\mathbb{R}^d$  and for all  $x, y \in \mathbb{R}^d$ ,

$$y = \mathbf{p}_\phi(t, x) \Leftrightarrow x \in (1-t)y + t\partial\phi(y). \quad (4-11)$$

4. the velocity  $v$  introduced in (3-3) can be defined from  $\mathbf{p}_\phi$  by:

$$v_\phi(t, x) = \frac{x - \mathbf{p}_\phi(t, x)}{t}, \quad (4-12)$$

for all  $t \in ]0, 1[$  and  $x \in \mathbb{R}^d$ .

*Proof:* The first point simply results from the non-expansivity of the proximal operator (see Definition 4.2). The second point can be established by observing that the Legendre transform  $(\phi_t)^*$  of  $\phi_t$  can be written in the form of a Moreau envelope. Indeed, through the property (4-2) of the inf-convolution we have  $(\phi_t)^* = {}^{1-t}(t\phi)^*$ . Using the relation (4-8) of the Definition 4.3, we can then deduce that for all  $t \in ]0, 1[$ ,  $(\phi_t)^*$  is of class  $C^1$  on  $\mathbb{R}^d$ , and

$$\nabla_x(\phi_t)^* = \frac{\text{id} - \text{Prox}_{(1-t)(t\phi)^*}}{1-t} = \text{Prox}_{\frac{t}{1-t}\phi} \left( \frac{\cdot}{1-t} \right) = \mathbf{p}_\phi(t, \cdot). \quad (4-13)$$

The third point is immediately deduced from the characterization (4-4) of the proximal operator. Finally, the fourth point comes by combining relations (4-10) and (3-3).  $\square$

**Remark 4.1.** In the following, when there is no ambiguity on  $\phi$ , we will use  $v$  to denote the velocity field  $v_\phi$ .

We now recall that the field of trajectories  $X$ , defined for all  $t \in ]0, 1[$  in (3-2), is

$$X(t, \cdot) = \nabla_x(\phi_t) = (1-t)\text{id} + t\nabla\phi.$$

Combining this relation with (4-10), we can observe that  $\mathbf{p}_\phi(t, \cdot)$  formally represents the reciprocal of the characteristic traces  $X(t, \cdot) = \nabla_x(\phi_t)$ .

In the general case (i.e. under the hypothesis  $(\Gamma_1)$ ),  $\mathbf{p}_\phi(t, \cdot)$  is not injective on  $\mathbb{R}^d$ : The operator  $\mathbf{p}_\phi(t, \cdot)$  thus repairs the "breaks" that can be generated by a transport plan. Indeed  $\mathbf{p}_\phi(t, \cdot)$  re-concentrates the area  $s$  generated by diffusion (by the characteristic trajectories  $X(t, \cdot)$ ) of the break points on these same points. Thus  $\mathbf{p}_\phi(t, \cdot)$  can be bijective only in the case where there are no "breaks" in the transport plan.

**Remark 4.2.** The "break" points of the transport plan correspond to the points where the potential  $\phi$  is not differentiable. Although Theorem 4.1 attests that the set of such points is negligible, the diffusion of these breaks, and in particular the torsion of the velocity field at these points in  $t = 0$  (or  $t = 1$  if we consider the points of irregularity of  $\phi^*$ ) is not. Indeed, the torsions of the velocity field in the neighborhood of break points may **prevent from having  $H^1$  regularity of the velocity field at these points**. Notice that a  $H^1$  regularity of the potential  $\phi$  would have greatly simplified the study discussed in section 6 on the uniqueness of the saddle points of the Lagrangian  $\mathbf{L}$ . Unfortunately, such regularity can not be assumed in general.

Next, we can deduce from (4-12) and the first property in 4.4, that for  $t \in ]0, 1[$  fixed, the velocity field  $v(t, \cdot)$  is lipschitz on  $\mathbb{R}^d$ . It is also possible to define a Lipschitz constant that is only time dependent so that it does not rely on space nor on  $\phi$ . The Lipschitz constant

$$L_t = 2/t(1-t) \quad (4-14)$$

is namely always valid on  $\mathbb{R}^d$  (for the Euclidean norm  $|\cdot|$ ), whatever  $\phi$  is. The field of velocity  $v$  is therefore continuous and almost everywhere Fréchet-differentiable in space (by Rademacher's Theorem 4.1), and thus  $\|\nabla_x v(t, x)\|$  is additionally **uniformly bounded** by  $L_t = 2/t(1-t)$  for almost all  $x \in \mathbb{R}^d$ , where  $\|\cdot\|$  denotes the subordinate norm to  $|\cdot|$ .

Using the reformulation of  $v$  in the definition 4.4, we finally deduce the following property on the velocity field:

**Proposition 4.1.** *We assume that  $\phi$  satisfies the property  $(\Gamma_1)$ . For every  $t \in ]0, 1[$ , and for every  $y \in \mathbb{R}^d$  such that  $\phi$  is a Fréchet-differentiable in  $y$  (for almost all  $y \in \mathbb{R}^d$ ), we have :*

$$\nabla\phi(y) - y = \partial_t X(t, y) = v(t, X(t, y)), \quad (4-15)$$

with  $X(t, \cdot) = \nabla\phi_t = \nabla[(1-t)\text{id} + t\nabla\phi]$ .

*Proof:* Let us take  $y \in \mathbb{R}^d$  such that  $\phi$  is differentiable at  $y$  (i.e.  $\partial\phi(y) = \{\nabla\phi(y)\}$ ). Note that according to (4-11), we have  $y = \mathbf{p}_\phi(t, (1-t)y + t\nabla\phi(y)) = \mathbf{p}_\phi(t, X(t, y))$ . The equation (4-15) can be deduced immediately from (4-12).  $\square$

The above proposition can also be reformulated as follows: given that  $X(t, \cdot) = \nabla\phi_t$  and  $\mathbf{p}_\phi(t, \cdot) = \nabla_x(\phi_t)^*$  (See the second point of the definition 4.4): for any  $t \in ]0, 1[$ , and for all  $x \in \mathbb{R}^d$  such that  $\phi$  Fréchet-differentiable in  $x$ , we have  $(\nabla_x(\phi_t)^* \circ \nabla\phi_t)(x) = x$ .

## 4.2 Velocity field control

In this subsection, we will show some properties of the velocity field  $v$  defined in (4-12). In particular, we will be interested in the fact that  $v$  is in the space  $L_{loc}^\infty([0, 1] \times \Omega)^d$ , so that it satisfies the property (P2').

**Proposition 4.2.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Let a velocity field  $v$  be defined with respect to  $\phi$  as in (4-12). We then have  $v \in L_{loc}^\infty([0, 1] \times \mathbb{R}^d)^d$ . More precisely, for any bounded open set  $\omega \subset \mathbb{R}^d$ , if we define  $M = \sup_{x \in \omega} |\partial\phi(x)|$  and  $M^* = \sup_{x \in \omega} |\partial\phi^*(x)|$ , we have*

$$\sup_{(t,x) \in ]0,1[ \times \omega} |v(t, x)| \leq 5(\max\{M, M^*\} + \sup(\omega)). \quad (4-16)$$

Moreover, there exists  $\psi \in W_{loc}^{1,\infty}([0, 1] \times \Omega)$ , such that  $v = \nabla_x \psi$ .

*Idea of proof:* We will show that  $v(t, \cdot)$  is uniformly bounded on  $\omega$  in the neighborhood of  $t = 0$ . Take for example  $t \in ]0, 1/2]$  and  $y \in \omega$ , and let  $x \in (1-t)y + t\partial\phi(y)$ . According to (4-11), we have  $\mathbf{p}_\phi(t, x) = y$ , so  $v(t, x) = (x - y)/t \in \partial\phi(y) - y$ .

We have already seen in (4-14) that for the Euclidean norm  $|\cdot|$ ,  $v(t, \cdot)$  is  $2/t(1-t)$ -Lipschitz in space on  $\mathbb{R}^d$  for  $t \in ]0, 1[$ , so

$$|v(t, y) - v(t, x)| \leq \frac{2}{t(1-t)} |x - y| \leq 4 \left| \frac{x - y}{t} \right| = 4|v(t, x)|. \quad (4-17)$$

We have  $|v(t, y)| \leq 5|v(t, x)| \leq 5(M + \sup(\omega))$  (by (4-17)) for all  $t \in ]0, 1/2]$  and all  $y \in \omega$ , with  $M = \sup_{x \in \omega} |\partial\phi(x)|$ . The same argument can be used in the neighborhood of  $t = 1$  on  $[1/2, 1[$ .

For the second point, we recall that according to the last point of the definition 4.4, we have  $v(t, \cdot) = (\text{id} - \nabla_x(\phi_t^*)/t)$  with  $\phi_t = (1-t)|\cdot|^2/2 + t\phi$ . Thus,  $v = \nabla_x\psi$  in the sense of distributions, with  $\psi(t, x) = (1/t) (|\cdot|^2/2 - (\phi_t)^*)$ , for all  $t \in ]0, 1[$  and for all  $x \in \mathbb{R}^d$ . We thus deduce that  $v \in L^\infty([0, 1] \times \mathbb{R}^d)^d$  and also  $\psi \in W_{loc}^{1,\infty}([0, 1] \times \mathbb{R}^d)$  by applying the Lemma 5.2.  $\square$

In this case,  $M$  and  $M^*$  are finite. Indeed, as  $\phi$  is assumed to satisfy the property  $(\Gamma_1)$ ,  $\phi$  and  $\phi^*$  are assumed to be finite and convex on  $\mathbb{R}^d$  and therefore locally Lipschitz, in particular Lipschitz on  $\omega$ . Thus,  $\partial\phi$  and  $\partial\phi^*$  are uniformly bounded on  $\omega$ . Note also that if  $\phi$  verifies the property  $(\Gamma_1)$ , then, as  $(\phi^*)^* = \phi$ ,  $\phi^*$  also satisfies this property.

As we shall see later in the proposition 4.5, if for all  $t \in ]0, 1[$  the interpolated transport plans  $\nabla\phi_t$  would had been reversal, i.e. if  $\phi$  had been of class  $C^1$  with Lipschitz gradient, then we would have had  $\mathbf{p}_\phi(t, \cdot) = (\nabla\phi_t)^{-1} = \nabla(\phi_t)^*$ , and then by (4-12)  $v(t, \cdot) = \nabla\phi((\nabla\phi_t)^{-1}) - (\nabla\phi_t)^{-1}$ . The field  $v$  would then have been extensible by continuity in  $t = 0$  and  $t = 1$ : the result of the proposition 4.2 would then have been obvious.

However, in the general case, a transport can induce a change in topology between the supports of the initial and final masses, that is to say admitting "breaks" and therefore points of non-regularity for the potential  $\phi$ .

With respect to the initial saddle point problem, we have  $q = (-(1/2)|v|^2, v) \in L^\infty(Q)^{d+1} \subset L^2(Q)^{d+1}$ . As already stated in (4-14), for every  $t \in ]0, 1[$ ,  $v(t, \cdot)$  is continuous and Lipschitz on  $\mathbb{R}^d$  (by providing  $\mathbb{R}^d$  with Euclidean norm, one can take  $2/t(1-t)$  as the Lipschitz constant). The field  $v(t, \cdot)$  is therefore Lipschitz on  $\mathbb{R}^d$ , for a Lipschitz constant independent of  $t$  on any interval  $[\alpha, \beta] \subset ]0, 1[$ . One can for instance consider the constant  $M_{\alpha,\beta} = \sup_{[\alpha,\beta]} 2/t(1-t)$ . It is therefore possible to apply the Cauchy-Lipschitz Theorem on  $[\alpha, \beta]$ . Then, for every  $x \in \mathbb{R}^d$  and  $t \in ]0, 1[$ , the Cauchy problem

$$\begin{cases} y'_{t,x} = v(\cdot, y_{t,x}) \\ y_{t,x}(t) = x, \end{cases} \quad (4-18)$$

admits a unique maximum solution over any interval  $]\alpha, \beta[$ ,  $0 < \alpha < t < \beta < 1$ . We can then easily prove that there exists a unique solution defined on  $]0, 1[$  and that it can be written  $y_{t,x}(s) = (s-t)v(t, x) + x$  for all  $s \in ]0, 1[$ . Indeed, such a solution satisfies  $y_{t,x}(t) = x$ , and  $y'_{t,x}(s) = v(t, x) = v(s, (s-t)v(t, x) + x) = v(s, y_{t,x}(s))$ , as stated in the next Proposition 4.3. It should also be noticed that the problem of Cauchy:

$$\begin{cases} y' = v(\cdot, y) \\ y(t) = X(t, x) = (1-t)x + t\nabla\phi(x), \end{cases} \quad (4-19)$$

has  $y(s) = (s-t)v(t, X(t, x)) + X(t, x) = X(s, x)$  as unique solution on  $]0, 1[$ .

We also want to show that the  $v$  field is locally Lipschitz on  $]0, 1[ \times \mathbb{R}^d$ , i.e. in **time-space** (and not just in space). This is the first point of the properties  $(\mathcal{B})$  on  $(0, 1) \times \mathbb{R}^d$ .

**Proposition 4.3.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Then for all  $t, s \in ]0, 1[$ , and for all  $x \in \mathbb{R}^d$ ,*

$$v(t, x) = v(s, (s-t)v(t, x) + x).$$

*Idea of proof:* This can be shown using the properties (4-11) and (4-12) of the operator  $\mathbf{p}$ .  $\square$

In the above proof, the hypothesis  $(\Gamma_1)$  is only useful for the conditions on  $\phi$ , and not for  $\phi^*$ . In fact, we here only need this assumption to make  $\phi$  admitting at all points of  $\mathbb{R}^d$  a non-empty and compact sub-differential.

**Proposition 4.4.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Then  $v$  is locally Lipschitz on the space  $]0, 1[ \times \mathbb{R}^d$ .*



*Idea of proof:* For all  $(t_1, x_1), (t_2, x_2) \in [\alpha, \beta] \times \mathbb{R}^d$ , we have, according to Proposition 4.3, the relation  $v(t_1, x_1) = v(t_2, (t_2 - t_1)v(t_1, x_1) + x_1)$  (one reduces to the same time  $t_2$ ). We can then conclude by the Lipschitz property (in space) of the field  $v(t, \cdot)$ , with the Lipschitz constant  $2/t(1 - t)$  for the Euclidean norm (see (4-14)).  $\square$

### 4.3 Preamble on the regularity of the velocity field

One of the assertions to verify in order to establish the Theorem 3.1, is to prove that every velocity field  $v$  defined by

$$v(t, x) = \frac{x - \mathbf{p}_\phi(t, x)}{t} = \frac{1}{t} \left( x - \text{Prox}_{\frac{t}{1-t}\phi} \left( \frac{x}{1-t} \right) \right), \quad (4-20)$$

with  $\phi$  satisfying  $(\Gamma_1)$ , is in line with properties  $(\text{II})$  and  $(\mathcal{B})$ .

The first point of the properties  $(\text{II})$  was partially established by the Proposition 4.2 (that is  $v \in L_{loc}^\infty([0, 1] \times \mathbb{R}^d)$ ), and will be fully established by the Lemma 5.2. The first point of the properties  $(\mathcal{B})$  has been established by the Proposition 4.4.

We will therefore try to establish that  $v$  also satisfies the remaining properties, i.e. the second point of the properties  $(\text{II})$ , and the last two points of properties  $(\mathcal{B})$  (sub-section 4.7).

As we have already mentioned after the theorem 3.1, properties  $(\mathcal{B})$  will no longer be used in the rest of our study. However, their demonstration, as well as that of the Burgers equation for  $v$ , will be done in parallel with other results concerning **the control of the gradient of regularized velocity fields**, and will allow the control of the solutions of the problem of Transport generated by the field  $v$ . These parallel results will then prove important in the proofs (based on the characteristic method) of the main results of the sub-section 6.2 and the section 7, relating to uniqueness The transport of the density  $\rho$  and the characterization of the velocity field  $v$  for optimal transport.

Moreover, the properties  $(\mathcal{B})$  illustrate one of the difficulties relative to the field  $v$  that we will have to circumvent, namely that if it is infinitely close to a regularity  $H^1$  on  $[0, 1] \times \mathbb{R}^d$ , it is not generally the case (for example in the case of a mass intersecting in two parts, the velocity field associated with the optimal transport is not in the space  $H^1$  in the instant moment  $t = 0$ ), which will complicate our study somewhat in the section 6 (we will return).

As we have already mentioned above, this non-regularity  $H^1$  is partly due to the possible "breaks" in the continuous transport scheme in  $t = 0$  (or conversely to possible connections in  $t = 1$ ). Such breaks correspond to the points of non-differentiability of the potential  $\phi$  and the connections in  $t = 1$  are linked to the non-differentiability points of the Legendre transform  $\phi^*$  of the potential  $\phi$ . However, because of the symmetry of the problem, we will not need to deal with the connection problems in  $t = 1$ .

We will start by showing these different properties in an "ideal" framework, that is to say without breaks: this will constitute the framework of the sub-section 4.4. For this purpose, we first consider that the potential  $\phi$  is fairly regular, more precisely that it satisfies the property  $(\Gamma_2)$  that we enumerate below:

$\phi$  satisfies the property  $(\Gamma_2)$  if and only if:

**Hypothesis  $(\Gamma_2)$ .**  $\phi$  satisfies the property  $(\Gamma_1)$ , is of class  $C^1$ , and  $\nabla_x \phi$  is Lipschitz (i.e.  $\phi \in C^{1,1}(\mathbb{R}^d)$ ).

We recall that  $\phi$  verifies the property  $(\Gamma_1)$  if and only if  $\phi$  and  $\phi^*$  are convex, continuous and admit a minimum on  $\mathbb{R}^d$ .

Once this is done, we will establish these properties in the general framework of the optimal transport (i.e. for  $\phi$  satisfying the property  $(\Gamma_1)$ ), going back to the regular case via a

regularization by Envelope of the potential  $\phi$ . This will be the main object of the sub-section 4.5.

For this next study we will often need to call upon two fundamental results, which we present below: the Rademacher Theorem (differentiability of a locally Lipschitz function).

**Theorem 4.1** (The Rademacher Theorem ([10] p.81)). *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$  be a locally Lipschitz function. Then  $f$  is  $\mathcal{L}^n$ -almost everywhere Fréchet-differentiable (and its differential in the sense of Fréchet coincides with its differential in the sense of distributions).*

Let us give at once an example of the application of Rademacher Theorem (an example which will be useful to our problem):

**Example 4.1.** *We know that proposition 4.4 is valid under the  $(\Gamma_1)$  property (and therefore also under the  $(\Gamma_2)$  property), we know that  $v$  is locally Lipschitz on  $]0, 1[ \times \mathbb{R}^d$ . Thus, according to Rademacher's Theorem ref rademacher,  $v$  is almost everywhere differentiable on  $]0, 1[ \times \mathbb{R}^d$ , and its differential corresponds to its derivative in the sense of distributions. In particular, it is observed that*

$$v \cdot \nabla_x v = \frac{1}{2} \nabla_x |v|^2, \quad (4-21)$$

*in the sense  $\mathcal{L}^{d+1}$ -almost everywhere, as in the sense of distributions.*

#### 4.4 Burgers Equation and results of regularity : case $(\Gamma_2)$

In this subsection, we will assume that  $\phi$  satisfies the hypothesis  $(\Gamma_2)$  in order to show that, in the "ideal" case (without breaks), the velocity field  $v$  satisfies the Burgers equation in the sense of distributions, i.e.

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0,$$

As well as the control of the gradient of the velocity field:  $\nabla_{t,x} v \in L_{loc}^\infty([0, 1[, L^1(\mathbb{R}^d))$  (the properties stated in the different results will in fact be a little more precise, in order to better correspond to our future objectives).

The operator  $\mathbf{p}_\phi$  (definition 4.4) corresponds in some ways to a spatial "reciprocal" of the operator  $X(t, \cdot) = \nabla_x(\phi_t) = (1 - t) \text{id} + t \nabla \phi$ . "In some ways" because,  $X(t, \cdot)$  is generally not invertible. This will however be the case under the assumption  $(\Gamma_2)$ .

**Proposition 4.5.** *Under the property  $(\Gamma_2)$ ,  $\mathbf{p}_\phi$  satisfies (that is to say in addition to the properties already specified in the definition 4.4, for all  $t \in [0, 1[$ , the following properties:*

1.  $\mathbf{p}_\phi(t, \cdot)$  is bijective on  $\mathbb{R}^d$  and for all  $x, y \in \mathbb{R}^d$ ,

$$y = \mathbf{p}_\phi(t, x) \Leftrightarrow x = (1 - t)y + t \nabla \phi(y). \quad (4-22)$$

2.  $\mathbf{p}_\phi(0, \cdot) = \text{id}$ .

3. the velocity  $v$  defined (3-3) can be defined from  $\mathbf{p}_\phi$  by :

$$v(t, x) = \nabla \phi(\mathbf{p}_\phi(t, x)) - \mathbf{p}_\phi(t, x) = \frac{x - \mathbf{p}_\phi(t, x)}{t}, \quad (4-23)$$

for all  $(t, x) \in ]0, 1[ \times \mathbb{R}^d$ ,

4.  $v$  can be continuously extended on  $[0, 1[ \times \mathbb{R}^d$  (i.e. in  $t = 0$ ) and

$$v(0, x) = \nabla\phi(x) - x, \quad (4-24)$$

for all  $x \in \mathbb{R}^d$ .

*Proof:* The potential  $\phi$  is assumed to be of class  $C^1$ , therefore differentiable at every point  $x \in \mathbb{R}^d$ , and by convexity we have  $\partial\phi(x) = \{\nabla\phi(x)\}$ , hence the first point. We thus deduce immediately the second (taking  $t = 0$ ), and the third point:

$$x = (1-t)\mathbf{p}_\phi(t, x) + t\nabla\phi(\mathbf{p}_\phi(t, x)) \Leftrightarrow \nabla\phi(\mathbf{p}_\phi(t, x)) - \mathbf{p}_\phi(t, x) = \frac{x - \mathbf{p}_\phi(t, x)}{t} = v(t, x). \quad (4-25)$$

For the last point, let  $x \in \mathbb{R}^d$  and a  $(t_n, x_n)_n \in (Q)^\mathbb{N}$  converging to  $(0, x)$ . For  $a$  a minimum of  $\phi$ , we have  $\mathbf{p}_\phi(t, (1-t)a) = a$  for all  $t \in [0, 1[$ : the lipschitz character of  $\mathbf{p}_\phi$ . Then let us prove by estimating the distance from  $\mathbf{p}_\phi$  to  $a$  that the sequence  $(\mathbf{p}_\phi(t_n, x_n))_n$  is bounded, same as the sequence  $(\nabla\phi(\mathbf{p}_\phi(t_n, x_n)))_n$  (by continuity of  $\nabla\phi$ ). The right-hand side of the equivalence (4-25) then allows us to prove that  $(\mathbf{p}_\phi(t_n, x_n))_n$  converges to  $x$  (so continuity of  $\mathbf{p}_\phi$  in  $(0, 0)$ ). The second term of this equivalence allows us to conclude.  $\square$

We note that  $\phi_t = (1/2)(1-t)|\cdot|^2 + t\phi$  is of class  $C^1$ , strictly convex and superlinear (for  $\phi$  is convex and  $|\cdot|^2$  is strictly convex and superlinear). Thus, for all  $t \in ]0, 1[$ ,  $X(t, \cdot) = (1-t)\text{id} + t\nabla\phi = \nabla\phi_t$  is bijective of reciprocal  $\mathbf{p}_\phi(t, \cdot) = \nabla_x(\phi_t)^*$ . Let's look at the first result announced at the beginning of this subsection on the  $v$  field: the Burgers equation.

**Proposition 4.6.** *Under the property  $(\Gamma_2)$ ,  $v$  satisfies (3-7), namely:*

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0$$

in the sense of distributions.

*Idea of proof:* We can note that, if  $\phi$  is on class  $C^1$ , then by deriving the advection relation  $\nabla_x \phi - \text{id} = \partial_t \nabla_x \phi_t = v(t, \nabla_x \phi_t)$ , we then obtain (see for justification the example 4.1):  $0 = \partial_{tt} \nabla_x \phi_t = (\partial_t v + v \cdot \nabla_x v)(\nabla_x \phi_t)$  (recall :  $\phi_t = (1-t)|\cdot|^2/2 + t\phi$ ).  $\square$

In order to prove the second regularity result on  $v$  announced at the beginning of this subsection, which will be stated in the Proposition 4.8, we will present now intermediate results on the potential  $\phi$ .

**Proposition 4.7.** *We assume that  $\phi$  satisfies the property  $(\Gamma_2)$ . Let  $R' > R > 0$  and  $a \in \mathbb{R}^d$  such that  $\phi(a) = \inf_{\mathbb{R}^d} \phi$ . Then there exists  $t_0 \in ]0, 1[$  such that for all  $t \in [0, t_0]$ ,*

$$\mathbf{p}_\phi(t, B(a, R)) \subset \mathbf{p}_\phi(t_0, B(a, R')). \quad (4-26)$$

More precisely, it is sufficient to satisfy the property (4-26) that  $t_0$  satisfies the condition:

$$t_0 < \min \left\{ \frac{1}{2}, \frac{R' - R}{M + 2|a|} \right\} \quad \text{avec} \quad M = \sup_{x \in B(a, 2(R+|a|))} |\partial\phi(x)|. \quad (4-27)$$

*Idea of proof:* Let  $x \in B(a, R)$  and  $t, t_0 \in [0, 1[$  such that  $t_0 > 0$  and  $t \in [0, t_0]$ .  $\mathbf{p}_\phi(t, x) \in \mathbf{p}_\phi(t_0, B(a, R'))$  if and only if there exists  $y \in B(a, R')$  such that  $\mathbf{p}_\phi(t_0, y) = \mathbf{p}_\phi(t, x)$ . According to the definition of  $\mathbf{p}_\phi$  (by the equivalence (4-22)), for  $y \in \mathbb{R}^d$ :

$$\mathbf{p}_\phi(t_0, y) = \mathbf{p}_\phi(t, x) \Leftrightarrow y = (1-t_0)\mathbf{p}_\phi(t, x) + t_0\nabla\phi(\mathbf{p}_\phi(t, x)). \quad (4-28)$$

Let us take  $y = (1 - t_0) \mathbf{p}_\phi(t, x) + t_0 \nabla \phi(\mathbf{p}_\phi(t, x))$  and look for a sufficient condition on  $t_0$  for  $y \in B(a, R')$ . First, let us recall that, from the (4-4),  $\mathbf{p}_\phi(t, (1 - t)a) = a$ , and therefore that for every  $x \in B(a, R)$ ,

$$|\mathbf{p}_\phi(t, x) - a| = |\mathbf{p}_\phi(t, x) - \mathbf{p}_\phi(t, (1 - t)a)| \leq \frac{1}{1 - t} |x - (1 - t)a| \leq \frac{1}{1 - t_0} (R + |a|).$$

By taking  $t_0 \leq 1/2$ , we thus have  $\mathbf{p}_\phi(t, x) \in B(a, 2(R + |a|))$ . Now, under the property  $(\Gamma_2)$ ,  $\phi$  is of class  $C^1$  on  $\mathbb{R}^d$  and therefore locally lipschitzian, and thus lipschitzian on  $B(a, 2(R + |a|))$ . We can therefore ask

$$M = \sup_{x \in B(a, 2(R + |a|))} |\partial \phi(x)| < +\infty.$$

For all  $t \in [0, t_0]$  and  $x \in B(a, R)$ , and  $y = (1 - t_0) \mathbf{p}_\phi(t, x) + t_0 \nabla \phi(\mathbf{p}_\phi(t, x))$ :

$$\begin{aligned} |y - a| &\leq (1 - t_0) |\mathbf{p}_\phi(t, x) - a| + t_0 |\nabla \phi(\mathbf{p}_\phi(t, x)) - a| \\ &\leq (1 - t_0) |\mathbf{p}_\phi(t, x) - \mathbf{p}_\phi(t, (1 - t)a)| + t_0 (|\nabla \phi(\mathbf{p}_\phi(t, x))| + |a|) \\ &\leq \frac{1 - t_0}{1 - t} |x - (1 - t)a| + t_0 (M + |a|) \leq |x - a| + t|a| + t_0 (M + |a|) \\ &\leq R + t_0 (M + 2|a|). \end{aligned}$$

Even if  $t_0 < \min\{1/2, (R' - R)/(M + 2|a|)\}$ , then for all  $t \in [0, t_0]$  and  $x \in B(a, R)$ ,  $y = (1 - t_0) \mathbf{p}_\phi(t, x) + t_0 \nabla \phi(\mathbf{p}_\phi(t, x)) \in B(a, R')$  i.e.  $\mathbf{p}_\phi(t_0, y) = \mathbf{p}_\phi(t, x)$ , and so  $\mathbf{p}_\phi(t, B(a, R)) \subset \mathbf{p}_\phi(t_0, B(a, R'))$ .  $\square$

**Lemma 4.1.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_2)$ . For every  $t \in [0, 1[$ ,  $\mathbf{p}_\phi(t, \cdot)$  is almost everywhere differentiable on  $\mathbb{R}^d$  (its differential coincides with its derivative in the sense of distributions). Moreover, for almost every  $x \in \mathbb{R}^d$ ,  $\nabla \phi$  is differentiable in  $\mathbf{p}_\phi(t, x)$  and*

$$\nabla_x \mathbf{p}_\phi(t, x) = (t \text{Hess}(\phi)(\mathbf{p}_\phi(t, x)) + (1 - t)I)^{-1} \quad (4-29)$$

where  $I \in \mathcal{M}_d(\mathbb{R})$  designates the identity matrix.

*Proof:* Let  $t \in [0, 1[$ . The operator  $\mathbf{p}_\phi(t, \cdot)$  is Lipschitz and bijective, and from (4-22) its reciprocal is  $\mathbf{p}_\phi(t, \cdot)^{-1} = t \nabla \phi + (1 - t) \text{id} = X(t, \cdot)$ . Recall that by hypothesis  $\nabla \phi$  is assumed to be Lipschitz. According to the Rademacher Theorem 4.1,  $\nabla \phi$  et  $\mathbf{p}_\phi(t, \cdot)$  are almost everywhere differentiable on  $\mathbb{R}^d$  and their differentials coincide with their derivatives in the sense of distributions. Thus the set  $F$  of the points of  $\mathbb{R}^d$  where  $\nabla \phi$  is not differentiable is zero measure (for the Lebesgue measure). The fact that  $\nabla \phi$  is assumed to be lipschitzian implies that  $X(t, \cdot)$  is also, and therefore  $\mathcal{L}^d(X(t, F)) = 0$  ([10] p. 75) :  $X(t, F)$  is the set of points  $x \in \mathbb{R}^d$  for which  $\nabla \phi$  is not differentiable in  $\mathbf{p}_\phi(t, x)$ , this means that  $\nabla \phi$  is differentiable in  $\mathbf{p}_\phi(t, x)$  for almost all  $x \in \mathbb{R}^d$ . Thus, for almost every  $x \in \mathbb{R}^d$ ,  $\mathbf{p}_\phi(t, \cdot)$  is differentiable in  $x$ ,  $\nabla \phi$  is differentiable in  $\mathbf{p}_\phi(t, x)$  et  $I = (t \text{Hess}(\phi)(\mathbf{p}_\phi(t, x)) + (1 - t)I) \nabla_x \mathbf{p}_\phi(t, x)$ . The potential  $\phi$  being convex,  $\text{Hess}(\phi)(\mathbf{p}_\phi(t, x))$  is symmetric positive, and hence by coercivity,  $t \text{Hess}(\phi)(\mathbf{p}_\phi(t, x)) + (1 - t)I$  is symmetric definite positive and therefore invertible in  $\mathcal{M}_2(\mathbb{R})$ , which concludes the proof.  $\square$

The fact that  $\nabla \phi$  is **globally** lipschitzian on  $\mathbb{R}^d$  is very important here because it ensures that  $\mathbf{p}_\phi(t, \cdot)$  does not concentrate too large areas of  $\mathbb{R}^d$ . Indeed, if  $\nabla \phi$  does not have such regularity, the set of  $x$  in which  $\phi$  is not twice differentiable could not be of measure zero.

Note that for every  $t \in [0, 1[$ , the operator  $\mathbf{p}_\phi(t, \cdot)$  proves to be almost everywhere differentiable on  $\mathbb{R}^d$ , whatever the regularity Of  $\phi$  (because a proximal operator is always Lipschitz). The additional property brought here by the regularity of  $\phi$  is in fact the bijectivity of the operator

$\mathbf{p}_\phi(t, \cdot)$ . The globally Lipschitz character of  $\nabla\phi$  yet allows us to explicate almost everywhere  $\nabla_x \mathbf{p}_\phi(t, \cdot)$  as a function of  $\mathbf{p}_\phi(t, \cdot)$ .

We consider the norm  $|\cdot|_1$  on  $\mathcal{M}_d(\mathbb{R})$  defined on  $|A|_1 = \sum_{i,j} |a_{ij}|$ .

We now have all the elements to state the following proposition, which is one of the main results of this section: it concerns the control of the gradient of the velocity field  $v$ . This control, as we have already mentioned in the previous subsection, will be exploited later to control the solutions of the transport problem generated by the  $v$  field (see the uniqueness results of the section 6. This Proposition is valid for  $v = v_\phi$  defined from a potential  $\phi$  verifying the  $(\Gamma_1)$  property, but as for all the results of this sub-Section, we will demonstrate this first by assuming that the potential  $\phi$  satisfies the property  $(\Gamma_2)$ , that is to say, let us recall, that  $\phi$  is assumed to be convex, of class  $C^1$ , reduced to  $\mathbb{R}^d$ , and its spatial gradient  $\nabla_x \phi$  is assumed to be Lipschitzian. We then generalize the proof in the following subsection (i.e. for a potential  $\phi$  satisfying the property  $(\Gamma_1)$ ).

**Proposition 4.8.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_2)$ . Let  $R' > R > 0$  and  $a \in \mathbb{R}^d$  such that  $\phi(a) = \inf_{\mathbb{R}^d} \phi$ . Then there exist constants  $C$  and  $C'$  (independent of  $\phi$ ,  $a$ ,  $R$  and  $R'$ ) such that for all  $t_0 \in ]0, 1[$  satisfying the condition 4-27, we have the property:*

$$\forall t \in ]0, t_0], \int_{B(a,R)} |\nabla_x v(t, x)|_1 dx \leq C \int_{B(a,R')} |\nabla_x v(t_0, x)|_1 dx \leq \frac{C'}{t_0(1-t_0)} \mathcal{L}^2(B(a, R')) \quad (4-30)$$

and so  $\nabla_x v \in L^\infty([0, t_0], L^1(B(a, R)))$ .

*Proof:* Let  $t \in ]0, 1[$ . Remember that for all  $x \in \mathbb{R}^d$ ,  $v(t, x) = \nabla\phi(\mathbf{p}_\phi(t, x)) - \mathbf{p}_\phi(t, x)$ . According to the Lemma 4.1, for almost all  $x \in \mathbb{R}^d$ ,  $v(t, \cdot)$  is differentiable on  $x$  and

$$\begin{aligned} \nabla_x v(t, x) &= \nabla_x \mathbf{p}_\phi(t, x) (\text{Hess}(\phi)(\mathbf{p}_\phi(t, x)) - I) \\ &= (t \text{Hess}(\phi)(\mathbf{p}_\phi(t, x)) + (1-t)I)^{-1} (\text{Hess}(\phi)(\mathbf{p}_\phi(t, x)) - I). \end{aligned} \quad (4-31)$$

The  $1/t(1-t)$ -Lipschitz character of  $v(t, \cdot)$  (to a multiplicative constant), already mentioned above, implies  $|\nabla_x v(t, \cdot)|_1$  is bounded by  $c/t(1-t)$  ( $c$  a constant depending only on the chosen norms). Thus  $|\nabla_x v(t, \cdot)|_1 \in L^\infty(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ . Since the function  $X(t, \cdot) = t\nabla\phi + (1-t)\text{id}$  is Lipschitz and bijective (and  $\mathbf{p}_\phi(t, \cdot)$  is its reciprocal), we therefore can apply the Generalized Variable Change Theorem ([10] p. 117) and we obtain:

$$\begin{aligned} \int_{B(a,R)} |\nabla_x v(t, x)|_1 dx &= \int_{\mathbf{p}_\phi(t, B(a,R))} |\det(t \text{Hess}(\phi)(y) + (1-t)I)| \\ &\quad \times |(t \text{Hess}(\phi)(y) + (1-t)I)^{-1} (\text{Hess}(\phi)(y) - I)|_1 dy. \end{aligned} \quad (4-32)$$

For every  $y \in \mathbb{R}^d$  where  $\nabla\phi$  is differentiable, the  $\text{Hess}(\phi)$  matrix is symmetric positive and can therefore be diagonalised in an orthonormal basis: let us consider  $\lambda_1(y), \dots, \lambda_d(y) \geq 0$  the eigenvalues associated with  $\text{Hess}(\phi)$  in such an orthonormal basis. Considering the equivalence between the  $|\cdot|_1$  norm and the Frobenius norm, we obtain the following relation:

$$\begin{aligned} C_1 |(t \text{Hess}(\phi)(y) + (1-t)I)^{-1} (\text{Hess}(\phi)(y) - I)|_1 &\leq \sum_{i=1}^d \left| \frac{\lambda_i(y) - 1}{t\lambda_i(y) + (1-t)} \right| \\ &\leq C_2 |(t \text{Hess}(\phi)(y) + (1-t)I)^{-1} (\text{Hess}(\phi)(y) - I)|_1, \end{aligned} \quad (4-33)$$

where the constants  $C_1$  and  $C_2$  depend only on the constants of equivalence between the  $|\cdot|_1$  norm and the Frobenius norm. Moreover, by invariance of the determinant by a similar transformation,

we get for all  $t \in [0, 1[$  and almost all  $y \in \mathbb{R}^d$ :

$$\det(t \text{Hess}(\phi)(y) + (1-t)I) = (t\lambda_1(y) + (1-t)) \cdots (t\lambda_d(y) + (1-t)). \quad (4-34)$$

By injecting (4-33) and (4-34) into (4-32), we so obtain, for every  $t \in ]0, 1[$  (and every  $R > 0$ ):

$$\begin{aligned} C_1 \int_{B(a,R)} |\nabla_x v(t, x)|_1 dx &\leq \sum_{j=1}^d \int_{\mathbf{p}_\phi(t, B(a,R))} |\lambda_j(y) - 1| \prod_{\substack{i=1 \\ i \neq j}}^d (t\lambda_i(y) + (1-t)) \\ &\leq C_2 \int_{B(a,R)} |\nabla_x v(t, x)|_1 dx \end{aligned} \quad (4-35)$$

recalling that  $\lambda_1(y), \dots, \lambda_d(y) \geq 0$ . The constants  $C_1$  and  $C_2$  are independent of  $t$  and  $R$ . Let  $t_0 \in ]0, 1[$  verifying the condition (4-27) of the Proposition 4.7. Hence, by this Proposition, we have  $\mathbf{p}_\phi(t, B(a, R)) \subset \mathbf{p}_\phi(t_0, B(a, R'))$ . The condition (4-27) also gives  $t_0 \leq 1/2$ , and so for any  $t \in ]0, t_0]$ , we have  $1-t \leq 1 \leq 2(1-t_0)$ . We have  $\lambda_i(y) \geq 0$  ( $i = 1, \dots, d$ ) for almost all  $y \in \mathbb{R}^d$ , and thus  $0 \leq t\lambda_i(y) + (1-t) \leq 2(t_0\lambda_i(y) + (1-t_0))$ . Thus, thanks to the inequality (4-35), we can conclude:

$$\begin{aligned} \int_{B(a,R)} |\nabla_x v(t, x)|_1 dx &\leq \frac{1}{C_1} \sum_{j=1}^d \int_{\mathbf{p}_\phi(t, B(a,R))} |\lambda_j(y) - 1| \prod_{\substack{i=1 \\ i \neq j}}^d (t\lambda_i(y) + (1-t)) \\ &\leq \frac{2^d}{C_1} \sum_{j=1}^d \int_{\mathbf{p}_\phi(t_0, B(a,R'))} |\lambda_j(y) - 1| \prod_{\substack{i=1 \\ i \neq j}}^d (t_0\lambda_i(y) + (1-t_0)) \\ &\leq \frac{2^d C_2}{C_1} \int_{B(a,R')} |\nabla_x v(t_0, x)|_1 dx \end{aligned} \quad (4-36)$$

Finally, as already mentioned at the beginning of proof, there exists a constant  $c > 0$  such that  $|\nabla_x v(t_0, \cdot)|_1$  is bounded by  $c/t_0(1-t_0)$ , which completes the proof of the Proposition.  $\square$

The proof of this Proposition also provides us with evidence for another Lemma concerning the control of  $v$  which will be useful in the following: it will in fact be the combination of the different results of this section concerning the control of the gradient of the field  $v$  which will allow us to control the solutions of the transport problem and thus obtain our future uniqueness results.

**Lemma 4.2.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_2)$ . Then there exists a constant  $C > 0$  such that for all  $t \in ]0, 1[$ ,*

$$\|\nabla_x v(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{1-t} \left( \|\text{Hess}(\phi)\|_{L^\infty(\mathbb{R}^d)} + d \right)$$

(where the norm  $L^\infty$  is taken from  $\|\cdot\|$ : the subordinate norm to the Euclidean norm  $|\cdot|$  of  $\mathbb{R}^d$ ).

*Proof:* By taking the inequality (4-33) and the equivalence between the  $|\cdot|_1$  and Frobenius norms, we obtain, for every  $y \in \mathbb{R}^d$  where  $\nabla\phi$  is differentiable:

$$\begin{aligned} C_1|(t\text{Hess}(\phi)(y) + (1-t)I)^{-1}(\text{Hess}(\phi)(y) - I)|_1 &\leq \sum_{i=1}^d \left| \frac{\lambda_i(y) - 1}{t\lambda_i(y) + (1-t)} \right|_1 \leq \frac{1}{1-t} \left( d + \sum_{i=1}^d \lambda_i(y) \right) \\ &\leq \frac{C_2}{1-t} (|\text{Hess}(\phi)(y)|_1 + d). \end{aligned} \quad (4-37)$$

We can then conclude by injecting the equation (4-31) into this last inequality.  $\square$

#### 4.5 Burgers Equation and results of regularity : case $(\Gamma_1)$

We will now place ourselves in a more general framework and extend the results of the previous subsection to potentials  $\phi$  which no longer satisfies the property  $(\Gamma_1)$ , that is, in this subsection  $\phi$  will possess non-differentiability points **causing breaks in the transport plane**.

In a first time, we will show that the result of the Proposition 4.6, which one prove that  $v$  satisfies the Burgers equation (3-7) namely:

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0,$$

remains valid for  $\phi$  satisfying only the property  $(\Gamma_1)$ .

In a second time, we will show, in particular by invoking the Proposition 4.8, that in general terms, that for all  $0 < R < R'$  and  $a \in \mathbb{R}^d$  such that  $\phi(a) = \inf_{\mathbb{R}^d} \phi$ , and for all  $0 < t_0 < \min\{1/2, (R' - R)/(M + 2|a|)\}$ , we have  $\nabla_x v \in L^\infty([0, t_0], L^1(B(a, R)))$ , And more particularly that there exists a constant  $C' > 0$  such that, for all  $t \in [0, t_0]$ ,

$$\int_{B(a, R)} |\nabla_x v(t, x)|_1 dx \leq \frac{C'}{t_0(1-t_0)} \mathcal{L}^2(B(a, R')).$$

In the context of our optimal transport problem, we proved at Proposition 3.2, that we can assume that the potential  $\phi$  satisfies the property  $(\Gamma_1)$ . We will show, by regularizing  $\phi$ , that  $\nabla_x v$  is uniformly integrable in the neighborhood of  $t = 0$ , then we will argue symmetrically to show that it is the same in the neighborhood of  $t = 1$ . For this regularization, we will use the Moreau envelope (introduced in the Definition 4.3)  $\gamma\phi$  of  $\phi$  for all  $\gamma > 0$  whose definition we recall: for all  $x \in \mathbb{R}^d$ ,

$$\gamma\phi(x) = \inf_{y \in \mathbb{R}^d} \frac{1}{2\gamma} |x - y|^2 + \phi(y) = \frac{1}{2\gamma} |x - \text{Prox}_{\gamma\phi}(x)|^2 + \phi(\text{Prox}_{\gamma\phi}(x)), \quad (4-38)$$

For all  $\gamma > 0$ , we also define the velocity field  $v_\gamma$  for all  $t \in ]0, 1[$  and  $x \in \mathbb{R}^d$  by

$$v_\gamma(t, x) = v_{\gamma\phi}(t, x) = \nabla\phi(\mathbf{p}_{\gamma\phi}(t, x)) - \mathbf{p}_{\gamma\phi}(t, x) = \frac{x - \mathbf{p}_{\gamma\phi}(t, x)}{t}, \quad (4-39)$$

with we recall,

$$\mathbf{p}_{\gamma\phi}(t, x) = \text{Prox}_{\frac{t}{1-t}\gamma\phi} \left( \frac{x}{1-t} \right).$$

We also recall that  $\gamma\phi$  is of class  $C^1$  and that for all  $x \in \mathbb{R}^d$ ,

$$\nabla(\gamma\phi)(x) = \frac{x - \text{Prox}_{\gamma\phi}(x)}{\gamma} \in \partial\phi(\text{Prox}_{\gamma\phi}(x)) \quad (\text{by (4-4)}). \quad (4-40)$$



The potential  $\gamma\phi$  is then  $\gamma^{-1}$ -Lipschitz.

First, we have to prove that if  $\phi$  satisfies the property  $(\Gamma_1)$ , then  $\gamma\phi$  satisfies the  $(\Gamma_2)$  property, and thus can be returned via the  $\phi_\gamma$ , to the cases treated in the sub-section 4.4 (i.e. for  $\phi$  and thus of transport plans without breaks).

**Lemma 4.3.** *If  $\phi$  satisfies the  $(\Gamma_1)$  property, then  $\gamma\phi$  satisfies the  $(\Gamma_2)$  property for all  $\gamma > 0$*

*Idea of proof:* Note that on one side  $(\gamma f)^* = f^* + \frac{\gamma}{2} |\cdot|^2$  (and then if  $f^*$  is in a Hölder space  $C^{1,1}$  and admits a minimum on  $\mathbb{R}^d$ , the same holds for  $(\gamma f)^*$ . And note on the other hand that the functions  $\phi$  and  $\gamma\phi$  have the same minima on  $\mathbb{R}^d$ .  $\square$

We now report convergence results for  $\nabla\gamma\phi$  and  $(v_\gamma)_\gamma$ .

**Lemma 4.4.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Then the family  $(\nabla\gamma\phi)_{\gamma>0}$  is locally bounded, uniformly with respect to  $\gamma > 0$ . Moreover, for all  $x \in \mathbb{R}^d$ , the set of the adhesion values of the family  $(\nabla\gamma\phi(x))_\gamma$ , when  $\gamma > 0$  tends to 0 is included in  $\partial\phi(x)$ .*

*In particular, if  $\phi$  is differentiable in  $x$ , then  $\nabla\gamma\phi(x)$  converges to  $\nabla\phi(x)$  when  $\gamma > 0$  tends to 0. The functions  $\nabla\gamma\phi$  converge simply almost everywhere to  $\nabla\phi$  when  $\gamma$  tends to 0.*

*Proof:* We can easily show that for every  $a$  minimum of  $\phi$  on  $\mathbb{R}^d$  and  $r > 0$ , we have  $\nabla(\gamma\phi)(B(a, r)) \subset \partial\phi(B(a, r))$ : it is enough to jointly use the relation (4-9) and the inclusion  $\text{Prox}_{\gamma\phi}(B(a, r)) \subset B(a, r)$ , this one resulting from the non-expansiveness of the operators  $\text{Prox}_{\gamma\phi}$  and the fact that  $a$  is a fixed point for these operators, i.e.  $\text{Prox}_{\gamma\phi}(a) = a$  (property resulting from the relation (4-4)). The  $\partial\phi(B(a, r))$  (union of the subdifferentials of  $\phi$  on  $B(a, r)$ ) is bounded in  $\mathbb{R}^d$  (because of the convex and locally Lipschitzian character of the potential  $\phi$ ), this proves the first point of our Lemma. The second point can be deduced immediately by applying the basic definition of the subdifferential to any sequence  $[\nabla\gamma_n\phi(x)]_n$  converging in  $\mathbb{R}^d$  ( $\gamma_n \rightarrow 0$ ). By the first point, we know that the family  $(\nabla\gamma\phi(x))_{\gamma>0}$  is bounded in  $\mathbb{R}^d$ , and this for all  $x \in \mathbb{R}^d$ . We can finally deduce from these two points that, if  $\phi$  is differentiable in  $x$ , then from any subsequence of  $[\nabla\gamma_n\phi(x)]_n$  (with  $\gamma_n \rightarrow 0$ ), we can extract a subsequence converging to  $\nabla\phi(x)$ : then it suffices to conclude with the absurd.  $\square$

**Proposition 4.9.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . The family  $(v_\gamma)_{\gamma>0}$  (with  $v_\gamma = v_{\gamma\phi}$ ) satisfies the following convergence properties:*

1. *For every bounded open set  $\omega$  in  $\mathbb{R}^d$  and any  $\gamma_0 > 0$ ,  $(v_\gamma)_{\gamma_0 \geq \gamma > 0}$  is uniformly bounded on  $]0, 1[ \times \omega$  by a constant independent of  $\gamma \in ]0, \gamma_0]$ .*
2. *For every bounded open set  $\omega$  in  $\mathbb{R}^d$  and any  $\gamma_0 > 0$ , there exists a constant  $M$  independent of  $\gamma \in ]0, \gamma_0]$ , such that*

$$\forall \gamma \in ]0, \gamma_0], \forall t \in ]0, 1[, \|v(t, \cdot) - v_\gamma(t, \cdot)\|_{L^\infty(\omega)} \leq M \frac{\gamma}{t}. \quad (4-41)$$

*We can deduce in particular from this relation that  $v_\gamma$  converges simply to  $v$  on  $]0, 1[ \times \mathbb{R}^d$  when  $\gamma$  tends to 0.*

3. *For every  $1 < p < +\infty$  and every bounded open set  $\omega$  in  $\mathbb{R}^d$ ,  $v_\gamma$  converges to  $v$  in  $L^p((0, 1) \times \omega)$  when  $\gamma$  tends to 0.*
4. *For every  $1 < p < +\infty$  and any open set bounded  $\omega$  in  $\mathbb{R}^d$ , and for all  $t \in ]0, 1[$ ,  $v_\gamma(t, \cdot)$  converges weakly in  $W^{1,p}(\omega)$  to  $v(t, \cdot)$  when  $\gamma$  tends to 0.*

*Brief proof:* For the first point (bound of the family  $v_\gamma$ ), according to Proposition 4.2, we have  $\|v_\gamma\|_{L^\infty(]0,1[ \times \omega)} \leq 5 (\max\{M_\gamma, M_\gamma^*\} + \sup(\omega))$ , with  $M_\gamma = \|\nabla^\gamma \phi\|_{L^\infty(\omega)}$  and  $M_\gamma^* = \|\partial(\gamma\phi)^*\|_{L^\infty(\omega)}$ . According to Lemma 4.4, the constants  $M_\gamma$  are uniformly bounded with respect to the  $\gamma > 0$ . The same holds for the constants  $M_\gamma^*$  with respect to the  $\gamma \in ]0, \gamma_0]$  by noticing, according to the property (4-2) of the inf-convolution, that  $\partial(\gamma\phi)^* = \partial\phi^* + \gamma \text{id}$ . For the second point, i.e. the inequality (4-41), according to (4-13), for all  $t \in ]0, 1[$ , we have the relation  $\nabla_x(\phi_t)^* = (\text{id} - \text{Prox}_{(1-t)(t\phi)^*})/(1-t)$ . In addition, by applying twice the fundamental relation of the proximal operator (4-4) and the property (4-2), we have

$$\text{Prox}_{(1-t)(t\gamma\phi)^*} = \text{Prox}_{(1-t)(t\phi)^* + (1-t)\frac{\gamma}{2t}|\cdot|^2} = \text{Prox}_{(1-t)(t\phi)^*}(\text{id} - \gamma(1-t)[\text{id} + (1-t)v_\gamma(t, \cdot)]).$$

Thus, according to (4-10) and (4-12), and by non-expansiveness of the proximal operator:

$$|v_\gamma(t, \cdot) - v(t, \cdot)| = \frac{1}{t(1-t)} |\text{Prox}_{(1-t)(t\gamma\phi)^*} - \text{Prox}_{(1-t)(t\phi)^*}| \leq \frac{\gamma}{t} |\text{id} + (1-t)v_\gamma(t, \cdot)|.$$

We thus conclude the relation (4-41) by the fact that the fields  $v_\gamma$  are uniformly bounded on  $]0, 1[ \times \omega$  independently of  $\gamma \in ]0, \gamma_0]$ . The third property is immediately deduced from the two preceding ones by invoking the Dominated Convergence Theorem. For the last point, we also apply the Dominated Convergence Theorem to prove that  $v_\gamma(t, \cdot)$  converges to  $v(t, \cdot)$  in  $L^p(\omega)$  when  $\gamma$  tends to 0: note that the first two properties are valid on **all**  $]0, 1[ \times \mathbb{R}^d$  and not only almost everywhere modulo some measure. Thus, any subsequence of  $(v_\gamma(t, \cdot))_\gamma$  which converge weakly in  $W^{1,p}(\omega)$  get  $v(t, \cdot)$  for limit (reasoning within the sense of the distributions). Let us recall an  $n$ -th time that the  $v_\gamma(t, \cdot)$  and  $v(t, \cdot)$  are  $c/t(1-t)$ -Lipschitz, where  $c$  is a constant depending only on the norm chosen on  $\mathbb{R}^d$  (hence independent of  $\phi$  and  $\gamma$ ), hence  $\|\nabla_x v_\gamma(t, \cdot)\|_{L^\infty(\omega)} \leq c/t(1-t)$ , for all  $\gamma > 0$ . We can then reason by extracting subsequences of  $(v_\gamma(t, \cdot))_\gamma$  converging weakly to  $v(t, \cdot)$  in  $W^{1,p}(\omega)$  (reflexive space if  $1 < p < +\infty$ ): if there was a subsequence of  $(v_\gamma(t, \cdot))_\gamma$  which would not converging to  $v(t, \cdot)$ , we could extract a sub-subsequence converging to  $v(t, \cdot)$ , which would be absurd ...  $\square$

The first two points of this latter proposition imply the following corollary:

**Corollary 4.1.** *We assume that  $\phi$  satisfies the property  $(\Gamma_1)$ . For every bounded open set  $\omega$  in  $\mathbb{R}^d$  and any  $\gamma_0 > 0$ , there exists a constant  $C$ , independent of  $\gamma$ , such that the family  $(v_\gamma)_{\gamma > 0}$  satisfies:*

$$\forall \alpha \in [0, 1], \forall \gamma \in ]0, \gamma_0], \forall t \in ]0, 1[, \|v(t, \cdot) - v_\gamma(t, \cdot)\|_{L^\infty(\omega)} \leq C \left(\frac{\gamma}{t}\right)^\alpha. \quad (4-42)$$

*Proof:* According to the first point of the Proposition 4.9, there exists a constant  $K$ , independent of  $\gamma \in ]0, \gamma_0]$ , such that  $\|v(t, \cdot) - v_\gamma\|_{L^\infty(]0,1[ \times \omega)} \leq K$ . According to (4-41), we then have for all  $\alpha \in [0, 1]$ ,

$$\|v(t, \cdot) - v_\gamma(t, \cdot)\|_{L^\infty(\omega)} \leq M^\alpha K^{1-\alpha} \left(\frac{\gamma}{t}\right)^\alpha \leq \max\{M, K\} \left(\frac{\gamma}{t}\right)^\alpha. \quad (4-43)$$

$\square$

We now have enough elements to prove in the general case, that is to say for  $\phi$  satisfying the property  $(\Gamma_1)$  (thus for a transport plane admitting possible "breaks"), the announced results At the beginning of the section concerning the Burgers equation, and in a second the control and regularity results on  $\nabla v$  and the  $\nabla v_\gamma$ .

**Proposition 4.10.** *With the property  $(\Gamma_1)$ ,  $v$  satisfies (3-7), that is to say:*

$$\partial_t v + \frac{1}{2} \nabla_x |v|^2 = 0,$$

or equivalently,  $\partial_t v + v \cdot \nabla_x v = 0$ , in the distribution sense.

*Idea of proof:* We apply the proposition 4.9 in the weak formulation of the Burgers equation.  $\square$

**Proposition 4.11.** *We assume that  $\phi$  satisfies the property  $(\Gamma_1)$ . Let  $R' > R > 0$  and  $a \in \mathbb{R}^d$  such that  $\phi(a) = \inf_{\mathbb{R}^d} \phi$  and let  $M = \sup_{x \in B(a, 2(R+|a|))} |\partial\phi(x)|$ . Then there exists an constant  $C' > 0$  – independent of  $\phi$ ,  $\gamma\phi$  (and then of  $\gamma$ ),  $a$ ,  $R$  and  $R'$  – such that for all  $t_0 \in ]0, 1[$  satisfying the condition  $t_0 < \min\{1/2, (R' - R)/(M + 2|a|)\}$ , and by setting  $v_0 = v$ , we have the property:*

$$\forall \gamma \geq 0, \forall t \in ]0, t_0], \int_{B(a, R)} |\nabla_x v_\gamma(t, x)|_1 dx \leq \frac{C'}{t_0(1-t_0)} \mathcal{L}^2(B(a, R')). \quad (4-44)$$

thus  $\nabla_x v_\gamma \in L^\infty([0, t_0], L^1(B(a, R)))$ , for all  $\gamma \geq 0$ .

*Idea of proof:* As already mentioned in the Lemma 4.4, we have the inclusion  $\nabla(\gamma\phi)(B(a, r)) \subset \partial\phi(B(a, r))$ , and thus, by setting  $r = 2(R + |a|)$ , we have  $M_\gamma = \sup_{B(a, r)} |\nabla\gamma\phi| \leq M$ . Thus, a  $t_0$  verifying the hypothesis of the statement (independent of  $\gamma$ ), also satisfies the hypothesis of the Proposition 4.8 whatever  $\gamma > 0$ . We can therefore apply this last proposition for such a  $t_0$  for all  $v_\gamma$  and we so get (4-44). For the case  $\gamma = 0$  (i.e.  $v_0 = v$ ), it is enough to apply the fourth convergence result of the proposition 4.9: we can conclude by passage to the lower semi-continuous weak limit.  $\square$

## 4.6 Additional results for unicity results to come

The two results of this subsection are results of control of the regularized velocity fields  $v_\gamma$ . These will be useful for studying the properties of uniqueness of saddle points (see section 6).

**Corollary 4.2** (Corollary of the Proposition 4.11). *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set and  $0 < t_m < 1$ . Then there exists a constant  $K > 0$  such that for every  $\gamma > 0$  and any  $t \in [0, t_m]$ , we have*

$$\int_{\Omega} |\nabla_x v_\gamma(t, x)|_1 dx \leq K. \quad (4-45)$$

*Proof:* For  $t < t_0$ , we apply the Proposition 4.11; and for  $t_m \geq t > t_0$  we use the fact that the term  $|\nabla_x v(t_0, \cdot)|_1$  is bounded by  $c/t(1-t)$ , for  $c$  a constant depending only of the chosen norm (here from  $|\cdot|_1$ ).  $\square$

We shall end with the following Lemma:

**Lemma 4.5.** *Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  convex verifying the property  $(\Gamma_1)$ . Then there exists a constant  $c$ , independent of  $\gamma$ , such that for every  $1 \geq \gamma > 0$ , all  $t \in ]0, 1[$  and almost all  $x \in \mathbb{R}^d$  where  $v_\gamma(t, \cdot)$  is differentiable,*

$$\|\nabla_x v_\gamma(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c}{\gamma(1-t)}.$$

*Proof:* The relation (4-8) of the Definition 4.3, as well as the non-expansiveness of the operator  $\text{id} - \text{Prox}_{\gamma f}$  (Definition 4.2), let us permit to assert that there exists a constant  $C_0$ , independent of  $\gamma$ , such that  $\|\text{Hess}(\gamma\phi)\|_{L^\infty(\mathbb{R}^d)} \leq C_0/\gamma$ . We then conclude by the Lemma 4.2.  $\square$

## 4.7 An independent but notable result

It is interesting to note that the result of the Proposition 4.11 can be generalized and that we can show by symmetry in time that  $\nabla_{t,x}v \in L^\infty((0,1), L^1(\Omega))$  for every bounded open set  $\Omega$ .

This result is in no way useful for the results of existence and uniqueness to follow, that prepare the other results of section. It is nevertheless quite notable for us to enunciate it: it offers a result of regularity and control of the velocity field of an isotropic optimal transport for the distance  $L^2$ . These properties complete the properties  $(\mathcal{B})$ . Because of its independence, we will only generalize the Proposition 4.11 for the case  $\gamma = 0$  (i.e.  $v_0 = v$ ).

**Theorem 4.2.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Let  $\Omega$  be a bounded open set of  $\mathbb{R}^d$ . Then  $\nabla_{t,x}v \in L^\infty((0,1), L^1(\Omega))$ , in other words there exists a constant  $K > 0$  such that for all  $t \in ]0, 1[$ ,*

$$\int_{\Omega} |\nabla_{t,x}v(t, x)|_1 dx \leq K. \quad (4-46)$$

*Idea of proof:* We symmetrize the result of the Proposition 4.11 (case  $\gamma = 0$ ) on the edges in time (respectively in the neighborhood of  $t = 0$  and  $t = 1$ ) and we apply the markup  $c/t(1-t)$  in the middle.  $\square$

We can extend this result:

**Corollary 4.3.** *We suppose that  $\phi$  satisfies the property  $(\Gamma_1)$ . Let  $\Omega$  be a bounded open set. For all  $p, q \geq 1$  such that  $1/p + 1/q > 1$ , we have  $\nabla_{t,x}v \in L^p((0,1), L^q(\Omega))$ . And in particular  $v \in W^{1,p}((0,1) \times \Omega)$  for all  $1 \leq p < 2$ .*

*Idea of proof:* We mix the result of the Theorem 4.2 with the markup by  $c/t(1-t)$  (in other words we partially bounded from above  $|\nabla_x v(t_0, \cdot)|_1$  in order to be able to apply the Theorem 4.2). For the particular case  $W^{1,p}$ , it is enough to take  $p = q < 2$ .  $\square$

Theorem 4.2 and its Corollary 4.3 give the most consistent assumptions of regularity that one can have in general for the velocity field  $v$ . Indeed, according to the Corollary 4.3, a velocity field  $v$  defined with respect to a potential  $\phi$  (verifying the property  $(\Gamma_1)$ ), let us recall it, by the relation (4-12), that is to say for all  $t \in ]0, 1[$  by

$$v(t, \cdot) = v_\phi(t, \cdot) = \frac{\text{id} - \mathbf{p}_\phi(t, \cdot)}{t} = \frac{1}{t} (\text{id} - (t\phi + (1-t)\text{id})^*), \quad (4-47)$$

is an element of  $W_{loc}^{1,p}([0,1] \times \mathbb{R}^d)$  (that is to say that its restriction to any bounded open set  $\Omega$  of  $\mathbb{R}^d$  is an element of  $W^{1,p}((0,1) \times \Omega)$ ) and this for all  $p < 2$ . The question that arises naturally is to know if  $v = v_\phi$  could not in reality be an element of  $H_{loc}^1([0,1] \times \mathbb{R}^d)$ . In general, this is not the case (take Caffarelli's example of the mass cut in two).

## 5 Existence

In order to prove the existence of a saddle point for the Lagrangian  $\mathbf{L}$ , we have given ourselves for first objective to build a couple density-velocity field  $(\rho, v)$  satisfying conditions  $(\mathbf{I}')$ . The velocity field  $v = v_\phi$ , defined in (4-12), immediately satisfies the properties  $(\text{P2}')$  and  $(\text{P3}')$ , according respectively from Proposition 4.2 and Proposition 4.10. We now have to build a density  $\rho = \rho_\phi$ , satisfying the property  $(\text{P1}')$  ( $\rho \in L^2(Q)$ ), and such that the couple  $(\rho_\phi, v_\phi)$  satisfies the condition  $(\text{P4}')$  (conservation of the mass). The candidate density, as already announced in (3-4), will of course be the density of the McCann interpolation between  $\rho_0 \mathcal{L}^d$  and  $\rho_1 \mathcal{L}^d$ . Before starting, let us define more precisely the notion of "measure push forward", which we briefly introduced in the subsection 2.1.

**Properties (C).** Let  $\Omega$  be an open set of  $\mathbb{R}^d$ , and let  $\nu$  be a measure on the Lebesgue tribe of  $\Omega$ . We said that  $\nu$  satisfies the properties (C) if and only if  $\nu(K) < +\infty$  for any compact  $K \subset \Omega$ ;  $\nu(E) = \inf\{\nu(V), E \subset V, V \text{ open set}\}$  for every Lebesgue-measurable set  $E$  of  $\Omega$ ;  $\nu(E) = \sup\{\nu(K), K \subset E, K \text{ compact}\}$  for any  $E$  open set and for any Lebesgue-measurable set  $E$  of  $\Omega$  such that  $\nu(E) < +\infty$ .

**Proposition 5.1.** We give  $\mathbb{R}^d$  of the tribe of Lebesgue. Let  $\mu$  be a positive measure  $\sigma$ -finished on the Lebesgue, and  $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$  measurable (always on the Lebesgue tribe). We suppose that the measure  $\mu$  is finite. Then there exists a positive measure on the Lebesgue tribe  $\nu$  on  $\mathbb{R}^d$ , satisfying the properties (C), such that

$$\forall f \in C_c^0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f d\nu = \int_{\mathbb{R}^d} f(Tx) d\mu. \quad (5-1)$$

Moreover, for every Lebesgue-measurable set  $A \subset \mathbb{R}^d$ , we have  $\nu(A) = \mu(T^{-1}(A))$ . We then say that  $\nu$  is **the thrust of the measure**  $\mu$  by the operator  $T$ , denoted  $\nu = T\#\mu$ .

The equation  $\nu(A) = \mu(T^{-1}(A))$  translates the fact that the  $\nu$  **conserve mass** measured by  $\mu$ :  $\nu$  gives to any displaced, deformed, contracted or dilated area by the operator  $T$  the same mass than given by  $\mu$  before applying the operator. The notion of measurement thrust therefore translates a property of **conservative mass transport**. This is partly at the origin of the idea of a dynamic formulation of the optimal transport problem. As stated in the introduction, this dynamic formulation implies that we replace the "optimal conservative assignment" approach with that of an "optimal conservative displacement" (we study the evolution of a density  $\rho$  between  $\rho_0$  and  $\rho_1$  on a time scale  $[0, 1]$ ). The natural candidate density that we consider is therefore the one formed by the set of intermediate measurements between  $\rho_0\mathcal{L}^d$  and  $\rho_1\mathcal{L}^d = (\nabla\phi\#\rho_0\mathcal{L}^d)$ , which can be assimilated to a series of "optimal micro-transports" along the time scale  $[0, 1]$ : the interpolation density of McCann, which have been already defined in (3-4), defined at each instant  $t$  by the density  $\rho_t = \rho_t^\phi$  of the measure

$$\rho_t\mathcal{L}^d = \rho_t^\phi\mathcal{L}^d = [(1-t)\text{id} + t\nabla\phi]\#(\rho_0\mathcal{L}^d) = \nabla\phi_t\#(\rho_0\mathcal{L}^d). \quad (5-2)$$

The following proposition ensures that it is possible to choose the representatives of each of these densities  $\rho_t$  so that the density  $(t, x) \mapsto \rho_t(x)$  is measurable and such that the weak formulation of the measure push forward (5-1) remains valid for test functions which are only measurable: indeed, within the framework of the Benamou-Brenier algorithm, the test functions involved in this weak formulation will be of type  $L^p$ .

**Proposition 5.2.** Let  $\rho_0 \in L^1(\mathbb{R}^d, d\mathcal{L}^d)$  be a compact support, such that  $\rho_0 \geq 0$ , and  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the  $(\Gamma_1)$  property. Then, for all  $t \in [0, 1[$  there exists a positive measure  $\nu_t$  on the Lebesgue tribe of  $\mathbb{R}^d$ , with support in  $(t\nabla\phi + (1-t)\text{id})(\text{supp}(\rho_0))$  (bounded), satisfying the various properties in the Proposition 5.1, such that  $\nu_t = (t\nabla\phi + (1-t)\text{id})\#(\rho_0\mathcal{L}^d)$ , i.e.

$$\forall f \in C_c^0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f d\nu_t = \int_{\mathbb{R}^d} f(t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x) \quad (5-3)$$

Moreover,  $\nu_t$  is absolutely continuous with respect to the Lebesgue measure  $\mathcal{L}^d$  on  $\mathbb{R}^d$  (i.e.  $\nu_t \ll \mathcal{L}^d$ ), in other words, there exists  $\rho_t \in L^1(\mathbb{R}^d, \mathcal{L}^d)$  such that  $\nu_t = \rho_t\mathcal{L}^d$ . It is also possible, for all  $t \in [0, 1[$ , to choose a representative of  $\rho_t$  in such a way that  $(t, x) \mapsto \rho_t(x)$  is measurable on  $[0, 1] \times \mathbb{R}^d$ .

Finally, the following properties are satisfied:

1. For all  $h \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$  and  $t \in [0, 1[$ , we have  $h \circ (t\nabla\phi + (1-t)\text{id}) \in L_{loc}^\infty(\mathbb{R}^d, \mathcal{L}^d)$ , and

$$\int_{\mathbb{R}^d} h d\nu_t = \int_{\mathbb{R}^d} h(x)\rho_t(x) d\mathcal{L}^d(x) = \int_{\mathbb{R}^d} h(t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x).$$

2. For all  $h \in L_{loc}^\infty([0, 1[\times\mathbb{R}^d, \mathcal{L}^{d+1})$ , the function  $(t, x) \mapsto h(t, t\nabla\phi(x) + (1-t)x)$  is in the space  $L_{loc}^\infty([0, 1[\times\mathbb{R}^d)$ , and

$$\int_0^1 \int_{\mathbb{R}^d} h(t, x)\rho_t(x) d\mathcal{L}^d(x) d\mathcal{L}(t) = \int_0^1 \int_{\mathbb{R}^d} h(t, t\nabla\phi(x) + (1-t)x)\rho_0(x) d\mathcal{L}^d(x) d\mathcal{L}(t).$$

3. For all  $h \in C_c^0([0, 1[\times\mathbb{R}^d)$ ,  $t \mapsto \int_{\mathbb{R}^d} h(t, \cdot) d\nu_t = \int_{\mathbb{R}^d} h \rho_t d\mathcal{L}^d$  is continuous on  $[0, 1[$ , in other words  $t \mapsto \nu_t$  is continuous on  $[0, 1[$  in  $\mathcal{D}'(\mathbb{R}^d)$ .

We shall not present here these last two propositions because of the length and the technicality of their proofs, which are useless to the understanding of our purpose. However, these statements are not superfluous. Indeed, without considering the extension to a class of measurable functions of a property verified originally by continuous functions, a property true "almost everywhere" for a measurable function, such as for exemple the boundary of a function  $L^\infty$ , is no longer necessarily true when we compose this function on the right with another one: we must, for example, ensure that the image of a set negligible by the other function, here  $t\nabla\phi + (1-t)\text{id}$ , remains negligible. Similarly, the fact that for every  $t \in [0, 1[$ , there exists a measurable spatial density  $\rho_t$  for  $\mathcal{L}^d$  for a measure  $\nu_t$  does not necessarily ensure the possibility to choose for each  $t$  a representative  $\tilde{\rho}_t$  of  $\rho_t$  such that the spatio-temporal density  $(t, x) \mapsto \tilde{\rho}_t(x)$  is measurable for  $\mathcal{L}^{d+1}$ . This proposition therefore justifies the use of certain theorems, in particular that of Fubini. For exact proof of the propositions 5.1 et 5.2, we refer to Appendix B of [14], which is exclusively devoted to them.

To return to our problem of existence of a saddle point, we now have to prove that our candidate density, the McCann interpolation density between  $\rho_0\mathcal{L}^d$  and  $\rho_1\mathcal{L}^d$ , verifies the (P1') property, namely is an element of  $L^2(Q)$ .

**Lemma 5.1.** *Let  $1 < p < +\infty$ . Let  $\Omega$  an open set of  $\mathbb{R}^d$ , and  $\rho_0, \rho_1 \in L^p(\Omega)$  two densities of  $\Omega$  with bounded support, et let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  convex such that  $\rho_1\mathcal{L}^d = (\nabla\phi) \# (\rho_0\mathcal{L}^d)$ . Let  $t \mapsto \rho_t$  the interpolation density of McCann between  $\rho_0$  and  $\rho_1$  such that defined in (5-2), and such that existence (and the space-time mesurability) is justified by the Proposition 5.2. Then  $t \mapsto \rho_t \in C^0([0, 1], L^p(\Omega))$  (**strong continuity**).*

*Proof:* For all  $1 < p < +\infty$ , we introduce the functional  $\mathcal{F}_p : \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty]$  defined for all  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  ( $\mathcal{P}_2(\mathbb{R}^d)$  is defined as the space of probability measures  $\mu$  on  $\mathbb{R}^d$  satisfying  $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < +\infty$ ).

$$\mathcal{F}_p(\mu) = \begin{cases} \int_{\mathbb{R}^d} |f(x)|^p d\mathcal{L}^d(x) & \text{if } \mu = f.\mathcal{L}^d \in \mathcal{P}_2(\mathbb{R}^d), \\ +\infty & \text{else.} \end{cases} \quad (5-4)$$

Such a functional one has been classified in [18] under the term "internal energy" of the space  $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ . This function is "geodesically convex" on the space  $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ , in other words it is convex along the geodesics of this space, which are the interpolations of McCann. Thus, the function  $\Lambda_p : [0, 1] \rightarrow [0, +\infty]$ , defined for all  $t \in [0, 1]$  by  $\Lambda_p(t) = \mathcal{F}_p(\rho_t\mathcal{L}^d)$ , for  $t \mapsto \rho_t$  defined in (5-2), is convex on  $[0, 1]$ , and is here finite in  $t = 0$  and  $t = 1$  (because  $\rho_0, \rho_1 \in L^p(\Omega)$ , and  $\rho_0, \rho_1 \in \mathcal{P}_p(\mathbb{R}^d)$ ): it is hence finite and bounded on all the interval  $[0, 1]$ . It is therefore clear, by



definition of  $\Lambda_p$ , that for every  $t \in [0, 1]$ ,  $\rho_t \in L^p(\Omega)$ . Moreover,  $t \mapsto \|\rho_t\|_{L^p(\Omega)}$  is bounded on  $[0, 1]$  by a constant  $M$ . It is also clear, from the Proposition 5.2, that  $t \mapsto \rho_t$  is weakly continuous by  $[0, 1]$  in  $L^p(\Omega)$ : it is sufficient to reason by density of the space of the test functions  $C_c^0(\Omega)$  in  $L^q(\Omega)$  (for  $q \in ]1, +\infty[$  such that  $1/p + 1/q = 1$ ). Since the function  $\Lambda_p$  is convex and finite on  $[0, 1]$ ,  $t \mapsto \Lambda(t) = \|\rho_t\|_{L^p(\Omega)}^p$  is continuous on  $]0, 1[$  (open), and admits a right limit in  $t = 0$  and a left limit in  $t = 1$ . Thus, for any  $t_0 \in ]0, 1[$ , we have  $\lim_{t \rightarrow t_0^+} \|\rho_t\|_{L^p(\Omega)}^p = \|\rho_{t_0}\|_{L^p(\Omega)}^p$ , and, with respect to the right limit of  $\Lambda$  in  $t = 0$  and its left limit in  $t = 1$ , we have

$$\lim_{t \rightarrow 0^+} \Lambda(t) = \lim_{t \rightarrow 0^+} \|\rho_t\|_{L^p(\Omega)}^p \leq \Lambda(0) = \|\rho_0\|_{L^p(\Omega)}^p, \text{ et } \lim_{t \rightarrow 1^-} \Lambda(t) = \lim_{t \rightarrow 1^-} \|\rho_t\|_{L^p(\Omega)}^p \leq \Lambda(1) = \|\rho_1\|_{L^p(\Omega)}^p.$$

Therefore, for all  $t_0 \in [0, 1]$ , we have  $\limsup_{t \rightarrow t_0} \|\rho_t\|_{L^p(\Omega)} \leq \|\rho_{t_0}\|_{L^p(\Omega)}$ .

Finally, we can conclude (see for example Proposition 3.30 of [21]) that the application  $t \mapsto \rho_t$  is strongly continuous from  $[0, 1]$  in  $L^p(\Omega)$ .  $\square$

Conversely, one could rigorously characterize the McCann interpolation by the relation

$$\forall \varphi \in C_c^0([0, 1] \times \Omega), \int_{(0,1) \times \mathbb{R}^d} h \rho \, dx \otimes dt = \int_0^1 \int_{\mathbb{R}^d} h(t, t\nabla\phi(x) + (1-t)x) \rho_0(x) \, dx \, dt. \quad (5-5)$$

Indeed, by reasoning with Fubini's Theorem and Lemma 5.1 (for extreme bounds  $t_{min} = 0$  and  $t_{max} < 1$ ), it can be proved that for any density  $\rho$  verifying (5-5), there exists a family of density  $(\rho_t)_{t \in [0,1]}$  as defined in the proposition 5.2, such that  $t \mapsto \rho_t \in C^0([0, 1[, L^p(\mathbb{R}^d))$  and such that  $\rho(t, x) = \rho_t(x)$  for almost all  $(t, x) \in [0, 1[ \times \mathbb{R}^d$  (see Lemma 4.1-5 of [14]).

By Brenier's Theorem [20] p. 66), we have  $\text{supp}(\rho_1) = \overline{\nabla\phi(\text{supp}(\rho_0))}$  (this property is maintained by the re-precision of the potential  $\phi$ , satisfying the property  $(\Gamma_1)$ , of the Proposition 3.2. Thus,  $\Omega$  being a convex open set of  $\mathbb{R}^d$  containing  $\text{supp}(\rho_0)$  and  $\text{supp}(\rho_1)$ , for all  $t \in [0, 1]$ , we have the inclusion  $(t\nabla\phi + (1-t)\text{id})(\text{supp}(\rho_0)) \subset \Omega$ . The weak formulation of the McCann interpolation (5-5) clearly shows that if a test function  $h$  has its support disjoint of  $\overline{Q}$ , then  $\int_0^1 \int_{\mathbb{R}^d} h \rho \, dx \, dt = 0$ : the support of  $\rho : (t, x) \mapsto \rho_t(x)$  is therefore (strictly) included in the set  $[0, 1] \times \Omega \subset \overline{Q}$ .

We have now proved that the candidate density  $\rho : (t, x) \mapsto \rho_t(x)$ , defined in (5-2), satisfies the condition (P1'). The above paragraph ensures that the component  $\mu = (\rho, \rho v)$  is zero in the neighborhood of the edges in space, and thus verifies the Neumann conditions implicitly included in the weak conservation formulation of mass, i.e. the condition (P4') (voir (3-5)). The following proposition aims to prove that the pair  $(\rho, v)$  satisfies the condition (P4').

**Proposition 5.3.** *Let  $\Omega$  be a convex open set of  $\mathbb{R}^d$ . Let  $\rho_0$  be a probability density and  $\mu_1$  a probability measure such that  $\text{supp}(\rho_0), \text{supp}(\mu_1) \subset \Omega$ , and such that there exists  $\phi : \mathbb{R}^d \mapsto \mathbb{R}$  satisfying the property  $(\Gamma_1)$ , and  $\mu_1 = \nabla\phi \# (\rho_0 \mathcal{L}^d)$ . Let  $\rho =: (t, x) \mapsto \rho_t(x)$  and  $v$  as defined respectively in (5-2) and (4-12). Then the couple  $(\rho, v)$  satisfies (3-5) (conservation of the mass) i.e.*

$$\forall h \in C_c^\infty([0, 1] \times \Omega), \int_{[0,1] \times \Omega} (\partial_t h + v \cdot \nabla_x h) \rho \, dx \, dt + \int_{\Omega} h(0, \cdot) \rho_0 \, dx - \int_{\Omega} h(1, \cdot) \, d\mu_1 = 0. \quad (5-6)$$

*Proof:* We recall that by the Proposition 4.1, for all  $t \in ]0, 1[$ , and for almost all  $x \in \mathbb{R}^d$ ,  $\partial_t X(t, x) = v(t, X(t, x))$ , therefore for all  $h \in C_c^\infty([0, 1] \times \Omega)$ ,

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}^d} (\partial_t h + v \cdot \nabla_x h) \rho \, dx \, dt &= \int_0^1 \int_{\mathbb{R}^d} (\partial_t h(t, X(t, \cdot)) + \partial_t X(t, \cdot) \cdot \nabla_x h(t, X(t, \cdot))) \rho_0 \, dx \, dt \\ &= \int_0^1 \frac{d}{dt} \int_{\mathbb{R}^d} h(t, X(t, \cdot)) \rho_0 \, dx \, dt = \int_{\mathbb{R}^d} h(1, \nabla_x \phi(x)) \rho_0(x) \, dx - \int_{\mathbb{R}^d} h(0, x) \rho_0(x) \, dx. \end{aligned} \quad (5-7)$$



Since  $\mu_1 = \nabla\phi\#(\rho_0\mathcal{L}^d)$ , we have  $\int_{\mathbb{R}^d} h(1, \nabla_x\phi)\rho_0 dx = \int_{\Omega} h(1, \cdot) d\mu_1$ . The fact that  $v \in L^\infty(Q)$  (Proposition 4.2) ensures that the integrals are well defined. We can thus conclude because of the inclusion  $\text{supp}(\rho) \subset [0, 1] \times \Omega$ .  $\square$

Under the hypotheses of Proposition 5.3, taking  $\mu_1$  as  $\mu_1 = \rho_1\mathcal{L}^d$ , and assuming moreover that  $\rho_0, \rho_1 \in L^p(\mathbb{R}^d)$  (and then  $\rho \in L^p((0, 1) \times \mathbb{R}^d)$  according to the Lemma 5.1) for  $p \geq 2$ . Then, taking  $q \leq 2$  such that  $1/p + 1/q = 1$ , the weak relation (5-6) extends to the test functions  $h \in W^{1,q}((0, 1) \times \Omega)$ . Indeed, the space  $C_c^\infty([0, 1] \times \Omega)$  is dense in  $W^{1,q}((0, 1) \times \Omega)$  for the associate norm and, in addition, the trace operator extends from  $C_c^\infty([0, 1] \times \Omega)$  to  $W^{1,q}((0, 1) \times \Omega)$  in  $t = 0$  et  $t = 1$  when we take  $L^q(\Omega)$  as the space of arrival, which is valid here since  $\rho_0, \rho_1 \in L^p(\Omega)$ .

Since the conditions (I') are now established for  $(\rho, v)$ , we are able to compile them to prove the Theorem 3.1), that is to say to build a triplet  $(\psi, q, \mu)$  satisfying the properties (I). Let us recall that for this, we have chosen to set  $\mu = (\rho, \rho v)$  et  $q = (-(1/2)|v|^2, v)$ . Let us start with a first result to clearly establish the condition (P3) on the term  $q$ : that is say that for such a  $q$ , there then exists  $\psi^* \in W_{loc}^{1,\infty}([0, 1] \times \Omega)$  such that  $q = \nabla_{t,x}\psi^*$ . Note that the velocity field, in addition to satisfying the properties (P2') and (P3'), can be written as  $v = \nabla_x\psi$  (see (3-6)). The triplet  $(\psi^*, q, \mu)$  will then be our saddle point, and we will show in the proof of the Theorem 3.1 that it also verifies the properties (P1) and (P2).

**Lemma 5.2.** *Let us consider a field of velocity  $v \in L^\infty(Q)^d$  (property (P2')), satisfying the property (P3') (the Burgers equation in the sense of distributions) for which there exists a potential  $\psi \in L_{loc}^1(Q)$ , such that  $v = \nabla_x\psi$ .*

*Then there exists a potential  $\psi^* \in W^{1,\infty}(Q)$ , satisfying in sense of distributions the Hamilton-Jacobi equation  $\partial_t\psi^* + (1/2)|\nabla_x\psi^*|^2 = 0$  and for which, by setting  $q = (-(1/2)|v|^2, v)$ , we have  $q = \nabla_{t,x}\psi^*$ . The property (P3) is then satisfied.*

*Proof:* In the sense of distributions, we have

$$\partial_t v + \frac{1}{2}\nabla_x|v|^2 = 0 \Leftrightarrow \partial_t(\nabla_x\psi) + \frac{1}{2}\nabla_x|v|^2 = 0, \Leftrightarrow \nabla_x \left( \partial_t\psi + \frac{1}{2}|v|^2 \right) = 0. \quad (5-8)$$

There exists a distribution  $T$  depending only on  $t \in ]0, 1[$ , such that  $\partial_t\psi + \frac{1}{2}|v|^2 = T$  (see, for example, Theorem 2.16 in [21]). We set  $\psi^* = \psi - G$ , where  $G$  is a primitive distribution of  $T$  on  $]0, 1[$  (and always depends only of  $t$ ). We then verify, in the sense of the distributions,  $\nabla_x\psi^* = \nabla_x\psi = v \in L^\infty(Q)^d$ , and  $\partial_t\psi^* = -(1/2)|v|^2 \in L^\infty(Q)$ . We recall that the open set  $\Omega$  is assumed to be regular: we so have  $\nabla_{t,x}\psi^* \in L^\infty(Q)^{d+1}$ . We then have  $\psi^* \in W^{1,\infty}(Q)$  (to make sure, see Lemma 4.1-11 of [14]). Thus, we have  $\psi^* \in W^{1,\infty}(Q)$ .  $\square$

We are now able to demonstrate the main result of this section, that is the Theorem 3.1 establishing **the existence of a saddle point** for the Lagrangian **L**.

*Proof of the Theorem 3.1:* Let us remember that a element  $(\psi, q, \mu)$  of  $\mathbf{L}^{ps}(\rho_0, \rho_1, \Omega)$  must satisfies  $(\psi, q, \mu) \in Sg$ , as well as the properties (I). First, by the Proposition 4.2, we know that  $v_\phi \in L^\infty(Q)^d$  (property (P2')) and, by the Lemma 5.1, that  $t \mapsto \rho_\phi(t, \cdot) = X_\phi(t, \cdot)\#\rho_0 \in C^0([0, 1], L^2(\Omega))$  (property (P1')).

Then  $\mu_\phi = (\rho_\phi, \rho_\phi v_\phi)$ ,  $q_\phi = (-(1/2)|v_\phi|^2, v_\phi) \in L^2(Q)$  (or at least the restrictions to  $Q$  of  $\mu_\phi$  and  $q_\phi$ ). Moreover, we have shown above that  $\text{supp}(\rho_\phi) \subset [0, 1] \times \Omega$ , then  $\text{supp}(\mu_\phi) \subset [0, 1] \times \Omega$ : The homogeneous Neumann conditions on the space edges of  $\mu$  are thus verified.

Moreover, by setting  $\mu_\phi = (\rho_\phi, \rho_\phi v_\phi)$  and  $q_\phi = (-(1/2)|v_\phi|^2, v_\phi)$ , the condition (P1) is naturally verified. Indeed, for all  $q' = (a, b) \in \mathcal{P}$  (the paraboloid defined in (3-1) p. (3-1)), we

have

$$\begin{aligned} \langle \mu_\phi, q' - q_\phi \rangle &= \int_0^1 \int_\Omega (a\rho_\phi + b \cdot v_\phi \rho) \, dx \, dt - \int_0^1 \int_\Omega \frac{1}{2} |v_\phi|^2 \rho_\phi \, dx \, dt \\ &\leq \int_0^1 \int_\Omega \left( a + \frac{1}{2} |b|^2 + \frac{1}{2} |v_\phi|^2 \right) \rho_\phi \, dx \, dt - \int_0^1 \int_\Omega \frac{1}{2} |v_\phi|^2 \rho_\phi \, dx \, dt \leq 0. \end{aligned}$$

The condition (P2) results as for it from the Proposition 5.3 (property (P4')). Finally,  $v_\phi$  verifies the condition of the Lemma 5.2: indeed, according to Proposition 4.10,  $v_\phi$  satisfies in the sense of distributions the Burgers equation (property (P3')). Moreover, by (3-6), we know that  $v_\phi$  derives from a spatial potential.

Application of Lemme 5.2 permits us to conclude to the existence of one  $\psi_\phi \in W_{loc}^{1,\infty}([0, 1] \times \Omega)$  such that  $q_\phi = \nabla_{t,x} \psi_\phi$  and thus to verify the condition (P3), and also that the triplet  $(\psi_\phi, q_\phi, \mu_\phi)$  is indeed an element of  $Sg$ . This concludes our proof.  $\square$

## 6 Unicity

### 6.1 Unicity of the velocity field on the density support

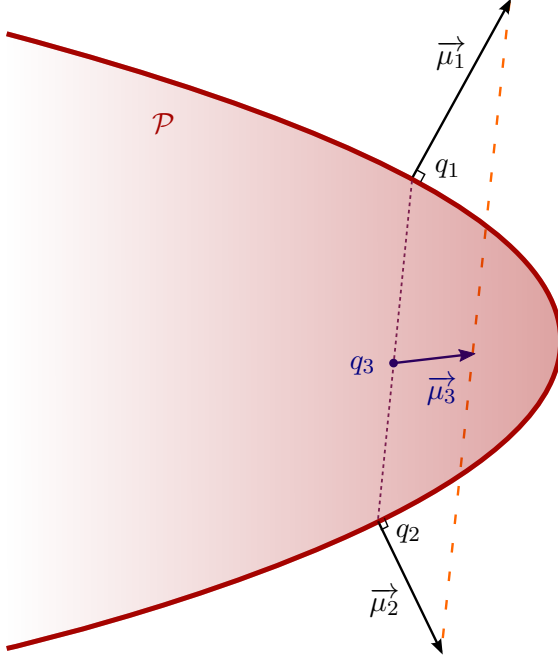
We start by studying the problem of the uniqueness of the velocity field on the support of the different potentially candidate densities, that is to say that for all the saddle points of  $\mathbf{L}$ , denoted by  $(\psi^*, q^*, \mu^*) = (\psi^*, q^*, (\rho^*, m^*))$ , the densities  $\rho^*$  are propagated according to the same velocity field  $v$ .

**Lemma 6.1.** *If  $(\psi^*, q^*, \mu^*)$  is an element of  $\mathbf{L}^{ps}(\rho_0, \rho_1, \Omega)$  (with  $\Omega$  a bounded convex open set of  $\mathbb{R}^d$ , and  $\rho_0, \rho_1 \in L^2(\mathbb{R}^d)$  two densities which supports are included in  $\Omega$ ), with  $\mu^* = (\rho^*, m^*)$ .*

*Then, for all  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$ , by setting  $v = v_\phi$  as defined at (4-12), for all  $t \in ]0, 1[$ , we have  $m^* = \rho^* v_\phi$ .*

*Idea of proof:* We will give a "schematic" proof of the uniqueness of the velocity field on the union of supports of the candidate densities, which is based on the convexity of the set of saddle points and the strict convexity of the paraboloid  $\mathcal{P} = \{(a, b) \in \mathbb{R} \times \mathbb{R}^d, a + |b|^2/2 \leq 0\}$ . For a more rigorous proof we refer again to [14] (chapter 4).

We assume  $(\psi_1, q_1, \mu_1)$  and  $(\psi_2, q_2, \mu_2)$  two stool points of  $\mathbf{L}$ . The fields  $\mu_1$  and  $\mu_2$  are both orthogonal (in the sense of the canonical scalar product of  $L^2$ ) to the hyperparaboloid defined by  $\tilde{\mathcal{P}} = \{(\tilde{a}, \tilde{b}) \in L^2(Q) \times L^2(Q)^d, \tilde{a} + |\tilde{b}|^2/2 \leq 0\}$  respectively at the fields  $q_1$  and  $q_2$ . We will see later in the 8 (see the page 41), that the set of saddle points of  $\mathbf{L}$  is convex: the point  $(1/2)[(\psi_1, q_1, \mu_1) + (\psi_2, q_2, \mu_2)]$  is also a saddle point of  $\mathbf{L}$ . The field  $(1/2)(\mu_1 + \mu_2)$  is also orthogonal to  $\tilde{\mathcal{P}}$  at the field in the  $(1/2)(q_1 + q_2)$ .



Let a point  $(t_0, x_0) \in [0, 1] \times \Omega$  such that the vectors  $\mu_1(t_0, x_0)$ ,  $\mu_2(t_0, x_0)$  as well as the vector  $(1/2)(\mu_1 + \mu_2)(t_0, x_0)$  are all three orthogonal to the paraboloid  $\mathcal{P}$  respectively at the points  $q_1(t_0, x_0)$ ,  $q_2(t_0, x_0)$  and  $(1/2)(q_1 + q_2)(t_0, x_0)$  (which is indeed the case for  $\mathcal{L}^d$ -almost all point  $(t, x) \in [0, 1] \times \Omega$ ); and such that  $\mu_1(t_0, x_0) \neq 0$  or  $\mu_2(t_0, x_0) \neq 0$ : thus  $(1/2)(\mu_1 + \mu_2)(t_0, x_0) \neq 0$  (due to the orthogonality of the vectors  $\mu_1(t_0, x_0)$  and  $\mu_2(t_0, x_0)$  at the paraboloid  $\mathcal{P}$ ).

If we have  $q_1(t_0, x_0) \neq q_2(t_0, x_0)$ , the point  $(1/2)(q_1 + q_2)(t_0, x_0)$  would then be strictly inside the paraboloid  $\mathcal{P}$ , because of its strict convexity: the vector  $(1/2)(\mu_1 + \mu_2)(t_0, x_0)$  would then necessarily be zero, which would contradict the above assumption about this vector.

Ainsi, on a  $q_1(t_0, x_0) = q_2(t_0, x_0)$ . Les vecteurs  $\mu_1(t_0, x_0)$  et  $\mu_2(t_0, x_0)$  sont donc tous deux orthogonales au paraboloïde  $\mathcal{P}$  au même point  $q_1(t_0, x_0)$  et donc proportionnels au vecteur  $(1, b_1(t_0, x_0))$ .

Thus, we have  $q_1(t_0, x_0) = q_2(t_0, x_0)$ . The vectors  $\mu_1(t_0, x_0)$  and  $\mu_2(t_0, x_0)$  are therefore both orthogonal to the paraboloid  $\mathcal{P}$  at the same point  $q_1(t_0, x_0)$  and therefore proportional to the vector  $(1, b_1(t_0, x_0))$ .

We then have  $\mu_k(t_0, x_0) = (\rho_k(t_0, x_0), m_k(t_0, x_0)) = (\rho_k(t_0, x_0), \rho_k(t_0, x_0)b_1(t_0, x_0))$ , for  $k = 1, 2$ , and so for  $\mathcal{L}^d$ -almost all  $(t_0, x_0) \in \text{supp}(\mu_1) \cup \text{supp}(\mu_2)$ . The fields  $\mu_1$  and  $\mu_2$  therefore share the same velocity field (i.e. the field  $b_1$ ) on  $\text{supp}(\mu_1) \cup \text{supp}(\mu_2)$ .  $\square$

**Let us resume:** for any  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$  fixed and for every saddle point  $(\psi^*, q^*, \mu^*)$  of  $\mathbf{L}$ ,  $\rho^*$  is associated with the same velocity field  $v_\phi$  with  $\partial_t \rho^* + \text{div}_x(\rho^* v_\phi) = 0$  (because  $m^* = \rho^* v_\phi$ ), with the initial and final conditions  $\rho^*(0, \cdot) = \rho_0$  and  $\rho^*(1, \cdot) = \rho_1$ . In other words, for all  $h \in H^1(Q)$ , we have

$$\int_0^1 \int_\Omega (\partial_t h(t, x) + v_\phi(t, x) \cdot \nabla_x h(t, x)) \rho^*(t, x) dx dt + \int_\Omega h(0, \cdot) \rho_0 - \int_\Omega h(1, \cdot) \rho_1 = 0.$$

We now prove that there exists a unique  $\rho^*$  which satisfies these conditions, i.e.  $\rho^*(t, \cdot) = X_\phi(t, \cdot) \# \rho_0$  (and this for all  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$ ). We will use the method of the characteristics, based on the Proposition 4.3.

## 6.2 Unicity of density in $L^2$

In the previous subsection, we showed that the velocity field  $v$  corresponding to the displacement of the densities was identical on the union of the supports of the candidate densities, and could be written in an explicit form, i.e. under the form (4-12). This result implies in particular that, if one could show the uniqueness of the density  $\rho$ , we could also conclude the uniqueness of the momentum  $m$  (because  $m = \rho v$ ). However, it is indeed this uniqueness of the velocity field which

will allow us, via a method of characteristics, to conclude to the uniqueness of the density and its explicitation (in other words that it is indeed the temporal interpolation of McCann between the densities  $\rho_0$  and  $\rho_1$ ). The result on which the uniqueness of the density will be explicitly based is the Proposition 6.1 which we enunciate below.

**Proposition 6.1.** *Let a convex potential  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying the property  $(\Gamma_1)$ ,  $\rho_0 \in L^2(\mathbb{R}^d)$ , and a velocity field  $v = v_\phi$  defined from  $\phi$  like in (4-12). Let a density  $\rho \in L^2((0, 1) \times \mathbb{R}^d)$ , with bounded support in  $[0, 1] \times \mathbb{R}^d$ , such that*

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) = 0, \\ \rho(0, \cdot) = \rho_0 \end{cases}$$

(in the distributions sense), i.e.

$$\forall h \in C_c^\infty([0, 1] \times \mathbb{R}^d), \int_0^1 \int_{\mathbb{R}^d} (\partial_t h + v \cdot \nabla_x h) \rho \, dx \, dt + \int_{\mathbb{R}^d} h(0, \cdot) \rho_0 \, dx = 0. \quad (6-1)$$

Then  $\rho(t, \cdot) = \rho_\phi(t, \cdot) = (t \nabla \phi + (1 - t) \operatorname{id}) \# \rho_0$  for almost all  $t \in [0, 1]$ . In other words:

$$\forall \varphi \in C_c^0([0, 1] \times \mathbb{R}^d), \int_0^1 \int_{\mathbb{R}^d} \varphi \rho \, dx \, dt = \int_0^1 \int_{\mathbb{R}^d} \varphi(t, t \nabla \phi(x) + (1 - t)x) \rho_0(x) \, dx \, dt. \quad (6-2)$$

Moreover  $t \mapsto \rho(t, \cdot) \in C^0([0, 1[, L^2(\mathbb{R}^d))$ .

Let us begin by explaining our approach. Let  $(\psi, q, \mu)$  be a saddle point of  $\mathbf{L}$  as defined in (2-10), and let  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$  (thus satisfying the property  $(\Gamma_1)$  on  $\mathbb{R}^d$ ). The latter satisfies the properties (I), which implies in particular the weak conservation of the mass relation  $G(h) + \langle \mu, \nabla_{t,x} h \rangle = 0$  for all  $h \in H^1(Q)$ , as well as the linear coupling density-momentum:  $\mu = (\rho, m) = (\rho, \rho v) \in L^2(Q)$ , with  $v$  defined as in (4-12) (see Lemma 6.1) and satisfying the properties (II) and (B) (see the conclusion of the section 4), whence  $v = v_\phi \in \mathcal{H} = \bigcap_{1 \leq p < 2} W^{1,p}(Q)$ .

From these properties, we deduce that for every  $h \in H^1(Q)$ :

$$\int_0^1 \int_{\Omega} (\partial_t h + v \cdot \nabla_x h) \rho \, dx \, dt = \int_{\Omega} h(1, \cdot) \rho_1 \, dx - \int_{\Omega} h(0, \cdot) \rho_0 \, dx. \quad (6-3)$$

Let  $\varphi \in C_c^\infty(Q)$  such that  $\operatorname{supp}(\varphi) \subset ]0, 1[ \times \Omega \subset Q$ . By solving the transport problem in  $v$  and  $\varphi$  by a characteristics method, there is a function  $\tilde{h} \in \mathcal{H}$  defined for any  $(t, x) \in ]0, 1[ \times \mathbb{R}^d$  by:

$$\tilde{h}(t, x) = - \int_t^1 \varphi(s, (s - t)v(t, x) + x) \, ds, \quad (6-4)$$

which one verifies  $\partial_t \tilde{h} + v \cdot \nabla_x \tilde{h} = \varphi$  and  $\nabla_{t,x} \tilde{h} = {}^t(\nabla_{t,x} v) \alpha + \beta$ , with  $\alpha \in L^\infty(\overline{Q})^d$ , and  $\beta \in L_{loc}^\infty(\overline{Q})^{d+1}$  (since  $\nabla_{t,x} v$  is of size  $d \times (d + 1)$ ). Moreover, we have

$$\tilde{h}(1, \cdot) = 0, \quad \text{et} \quad \tilde{h}(0, \cdot) = - \int_0^1 \varphi(t, X(t, \cdot)) \, dt, \quad \text{avec} \quad X(t, \cdot) = t \nabla \phi + (1 - t) \operatorname{id} \quad (6-5)$$

The function  $\tilde{h}$  is an element of the space  $\mathcal{H}$  and therefore of every space  $W^{1,p}(Q)$  (at least its restriction to  $\Omega$ ) for all  $1 \leq p < 2$ , it has a trace in  $L^2(\Omega)$  at  $t = 0$  and  $t = 1$ , whatever the dimension  $d$  of the transport space: it is enough to choose  $p$  close to 2. However, under conditions (6-5), the relation

$$\tilde{h}(0, \cdot) = - \int_0^1 \varphi(t, X(t, \cdot)) \, dt = - \int_0^1 \varphi(t, t \nabla \phi + (1 - t) \operatorname{id}) \, dt$$

is also not obvious to show (since in general  $v$  does not extend continuously in 0), even if it is very intuitive. Indeed, we have for all  $t \in ]0, 1[$ ,

$$\tilde{h}(t, X(t, x)) = - \int_t^1 \varphi(s, (s-t)v(t, X(t, x)) + X(t, x)) ds = - \int_t^1 \varphi(s, X(s, \cdot)) ds, \quad (6-6)$$

given that  $X(0, \cdot) = \text{id}$ . However, since  $X(t, \cdot)$  is generally not invertible, and  $\nabla\phi$  is generally non-continuous, the relation 6-5 can not be deduced immediately from relation 6-6.

Two cases would have, if they had taken place, facilitated the demonstration: Firstly, if we could have had  $v \in H_{loc}^1(\overline{Q})$ , we would have had :

$$\|\nabla_{t,x}\tilde{h}\|_D^2 \leq \|\alpha\|_{L^\infty(D)}^2 \|\nabla_{t,x}v\|_{L^2(D)}^2 + \mathcal{L}^d(\omega) \|\beta\|_{L^\infty(D)}^2 < +\infty,$$

on any domain  $D = (0, 1) \times \omega$ , with  $\omega$  a bounded open set ( $\omega \subset \mathbb{R}^d$ ). Hence we deduce that  $\tilde{h} \in H_{loc}^1(\overline{Q})$  and thus  $\tilde{h} \in H^1(Q)$  (or at least the restriction from  $\tilde{h}$  to  $\Omega$ ). In a second time, if we could have had  $\rho \in L^q(Q)$  with  $q > 2$  (rather than  $\rho \in L^2(Q)$ ), it would have had been possible to extend the conservation relation of mass (6-3) to all  $h \in W^{1,p}(Q)$  (instead of  $h \in H^1(Q)$ ) for  $1 \leq p < 2$  verifying  $1/p + 1/q = 1$ , and thus for any  $h \in \mathcal{H}$ . In both cases, it would have been possible to inject  $\tilde{h}$  into the relation (6-3):

$$\begin{aligned} \int_0^1 \int_\Omega \varphi(t, x) \rho(t, x) dx dt &= \int_0^1 \int_\Omega \left( \partial_t \tilde{h}(t, x) + v(t, x) \cdot \nabla_x \tilde{h}(t, x) \right) \rho(t, x) dx dt \\ &= \int_\Omega \tilde{h}(1, x) \rho_1(x) dx - \int_\Omega \tilde{h}(0, x) \rho_0(x) dx = - \int_\Omega \tilde{h}(0, x) \rho_0(x) dx \\ &= \int_\Omega \int_0^1 \varphi(t, X(t, x)) dt \rho_0(x) dx = \int_0^1 \int_\Omega \varphi(t, X(t, x)) \rho_0(x) dx dt. \end{aligned} \quad (6-7)$$

And this for any  $\varphi \in C_c^\infty(Q)$  such that  $\text{supp}(\varphi) \subset ]0, 1[ \times \Omega$ . This would have allowed us to conclude that  $\rho$  is the same as the one constructed in the Theorem 3.1 (existence of a saddle point for  $\mathbf{L}$ ), that is to say

$$\rho(dx \otimes dt) = (\text{id}_\tau, X) \# (\rho_0 dx \otimes dt), \text{ avec } \text{id}_\tau : (t, x) \mapsto t \quad (6-8)$$

(even if it means to have to extend to the entire space  $Q$  to properly define the measure push forward in agreement with the notion of measure push forward as defined in the Proposition 5.2).

**Unfortunately this will not be the case:** we are unable to show either of these two cases. However, we do not abandon "the idea", that is to say to treat the problem more or less by the method of the characteristics.

To cling to the first of these two cases, we will approach the velocity field  $v$  (associated with the transport plane  $\nabla\phi$ ) by velocity fields  $v_\gamma = v_{\gamma\phi}$  associated with regularized transport plans: that is to say transport plans of type  $\nabla\gamma\phi$ , où  $\gamma\phi$ , where  $\gamma\phi$  denotes the  $\gamma$ -regularization by a Moreau envelope of the potential  $\phi$  (see Definition 4.3). This regularization has in particular the property of erasing the fractures of the transport plan, which ones are responsible for the fact that  $v$  does not have regularity  $H^1$  in the neighborhood of  $t = 0$  (the neighborhood of  $t = 1$  do not causing problems because, by construction,  $\tilde{h}$  is uniformly zero on this neighborhood).

The convergence of the relation (6-3) injected by the  $\tilde{h}_\gamma$  (solutions associated with  $v_\gamma$ ) to a relation (6-8) type will call on the results developed in the section 4, especially those in the

subsections 4.5 and 4.6. Moreover, to prove this convergence, we will also cling to the second evoked case by calling upon the density of the spaces  $L^q(Q)$  ( $q > 2$ ) in  $L^2(Q)$  (see Lemma 6.2).

In addition, the trace explicitness problem of  $\tilde{h}_\gamma$  discussed above will not arise because the  $v_\gamma$  extend by continuity in  $t = 0$ : it is then similarly for the  $\tilde{h}_\gamma$  (see Proposition 4.5). The proof of the possibility of such an extension follows the same reasoning than those one illustrated by the relation (6-6): indeed, in this case  $X_\gamma(t, \cdot)$  will be invertible and  $\nabla^\gamma \phi$  will be continuous.

**In summary**, by a characteristics method, it is possible to construct some functions  $\tilde{h}_\gamma \in H^1(\bar{Q})$ , uniformly zero in the neighborhood of  $t = 1$  (independently of  $\gamma$ ), such that:

$$\partial_t \tilde{h}_\gamma + v_\gamma \cdot \nabla_x \tilde{h}_\gamma = \varphi = \partial_t \tilde{h}_\gamma + v \cdot \nabla_x \tilde{h}_\gamma + (v_\gamma - v) \cdot \nabla_x \tilde{h}_\gamma,$$

et avec  $\tilde{h}_\gamma(0, \cdot)$  convergeant vers  $-\int_0^1 \varphi[t, (1-t) \text{id} + t \nabla_x \phi] dt$  dans  $L^2(\mathbb{R}^d)$  lorsque  $\gamma$  tend vers 0.

En injectant une telle fonction  $\tilde{h}_\gamma$  dans (6-3), on obtient

and such that  $\tilde{h}_\gamma(0, \cdot)$  converges to  $-\int_0^1 \varphi[t, (1-t) \text{id} + t \nabla_x \phi] dt$  in  $L^2(\mathbb{R}^d)$  when  $\gamma$  tends to 0. By injecting such a function  $\tilde{h}_\gamma$  in (6-3), one obtains

$$\int_0^1 \int_{\mathbb{R}^d} \varphi \rho dx dt = \int_0^1 \int_{\mathbb{R}^d} \varphi[t, (1-t)x + t \nabla_x \phi(x)] \rho_0(x) dx dt + R_\gamma(\varphi), \quad \text{with} \quad (6-9)$$

$$R_\gamma(\varphi) = \int_0^1 \int_{\mathbb{R}^d} (v_\gamma - v) \cdot \nabla_x \tilde{h}_\gamma \rho dx dt - \int_{\mathbb{R}^d} \tilde{h}_\gamma(0, \cdot) \rho_0 dx - \int_0^1 \int_{\mathbb{R}^d} \varphi[t, (1-t) \text{id} + t \nabla_x \phi] \rho_0 dx dt. \quad (6-10)$$

It is therefore necessary, to prove the Proposition 6.1, to prove that the  $R_\gamma(\varphi)$  converges to 0 when  $\gamma$  tends to 0: to do this, as already mentioned above, we will use the complementary results of the subsection 4.6.

The  $\tilde{h}_\gamma$  are defined with respect to  $v_\gamma$  by (6-4). We can then prove that we have

$$\left| \nabla_x \tilde{h}_\gamma(t, x) \right| \leq (|\nabla_x v_\gamma(t, x)| + 1) \|\nabla_x \varphi\|_{L^\infty([0,1] \times \mathbb{R}^d)}, \quad \text{for almost all } (t, x) \in ]0, 1[ \times \mathbb{R}^d \quad (6-11)$$

(for more details see [14] subsection 4.2.5), we specifically detail the fact that  $\tilde{h}_\gamma \in H^1(\bar{Q})$ .

The potential  $\gamma \phi$  verifying the property  $(\Gamma_1)$  (Lemma 4.3), and according to (4-24),  $v_\gamma$  is extended by continuity in  $t = 0$ : so it is the same for  $\tilde{h}_\gamma$  (which is thus continuous on  $[0, 1[ \times \bar{\Omega}$ ), and for all  $x \in \mathbb{R}^d$ ,

$$\tilde{h}_\gamma(0, x) = - \int_0^1 \varphi(s, s v_\gamma(0, x) + x) ds = - \int_0^1 \varphi(s, s \nabla^\gamma \phi(x) + (1-s)x) ds \quad (6-12)$$

(coinciding with the trace  $L^2$  of  $\tilde{h}_\gamma$  in  $t = 0$ ). According to the Lemma 4.4,  $\nabla^\gamma \phi(x)$  converges for almost all  $x \in \Omega$  to  $\nabla \phi(x)$  (for all  $x$  where  $\phi$  is differentiable). Moreover, the term  $\varphi(s, s \nabla^\gamma \phi(x) + (1-s)x)$  is uniformly bounded by  $\|\varphi\|_{L^\infty}$  for all  $(s, x) \in ]0, 1[ \times \Omega$ . Thus, by dominated convergence, we have

$$r_\gamma(\varphi) = \int_\Omega \tilde{h}_\gamma(0, \cdot) \rho_0 dx + \int_0^1 \int_\Omega \varphi(s, s \nabla^\gamma \phi(x) + (1-s)x) \rho_0(x) dx ds \xrightarrow{\gamma \rightarrow 0} 0. \quad (6-13)$$

With the mark-up (6-11), we thus have

$$|R_\gamma(\varphi)| \leq |r_\gamma(\varphi)| + \|\nabla_x \varphi\|_{L^\infty} \left( \int_0^{t_m} \int_\Omega |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt + \int_0^{t_m} \int_\Omega |v - v_\gamma| \cdot |\rho| dx dt \right). \quad (6-14)$$

Proposition 4.9 tells us that  $|v - v_\gamma|$  is uniformly bounded and simply converges to 0 on  $]0, 1[ \times \Omega$  when  $\gamma$  tends to 0. Thus, since  $\rho \in L^2((0, 1) \times \Omega)$ , we conclude via the Dominated Convergence Theorem that the term  $\int_0^{t_m} \int_\Omega |v - v_\gamma| \cdot |\rho| dx dt$  converges to 0.

Finally, to complete the proof of Proposition 6.1, we have to prove that the term  $\int_0^{t_m} \int_\Omega |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt$  converges to 0, which is the subject of the following Lemma.

**Lemma 6.2.** *Let  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$  be a convex potential verifying the  $(\Gamma_1)$  property, and a velocity field  $v = v_\phi$  defined with respect to  $\phi$  as in (4-12), and  $0 < t_m < 1$ . Let  $\rho \in L^2((0, 1) \times \mathbb{R}^d)$ , with bounded support into  $[0, 1] \times \mathbb{R}^d$ , and  $\Omega \subset \mathbb{R}^d$  a bounded open set such that  $\text{supp}(\rho) \subset [0, 1] \times \Omega \subset \bar{Q}$ . For any  $\gamma > 0$ , we define  $v_\gamma = v_{\gamma\phi}$ , where  $\gamma\phi$  is the Moreau envelope of  $\phi$  by the parameter  $\gamma$ , see Definition 4.3. Then we have the result of convergence:*

$$\int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt \xrightarrow{\gamma \rightarrow 0} 0. \quad (6-15)$$

*Proof:* We will begin by showing the following increase:

$$\forall \gamma \in ]0, 1], \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt \leq M \|\rho\|_{L^2(Q)}, \quad (6-16)$$

with  $M$  a constant. To do this, we will proceed to a Chasles division in time of type  $\int_0^{t_m} = \int_0^\gamma + \int_\gamma^{t_m}$ . Recall that for  $t \in ]0, 1[$ , the fields  $v_\gamma(t, \cdot)$  and  $v(t, \cdot)$  are  $c/t(1-t)$ -Lipschitz sur  $\mathbb{R}^d$ , with  $c$  independent of  $\phi$  and  $\gamma$ . For example, by the Lemma 4.5, we have  $\|\nabla_x v_\gamma(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq (c/1-t) \min\{1/\gamma, 1/t\}$ . In the first term, we will thus have  $t \leq \gamma$ , and we will proceed to the bounded from above  $|\nabla_x v_\gamma| \leq \sqrt{c} |\nabla_x v_\gamma|^{1/2} / (\sqrt{\gamma}\sqrt{1-t})$ . In the second, we will have  $t \geq \gamma$ , and we will so proceed to the bounded from above  $|\nabla_x v_\gamma| \leq \sqrt{c} |\nabla_x v_\gamma|^{1/2} / (\sqrt{t}\sqrt{1-t})$ . For the various weighted upper borns, we will invoke the Corollaries 4.1 and 4.2, with keeping the same notations for the constants and parameters involved in these utterances (the parameter  $\alpha$  and the constants  $C$  and  $K$ ).

For the first term, we choose  $\alpha = 0$ . We have:

$$\begin{aligned} \int_0^\gamma \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt &\leq \frac{C\sqrt{c}}{\sqrt{\gamma}} \int_0^\gamma \frac{1}{\sqrt{1-t}} \int_\Omega |\nabla_x v_\gamma(t, \cdot)|^{\frac{1}{2}} \cdot |\rho(t, \cdot)| dx dt \\ &\leq \frac{C\sqrt{c}}{\sqrt{1-t_m}\sqrt{\gamma}} \int_0^\gamma \left( \int_\Omega |\nabla_x v_\gamma(t, \cdot)| dx \right)^{\frac{1}{2}} \cdot \left( \int_\Omega |\rho|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \frac{C\sqrt{Kc}}{\sqrt{1-t_m}\sqrt{\gamma}} \int_0^\gamma \left( \int_\Omega |\rho|^2 dx \right)^{\frac{1}{2}} dt \leq \frac{M}{2\sqrt{\gamma}} \cdot \sqrt{\gamma} \|\rho\|_{L^2(Q)} = \frac{M}{2} \|\rho\|_{L^2(Q)}, \end{aligned} \quad (6-17)$$

with  $M = (2C\sqrt{Kc})/\sqrt{1-t_m}$ . For the second term, we choose  $\alpha = 1/2$ . We then have:

$$\begin{aligned} \int_\gamma^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt &\leq C\sqrt{c}\sqrt{\gamma} \int_\gamma^{t_m} \frac{1}{t\sqrt{1-t}} \int_\Omega |\nabla_x v_\gamma|^{\frac{1}{2}} \cdot |\rho| dx dt \\ &\leq \frac{C\sqrt{c}}{\sqrt{1-t_m}} \sqrt{\gamma} \int_\gamma^{t_m} \frac{1}{t} \left( \int_\Omega |\nabla_x v_\gamma(t, \cdot)| dx \right)^{\frac{1}{2}} \cdot \left( \int_\Omega |\rho|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \frac{C\sqrt{Kc}}{\sqrt{1-t_m}} \sqrt{\gamma} \left( \int_\gamma^{t_m} \frac{1}{t^2} dt \right)^{\frac{1}{2}} \cdot \|\rho\|_{L^2(Q)} \leq \frac{M}{2} \sqrt{\gamma} \sqrt{\frac{1}{\gamma} - \frac{1}{t_m}} \cdot \|\rho\|_{L^2(Q)} \leq \frac{M}{2} \|\rho\|_{L^2(Q)}, \end{aligned} \quad (6-18)$$



Thus, by summing (6-17) and (6-18), we obtain the inequality (6-16).

This upper born is of course not sufficient to prove (6-15). In fact we are "at the limit" of obtaining this convergence: if, in previous calculations, we had could, rather than perform the Cauchy-Schwarz inequalities, perform a Hölder inequality integrating slightly less the gradient of the field  $v$ , and therefore a little more the density  $\rho$ , we could have obtained this convergence. More precisely, assuming that  $\rho \in L^p(Q)$ , with  $p > 2$ , rather than  $\rho \in L^2(Q)$ , we then obtain the result of convergence (6-15). Indeed, with taking  $1 < q < 2$  such that  $1/p + 1/q = 1$ , note that:

$$|\nabla_x v(t, x)| = |\nabla_x v(t, x)|^{\frac{1}{p} + \frac{1}{q}} \leq \left( \frac{c}{t(1-t)} \right)^{\frac{1}{p}} |\nabla_x v(t, x)|^{\frac{1}{q}}. \quad (6-19)$$

For all  $0 < \alpha < 1$ , we then have:

$$\begin{aligned} \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt &\leq C \int_0^{t_m} \left( \frac{\gamma}{t} \right)^\alpha \left( \frac{c}{t(1-t)} \right)^{\frac{1}{p}} \int_{\Omega} |\nabla_x v_\gamma|^{\frac{1}{q}} \cdot |\rho| dx dt \\ &\leq C \left( \frac{c}{1-t_m} \right)^{\frac{1}{p}} \gamma^\alpha \int_0^1 \frac{1}{t^{\alpha+1/p}} \left( \int_{\Omega} |\nabla_x v_\gamma| dx \right)^{\frac{1}{q}} \cdot \left( \int_{\Omega} |\rho|^p dx \right)^{\frac{1}{p}} dt \\ &\leq C \left( \frac{c}{1-t_m} \right)^{\frac{1}{p}} K^{\frac{1}{q}} \gamma^\alpha \left( \int_0^1 \frac{1}{t^{q\alpha+q/p}} dt \right)^{\frac{1}{q}} \|\rho\|_{L^p(Q)}. \end{aligned} \quad (6-20)$$

Now, we have  $q < 2$ , and it follows that by fixing  $0 < \alpha < 1$  small enough, we can then have  $q(\alpha + 1) < 2 \Leftrightarrow q\alpha + q/p = q\alpha + q - 1 < 1$ : the term  $1/t^{q\alpha+q/p}$  is thus integrable on  $]0, 1[$ , and we then obtain the result of convergence (6-15).

Finally, we have to prove the convergence (6-15) in the case where we have only  $\rho \in L^2(Q)$ . Note that the intersection of the spaces  $L^p(Q)$ , for all  $p > 2$ , is dense into  $\rho \in L^2(Q)$ . Indeed, if  $\rho \in L^2(Q)$ , then for all  $0 < \lambda < 1$ , we have  $2/\lambda > 2$  and  $|\rho|^\lambda \in L^{2/\lambda}((0, 1) \times \Omega)$ . Moreover, it is easy to show, via the Dominated Convergence Theorem, that the family  $(|\rho|^\lambda)_{\lambda \in ]0, 1[}$  converges to  $|\rho|$  in  $L^2(Q)$  when  $\lambda$  tends to 1.

Let  $\epsilon > 0$ . For all  $\gamma > 0$  and all  $0 < \lambda < 1$ , we have the upper born:

$$\begin{aligned} \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt &\leq \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot \left| |\rho| - |\rho|^\lambda \right| dx dt \\ &\quad + \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho|^\lambda dx dt. \end{aligned} \quad (6-21)$$

By fixing  $\lambda \in ]0, 1[$ , such that  $\| |\rho| - |\rho|^\lambda \|_{L^2(Q)} \leq \epsilon/M$ , we thus have by (6-16):

$$\forall \gamma \in ]0, 1[, \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot \left| |\rho| - |\rho|^\lambda \right| dx dt \leq M \| |\rho| - |\rho|^\lambda \|_{L^2(Q)} \leq \epsilon. \quad (6-22)$$

We then can, by injecting the last inequality into (6-21), and by the convergence result obtained for  $|\rho|^\lambda \in L^p(Q)$ ,  $p = 2/\lambda > 2$ , set a rank  $\gamma_0 > 0$  such that for any  $0 < \gamma \leq \gamma_0$ ,

$$\int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho| dx dt \leq \epsilon + \int_0^{t_m} \int_{\mathbb{R}^d} |v - v_\gamma| \cdot |\nabla_x v_\gamma| \cdot |\rho|^\lambda dx dt \leq 2\epsilon. \quad (6-23)$$

□

With this last proof, we conclude the proof of the Proposition 6.1. To finish to prove the Theorem 3.2 which deals with the uniqueness of the component  $\mu = (\rho, m)$  shared by the saddle points of  $\mathbf{L}$ , e only have to show that a density  $\rho^*$  associated with one of these saddle points  $(\psi^*, q^*, \mu^*)$  verifies the conditions of application of the Proposition 6.1.

*Proof the Theorem 3.2:* Let  $(\psi^*, q^*, \mu^*)$  an element of  $\mathbf{L}^{ps}(\rho_0, \rho_1, \Omega)$  (i.e. a saddle point). According to the Proposition 3.1, we have  $G(h) + \langle \mu^*, \nabla_{t,x} h \rangle = 0$ , for all  $h \in H^1(Q)$ . Let  $\phi \in \Phi(\rho_0 \mathcal{L}^d, \rho_1 \mathcal{L}^d)$  (thus verifying the property  $(\Gamma_1)$ ). According to the Lemma 6.1, by defining  $v_\phi$  on  $]0, 1[ \times \mathbb{R}^d$  like in (4-12), we have  $m^* = \rho^* v_\phi$ . In other words, for all  $h \in H^1(Q)$ :

$$\int_Q (\partial_t h + v_\phi \cdot \nabla_x h) \rho^* dx dt + \int_\Omega h(0, \cdot) \rho_0 dx - \int_\Omega h(1, \cdot) \rho_1 dx = 0. \quad (6-24)$$

Let  $\bar{\rho}^* \in L^2((0, 1) \times \mathbb{R}^d)$  the extension by 0 of  $\rho^*$  on  $(0, 1) \times \mathbb{R}^d$ . Noting that for all  $h \in H_{loc}^1((0, 1) \times \mathbb{R}^d)$ , we have  $h_Q \in H^1(Q)$  and  $\nabla_{t,x} h_Q = (\nabla_{t,x} h)|_Q$ , the relation (6-24) is extensible from  $Q$  to the entire space  $(0, 1) \times \mathbb{R}^d$ . Thus, according to the Proposition 6.1, we then have the equivalence  $\bar{\rho}^*(t, \cdot) = \rho_\phi(t, \cdot) = (t \nabla \phi + (1-t) \text{id}) \# \rho_0$  for almost all  $t \in [0, 1]$ , with in addition  $t \mapsto \rho_\phi(t, \cdot) \in C^0([0, 1[, L^2(\mathbb{R}^d))$ .  $\square$

## 7 Characterization of an optimal transport velocity field

In this last section, we present a generalization of our study about the uniqueness of the component  $\mu$ : we want to use this study to try to characterize less formally an optimal transport velocity field. The result will be roughly the following: any density of  $L^2$ , with bounded support and continuously borne by a locally bounded velocity field, of which **the trajectories are all straight lines that never intersect**, that is to say, satisfying the properties (II) corresponds to an **optimal transport** (a McCann interpolation) and is **the only solution** for such a displacement.

For a convex open set  $\Omega$  of  $\mathbb{R}^d$  (not assumed bounded here), Let us define the space  ${}^bL_+^2((0, 1) \times \Omega)$  of densities  $\rho \in L^2((0, 1) \times \Omega)$  which are non-negatives and with compact supports into  $[0, 1] \times \Omega$ .

**Theorem 7.1.** *Let  $\Omega$  be a convex open set of  $\mathbb{R}^d$ . Let  $v^*$  a velocity field on  $\Omega$  satisfying the all properties (II), and let  $\rho_0 \in L^2(\Omega)$ , such that  $\rho_0 \geq 0$ , and such that  $\text{supp}(\rho_0)$  is bounded in  $\Omega$ . Let a density  $\rho^* \in {}^bL_+^2(Q)$  (space defined above) be a solution, in the sense of the distributions, of the system*

$$\begin{cases} \partial_t \rho + \text{div}_x(\rho v^*) = 0, \\ \rho(0, \cdot) = \rho_0, \end{cases} \quad (7-1)$$

*Then the density  $\rho^*$  is the unique solution of the system (7-1) in the space  ${}^bL_+^2((0, 1) \times \Omega)$ . Moreover,  $\rho^*$  is assimilated to an interpolation of McCann, and  $\rho^* \in C^0([0, 1[, L^2(\Omega)) \cap {}^bL_+^2(Q)$ : in other words, there exists a unique non-negative measure  $\nu_1$  defined on the Lebesgue tribe of  $\Omega$ , which support is bounded in  $\text{supp}(\rho_0) \cup [\bigcup_{t \in [0, 1]} \text{supp}(\rho^*(t, \cdot))]$ , satisfying the properties (C), and a convex function  $\phi$  on  $\mathbb{R}^d$  verifying the property  $(\Gamma_1)$ , such as  $\nu_1 = \nabla \phi \# (\rho_0 \mathcal{L}^d)$ , and*

$$\forall t \in [0, 1[, (\rho^*(t, \cdot) \mathcal{L}^d) = (\rho_\phi(t, \cdot) \mathcal{L}^d) = (t \nabla \phi + (1-t) \text{id}) \# (\rho_0 \mathcal{L}^d). \quad (7-2)$$

*The couple  $(\rho^*, v^*)$  the verifies the system:*

$$\begin{cases} \partial_t \rho^* + \text{div}_x(\rho^* v^*) = 0, \\ \rho^*(0, \cdot) = \rho_0, \quad \rho^*(1, \cdot) = \nu_1. \end{cases} \quad (7-3)$$

*that is, reformulated in the weak sense,*

$$\forall h \in C_c^\infty([0, 1] \times \Omega), \int_0^1 \int_\Omega (\partial_t h + v^* \cdot \nabla_x h) \rho^* dx dt + \int_\Omega h(0, \cdot) \rho_0 dx - \int_\Omega h(1, \cdot) d\nu_1 = 0. \quad (7-4)$$

Finally, the velocity field  $v^*$  satisfies the properties **(B)** (recalled just before the utterance of this Theorem) on  $\text{supp}(\rho^*)$  and, for all  $(t, x) \in \text{supp}(\rho^*)$ , we have  $v^*(t, x) = v_\phi(t, x)$  (with  $v_\phi$  always defined by (4-12)).

The proof resumes about the same elements as those developed in the sections 5 and 6, sometimes somewhat more technical, mainly due to the fact that the measure of arrival in this case (7-1) is no longer a density measure, of type  $\rho_1 \mathcal{L}^d$ , but simply a finite measure  $\nu_1$  satisfying the properties **(C)**. We will not give here the details of the evidence, just an idea of the process. For details, we refer again to (section 4.3).

The first step is simply to prove the existence, in the sense of the distributions, of the incoming measure  $\nu_1$ , as defined in the statement of the Theorem 7.1. For this purpose we use classical functional analysis tools [14] (in particular the Riesz [19] representation theorem). We also show that the weak formulation (7-4) is always valid for test functions taken from  $W_c^{1,\infty}([0, 1] \times \Omega)$ .

Then we consider  $\phi$ , an optimal transport potential between  $\rho_0 \mathcal{L}^p$  and  $\nu_1$  satisfying the property  $(\Gamma_1)$ , then we build "a saddle point", i.e. a triplet  $(\mu_\phi, q_\phi, \psi_\phi)$ , as we did in the section 5, and constructed in the Theorem 3.1 and the Lemme 5.2. In fact, the Brenier Theorem assumes density only for the starting density  $\rho_0 \mathcal{L}^d$ : the Proposition 3.2 is thus always valid, as well as the set of reasoning we have done with respect to the construction of the velocity field  $v_\phi$ . However, we can not obtain the membership of  $\rho_\phi$  to  $L^2((0, 1) \times \Omega)$  via the Lemma 5.1 (which requires  $\nu_1 = \rho_1 \mathcal{L}^d$  with  $\rho_1 \in L^2(\Omega)$ ). We can not extend the test functions of the weak formulation of the mass conservation for the  $(\rho_\phi, v_\phi)$  pair to the space  $H_{loc}^1((0, 1) \times \Omega)$ , as follows the Proposition 5.3 (note that the latter only considers density measurement for the initial one). Extending these test functions to the space  $W_{loc}^{1,\infty}((0, 1) \times \Omega)$ , which is necessary because the potential  $\psi_\phi$  is necessarily in this last space.

We then construct, in the same way, a second saddle point from the pair  $(\rho^*, v^*)$ , that is to say a triplet  $(\mu^*, q^*, \psi^*)$ , with  $\mu^* = (\rho^*, \rho^* v^*)$ ,  $q^* = (-(1/2)|v^*|^2, v^*)$  and  $\nabla_{t,x} \psi^* = q^*$  (Lemma 5.2). We can then, as well as for the Lemma 6.1, prove the uniqueness of the velocity field on the supports of  $\rho_\phi$  and  $\rho^*$ : in other words, in the case that interests us,  $\rho^* v^* = \rho^* v_\phi$ . Although our triplets  $(\mu_\phi, q_\phi, \psi_\phi)$  and  $(\mu^*, q^*, \psi^*)$  are no longer necessarily in  $L^2$ , and that we can no longer really speak of "projections" and "orthogonality" in the schematic proof that we have done for the Lemma 6.1, the reasoning remains globally the same and we reach the same conclusion (Lemma 4.3-14 de [14]). Thus, according to the Proposition 6.1, the density  $\rho^*$  verifies the relation (7-2), with  $t \mapsto \rho^*(t, \cdot) \in C^0([0, 1[, L^2(\Omega))$ .

Now, let us show that  $\rho^*$  is the only solution with bounded support of the system (7-1) in the space  ${}^bL_+^2((0, 1) \times \Omega)$ : assume there exist two solutions  $\rho^1, \rho^2$  of the system (7-1) in  ${}^bL_+^2((0, 1) \times \Omega)$ , and show that then  $\rho^1 = \rho^2$  in  $L^2((0, 1) \times \Omega)$ . Let us introduce  $\bar{\rho} = (\rho^1 + \rho^2)/2$ . Such a  $\bar{\rho}$  is also a solution of the (7-1) in  ${}^bL_+^2((0, 1) \times \Omega)$ . Therefore, from the above, there would exist a convex function  $\bar{\phi}$  of  $\mathbb{R}^d$  satisfying the property  $(\Gamma_1)$ , such that, by defining  $v_{\bar{\phi}}$  as in (4-12), we have  $\bar{\rho} v^* = \bar{\rho} v_{\bar{\phi}}$ , i.e.  $(\rho^1 + \rho^2) v^* = (\rho^1 + \rho^2) v_{\bar{\phi}}$ . The field  $v^*$  is then almost everywhere equal to the field  $v_{\bar{\phi}}$  on  $\text{supp}(\rho^1) \cup \text{supp}(\rho^2)$ , thus  $\rho^1 v^* = \rho^1 v_{\bar{\phi}}$  and  $\rho^2 v^* = \rho^2 v_{\bar{\phi}}$ . Therefore,  $\rho^1$  and  $\rho^2$  both satisfy the system (7-1), with replacing  $v^*$  by  $v_{\bar{\phi}}$ . According to the Proposition 6.1, we thus have  $\rho^1 = \rho^2 : t \mapsto \rho_{\bar{\phi}} = (t \nabla \bar{\phi} + (1-t) \text{id}) \# \rho_0$  in  $L^2((0, 1) \times \Omega)$ .

## 8 Convergence of the algorithm

The aim of this section is to demonstrate the weak convergence of the Benamou-Brenier algorithm, as well as the strong convergence of a relaxed version of the algorithm towards a saddle point of the Lagrangian  $\mathbf{L}$ . For this purpose, we will reformulate the problem of convergence of the algorithm towards one of these saddle points, into a more generic problem of convergence to

a fixed point of non-expansive operator. We start by identifying the saddle points of  $\mathbf{L}$  at the fixed points of an "iteration of the algorithm" operator.

**Proposition 8.1.** *There is an equivalence between a saddle point of  $\mathbf{L}$  (and thus of  $\mathbf{L}_r$  for all  $r > 0$ ) and a fixed point of the algorithm (defined in the subsection 2.3). In other words,  $(\psi, q, \mu)$  is a saddle point of  $\mathbf{L}$  if and only if it remains invariant for the Benamou-Brenier algorithm (here we take  $r > 0$ ).*

*Proof:* Let  $(\psi, q, \mu)$  be a saddle point of  $\mathbf{L}$ . We denote  $(\psi', q', \mu')$  the new triplet obtained after one iteration of the algorithm. Let us show that  $(\psi', q', \mu') = (\psi, q, \mu)$  in  $Sg$ . By taking the step A, and by the property (P2) of (I), by taking  $h = \psi' - \psi$ , we obtain  $\|\nabla_{t,x}(\psi' - \psi)\|^2 = 0$ . According to the inequality of Poincaré, we get  $\psi' = \psi$  in  $H^1((0, 1) \times \Omega)/\mathbb{R}$ . In step B, we look for the unique  $q'$  verifying  $\langle \mu + \nabla\psi' - q', p - q' \rangle \leq 0$ , for all  $p \in \tilde{\mathcal{P}}$ . As  $\psi = \psi'$ , by the properties (P1) and (P3) which characterize a saddle point,  $q$  is a good candidate and therefore the only one, hence  $q' = q$ . Finally,  $\nabla_{t,x}\psi' = \nabla_{t,x}\psi = q = q'$  and thus by the step C we finally have  $\mu' = \mu$ .

Finally, let  $(\psi, q, \mu)$  be a fixed point of the algorithm. Let us show that it is a saddle point of  $\mathbf{L}$ . Step C gives immediately  $\nabla_{t,x}\psi = q$ , and consequently step B gives  $\langle \mu, p - q \rangle \leq 0$  for all  $p \in \tilde{\mathcal{P}}$ , and step A gives  $G(h) + \langle \mu, \nabla_{t,x}h \rangle = 0$  for all  $h \in H^1(Q)$ . Since the three properties (I) are verified,  $(\psi, q, \mu)$  is therefore a saddle point of  $\mathbf{L}$ .  $\square$

It is now possible, according to the Proposition 8.1, to formally redefine our algorithm convergence probleme to a saddle point of Lagrangian  $\mathbf{L}$ , to a problem of convergence of a sequence of type  $x_{n+1} = \mathbf{M}x_n$  to a fixed point of the operator  $\mathbf{M}$  defined as the product of an iteration of each step of the algorithm (and taking for example  $x_n = (\psi_n, q_n, \mu_n)$ ). We will see in the subsection 8.2 that by placing in an appropriate Hilbert space, it is possible to characterize such an operator  $\mathbf{M}$  as a non-expansive type operator (1-Lipschitz).

As a preamble to this reformulation, the following subsection will be devoted to the formulation of some general results useful for our study in relation to non-expansive operators.

## 8.1 Some convergence results for non-expansive operators

For any non-expansive operator  $\mathbf{M}$ , we will state in this section a series of results allowing to obtain both **weak** convergence to a fixed point of  $\mathbf{M}$  of the iterative algorithm  $x_{n+1} = \mathbf{M}x_n$ , and the **strong** convergence of a relaxed version of this algorithm.

We begin by recalling some useful standard definitions for the sequel. Let  $(H, \langle \cdot, \cdot \rangle)$  an Hilbert space and let  $\mathbf{M} : H \rightarrow H$ .

**Definition 8.1** (Non-expansive Operator). *The operator  $\mathbf{M}$  is called **non-expansive** if and only if it is 1-Lipschitz, and **firmly non-expansive** (implies non-expansive) if and only if we have  $\|\mathbf{M}x - \mathbf{M}y\|^2 \leq \langle x - y, \mathbf{M}x - \mathbf{M}y \rangle$ , for all  $x, y \in H$ .*

*The operator  $\mathbf{M}$  is also called **quasi-firmly non-expansive** on a subset  $A$  of  $H$ , containing the set of fixed points of  $\mathbf{M}$ , if and only if, for any fixed point  $x^*$  of  $\mathbf{M}$ , we have  $\|x - x^*\|^2 - \|\mathbf{M}x - x^*\|^2 \geq \|x - \mathbf{M}x\|^2$  for all  $x \in A$ .*

Let us now consider two convergence results that can be applied, for the first one to the problem of the weak convergence of the algorithm, and for the second one, to the problem of the strong convergence of a relaxed version of the algorithm. The Theorem 8.1 is easily proved by classical tools of functional analysis (the proof in particular invoke the Opial Lemma []). The second one is a result due to H. Bauschke in [1]. The detailed proofs of those two theorems are in [14] (section 2.3 and appendix B).

**Theorem 8.1.** *Let  $\mathbf{M}$  be a non-expansive operator on  $H$ , and quasi-firmly non-expansive on  $\mathbf{M}(H)$  (the image of  $H$  by  $\mathbf{M}$ ). Assume that the set  $\text{Fix}(\mathbf{M})$  of the fixed points of  $\mathbf{M}$  is non-empty. Let  $(x_n)_n$  be a sequence of elements of  $H$  satisfying for every  $n \in \mathbb{N}$  the estimate:  $\|\mathbf{M}(x_n) - x_{n+1}\| \leq \epsilon_n$ , where  $(\epsilon_n)_n$  is a non-negative real sequence satisfying  $\sum_n \epsilon_n < +\infty$ . Then  $(x_n)_n$  weakly converges in  $H$  to a fixed point of  $\mathbf{M}$ .*

**Theorem 8.2** (H. Bauschke [1]). *Let  $\mathbf{M}$  be a non-expansive operator and assume that the set  $\text{Fix}(\mathbf{M})$  of the fixed points of  $\mathbf{M}$  is non-empty, and let  $(\lambda_n)_{n \geq 0}$  be a sequence of parameters of  $[0, 1[$ , converging to 0, and satisfying:  $\sum_n \lambda_n = +\infty$  and  $\sum_n |\lambda_{n+1} - \lambda_n| < +\infty$ . Let  $H$  be a Hilbert space, and  $\mathbf{M}$  a non-expansive operator over  $H$ . Given  $a$  and  $x_0$  in  $H$ , we define the sequence  $(x_n)$  by the recurrence  $x_{n+1} = \lambda_n a + (1 - \lambda_n)\mathbf{M}_n x_n$  ( $\forall n \geq 0$ ), where for all  $n \in \mathbb{N}$  is verified the estimate  $\|\mathbf{M}_n x_n - \mathbf{M}x_n\| \leq \epsilon_n$ , with  $\sum_n \epsilon_n < +\infty$ .*

*Then the sequence  $(x_n)_n$  converges strongly to  $P_{F_a}$  (where  $F = \text{Fix}(\mathbf{M})$  is the set of fixed points of  $\mathbf{M}$ : it is a closed convex set).*

The sequence  $(\epsilon_n)_n$  here represents the inevitable numerical errors inherent in the implementation of such an algorithm. It is assumed here that these errors are highly controlled, which is not very realistic in practice.

About the Theorem 8.2, it is easy to prove that the set of fixed points of a non-expansive operator is a closed convex set (see Lemma 2.3-7 of [14]).

## 8.2 Formulation of the Benamou-Brenier algorithm in terms of non-expansive operator

We will now be able to put into practice the convergence results set out in the previous section in the context of the Benamou-Brenier algorithm. It is sufficient to show that the iterations of the algorithm can be considered as the iteration of a certain non-expansive operator. Let the space  $H = L^2(Q)^{d+1} \times L^2(Q)^{d+1}$ , provided with the scalar product  $\langle (\mu_1, q_1), (\mu_2, q_2) \rangle_H = \langle \mu_1, \mu_2 \rangle_{L^2} + r^2 \langle q_1, q_2 \rangle_{L^2}$ . The space  $(H, \langle \cdot, \cdot \rangle)$  is then a Hilbert space.

Let  $\mathbf{B} : H \rightarrow H$  be the operator which associate to  $(\mu, q)$  the product  $(\mu', q')$  of the last two steps (B and C) of the algorithm Benamou-Brenier. Here  $\psi$  is ephemeral: indeed, if  $(\mu^*, q^*, \psi^*)$  is a saddle point of the Lagrangian  $\mathbf{L}$  (and thus  $\mathbf{L}_r$ ) defined in (2-9), then  $(\mu^*, q^*) = \mathbf{B}(\mu^*, q^*)$ . Conversely if  $(\mu^*, q^*)$  is a fixed point of  $\mathbf{B}$  then  $(\mu^*, q^*, \psi^*)$  is a saddle point of the Lagrangian (where  $\psi^*$  is the unique element of  $Sg$  which satisfies  $q^* = \nabla \psi^*$ ). The potential  $\psi$  therefore constitutes only a calculation step.

**Proposition 8.2.** *The operator  $\mathbf{B}$  is non-expansive on  $H$ , and quasi-firmly non-expansive on  $\mathbf{B}(H)$ .*

*Proof:*  $(\mu_1, q_1)$  and  $(\mu_2, q_2)$  being given, we determine  $(\mu'_1, q'_1) = \mathbf{B}(\mu_1, q_1)$  and  $(\mu'_2, q'_2) = \mathbf{B}(\mu_2, q_2)$  by the following iteration (recall: see subsection 2.3). For  $i = 1, 2$ , we look for:

- Step A : The unique  $\psi'_i \in (H^1/\mathbb{R})(Q)$  such that  $G(h) + \langle \mu_i, \nabla h \rangle_{L^2} + r \langle \nabla \psi'_i - q_i, \nabla h \rangle_{L^2} = 0$ , for all  $h \in (H^1/\mathbb{R})(Q)$ .
- Step B : The unique  $q'_i$  such that  $\langle \mu_i + r(\nabla \psi'_i - q'_i), p - q'_i \rangle_{L^2} \leq 0$ , for all  $p \in \tilde{\mathcal{P}}$ .
- Step C : We define  $\mu'_i$  by  $\mu'_i = \mu_i + r(\nabla_{t,x} \psi'_i - q'_i)$ .

Let us start by studying the non-expansivity of  $\mathbf{B}$ . Note that by injecting the equation of step C into step A and step B (for  $i = 1$  or  $i = 2$ ), we obtain the two new equations:

$$\forall h \in H^1(Q)/\mathbb{R}, \quad G(h) + \langle \mu'_i, \nabla h \rangle_{L^2} + r \langle q'_i - q_i, \nabla h \rangle_{L^2} = 0, \quad \text{et } p \in \tilde{\mathcal{P}}, \quad \langle \mu'_i, p - q'_i \rangle_{L^2} \leq 0. \quad (8-1)$$

Let us call them (8-1)<sup>1</sup> and (8-1)<sup>2</sup>. We set  $\bar{\mu}^{(l)} = \mu_2^{(l)} - \mu_1^{(l)}$ ,  $\bar{q}^{(l)} = q_2^{(l)} - q_1^{(l)}$  and  $\bar{\psi}^{(l)} = \psi_2^{(l)} - \psi_1^{(l)}$ .

Respectively, by taking  $(8-1)_{i=2}^1 - (8-1)_{i=1}^1$  and  $h = \bar{\psi}'$ , and by summing (Etape B) $_{i=2}$  with  $p = q'_1$ , and (Etape B) $_{i=1}$  with  $p = q'_2$ , we obtain respectively the two following relations:

$$\langle \bar{\mu}', \nabla \bar{\psi}' \rangle + r \langle \bar{q}' - \bar{q}, \nabla \bar{\psi}' \rangle = 0 \quad \text{and} \quad \langle \bar{\mu}', \bar{q}' \rangle \geq 0. \quad (8-2)$$

noted  $(8-2)^1$  and  $(8-2)^2$ . By summing these two relations, we then have  $\langle \bar{\mu}', \nabla \bar{\psi}' - \bar{q}' \rangle + r \langle \bar{q}' - \bar{q}, \nabla \bar{\psi}' \rangle \leq 0$ . By factoring the term  $|\bar{\mu}|^2 - |\bar{\mu}'|^2$ , we then obtain

$$\begin{aligned} |\bar{\mu}|^2 - |\bar{\mu}'|^2 &= \langle \bar{\mu} - \bar{\mu}', \bar{\mu} + \bar{\mu}' \rangle = -r \langle \nabla \bar{\psi}' - \bar{q}', 2(\bar{\mu} + r(\nabla \bar{\psi}' - \bar{q}')) - r(\nabla \bar{\psi}' - \bar{q}') \rangle \\ &= -2r \langle \bar{\mu}', \nabla \bar{\psi}' - \bar{q}' \rangle + r^2 |\nabla \bar{\psi}' - \bar{q}'|^2 \geq 2r^2 \langle \nabla \bar{\psi}', \bar{q}' - \bar{q} \rangle + r^2 |\nabla \bar{\psi}' - \bar{q}'|^2. \end{aligned} \quad (8-3)$$

$$\begin{aligned} \text{De plus, on a } \langle \nabla \bar{\psi}', \bar{q}' - \bar{q} \rangle &= \langle \nabla \bar{\psi}' - \bar{q}, \bar{q}' - \bar{q} \rangle + \langle \bar{q}, \bar{q}' - \bar{q} \rangle \\ &= (1/2) (|\bar{q}'|^2 - |\bar{q}|^2 + |\bar{q}' - \bar{q}|^2) + \langle \nabla \bar{\psi}' - \bar{q}, \bar{q}' - \bar{q} \rangle - |\bar{q}' - \bar{q}|^2 \\ &= (1/2) (|\bar{q}'|^2 - |\bar{q}|^2 + |\bar{q}' - \bar{q}|^2) + \langle \nabla \bar{\psi}' - \bar{q}', \bar{q}' - \bar{q} \rangle. \end{aligned} \quad (8-4)$$

By re-injecting  $(8-4)$  in  $(8-3)$ , on obtain:

$$\begin{aligned} |\bar{\mu}|^2 - |\bar{\mu}'|^2 &\geq r^2 \left( (|\bar{q}'|^2 - |\bar{q}|^2) + (|\bar{q}' - \bar{q}|^2 + 2\langle \nabla \bar{\psi}' - \bar{q}', \bar{q}' - \bar{q} \rangle + |\nabla \bar{\psi}' - \bar{q}'|^2) \right) \\ &\geq r^2 (|\bar{q}'|^2 - |\bar{q}|^2) + r^2 |\nabla \bar{\psi}' - \bar{q}'|^2, \end{aligned} \quad (8-5)$$

$$\text{i.e. } (|\bar{\mu}|^2 + r^2 |\bar{q}|^2) - (|\bar{\mu}'|^2 + r^2 |\bar{q}'|^2) \geq r^2 |\nabla \bar{\psi}' - \bar{q}'|^2 \geq 0. \quad (8-6)$$

Now, we have  $\|(\mu_1, q_1) - (\mu_2, q_2)\|_H = |\bar{\mu}|^2 + r^2 |\bar{q}|^2$  and  $\|\mathbf{B}(\mu_1, q_1) - \mathbf{B}(\mu_2, q_2)\|_H = |\bar{\mu}'|^2 + r^2 |\bar{q}'|^2$ : the operator  $\mathbf{B}$  is therefore non-expansive.

We now demonstrate the quasi-firmly non-expansiveness of  $\mathbf{B}$  on  $\mathbf{B}(H)$ . Let  $(\mu^*, q^*)$  a fixed point of  $\mathbf{B}$  (hence include in  $\mathbf{B}(H)$ ),  $x = (\mu, q) \in \mathbf{B}(H)$  let us define  $(\mu', q') = \mathbf{B}(\mu, q) = \mathbf{B}x$ . By using equation  $(8-6)$  with  $(\bar{\psi}, \bar{q}, \bar{\mu}) = (\psi - \psi^*, q - q^*, \mu - \mu^*)$  and  $(\bar{\psi}', \bar{q}', \bar{\mu}') = (\psi' - \psi^*, q' - q^*, \mu' - \mu^*)$ , we obtain:

$$(|\mu - \mu^*|^2 + r^2 |q - q^*|^2) - (|\mu' - \mu^*|^2 + r^2 |q' - q^*|^2) \geq r^2 |\nabla(\psi' - \psi^*) - (q - q^*)|^2. \quad (8-7)$$

$$\begin{aligned} \text{For the second member, } r^2 |\nabla(\psi' - \psi^*) - (q - q^*)|^2 &= r^2 |\nabla \psi' - q|^2 = |(\mu' - \mu) + r(q' - q)|^2 \\ &= |\mu' - \mu|^2 + 2r \langle \mu' - \mu, q' - q \rangle + r^2 |q' - q|^2, \end{aligned}$$

with  $\langle \mu' - \mu, q' - q \rangle \geq 0$ , according to the relation  $(8-2)^2$  (recalling that  $(\mu, q) \in \mathbf{B}(H)$ ). Thus, proceeding in the same way as in [11], we obtain:

$$(|\mu - \mu^*|^2 + r^2 |q - q^*|^2) - (|\mu' - \mu^*|^2 + r^2 |q' - q^*|^2) \geq |\mu' - \mu|^2 + r^2 |q' - q|^2,$$

that is to say,  $\|(\mu, q) - (\mu^*, q^*)\|_H^2 - \|(\mu', q') - (\mu^*, q^*)\|_H^2 \geq \|\mathbf{B}(\mu, q) - (\mu, q)\|_H^2$ .  $\square$

By applying the Theorem 8.1 to the operator  $\mathbf{B}$ , we are then able to prove the weak- $L^2$  convergence of the Benamou-Brenier algorithm to a fixed point of  $\mathbf{B}$  (i.e. to a saddle point of  $\mathbf{L}$ ) whose existence is justified by the Theorem 3.1. Similarly, by applying the Theorem 8.2, we can easily define a relaxed version of the algorithm with strong- $L^2$  convergence. This non-expansive operator approach has recently been used to show the convergence, weak or strong, of other algorithms, especially some splitting-proximal algorithms [5].

Proposition 8.2 also justifies the **convexity** (and closure) of the set of saddle points of  $\mathbf{L}$  (property invoked in the schematic proof of the Lemma 6.1). Indeed, as already mentioned above, the set of fixed points of a non-expansive operator is a closed convex set. The operator  $\mathbf{B}$  immediately gives us this convexity for the components  $\mu$  and  $q$ . The characteristic (P3) of the properties (I) as well as the linearity of the gradient operator  $\nabla_{t,x}$  transpose this convexity to the component  $\psi$ , and therefore to the set of saddle points of  $\mathbf{L}$ .



## 9 Conclusion and perspectives

The starting point of our work was to study the consistency of the Lagrangian augmented algorithm by Benamou-Brenier. We have shown in the section 8 the weak convergence of the Benamou-Brenier algorithm to a saddle point of the Lagrangian  $\mathbf{L}$  (see page 4), which models the dynamic formulation of the optimal transport problem, as proposed in [3]. This proof of convergence is based on a reformulation of the algorithm as an iterative sequence of a non-expansive operator  $\mathbf{B}$  (1-Lipschitz), whose fixed points are equivalent to the saddle points of  $\mathbf{L}$ . This formulation enabled us to exploit the literature associated with the theory of non-expansive operators, by which we were able to propose a relaxed version of the Benamou-Brenier algorithm, with strong convergence, to a saddle point of  $\mathbf{L}$ . However, numerical experiments did not reveal any real improvement in the relaxed versions, with respect to the speed of convergence and the accuracy of the calculations, compared to the original algorithm.

This convergence of the algorithm and its relaxed versions remain nevertheless conditioned to the existence of a saddle point for the Lagrangian  $\mathbf{L}$  (and therefore of a fixed point of the operator  $\mathbf{B}$ ). Therefore, in the sections 4 and 5, we have initially undertaken to prove the existence of such a saddle point, while also showing in section 6, the uniqueness of the evolution of the density and the momentum resulting from such transport. We have tried to prove all this under the most general possible conditions, especially in cases where the starting and arrival densities  $\rho_0$  and  $\rho_1$  cancel at certain areas in the transport domain (the existence of a saddle point in the case of strictly positive densities which have already been treated in [13]).

Such conditions imply in particular to be able to take into account the case where the number of connected components of the supports of the densities  $\rho_0$  and  $\rho_1$  are not the same. Such cases generally reveal non-regular optimal transport plans, which is why a large part of our work will have involved an in-depth study of the regularity and behavior of a velocity field associated with such transport plans. This study of the velocity field associated with an optimal transport plan will have been the main object of the section 4. This study will also have been the opportunity, specially in section 7, to try to characterize in general the properties of a velocity field associated with an optimal transport in the space  $L^2$  (see the Theorem 7.1 p. 37).

Our prospects for completing this work are quite broad. As for the Benamou-Brenier algorithm, and more generally the algorithms dealing with dynamic optimal transport, we would like to be able to highlight tools and methods of precise analysis of the convergence properties of these algorithms: stopping or distance criterias of the calculated steps with respect to the theoretical solution, theoretical information on the speed of convergence, etc...

In regards to the studies conducted in sections 4, 5 and 6, concerning the existence and uniqueness of solutions to the dynamic optimal transport problem in  $L^2$ , we would like to be able to extend them to cases of dynamic optimal transport operating in less classical environments, especially non-isotropic environments (see our article [15]), or even more generally within Riemannian varieties.

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