



HAL
open science

Combinatorial theory of permutation-invariant random matrices I: partitions, geometry and renormalization

Franck Gabriel

► **To cite this version:**

Franck Gabriel. Combinatorial theory of permutation-invariant random matrices I: partitions, geometry and renormalization. 2015. hal-01128574v2

HAL Id: hal-01128574

<https://hal.science/hal-01128574v2>

Preprint submitted on 1 Jul 2015 (v2), last revised 28 Oct 2016 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COMBINATORIAL THEORY OF PERMUTATION-INVARIANT RANDOM MATRICES I: PARTITIONS, GEOMETRY AND RENORMALIZATION.

by

Franck Gabriel

E-mail: franck.gabriel@normalesup.org

Abstract. — In this article, we define and study a new geometry and a new order on the set of partitions of an even number of objects. One of the definitions involves the partition algebra, a structure of algebra on the set of such partitions depending on an integer parameter N . Then we emulate the theory of random matrices in a combinatorial framework: for any parameter N , we introduce a family of linear forms on the partition algebras which allows us to define a notion of weak convergence similar to the convergence in moments in random matrices theory.

A renormalization of the partition algebras allows us to consider the weak convergence as a simple convergence in a fixed space. This leads us to the definition of a deformed partition algebra for any integer parameter N and to the definition of two transforms: the cumulants transform and the exclusive moments transform. Using an improved triangle inequality for the distance defined on partitions, we prove that the deformed partition algebras, endowed with a deformation of the linear forms converge as N go to infinity. This result allows us to prove combinatorial properties about geodesics and a convergence theorem for semi-groups of functions on partitions.

At the end we study a sub-algebra of functions on infinite partitions with finite support: a new addition operation and a notion of \mathcal{R} -transform are defined. We introduce the set of multiplicative functions which becomes a Lie group for the new addition and multiplication operations. For each of them, the Lie algebra is studied.

The appropriate tools are developed in order to understand the algebraic fluctuations of the moments and cumulants for converging sequences. This allows us to extend all the results we got for the zero order of fluctuations to any order.

Contents

1. Introduction.....	2
2. Partition algebra.....	5
3. Geometry on the set of partitions.....	13
4. Convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[P_k(N)]$	25
5. The deformed partition algebra.....	37
6. Refined geometry of the partition algebra.....	39
7. Consequences of the convergence of the deformed algebras.....	43
8. Geometric and combinatorial consequences of Theorem 4.1.....	46
9. Algebraic fluctuations.....	52

10. An introduction to the general \mathcal{R} -transform.....	61
11. Conclusion.....	74
References.....	74

1. Introduction

This article is the first of a self-contained set of three articles [8], [9] and [10] on a combinatorial method in random matrices theory based on a geometry on partitions and a new point of view on usual/free cumulants based on dualities between groups and sub-algebras of partitions. This general method allows us to work with random matrices which are invariant by conjugation by the symmetric group instead of the unitary or orthogonal group, besides, no more assumption about the factorization of moments is needed. The first article is about the combinatorial framework based on the partition algebra. In the second article we will apply this framework to random matrices, and the third one will put the emphasis on the random walks on the symmetric group and the link with the \mathfrak{S}_∞ -Yang-Mills measure.

This set of articles has to be considered as the continuation of what could be called the Gauge Theory School in random matrices. The article of F. Xu [18] is one of the pioneer work about this point of view on random matrices. Later, this point of view was developed by A. Sengupta [17], then highly improved by T. Lévy [12], [13], then it was used by two students of T. Lévy: A. Dahlqvist in [7] and G. Cébron [5], [4].

We wrote these articles as a lesson for graduate students with the intention that no special requirement is needed to understand them. The reader will find a new presentation and introduction to the random matrices theory. To achieve this, we only used the Gauge Theory School's papers, the seminal article for partition algebras [11], and the book [16] which, in some sense, we tried partially to generalize. Another point of view on random matrices which are invariant by conjugation by the symmetric group was given first by C. Male in his paper on traffics [14]. Yet, the goal was to develop the ideas of the Gauge Theory School and thus we did not use this article. In the forthcoming article [6], the author and his coauthor build connections between the notions developed here and the notions developed in [14]. In some sense, these articles can be seen also a bridge to go from the book [16] to the traffic interpretation of [14], traffics which have shown their importance in the study of random graphs [15]. At the moment the author was finishing these articles, he was informed of M. Capitaine and M. Casalis's work, [3], on their Schur-Weyl's interpretation of non-commutative free cumulants for unitary and orthogonal invariant random matrices.

The point of view developed in the three articles [8], [9] and [10] allows us to recover in a very simple way some famous theorems. The reader will also find in these articles a simple tool box in order to prove convergences of random matrices (for example random walks on the symmetric group). He will also find the tools in order to understand the algebraic fluctuations of moments of random matrices. Besides, this point of view allows us to define a general notion of freeness for matrices which are invariant by conjugation by the symmetric group and we construct the first non-commutative multiplicative Lévy processes for this notion of freeness. We will formulate two equivalent definitions of

this freeness: one based on cumulants, and the other on moments. This freeness notion is linked with a new \mathcal{R} -transform which generalizes the old known \mathcal{R} -transform. A Kreweeras complement is defined for partitions: this generalizes the notion already set for permutations. Amongst others, we will state a matricial Wick's theorem, which allows us to recover the Wick law for Gaussian Hermitian or symmetric matrices. We will also recover theorems about convergence of Hermitian Lévy processes proved in [2], [1] and unitary Lévy processes proved in [4]: we extend them to the symmetric and the orthogonal case. A new central limit theorem will be stated, which generalizes the non-commutative and the commutative central limit theorem. In the article [9], convergences of random walks on the symmetric group will be proved, and will be used in order to show that the Wilson loops of the \mathfrak{S}_N -Yang-Mills measure converge in probability when N goes to infinity. This will imply a result about some convergence of ramified coverings on the disk. We will also see how to inject the usual theory of probabilities in this framework. This last assertion shows that one could, in this framework, study the probabilistic fluctuations.

1.1. Renormalization and a physical point of view. — In this article, we emulate the theory of random matrices in a combinatorial framework. Given a partition p of a number of points, and an integer N , we consider (p, N) as a physical system involving N particles. When the number of points is even, by polarizing the points in two sets, we can consider (p, N) as a discrete time transformation operation. A partition p can be seen as an elementary evolution of a system of size N : we can define the composition of two partitions. Later in the paper, we consider these discrete-time transformations also as the Hamiltonian of continuous time transformations.

An evolution of a system of size N is a linear combination of elementary evolutions of size N . Thus, every transformation is uniquely characterized by a size N and by a finite number of coefficients which, as we will see in the article, are bare quantities. Two questions arise: how to describe a system of infinite size and how to renormalize the bare quantities. As one does for perturbative renormalization, the important idea is to consider observables: we define some observables, one for each partition. In Theorem 4.1, we show how the bare coefficient must be renormalized in order to have finite observables at the limit $N = \infty$.

Then, we show that, by using the same renormalization, the composition of two evolutions converge also: this is proved in Theorems 6.1 and 7.1. In Theorem 7.2, we consider continuous-time evolution transformations: we show that if the Hamiltonian is renormalized as we did for discrete time transformations, then the evolution converges. In Theorem 10.2, we characterize the Hamiltonian so that the factorization property of large N holds.

We study also the development in power of $1/N$ of systems of size N which converge to a continuous system.

The main novelty is to show that, even if one knows how to renormalize the bare constants, it does not seem interesting to define a vector space of infinite systems since all systems considered are defined in the same vector space whose basis is the set of partitions of $2k$ elements. In order to have an interesting space of infinite systems, one has to consider a renormalization of the algebras in which are defined the N -dimensional

systems: the limit defines a non-trivial algebra in which one can study continuous evolutions of continuous systems.

Let us remark that a consequence of our results is that, in our toy-model, given a continuous system, one has canonically a sequence of approximations by systems involving N particles.

1.2. Layout of the article. — Using the set \mathcal{P}_k of partitions of $2k$ elements as basis, one can define an algebra known as the partition algebra which definition depends on a parameter $N \in \mathbb{N}$: the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$. For a comprehensive study of this algebra, we recommend the article [11]. The main definitions are set in Section 2.

In Section 3, we define a geometry on the set of partitions of $2k$ elements which generalizes a well-known geometry on the symmetric group \mathfrak{S}_k . This geometry is defined by constructing a kind of Caley graph for \mathcal{P}_k . This geometry allows to define a new order on \mathcal{P}_k for which we construct the Hasse diagram and we compute the Möbius function. Using this new geometry, in Section 4 we define two notions of convergence of sequences which are shown to be equivalent. We define the notion of coordinate numbers, normalized moments, exclusive coordinate numbers and exclusive normalized moments. One of the results that we prove is that exclusive coordinate numbers and exclusive normalized moments are equal. In Section 5, a new deformed partition algebra is defined: $\mathbb{C}[\mathcal{P}_k(N, N)]$. These algebras are shown to converge to a new algebra: this is obtained by an improvement of the triangle inequality proved in Section 6 for the distance defined on the set of partitions of $2k$ elements. Let us remark that we define in the same section a Kreweras complement for partitions which generalizes the notion for permutations. We use these results in Section 7 in order to show that the multiplication is continuous for the notion of convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$. We also study the convergence of semi-groups in $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$. In Section 8, using the convergence of sequences defined in Section 4, we show how one can prove combinatorial results, for example, a new proof of the improved triangle inequality is given.

In Section 9, we develop the notion of algebraic fluctuations, and extend the results already proved for the zero order of fluctuations to any order.

In Section 10, we construct an algebra $\mathfrak{E}[\mathcal{P}]$ whose elements are functions on $\cup_{k \in \mathbb{N}} \mathcal{P}_k$. This algebra can be endowed with two special laws: \boxplus and \boxtimes . We study two subgroups of $\mathfrak{E}[\mathcal{P}]$ associated with the operations \boxplus and \boxtimes , the group of multiplicative invertible elements. These groups are Lie groups, the Lie algebras of these groups are studied. We also define the \mathcal{R}_A -transform, which generalizes the usual \mathcal{R} -transform and we define two other transformations linked with the notion of exclusive moments. To finish the article, we extend these definitions to the setting of higher order fluctuations.

1.3. Acknowledgements. — This work has been realized during my PhD at the LPMA which offered me the necessary liberty to complete this article. Many thanks to the researchers and administrative staff of the LPMA. I am grateful to my PhD advisor Thierry Lévy for supporting this research, for his helpful comments and corrections which led to improvements in this manuscript and for the useful discussions about mathematics and other subjects. I would like to express my special gratitude to Terence Tao, particularly for his blog which, during a period of doubts, made me enjoy maths again. This project began when I wanted to understand the link between the work of T.

Lévy and the formulation given at the Pims summer school by David Brydges of Wick's theorem, I am really thankful to him for this. Also many thanks are due to Antoine Dahlqvist who explained me the duality between permutations and partitions. I would like to thank Tom Halverson and Arun Ram for answering my questions about partitions. I am also grateful to Guillaume Cébron since I learned a lot thanks to his papers. Thanks to Patrick Gabriel for his interest in my work and the discussions related to this work that we had together, Marie-Françoise Gabriel and Catherine Lam for trying to correct the English in this manuscript. At last, a thought to all the people which are supporting researchers and whose names never appear in the acknowledgements.

2. Partition algebra

2.1. First definitions. — Let k and N be two positive integers. We will consider three different algebras $\mathbb{C}[\mathfrak{S}_k]$, $\mathbb{C}[\mathcal{B}_k(N)]$, $\mathbb{C}[\mathcal{P}_k(N)]$: respectively the symmetric algebra, the Brauer algebra, and the partition algebra. These algebras satisfy the inclusions:

$$\mathbb{C}[\mathfrak{S}_k] \subset \mathbb{C}[\mathcal{B}_k(N)] \subset \mathbb{C}[\mathcal{P}_k(N)].$$

Thus, we will first construct $\mathbb{C}[\mathcal{P}_k(N)]$ and we will see the two others algebras as sub-algebras of $\mathbb{C}[\mathcal{P}_k(N)]$. The reference article for the partition algebra is the article [11] of T.Halverson and A.Ram.

Let us consider $2k$ elements which we denote by: $1, \dots, k$ and $1', \dots, k'$. We define \mathcal{P}_k as the set of set partitions of $\{1, \dots, k\} \cup \{1', \dots, k'\}$. If $k = 0$, we consider $\mathcal{P}_k = \{\emptyset\}$. Let p be an element of \mathcal{P}_k . We will denote by p^1, \dots, p^r the blocks in p . The number of connected components $\text{nc}(p)$, the propagating number $\text{pn}(p)$ and the length $l(p)$ of p are defined respectively by:

$$\text{nc}(p) = r,$$

$$\text{pn}(p) = \# \{i, p^i \text{ contains both an element of } \{1, \dots, k\} \text{ and one of } \{1', \dots, k'\}\},$$

$$l(p) = k.$$

Any partition $p \in \mathcal{P}_k$ can be represented by a graph. For this we consider two rows: k vertices are in the top row, labeled by 1 to k from left to right and k vertices are in the bottom row, labeled from $1'$ to k' from left to right. Any edge between two vertices means that the labels of the two vertices are in the same block of the partition p . Examples are given in Figure 1 and 2.

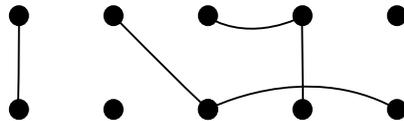


FIGURE 1. Partition $p_1 = \{\{1', 1\}\{2'\}\{2, 3', 5'\}\{3, 4, 4'\}\{5\}\}$.

The notion of tensor product of partitions will be also very useful.

Definition 2.1. — Let k and l be two positive integers. Let p be an element of \mathcal{P}_k and let p' be an element of \mathcal{P}_l . Let us consider two diagrams: one associated with p , another

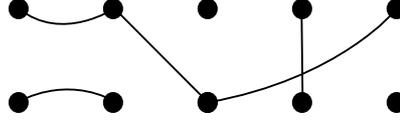


FIGURE 2. Partition $p_2 = \{\{1', 2'\}\{1, 2, 3', 5\}\{3\}\{4', 4\}\{5'\}\}$.

with p' . Let $p \otimes p'$ be the partition in \mathcal{P}_{k+l} associated with the diagram where one has put the diagram associated with p on the left of the diagram associated with p' .

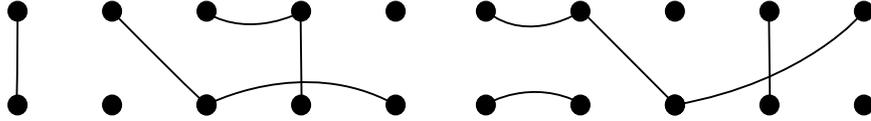


FIGURE 3. Partition $p_1 \otimes p_2$.

Let p_1 and p_2 be two elements of \mathcal{P}_k . We say that p_1 is *coarser* than p_2 if any two elements which are in the same block of p_2 are also in the same block of p_1 . This order is directed: for any partitions p_1 and p_2 in \mathcal{P}_k there exists a third partition p_3 which is coarser than p_1 and p_2 . For example, one can consider the partition $p_1 \vee p_2$ defined as follows.

Definition 2.2. — We define $p_1 \vee p_2$ as the partition in \mathcal{P}_k such that for any $i, j \in \{1, \dots, k\} \cup \{1', \dots, k'\}$, i and j are in the same block of $p_1 \vee p_2$ if and only if there exists $i = x_0, x_1, \dots, x_l = j$ such that for any $n \in \{0, \dots, l-1\}$, $x_n \in \{1, \dots, k\} \cup \{1', \dots, k'\}$ and the two elements x_n and x_{n+1} are in the same block of either p_1 or p_2 .

It is always interesting to have a graphical representation for the operations defined on partitions. One can recover a diagram representing $p_1 \vee p_2$ by putting a diagram representing p_2 over one representing p_1 .

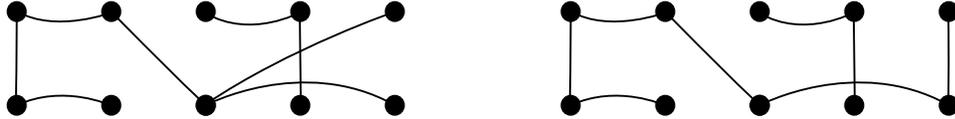


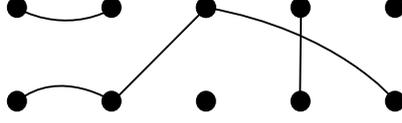
FIGURE 4. Two diagrams which represent $p_1 \vee p_2 = \{\{1, 1', 2, 2', 3', 5, 5'\}, \{3, 4, 4'\}\}$.

We will need also later of the infimum of p and p'

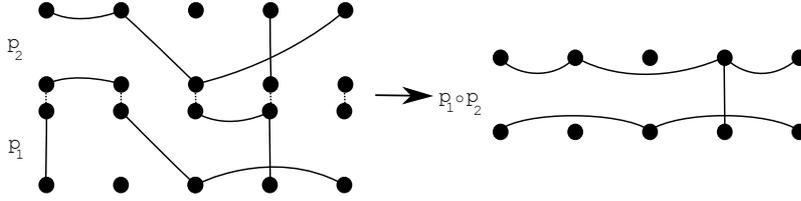
Definition 2.3. — Let p_1 and p_2 be two elements of \mathcal{P}_k . We define $p_1 \wedge p_2$ as the partition in \mathcal{P}_k such that for any $i, j \in \{1, \dots, k\} \cup \{1', \dots, k'\}$, i and j are in the same block of $p_1 \wedge p_2$ if and only if they are in the same block of p_1 and in the same block of p_2 .

Let us play a little with the graphical representation of p_1 and p_2 in order to define other natural operations on the set of partitions.

We will use later the transposition of a partition: it is the partition obtained by permuting the role of $\{1, \dots, k\}$ and $\{1', \dots, k'\}$. For example if $k = 3$, let $p = \{\{1, 1', 3'\}, \{2, 3\}, \{2'\}\}$, then ${}^t p = \{\{1', 1, 3\}, \{2', 3'\}, \{2\}\}$. For every diagram associated with p , the diagram obtained by flipping it according to a horizontal axis is a diagram associated with ${}^t p$. One can find an example in Figure 5

FIGURE 5. Partition ${}^t p_2$

An other thing we can do is to put one diagram representing p_2 above one diagram representing p_1 . Let us identify the lower vertices of p_2 with the upper vertices of p_1 . We obtain a graph with vertices on three levels, then erase the vertices in the middle level, keeping the edges obtained by concatenation of edges passing through the deleted vertices. Any connected component entirely included in the middle row is then removed. Let us denote by $\kappa(p_1, p_2)$ the number of such connected components. We obtain an other diagram associated with a partition denoted by $p_1 \circ p_2$. For any elements p_1 and p_2 of \mathcal{P}_k , the partition $p_1 \circ p_2$ does not depend on the choice of diagrams representing the partitions p_1 and p_2 .

FIGURE 6. Partition $p_1 \circ p_2$.

The set of Brauer elements and the set of permutations will be stable by this operation of concatenation.

Definition 2.4. — *The set of Brauer elements \mathcal{B}_k is the set of pair partitions in \mathcal{P}_k . The set of permutation \mathfrak{S}_k is the set of pair partitions in \mathcal{P}_k whose propagating number is equal to k .*

For any p_1 and p_2 in \mathcal{B}_k (resp. \mathfrak{S}_k), $p_1 \circ p_2 \in \mathcal{B}_k$ (resp. \mathfrak{S}_k). Let us define the three algebras $\mathbb{C}[\mathfrak{S}_k]$, $\mathbb{C}[\mathcal{B}_k(N)]$ and $\mathbb{C}[\mathcal{P}_k(N)]$.

Definition 2.5. — *The partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ is the associative algebra over \mathbb{C} with basis \mathcal{P}_k endowed with the multiplication defined by:*

$$\forall p_1, p_2 \in \mathcal{P}_k, p_1 p_2 = N^{\kappa(p_1, p_2)}(p_1 \circ p_2).$$

The Brauer algebra $\mathbb{C}[\mathcal{B}_k(N)]$ (resp. symmetric algebra $\mathbb{C}[\mathfrak{S}_k]$) is the sub-algebra of $\mathbb{C}[\mathcal{P}_k(N)]$ generated by the elements of \mathcal{B}_k (resp. the elements of \mathfrak{S}_k).

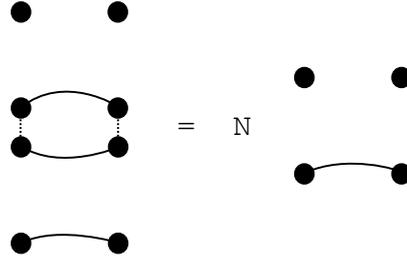


FIGURE 7. Example of a product which involves the counting of loops.

Notation 2.1. — In all the paper, A_k will stand either for \mathcal{P}_k or \mathcal{B}_k or \mathfrak{S}_k . Thus for any $N \in \mathbb{N}$, $\mathbb{C}[A_k(N)]$ will stand for $\mathbb{C}[\mathcal{P}_k(N)]$, $\mathbb{C}[\mathcal{B}_k(N)]$ or $\mathbb{C}[\mathfrak{S}_k(N)]$.

Let us remark that actually, the algebra $\mathbb{C}[\mathfrak{S}_k(N)]$ does not depend on N . We can see any permutation $\sigma \in \mathfrak{S}_k$ as a bijection from $\{1, \dots, k\}$ to itself: for any $i \in \{1, \dots, k\}$ there exists a unique $j \in \{1', \dots, k'\}$ such that $\{i, j'\} \in \sigma$, we set $\sigma(i) = j$. For any permutations σ_1 and σ_2 , the bijection associated with $\sigma_1 \sigma_2$ is the composition of the two bijections associated with σ_1 and σ_2 .

We can extend the operations of transposition, tensor product and multiplication on the partition algebra, by linearity or bi-linearity.

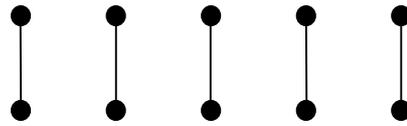
The sub-algebra $\mathbb{C}[\mathfrak{S}_k]$ is not only stable for the \circ operation. It also satisfies the following property which can be proved by looking at the propagating number.

Lemma 2.1. — Let $p, p' \in \mathcal{P}_k$, if $p \circ p' \in \mathfrak{S}_k$ then p and p' are in \mathfrak{S}_k .

Besides, for any partition $\sigma \in \mathfrak{S}_k$ and any $p \in A_k$, $\kappa(\sigma, p) = \kappa(p, \sigma) = 0$. Let us remark that, for any integer N , the algebras $\mathbb{C}[A_k(N)]$ have the same neutral element, denoted by id_k or id , for the product operation:

$$id_k = \{\{i, i'\}, i \in \{1, \dots, k\}\},$$

whose diagram for $k = 5$ is drawn in Figure 8. A consequence of Lemma 2.1 is that, as $id_k \in \mathfrak{S}_k$, the only invertible elements of $A_k(N)$, for the multiplication operation, are the permutations. The inverse of a permutation σ is $\sigma^{-1} = {}^t\sigma$.

FIGURE 8. The neutral element id_5 .

We will later need some special permutations.

Definition 2.6. — Let $I \subset \{1, \dots, k\}$: $I = \{i_1, \dots, i_l\}$ with $i_1 < \dots < i_l$. We define σ_I the permutation which sends i_j on j for any $j \in \{1, \dots, l\}$ and $i \notin I$ on $l+i-\#\{n, i_n < i\}$. This is the partition:

$$\sigma_I = \{\{i_j, j'\}, j \in \{1, \dots, l\}\} \cup \{\{i, (l+i-\#\{n, i_n < i\})'\}, i \notin I\}.$$

Definition 2.7. — The transposition $(1, 2)$ is the partition $\sigma_{\{2\}}$ in \mathcal{P}_2 defined by:

$$(1, 2) = \{\{1, 2'\}, \{2, 1'\}\}.$$

The Weyl contraction is the Brauer element in \mathcal{P}_2 defined by:

$$[1, 2] = \{\{1, 2\}, \{1', 2'\}\}.$$

These partitions are drawn in Figure 9.



FIGURE 9. The transposition $(1, 2)$ and the Weyl contraction $[1, 2]$.

Definition 2.8. — Let i, j be two distinct integers in $\{1, \dots, k\}$. The transposition (i, j) in \mathfrak{S}_k is:

$$(i, j) = \sigma_{\{i, j\}}^{-1}((1, 2) \otimes Id_{k-2})\sigma_{\{i, j\}} = \{\{i', j\}, \{i, j'\}\} \cup \{\{l, l'\}, l \notin \{i, j\}\}.$$

The set of transpositions on k elements is:

$$\mathcal{T}_k = \{(i, j), i, j \in \{1, \dots, k\}, i \neq j\}.$$

The Weyl contraction $[i, j]$ in \mathcal{B}_k is:

$$[i, j] = \sigma_{\{i, j\}}^{-1}([1, 2] \otimes Id_{k-2})\sigma_{\{i, j\}} = \{\{i, j\}, \{i', j'\}\} \cup \{\{l, l'\}, l \notin \{i, j\}\}.$$

Due to the remark we made after Lemma 2.1, the product does not depend on which $\mathbb{C}[\mathcal{B}_k(N)]$ one considers to define the product. We denote by \mathcal{W}_k the set of Weyl contractions in \mathcal{B}_k :

$$\mathcal{W}_k = \{[i, j], i, j \in \{1, \dots, k\}, i \neq j\}.$$

A notion linked with the tensor operation, which will be central in the asymptotic freeness results in the article [9], is the notion of irreducibility of partitions. Let p be in \mathcal{P}_k .

Definition 2.9. — A cycle of p is a block of $p \vee id$. The set of cycles of p is denoted by $\mathcal{C}(p)$. The number of cycles of p is denoted by $c(p)$. The partition p is composed if $c(p) > 1$. The partition p is irreducible if it is not composed. By convention, the empty partition is irreducible.

Let us consider the set of irreducible partitions.

Definition 2.10. — We will denote by $A_k^{(i)}$ the set of irreducible partitions of A_k .

It has to be noted that for any integer k :

$$\mathfrak{S}_k^{(i)} = \{\sigma^{-1}(1, \dots, k)\sigma, \sigma \in \mathfrak{S}_k\},$$

where $(1, \dots, k) \in \mathcal{S}_k$ is the k -cycle equal to $\sigma_{\{2, 3, \dots, k\}}$.

The partition p is composed if and only if there exist p_1 and p_2 two partitions non equal to the empty partition, and I a subset of $\{1, \dots, k\}$ such that $\#I = l(p_1)$, $l(p_2) = k - \#I$ and:

$$\sigma_I^{-1}(p_1 \otimes p_2)\sigma_I = p.$$

Let us define the decomposition of p into two partitions.

Definition 2.11. — *The set of decompositions of p into two partitions is:*

$$\mathfrak{F}_2(p) = \{(p_1, p_2, I), \sigma_I^{-1}(p_1 \otimes p_2)\sigma_I = p\}.$$

Let us remark that for any partition, even the irreducible partitions, $\mathfrak{F}_2(p) \neq \emptyset$. For example, if p is irreducible:

$$\mathfrak{F}_2(p) = \{(p, \emptyset, \{1, \dots, k\}), (\emptyset, p, \emptyset)\}.$$

Let also remark that $\mathfrak{F}_2(\emptyset) = \{(\emptyset, \emptyset, \emptyset)\}$.

We will need a notion of weak irreducibility later: this is based on the notions of extraction and restriction. For any partition p we have a lot of choice in order to represent p as a graph: the complete graph which represents p is the graph such that i and j , two elements of $\{1, \dots, k\} \cup \{1', \dots, k'\}$ are linked if and only if i and j are in the same block of p .

Definition 2.12. — *Let J be a subset of $\{1, \dots, k\} \cup \{1', \dots, k'\}$. Let us denote by J^s the symmetrization of J :*

$$J^s = J \cup \{j \in \{1', \dots, k'\}, \exists i \in J \cap \{1, \dots, k\}, j = i'\} \cup \{i \in \{1, \dots, k\}, i' \in J\}.$$

We define:

- *The extraction of p to J , denoted p_J . Let us take the complete graph which represents p , let us erase all the vertices which are not in J^s and all the edges which are not between two vertices in J^s and at last let us label the remaining vertices from left to right. This is the graph of p_J .*
- *The restriction of p to J , denoted $p|_J$. Let us take the complete graph which represents p , let us erase all the edges which are not between two vertices in J and let us connect each $i \notin J^s$ with i' . This is the graph of $p|_J$.*

By convention, if $J^s = \{1, \dots, k\} \cup \{1', \dots, k'\}$, then $p_J = \emptyset$ and $p|_J = id$.

Definition 2.13. — *The support of p is:*

$$S(p) = \{1, \dots, k\} \setminus \{i \in \{1, \dots, k\}, \{i, i'\} \in p\}.$$

The partition p is weakly irreducible if $p_{S(p)}$ is irreducible. In particular the permutation id_k is weakly irreducible.

2.2. Partitions and representation. — In this section, we define a natural action of the partition algebra (and by restriction of the Brauer and of the symmetric algebra) on $(\mathbb{C}^N)^{\otimes k}$. This action will be useful in order to translate combinatorial properties into linear algebraic properties.

Let N and k be two positive integers.

Definition 2.14. — For any $p \in \mathcal{P}_k$ and any k -uples (i_1, \dots, i_k) and $(i_{1'}, \dots, i_{k'})$ of elements of $\{1, \dots, N\}$, we set:

$$p_{i_1, \dots, i_k}^{i_{1'}, \dots, i_{k'}} = \begin{cases} 1, & \text{if for any two elements } r \text{ and } s \in \{1, \dots, k\} \cup \{1', \dots, k'\} \text{ which} \\ & \text{are in the same block of } p, \text{ one has } i_r = i_s, \\ 0, & \text{otherwise.} \end{cases}$$

We can now define the action of the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ on $(\mathbb{C}^N)^{\otimes k}$. Let (e_1, \dots, e_N) be the canonical basis of \mathbb{C}^N .

Definition 2.15. — The action of the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ on $(\mathbb{C}^N)^{\otimes k}$ is defined by the fact that for any $p \in \mathcal{P}_k$, for any $(i_1, \dots, i_k) \in \{1, \dots, N\}^k$:

$$p.(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{(i_{1'}, \dots, i_{k'}) \in \{1, \dots, N\}^k} p_{i_{1'}, \dots, i_{k'}}^{i_1, \dots, i_k} e_{i_{1'}} \otimes \dots \otimes e_{i_{k'}}.$$

This action defines a representation of the partition algebra $\mathbb{C}[\mathcal{P}_k(N)]$ on $(\mathbb{C}^N)^{\otimes k}$ which we denote by $\rho_N^{\mathcal{P}_k}$:

$$\rho_N^{\mathcal{P}_k} : \mathbb{C}[\mathcal{P}_k(N)] \mapsto \text{End} \left((\mathbb{C}^N)^{\otimes k} \right).$$

Let us define E_i^j be the matrix which sends e_j on e_i . Let p be a partition in \mathcal{P}_k . We can write the matrix of $\rho_N^{\mathcal{P}_k}(p)$ in the basis $(e_{i_1} \otimes \dots \otimes e_{i_k})_{(i_l)_{l=1}^k \in \{1, \dots, N\}^k}$:

$$(1) \quad \rho_N^{\mathcal{P}_k}(p) = \sum_{(i_1, \dots, i_k, i_{1'}, \dots, i_{k'}) / p_{i_{1'}, \dots, i_{k'}}^{i_1, \dots, i_k} = 1} E_{i_{1'}}^{i_1} \otimes \dots \otimes E_{i_{k'}}^{i_k}.$$

For example, if p is the transposition $(1, 2)$, then:

$$\rho_N^{\mathcal{P}_2}((1, 2)) = \sum_{a, b=1}^N E_a^b \otimes E_b^a.$$

We think that this presentation allows us to understand, in an easier way, the representation $\rho_N^{\mathcal{P}_k}$. We illustrate in Figure 10, how to find the partition which representation is given by a sum of the form (1). The partition p_1 used in Figure 10 is the partition drawn in Figure 1.

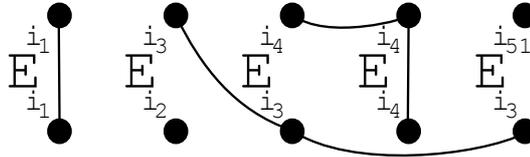


FIGURE 10. $\sum_{i_1, i_2, i_3, i_4, i_5} E_{i_1}^{i_1} \otimes E_{i_2}^{i_3} \otimes E_{i_3}^{i_4} \otimes E_{i_4}^{i_4} \otimes E_{i_5}^{i_5} = \rho_N^{\mathcal{P}_5}(p_1)$.

Let us suppose that $N \geq 2k$. Using Theorem 3.6 in [11], the application $\rho_N^{\mathcal{P}_k}$ is injective. Actually, if one considers only its restriction to the symmetric algebra or the

Brauer algebra, it is enough to ask for $N \geq k$. For $N = k - 1$ this result does not hold, this is a consequence of the Mandelstam's identity which asserts that:

$$\sum_{\sigma \in \mathfrak{S}_k} (-1)^{\epsilon(\sigma)} \rho_{k-1}^{\mathcal{P}_k}(\sigma) = 0,$$

where $\epsilon(\sigma)$ is the signature of σ .

Let us remark that the natural action of $\mathbb{C}[\mathcal{P}_k(N)]$ on $(\mathbb{C}^N)^{\otimes k}$ behaves well under the operation of product tensor.

Lemma 2.2. — *Let k and k' be two positive integers. Let $p \in \mathbb{C}[\mathcal{P}_k]$ and $p' \in \mathbb{C}[\mathcal{P}_{k'}]$. We have for any integer N :*

$$\rho_N^{\mathcal{P}_{k+k'}}(p \otimes p') = \rho_N^{\mathcal{P}_k}(p) \otimes \rho_N^{\mathcal{P}_{k'}}(p').$$

2.3. The exclusive basis of $\mathbb{C}[\mathcal{P}_k]$. — The basis used to define the partition algebra is quite natural, yet, it is not always very easy to work with. Indeed, if we look at the representation $\rho_N^{\mathcal{P}_k}$ of a partition, we see that the condition we used to define the delta function is not exclusive. It means that we did not use the following exclusive delta function:

$$(p_{i_1, \dots, i_k}^{i_1, \dots, i_k})^{ex} = \begin{cases} 1, & \text{if for any two elements } r \text{ and } s \in \{1, \dots, k\} \cup \{1', \dots, k'\}, \\ & i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } p, \\ 0, & \text{otherwise.} \end{cases}$$

By changing in Definition 2.15 the delta function defined in Definition 2.14 by this new exclusive delta function, we define a new function:

$$\tilde{\rho}_N^{\mathcal{P}_k} : \mathbb{C}[\mathcal{P}_k(N)] \rightarrow \text{End} \left((\mathbb{C}^N)^{\otimes k} \right).$$

Does it exist, for any partition $p \in \mathcal{P}_k$ an element $p^c \in \mathbb{C}[\mathcal{P}_k]$ such that for any integer N , $\rho_N^{\mathcal{P}_k}(p^c) = \tilde{\rho}_N^{\mathcal{P}_k}(p)$? The answer is given by the following definition, as explained by Equation (2.3) of [11].

Definition 2.16. — *We define the family $(p^c)_{p \in \mathcal{P}_k}$ as the only family of elements in $\mathbb{C}[\mathcal{P}_k]$ defined by the relation:*

$$p = \sum_{p' \text{ coarser than } p} p'^c.$$

The notion of being coarser defines a partial order on \mathcal{P}_k : the relation can be inverted. The family $(p^c)_{p \in \mathcal{P}_k}$ is well defined and it is a basis of the partition algebra $\mathbb{C}[\mathcal{P}_k]$. We will call $(p^c)_{p \in \mathcal{P}_k}$ the exclusive partition basis, it satisfies the following proposition.

Proposition 2.1. — *For any positive integers k and N , for any partition $p \in \mathcal{P}_k$,*

$$\rho_N^{\mathcal{P}_k}(p^c) = \tilde{\rho}_N^{\mathcal{P}_k}(p).$$

3. Geometry on the set of partitions

In this section, we define a new geometry on the set of partitions \mathcal{P}_k which generalizes some well-known geometry on the symmetric group. We will see three ways to construct a distance on \mathcal{P}_k : one will allow us to work with linear algebra, another to compute the distance in a combinatorial way, and the last one will use a graph which we will consider as the generalized Cayley graph of \mathcal{P}_k . We could have worked with partitions of $\{1, \dots, K\}$ in order to define the geometry, yet, as in the following work we only consider \mathcal{P}_k , we decided to state the definitions and results in this setting. The fact that we consider \mathcal{P}_k comes from the fact that we are interested in random matrices. Yet, most of the results and definitions would extend easily to the combinatorial setting behind the theory of random tensors which are invariant in law by the symmetric group.

Depending on the context, we will consider a partition either as an element of \mathcal{P}_k or as an element of $\text{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$ via the action defined in Definition 2.15. We remind the reader that (e_1, \dots, e_N) is the canonical base of \mathbb{C}^N . The family $\{e_{i_1} \otimes \dots \otimes e_{i_k}, (i_1, \dots, i_k) \in \{1, \dots, N\}^k\}$ is a basis of $\left(\mathbb{C}^N\right)^{\otimes k}$: let Tr^k be the trace with respect to this canonical basis. We do not renormalize it, thus $Tr^k\left(\text{Id}_{\left(\mathbb{C}^N\right)^{\otimes k}}\right) = N^k$. We can define the trace of a partition.

Definition 3.1. — *Let k and N be two positive integers, let p be a partition in \mathcal{P}_k . We define:*

$$Tr_N(p) = Tr^k\left(\rho_N^{\mathcal{P}_k}(p)\right).$$

For any integer N , we extend Tr_N by linearity to $\mathbb{C}[\mathcal{P}_k(N)]$.

Let us remark that, if one does not want to use the representation $\rho_N^{\mathcal{P}_k}$, one could have also define the trace by defining for any partition $p \in \mathcal{P}_k$,

$$(2) \quad Tr_N(p) = N^{\text{nc}(p \vee \text{id})}.$$

3.1. Definition of the geodesic order. — We can now define a distance on \mathcal{P}_k .

Proposition 3.1. — *Let N be a positive integer, let p and p' be two elements of \mathcal{P}_k . The number:*

$$d(p, p') = -\log_N\left(\frac{Tr_N({}^t p p')}{\sqrt{Tr_N({}^t p p) Tr_N({}^t p' p')}}\right)$$

does not depend on N : it is called the distance between p and p' .

The fact that $d(p, p')$ does not depend on N is a consequence of Lemma 3.1. Actually we have not prove yet that it is a distance, even if it is fairly easy to see that it satisfies the strict positivity property: it is a consequence of the Cauchy-Schwarz's inequality.

The easiest way to prove that $d(p, p')$ does not depend on N is to show that it is a combinatorial object.

Lemma 3.1. — For any p and p' in \mathcal{P}_k :

$$d(p, p') = \frac{1}{2}(\text{nc}(p) + \text{nc}(p')) - \text{nc}(p \vee p').$$

Proof. — This is a consequence of the following equality which holds for any p and p' in \mathcal{P}_k and any positive integer N :

$$(3) \quad \text{Tr}_N({}^t p p') = N^{\text{nc}({}^t p \circ p' \vee \text{id}) + \kappa({}^t p, p')} = N^{\text{nc}(p \vee p')},$$

which is a consequence of Equality (2) and the combinatorial equality:

$$\text{nc}({}^t p \circ p' \vee \text{id}) + \kappa({}^t p, p') = \text{nc}(p \vee p').$$

This latter equality can be understood by flipping the diagram of ${}^t p$ over the one of p' : the flip transposes ${}^t p$ thus we get the two diagrams of p and p' one over the other. By definition, the diagram constructed by putting a diagram representing p' over one representing p is associated with $p \vee p'$. \square

It remains to show that d satisfies the triangle inequality on the set of partitions \mathcal{P}_k . For that we will show that it is a geodesic distance on a graph.

Definition 3.2. — We define the weighted graph $\mathbb{G}_k = (\mathbb{V}_k, \mathbb{E}_k, w_k)$ such that:

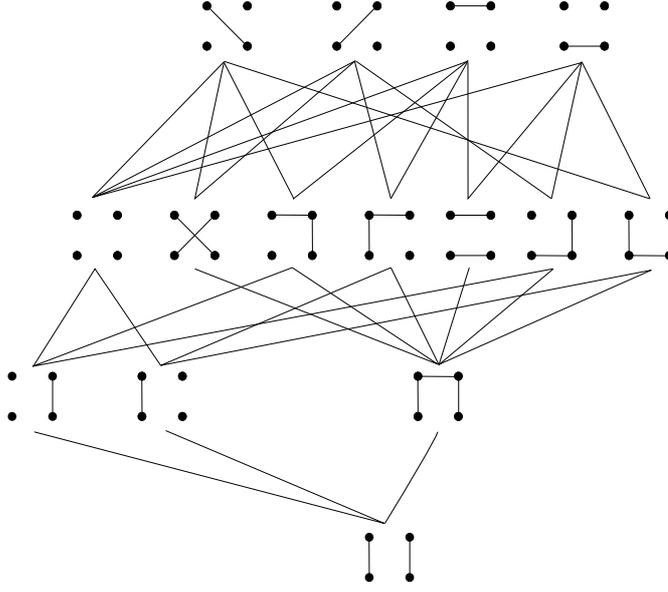
- the set of vertices \mathbb{V}_k is \mathcal{P}_k ,
- there exists an edge e in \mathbb{E}_k between p and p' two elements of \mathcal{P}_k if and only if:
 - one can go from one to the other by gluing two blocks. Let us suppose that we can go from p to p' . If p is the partition $\{p^1, \dots, p^r\}$ then there exist i and j , distinct, such that $p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{p^i \cup p^j\}$. The weight of the edge e is set to 0.5: $w_k(e) = 0.5$.
 - one can go from one to the other by permuting two elements of $\{1, \dots, k\} \cup \{1', \dots, k'\}$ which are in distinct blocks. Let us suppose that we can go from p to p' by permuting two elements. In this case, if p is the partition $\{p^1, \dots, p^r\}$, there exist $s, t \in \{1, \dots, k, 1', \dots, k'\}$ distinct and $i, j \in \{1, \dots, r\}$ distinct, such that $s \in p^i$, $t \in p^j$ and $p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{(p^i \setminus \{s\}) \cup \{t\}, (p^j \setminus \{t\}) \cup \{s\}\}$. The weight of the edge e is set to 1: $w_k(e) = 1$.

Remark 3.1. — The graph \mathbb{G}_k plays the role of the Cayley graph of \mathcal{P}_k . Actually, if one considers the subgraph \mathbb{S}_k obtained by restraining it to the vertices which are permutations, one really obtains the Cayley graph of the symmetric group \mathfrak{S}_k . The Cayley graph \mathbb{B}_k of \mathcal{B}_k is defined as the restriction of \mathbb{G}_k to the vertices which are in \mathcal{B}_k .

We gave this definition so that the reader can understand easily why this graph is a generalization of the usual Cayley graph. Yet, there is an other graph which will be used in Proposition 3.2. Let us define $\mathbb{G}'_k = (\mathbb{V}'_k, \mathbb{E}'_k, w'_k)$ such that:

- the set of vertices \mathbb{V}'_k is \mathcal{P}_k ,
- there exists an edge in \mathbb{E}'_k between p and p' two elements of \mathcal{P}_k if and only if one can go from one to the other by gluing two blocks,
- the weight function w'_k is constant equal to 1/2.

From now on, when we will consider the Caley graph for \mathcal{P}_k , we will consider this graph \mathbb{G}'_k . The graphs \mathbb{G}_k and \mathbb{G}'_k are interesting as they allow us to better understand the distance d .

FIGURE 11. The graph \mathbb{G}'_2 .

Proposition 3.2. — Let p and p' be two elements of \mathcal{P}_k . Let us define $C_{\mathbb{G}_k}(p, p')$ (resp. $C_{\mathbb{G}'_k}(p, p')$) the set of paths π in \mathbb{G}_k (resp. \mathbb{G}'_k) which begin in p and finish in p' . Let us define the geodesic distance on \mathbb{G}_k and on \mathbb{G}'_k between p and p' by:

$$d_{\mathbb{G}_k}(p, p') = \min_{\pi \in C_{\mathbb{G}_k}(p, p'), \pi = e_1 \dots e_l} w(e_1) + \dots + w(e_l),$$

$$d_{\mathbb{G}'_k}(p, p') = \min_{\pi \in C_{\mathbb{G}'_k}(p, p'), \pi = e_1 \dots e_l} w(e_1) + \dots + w(e_l).$$

We have the equalities:

$$d(p, p') = d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p').$$

Proof. — Let p and p' be two elements of \mathcal{P}_k . It is enough to prove that $d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p')$ and $d(p, p') = d_{\mathbb{G}'_k}(p, p')$.

First, let us show that $d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p')$. This assertion comes from the fact that one can permute two elements of $\{1, \dots, k\} \cup \{1', \dots, k'\}$ in the partition p by gluing two blocks of p and then splitting one block of the resulting partition. Indeed, let us suppose that $p = \{p^1, \dots, p^r\}$. Let $s, t \in \{1, \dots, k, 1', \dots, k'\}$, distinct, and let $i, j \in \{1, \dots, r\}$, distinct, such that $s \in p^i$ and $t \in p^j$. Then:

$$p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{(p^i \setminus \{s\}) \cup \{t\}, (p^j \setminus \{t\}) \cup \{s\}\}$$

can be obtained by:

1. gluing p_i and p_j ,
2. splitting $p_i \cup p_j$ in two: $(p^i \setminus \{s\}) \cup \{t\}$ and $(p^j \setminus \{t\}) \cup \{s\}$.

The weight of this path is equal to $0.5 + 0.5 = 1$. Thus, to compute the distance $d_{\mathbb{G}_k}(p, p')$, it is enough to look only at paths in \mathbb{G}'_k : $d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p')$.

Then, let us show that $d(p, p') = d_{\mathbb{G}'_k}(p, p')$. For this, let us see what happens to the distance $d(p, p')$ between p and p' when one moves from p' to one neighborhood of p' in \mathbb{G}'_k . Suppose that we glue two blocks of p' , then $\text{nc}(p)$ is constant, $\text{nc}(p')$ decreases by 1 and $\text{nc}(p \vee p')$ stays constant or decreases by 1. In this case $d(p, p')$ will increase or decrease by 0.5. Suppose now that we cut one block of p' , then $\text{nc}(p)$ is constant, $\text{nc}(p')$ increases by 1 and $\text{nc}(p \vee p')$ stays constant or increases by 1. In this case $d(p, p')$ will also increase or decrease by 0.5.

Thus a gluing/cutting can at most increase the value of $d(p, p')$ by 0.5. It implies that $d(p, p') \leq d_{\mathbb{G}'_k}(p, p')$. We have to show now that $d_{\mathbb{G}'_k}(p, p') \leq d(p, p')$. Let us remark that $p \vee p'$ is coarser than p : we can go from p to $p \vee p'$ by doing $\text{nc}(p) - \text{nc}(p \vee p')$ gluing of blocks. The same holds for p' : we can go from p' to $p \vee p'$ by doing $\text{nc}(p') - \text{nc}(p \vee p')$ gluing of blocks. Thus one can go from p to $p \vee p'$ and then from $p \vee p'$ to p' in $\text{nc}(p) + \text{nc}(p') - 2\text{nc}(p \vee p')$ steps in \mathbb{G}'_k . Thus $d_{\mathbb{G}'_k}(p, p') \leq \frac{1}{2} [\text{nc}(p') + \text{nc}(p) - 2\text{nc}(p \vee p')] = d(p, p')$. \square

The function $d_{\mathbb{G}'_k}$ is a geodesic distance on a graph: it is thus a distance. As we have just shown that $d = d_{\mathbb{G}'_k}$, the next corollary is immediately proved.

Corollary 3.1. — *The function $d : \mathcal{P}_k \times \mathcal{P}_k \rightarrow \mathbb{R}^+$ is a distance.*

Lemma 3.2. — *The restriction of d to the permutation group is quite usual:*

$$d(\sigma, \sigma') = k - \text{nc}(\sigma^{-1}\sigma'),$$

for any $\sigma, \sigma' \in \mathfrak{S}_k$. This distance is in fact the geodesic distance on the Cayley graph \mathcal{S}_k of \mathfrak{S}_k . By Lemma 6.26 of [12], the restriction of the distance d to \mathcal{B}_k is also the geodesic distance on the Cayley graph \mathbb{B}_k of \mathcal{B}_k .

Using this distance, we can define a notion of set-geodesic for any of the three sets of partitions we are interested in. We remind the reader that the notation A_k was settled in Notation 2.1.

Definition 3.3. — *Let $p \in A_k$, the set-geodesic $[id, p]_{A_k}$ is defined by:*

$$[id, p]_{A_k} = \{p' \in A_k, d(id, p) = d(id, p') + d(p', p)\}.$$

A geodesic in a graph between two vertices p and p' is a path in this graph which length is equal to the geodesic distance. Using Proposition 3.2 and Lemma 3.2, one shows that for any $p \in A_k$, the set-geodesic $[id, p]_{A_k}$ is the union of the geodesics between id and p in the Cayley graph of A_k .

The distance on A_k allows us to define a new partial order on A_k .

Definition 3.4. — *Let p and p' be elements of A_k , we write that $p' \leq p$ if and only if $d(id, p) = d(id, p') + d(p', p)$.*

This is a partial order as the restriction of d to $A_k \times A_k$ is a distance.

3.2. Characterization of the order. — Let us define a notion of defect. Let p and p' in \mathcal{P}_k .

Definition 3.5. — We define the defect of p' from not being on the set-geodesic $[id, p]_{\mathcal{P}_k}$ by:

$$\text{df}(p', p) = d(id, p') + d(p', p) - d(id, p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p \vee p') + \text{nc}(p \vee id).$$

A simple but very useful lemma is the following.

Lemma 3.3. — If p is coarser than p' , then:

$$\text{df}(p', p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p) + \text{nc}(p \vee id).$$

If p is finer than p' then:

$$\text{df}(p', p) = \text{nc}(p \vee id) - \text{nc}(p' \vee id).$$

Proof. — This is a simple calculation, where one has to use the fact that $\text{nc}(p \vee p') = \text{nc}(p)$ if p is coarser than p' and $\text{nc}(p \vee p') = \text{nc}(p')$ if p is finer than p' . \square

We will now characterize the order by constructing the Hasse diagram of (\mathcal{P}_k, \leq) . For this, we define the notion of pivotal block for a partition $p \in \mathcal{P}_k$.

Definition 3.6. — We define the set of pivotal blocks for p as the set of blocks b of p such that there exists a way to cut b into two blocks in order to cut a cycle of p into two cycles. We denote by $\text{Piv}(p)$ the set of pivotal blocks for p .

For example, if $p = \{\{1, 2, 1', 2'\}\} \in \mathcal{P}_2$, the block $\{1, 2, 1', 2'\}$ is a pivotal block for p since we can cut it in order to get the new partition $\{\{1, 1'\}, \{2, 2'\}\}$ which has one more cycle.

Definition 3.7. — We denote by $\Delta(p)$ the set of all partitions p' which are obtained by cutting in p a pivotal block for p into two blocks in such way that p' has one more cycle than p . This defines a function Δ from \mathcal{P}_k to the subsets of \mathcal{P}_k , namely $\mathcal{P}(\mathcal{P}_k)$. The admissible splits of p are $\text{Sp}(p) = \bigcup_{k=0}^{\infty} \Delta^k(p)$.

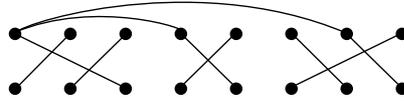


FIGURE 12. A partition p such that $(1, 2, 3)(4, 5)(6, 7, 8) \in \text{Sp}(p)$.

Lemma 3.4. — If p' is finer than p , the three following assertions are equivalent:

1. $p' \in [id, p]_{\mathcal{P}_k}$,
2. $\text{nc}(p') - \text{nc}(p' \vee id) = \text{nc}(p) - \text{nc}(p \vee id)$,
3. $p' \in \text{Sp}(p)$.

Proof. — The fact that the two first assertions are equivalent is a consequence of Lemma 3.3. Let us prove that the second and the third assertions are equivalent. If p' is finer than p then one can go from p to p' only by cutting blocks. At each step the number of blocks goes up by one and the number of cycle is either constant or goes up by one. Thus, if we want that the number of blocks minus the number of cycle is constant at the beginning and at the end it must be constant during all the path from p to p' . This means that at each step the number of cycles must go up by one, and thus we are cutting a pivotal block in a way to create one more cycle. This proves that the second condition is equivalent to $p' \in \text{Sp}(p)$. \square

Definition 3.8. — For any $p \in \mathcal{P}_k$, let us define $\text{Gl}_c(p)$ the set of partitions p' in \mathcal{P}_k such that p' is obtained by gluing blocks of p which are in the same cycle of p . The set $\text{Gl}_c(p)$ is only:

$$\text{Gl}_c(p) = \{p' \in \mathcal{P}_k \mid \text{nc}(p \vee id) = \text{nc}(p' \vee id) \text{ and } p' \text{ is coarser than } p\}.$$

The following proposition is straightforward.

Lemma 3.5. — If p' is coarser than p , the two following assertions are equivalent:

1. $p' \in [id, p]_{\mathcal{P}_k}$,
2. $\text{nc}(p' \vee id) = \text{nc}(p \vee id)$,
3. $p' \in \text{Gl}_c(p)$.

Lemmas 3.4 and 3.5 allow us to prove the following characterization of the geodesic order.

Theorem 3.1. — The partition p' is in $[id, p]_{\mathcal{P}_k}$ if and only there exists $p'' \in \mathcal{P}_k$ such that the two following conditions hold:

1. $p'' \in \text{Gl}_c(p)$,
2. $p' \in \text{Sp}(p'')$.

If so, then $p'' = p \vee p'$.

Thus, if one defines the set of admissible gluings $\text{Sp}^{-1}(p)$ as the set of all partitions p' such that $p \in \text{Sp}(p')$, then $p' \in [id, p]_{\mathcal{P}_k}$ if and only if $\text{Gl}_c(p) \cap \text{Sp}^{-1}(p') \neq \emptyset$, and if so $\text{Gl}_c(p) \cap \text{Sp}^{-1}(p') = \{p \vee p'\}$.

An other formulation in order to state this theorem is the following theorem which allows us to construct the Hasse diagram for the geodesic order.

Theorem 3.2. — We define the graph $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$ such that:

- the set of vertices \mathcal{V}_k is \mathcal{P}_k ,
- there exists an oriented edge from p to p' if one can go from p to p' by gluing two blocks of p which are not in the same cycle of p or if one can go from p' to p by gluing two blocks which are in the same cycle of p' .

The graph \mathcal{G}_k is the Hasse diagram of the geodesic order on \mathcal{P}_k : $p' \in [id, p]_{\mathcal{P}_k}$ if and only if there exists a path from p' to p in \mathcal{G}_k .

Remark 3.2. — Let $\{p, p'\}$ be an edge in \mathcal{G}'_k , the Caley graph of \mathcal{P}_k . As the Caley graph is not oriented, we can always suppose that one can go from p to p' by gluing two blocks of p . If the two blocks we glue are not in the same cycle of p , then the oriented

edge (p, p') is in the Hasse diagram for the geodesic order. If not, then it means that (p', p) is in the Hasse diagram for the geodesic order.

Thus, for any non-oriented edge e of \mathbb{G}'_k , there exists an orientation e^+ of e such that e^+ is an edge of the Hasse diagram for the geodesic order and $(e^+)^{-1}$ is not: the Hasse diagram is obtained by choosing a good orientation on the edges of the Cayley graph \mathbb{G}'_k . For \mathcal{P}_2 , the Hasse diagram is obtained by considering the graph drawn in Figure 3.1 and orienting the edges from down to top.

Proof of Theorem 3.1. — Using Lemma 3.8, we see that $p' \in [id, p]_{\mathcal{P}_k}$ if and only if $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$ and $p' \in [id, p]_{\mathcal{P}_k}$, thus, if and only if $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$ and:

$$\text{df}(p', p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p \vee p') + \text{nc}(p \vee id) = 0,$$

which is equivalent to $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$ and:

$$\text{df}(p', p \vee p') = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p \vee p') + \text{nc}(p \vee p' \vee id) = 0,$$

which is again equivalent to $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$ and $p' \in [id, p \vee p']_{\mathcal{P}_k}$. Using Lemmas 3.4 and 3.5, we get that $p' \in [id, p]_{\mathcal{P}_k}$ if and only if:

$$p \vee p' \in \text{Gl}_c(p) \cap \text{Sf}^{-1}(p').$$

Let us prove that $\#(\text{Gl}_c(p) \cap \text{Sf}^{-1}(p')) \leq 1$. Let $p'' \in \text{Gl}_c(p) \cap \text{Sf}^{-1}(p')$: this implies that:

$$\begin{aligned} \text{nc}(p'' \vee id) &= \text{nc}(p \vee id) \\ \text{nc}(p'') - \text{nc}(p'' \vee id) &= \text{nc}(p') - \text{nc}(p' \vee id). \end{aligned}$$

Thus $\text{nc}(p'') - \text{nc}(p \vee id) = \text{nc}(p') - \text{nc}(p' \vee id)$. Yet the fact that $\text{df}(p', p) \geq 0$ tells us that:

$$\text{nc}(p \vee p') - \text{nc}(p \vee id) \leq \text{nc}(p') - \text{nc}(p' \vee id).$$

Since p'' is coarser than p and than p' , $\text{nc}(p \vee p') \geq \text{nc}(p'')$. Thus:

$$\text{nc}(p'') - \text{nc}(p'' \vee id) \leq \text{nc}(p \vee p') - \text{nc}(p'' \vee id) \leq \text{nc}(p') - \text{nc}(p' \vee id).$$

Since the left and right hand sides are equal, we have that $\text{nc}(p \vee p') = \text{nc}(p'')$ and since p'' is coarser than $p \vee p'$, we obtain that $p'' = p \vee p'$. \square

3.2.1. Properties of the admissible splitting. — Using the definition of the admissible splittings, the following proposition is straightforward.

Proposition 3.3. — *Let $p \in \mathcal{P}_k$, the set $\text{Sp}(p) \cap \mathcal{B}_k$ is either empty or has one element.*

This last proposition leads us to the following definition.

Definition 3.9. — *For any positive integer k , we define:*

$$\begin{aligned} \overline{\mathcal{S}}_k &= \{p \in \mathcal{P}_k \mid \#(\text{Sp}(p) \cap \mathcal{S}_k) = 1\}, \\ \overline{\mathcal{B}}_k &= \{p \in \mathcal{P}_k \mid \#(\text{Sp}(p) \cap \mathcal{B}_k) = 1\}. \end{aligned}$$

For any $p \in \overline{\mathcal{B}}_k$, we denote by $\text{Mb}(p)$ the unique element in $\text{Sp}(p) \cap \mathcal{B}_k$.

An other important lemma is the following.

Lemma 3.6. — *Let p be a partition in \mathcal{P}_k which does not have any pivotal block, then $\text{Sp}(p) = \{p\}$. In particular, if $p \in \mathcal{B}_k$, p does not have any pivotal block, thus:*

$$\text{Sp}(p) = \{p\}.$$

Proof. — The first assertion is a direct consequence of the definitions. It remains to show that if $p \in \mathcal{B}_k$, p does not have any pivotal block. We can suppose that p is irreducible. Let us suppose that p has a pivotal block that we will denote by c . We can always suppose that c is of the form $\{i, (i+1)'\}$ or $\{i, i+1\}$ since we can shuffle the columns of p and take the transpose of p . With the same argument, we can suppose that when one cuts the block c , the new partition we get has the form $p_1 \otimes p_2$ with $p_1 \in \mathcal{P}_i$. Here is the contradiction: the partition p_1 must be composed of blocks of size two except one block which is equal to $\{i\}$. This is not possible since p_1 must be a partition of $2i$ elements. \square

3.3. Computation of the Möbius function for the geodesic order. — Since we have defined an order on \mathcal{P}_k , namely the geodesic order, it would be interesting to compute the Möbius function. In order to do so, we need to define two orders.

Definition 3.10. — *Let p and p' be in \mathcal{P}_k . We say that:*

- p' is coarser-compatible than p if and only if $p' \in \text{Gl}_c(p)$. We denote this by $p' \dashv p$.
- p' is finer-admissible than p if and only if $p' \in \text{Sp}(p)$. We denote this by $p' \sqsupset p$.

Both notions define a partial order on \mathcal{P}_k .

We used the symbols \dashv since when one reads it from right to left, one sees two segments which become one segment ; we are gluing two blocks of p . For \sqsupset , when one reads it from right to left, one sees a segment which splits into two parts: we are splitting one block of p .

We can define the matrices of the partial order \leq, \dashv, \sqsupset .

Definition 3.11. — *The matrices of the partial orders \leq, \dashv, \sqsupset are:*

- for the geodesic order \leq : $G_{p,p'} = \delta_{p' \leq p}$,
- for the coarser-compatible order \dashv : $C_{p,p'} = \delta_{p' \dashv p}$,
- for the finer-admissible order \sqsupset : $S_{p,p'} = \delta_{p' \sqsupset p}$.

Using these definitions, we can translate the Theorem 3.1.

Theorem 3.3. — *The partial orders \leq, \dashv , and \sqsupset are linked by the following equality:*

$$G = CS.$$

Proof. — Let us consider p and p' in \mathcal{P}_k . Using Theorem 3.1, we have:

$$G_{p,p'} = \delta_{p' \leq p} = \sum_{p'' \in \mathcal{P}_k | p'' \in \text{Gl}_c(p), p' \in \text{Sp}(p'')} 1.$$

Thus:

$$G_{p,p'} = \sum_{p'' \in \mathcal{P}_k} \delta_{p'' \in \text{Gl}_c(p)} \delta_{p' \in \text{Sp}(p'')} = \sum_{p'' \in \mathcal{P}_k} \delta_{p'' \dashv p} \delta_{p' \sqsupset p''} = \sum_{p'' \in \mathcal{P}_k} C_{p,p''} S_{p'',p'}.$$

This allows us to finish the proof. \square

The notion of Möbius function is linked with the inverse of the matrix of the order.

Definition 3.12. — Let T be a finite set endowed with a partial order. Let M be the matrix of the order as we defined in Definition 3.11. The Möbius function is the function such that for any a and b in T ,

$$\mu(a, b) = (M^{-1})_{b,a}.$$

Our goal is to compute the Möbius function for (\mathcal{P}_k, \leq) . In order to compute the inverse of G , we need the following lemma.

Lemma 3.7. — Let p, p' and p'' be in \mathcal{P}_k . Let us suppose that $p' \dashv p''$ and $p'' \sqsupset p$, then $p'' = p \wedge p'$.

Let us remark that it is the "infimum version" of the last assertion of Theorem 3.1 which asserts that $p'' \dashv p$ and $p' \sqsupset p''$ if and only if $p'' = p \vee p'$.

Proof of Lemma 3.7. — Let p, p' and p'' be three partitions in \mathcal{P}_k which satisfy the hypotheses of Lemma 3.7. Using Lemmas 3.4 and 3.5, we know that:

$$\begin{aligned} \text{nc}(p' \vee id) &= \text{nc}(p'' \vee id) \\ \text{nc}(p'') - \text{nc}(p'' \vee id) &= \text{nc}(p) - \text{nc}(p \vee id). \end{aligned}$$

Thus by using the two last equalities:

$$\text{nc}(p'') - \text{nc}(p' \vee id) = \text{nc}(p) - \text{nc}(p \vee id).$$

Besides, $p' \leq p''$ and $p'' \leq p$ thus $p' \leq p$: this implies that $\text{df}(p, p') = 0$, thus:

$$\text{nc}(p') - \text{nc}(p' \vee id) + \text{nc}(p \vee id) - \text{nc}(p \vee p') = 0.$$

Using the two lattest equations, we get:

$$(4) \quad \text{nc}(p'') + \text{nc}(p \vee p') - \text{nc}(p') - \text{nc}(p) = 0.$$

Using the triangle inequality, we know that $d(p, p \wedge p') + d(p \wedge p', p'') - d(p, p') \geq 0$, thus:

$$(5) \quad \text{nc}(p \wedge p') + \text{nc}(p \vee p') - \text{nc}(p) - \text{nc}(p') \geq 0.$$

Since p'' is finer than p' and than p , p'' is finer than $p \wedge p'$: this implies that $\text{nc}(p \wedge p') \leq \text{nc}(p'')$. Using Equations (4) and (5), we get that $\text{nc}(p'') = \text{nc}(p \wedge p')$. Thus $p'' = p \wedge p'$. \square

Theorem 3.4. — Let p and p' in \mathcal{P}_k . We have:

$$(G^{-1})_{p,p'} = \delta_{p \wedge p' \sqsupset p} \delta_{p' \dashv p \wedge p'} \mu_f(p \wedge p', p) \mu_f(p \wedge p', p'),$$

where for any partition p_1 and p_2 such that p_1 is finer than p_2 :

$$\mu_f(p_1, p_2) = (-1)^{\text{nc}(p_1) - \text{nc}(p_2)} \prod_{i=3}^{\text{nc}(p_1)} ((i-1)!)^{r_i}$$

where r_i is the number of blocks of p_2 which contains exactly i blocks of p_1 .

Proof. — Using Theorem 3.3, we know that $G = CS$. Thus $G^{-1} = S^{-1}C^{-1}$. Let p and p' be two partitions in \mathcal{P}_k , we know that:

$$(G^{-1})_{p,p'} = \sum_{p'' \in \mathcal{P}_k} (S^{-1})_{p,p''} (C^{-1})_{p'',p'}.$$

Since S , respectively C , is the matrix of the order \sqsupset , respectively \sqsupseteq , for any partitions p_1 and p_2 in \mathcal{P}_k :

$$\begin{aligned} (S^{-1})_{p_1,p_2} &= \delta_{p_2 \sqsupset p_1} (S^{-1})_{p_1,p_2}, \\ (C^{-1})_{p_1,p_2} &= \delta_{p_2 \sqsupseteq p_1} (C^{-1})_{p_1,p_2}. \end{aligned}$$

This is due to the fact that the inverse of an upper triangular invertible matrix is still upper triangular. Thus:

$$(G^{-1})_{p,p'} = \sum_{p'' \in \mathcal{P}_k | p'' \sqsupset p, p' \sqsupseteq p''} (S^{-1})_{p,p''} (C^{-1})_{p'',p'}.$$

Using Lemma 3.7, we get that:

$$(G^{-1})_{p,p'} = (S^{-1})_{p,p \wedge p'} (C^{-1})_{p \wedge p',p'}.$$

It remains to compute $(S^{-1})_{p,p \wedge p'}$ and $(C^{-1})_{p \wedge p',p'}$.

Each time that one considers a partial order on a finite set, its matrix O can be written as $Id + \tilde{O}$, with \tilde{O} being a nilpotent matrix. Thus:

$$O^{-1} = \sum_{i=0}^{\infty} (-1)^i \tilde{O}^i,$$

this is the Rota's Formula for the Möbius inversion. Thus, for any p and p' in \mathcal{P}_k :

$$\begin{aligned} (S^{-1})_{p,p \wedge p'} &= \sum_{i=0}^{\infty} (-1)^i \sum_{(p_0, \dots, p_i) \in \mathcal{P}_k | p = p_0 \not\supset p_1 \not\supset \dots \not\supset p_i = p \wedge p'} \left[\prod_{l=0}^{i-1} S_{p_l, p_{l+1}} \right] \\ &= \sum_{i=0}^{\infty} (-1)^i \sum_{(p_0, \dots, p_i) \in \mathcal{P}_k | p = p_0 \not\supset p_1 \not\supset \dots \not\supset p_i = p \wedge p'} \left[\prod_{l=0}^{i-1} \delta_{p_{l+1} \sqsupset p_l} \right]. \end{aligned}$$

Yet, if $p \wedge p' \sqsupset p$, then for any positive integer i , for any $i+1$ -tuple (p_0, \dots, p_i) :

$$\prod_{l=0}^{i-1} \delta_{p_{l+1} \sqsupset p_l} = \prod_{l=0}^{i-1} \delta_{p_{l+1} \text{ finer than } p_l}.$$

Thus:

$$(S^{-1})_{p,p \wedge p'} = \delta_{p \wedge p' \sqsupset p} \sum_{i=0}^{\infty} (-1)^i \sum_{(p_0, \dots, p_i) \in \mathcal{P}_k | p = p_0 \not\supset p_1 \not\supset \dots \not\supset p_i = p \wedge p'} \left[\prod_{l=0}^{i-1} \delta_{p_{l+1} \text{ finer than } p_l} \right].$$

This implies that:

$$(S^{-1})_{p,p \wedge p'} = \delta_{p \wedge p' \sqsupset p} (F^{-1})_{p,p \wedge p'},$$

with F being the matrix such that for any $p_1, p_2 \in \mathcal{P}_k$, $F_{p_1, p_2} = \delta_{p_2 \text{ finer than } p_1}$. The inverse of this matrix is well known, for any $p_1, p_2 \in \mathcal{P}_k$ such that p_2 is finer than p_1 :

$$(F^{-1})_{p_1, p_2} = \mu_f(p_2, p_1),$$

where μ_f is the Möbius function for the order of being finer and is given in the statement of Theorem 3.4. Similar arguments allow us to compute the inverse of C and to obtain that:

$$(C^{-1})_{p \wedge p', p'} = \delta_{p' \dashv p \wedge p'} \mu_f(p \wedge p', p').$$

This allows us to obtain the desired formula for G^{-1} . \square

Theorem 3.5. — *The Möbius function for (\mathcal{P}_k, \leq) , denoted by μ_{\leq} , is given by:*

$$\forall p_1, p_2 \in \mathcal{P}_k, \mu_{\leq}(p_1, p_2) = \delta_{p_1 \dashv p_1 \wedge p_2} \delta_{p_1 \wedge p_2 \sqsupset p_2} \mu_f(p_1 \wedge p_2, p_1) \mu_f(p_1 \wedge p_2, p_2).$$

Let us remark that if $p_1 \dashv p_1 \wedge p_2$ then $p_1 \leq p_1 \wedge p_2$ and if $p_1 \wedge p_2 \sqsupset p_2$ then $p_1 \wedge p_2 \leq p_2$. Thus if both conditions hold, $p_1 \leq p_2$ by transitivity of the geodesic order. This is why we do not add the condition that $p_1 \leq p_2$ in the formula for the Möbius function μ_{\leq} .

Remark 3.3. — *Let us remark that, as a by-product of the proof of Theorem 3.4, we computed the matrices C^{-1} and S^{-1} , thus we know the Möbius functions for the orders \dashv, \sqsupset and \leq .*

3.4. Some properties of the geodesic order. —

3.4.1. Factorization of the geodesics. — The following property, known for \mathfrak{S}_k and \mathcal{B}_k , is still true for \mathcal{P}_k : a geodesic between id and $p_1 \otimes p_2$ must be the tensor product of a geodesic between id and p_1 and a geodesic between id and p_2 .

Lemma 3.8. — *Let $p \in A_k$, we have:*

$$[id, p]_{A_k} \simeq \prod_{C \in \mathcal{C}(p)} [id_{\frac{\#C}{2}}, p_C]_{A_{\frac{\#C}{2}}}.$$

In particular if p_1 and p_2 are two partitions, $p_1 \in A_{k_1}$ and $p_2 \in A_{k_2}$, then $p' \in [id, p_1 \otimes p_2]_{A_{k_1+k_2}}$ if and only if there exist $p'_1 \in A_{k_1}$ and $p'_2 \in A_{k_2}$ such that $p' = p'_1 \otimes p'_2$, $p'_1 \in [id, p_1]_{A_{k_1}}$ and $p'_2 \in [id, p_2]_{A_{k_2}}$.

3.4.2. No Brauer element is smaller than a permutation. — In the following lemma, we show that the geodesic in the Cayley graph of \mathcal{P}_k between two permutations either stay in the set of permutations or intersect $\mathcal{P}_k \setminus \mathcal{B}_k$. Using the fact that $[id, p]_{A_k}$ is the union of the geodesics between id and p in the Cayley graph of A_k , we get an equality between $[id, \sigma]_{\mathcal{B}_k}$ and $[id, \sigma]_{\mathfrak{S}_k}$.

Lemma 3.9. — *Let $\sigma \in \mathfrak{S}_k$, then $[id, \sigma]_{\mathcal{B}_k} = [id, \sigma]_{\mathfrak{S}_k}$.*

Proof. — We will do a proof by contradiction. Let $S \subset \mathfrak{S}_k$ be the set of permutations such that $[id, \sigma]_{\mathcal{B}_k} \neq [id, \sigma]_{\mathfrak{S}_k}$. Let $\sigma \in S$ be a permutation such that $d(id, \sigma) = \min_{\sigma' \in S} d(id, \sigma')$. Let us consider b an element of $\mathcal{B}_k \setminus \mathfrak{S}_k$ such that $b \in [id, \sigma]_{\mathcal{B}_k}$. There exists a geodesic in \mathcal{B}_k which goes through b and goes from id to σ . Let $b' \in \mathcal{B}_k$ be the unique element on this geodesic such that $d(id, b') = 1$. Let us remark that $b \in [b', \sigma]_{\mathcal{B}_k}$:

this implies that b' can not be a permutation. Indeed, if b' was a permutation, then $[b', \sigma]_{\mathcal{B}_k} \neq [b', \sigma]_{\mathfrak{S}_k}$ and thus, $[id, b'^{-1}\sigma]_{\mathcal{B}_k} \neq [id, b'^{-1}\sigma]_{\mathfrak{S}_k}$. Yet $d(id, b'^{-1}\sigma) = d(b', \sigma) = d(id, \sigma) - 1$. This would contradict the fact that $d(id, \sigma) = \min_{\sigma' \in \mathfrak{S}} d(id, \sigma')$. Thus b' must be an element of $\mathcal{B}_k \setminus \mathfrak{S}_k$. Since $d(id, b') = 1$, there exist i and j in $\{1, \dots, k\}$ such that b' is equal to the Weyl contraction $[i, j]$ in \mathcal{B}_k . Thus there exist i and j in $\{1, \dots, k\}$ such that $[i, j] \in [id, \sigma]_{\mathcal{B}_k}$. Using Theorem 3.1, this means that $[i, j] \vee \sigma \in \text{Gl}_c(\sigma)$ and $[i, j] \in \text{Sp}([i, j] \vee \sigma)$. The first condition tells us that i and j must be in the same cycle of σ , let us suppose so. The second condition implies that σ has only one cycle which is not trivial and it implies that:

$$\text{nc}(\sigma \vee [i, j]) - \text{nc}(\sigma \vee [i, j] \vee id) = \text{nc}([i, j]) - \text{nc}([i, j] \vee id) = 1.$$

Yet $\text{nc}(\sigma \vee [i, j]) - \text{nc}(\sigma \vee [i, j] \vee id) = 0$ thus $[i, j]$ can not be in $[id, \sigma]_{\mathcal{B}_k}$. This yields the contradiction. \square

This lemma is the key point which will allow us to explain in the second article [9] why processes on $U(N)$ and $O(N)$ have the same limit when one only considers usual moments.

3.4.3. Geodesics and tensor product. — For the last geometric proposition, we need to define the left and right parts of a partition p .

Definition 3.13. — Let k and l be two positive integers. Let $p \in A_{k+l}$, we denote by p_k^g the extraction of p to $\{1, \dots, k\}$ and p_k^d the extraction of p to $\{k+1, \dots, k+l\}$. The partition p_k^g is in \mathcal{P}_k and p_k^d is in \mathcal{P}_l .

Proposition 3.4. — Let k_1 and k_2 be two positive integers and let $k = k_1 + k_2$. Let p be an element of \mathcal{P}_k . Let p_1 and p_2 be respectively in \mathcal{P}_{k_1} and \mathcal{P}_{k_2} . We have equivalence between:

1. $p_1 \otimes p_2 \sqsupset p$,
2. $p_1 \sqsupset p_{k_1}^g$, $p_2 \sqsupset p_{k_2}^d$ and $p_{k_1}^g \otimes p_{k_2}^d \sqsupset p$.

Proof. — First of all, it is not difficult to see that the second condition implies the first one. Indeed, if $p_1 \sqsupset p_{k_1}^g$ and $p_2 \sqsupset p_{k_2}^d$ then $p_1 \otimes p_2 \sqsupset p_{k_1}^g \otimes p_{k_2}^d$. Thus, by transitivity of the order, if $p_{k_1}^g \otimes p_{k_2}^d \sqsupset p$, we get that $p_1 \otimes p_2 \sqsupset p$.

It remains to show that the first condition implies the second one. Let us remark that p is coarser than $p_1 \otimes p_2$ if and only if $p_{k_1}^g$ is coarser than p_1 and $p_{k_2}^d$ is coarser than p_2 . Thus it remains to prove that under the condition that p is coarser than $p_1 \otimes p_2$, $p_1 \otimes p_2 \in [id, p]_{A_k}$ implies that $p_1 \in [id, p_{k_1}^g]_{\mathcal{P}_{k_1}}$, $p_2 \in [id, p_{k_2}^d]_{\mathcal{P}_{k_2}}$ and $p_{k_1}^g \otimes p_{k_2}^d \in [id, p]_{A_k}$. Since for any partitions the defect between two partitions is always positive, the result would be a consequence of the following equality:

$$\text{df}(p_1 \otimes p_2, p) = \text{df}(p_1, p_{k_1}^g) + \text{df}(p_2, p_{k_2}^d) + \text{df}(p_{k_1}^g \otimes p_{k_2}^d, p).$$

Let us prove this equality: using Lemma 3.3:

$$\begin{aligned}
& \text{df}(p_1 \otimes p_2, p) - \text{df}(p_1, p_k^g) - \text{df}(p_2, p_k^d) - \text{df}(p_k^g \otimes p_k^d, p) \\
&= \text{nc}(p_1 \otimes p_2) - \text{nc}((p_1 \otimes p_2) \vee id) - \text{nc}(p) + \text{nc}(p \vee id) \\
&\quad - \text{nc}(p_1) + \text{nc}(p_1 \vee id) + \text{nc}(p_k^l) - \text{nc}(p_k^g \vee id) \\
&\quad - \text{nc}(p_2) + \text{nc}(p_2 \vee id) + \text{nc}(p_k^r) - \text{nc}(p_k^d \vee id) \\
&\quad - \text{nc}(p_k^g \otimes p_k^d) + \text{nc}((p_k^g \otimes p_k^d) \vee id) + \text{nc}(p) - \text{nc}(p \vee id) \\
&= 0,
\end{aligned}$$

since:

$$\begin{aligned}
\text{nc}(p_1 \otimes p_2) &= \text{nc}(p_1) + \text{nc}(p_2), \\
\text{nc}((p_1 \otimes p_2) \vee id) &= \text{nc}(p_1 \vee id) + \text{nc}(p_2 \vee id), \\
\text{nc}(p_k^g \otimes p_k^d) &= \text{nc}(p_k^g) + \text{nc}(p_k^d), \\
\text{nc}((p_k^g \otimes p_k^d) \vee id) &= \text{nc}(p_k^g \vee id) + \text{nc}(p_k^d \vee id).
\end{aligned}$$

This ends the proof. \square

4. Convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[P_k(N)]$

4.1. Coordinate numbers and moments. —

4.1.1. Definitions. — Let k be a positive integer, recall the notation A_k defined in Notation 2.1. For each integer N , we have defined an algebra $\mathbb{C}[A_k(N)]$. Let $(E_N)_{N \in \mathbb{N}}$ be a sequence such that for any positive integer N , $E_N \in \mathbb{C}[A_k(N)]$. For each integer N , the algebra $\mathbb{C}[A_k(N)]$, seen as a vector space has the same basis A_k . Thus, we could study the convergence of $(E_N)_{N \in \mathbb{N}}$ only from the vector space point of view by saying that the sequence $(E_N)_{N \in \mathbb{N}}$ converges if and only if the coordinates of E_N in the basis A_k converge. Actually, this convergence forgets the fact that $\mathbb{C}[A_k(N)]$ is an algebra which depends on an integer N . In order to define a better definition of convergence, we have to define the coordinate numbers of E in $\mathbb{C}[A_k(N)]$.

Definition 4.1. — *Let N be an integer. Let E be an element of $\mathbb{C}[A_k(N)]$. We define the numbers $(\kappa^p(E))_{p \in A_k}$ as the only numbers such that:*

$$E = \sum_{p \in A_k} \frac{\kappa^p(E)}{N^{\frac{-k + \text{nc}(p)}{2} + d(id, p)}} p.$$

The family $(\kappa^p(E))_{p \in A_k(N)}$ is called the coordinate numbers of E .

After Definition 4.4, we will explain how we get this definition, and why this definition is in fact the most natural thing one can do. We will need to use the following equality: for any integer k , for any $p \in A_k$,

$$(6) \quad \frac{-k + \text{nc}(p)}{2} + d(id, p) = \text{nc}(p) - \text{nc}(p \vee id).$$

This implies the following remark.

Remark 4.1. — For any integer k , any integer N , for any $E \in \mathbb{C}[\mathcal{P}_k(N)]$:

$$E = \sum_{p \in A_k} \frac{\kappa^p(E)}{N^{\text{nc}(p) - \text{nc}(p \vee \text{id})}} p.$$

We will consider the coordinate numbers as linear applications from $\mathbb{C}[A_k(N)]$ to \mathbb{R} :

$$\begin{aligned} \kappa^p &: \mathbb{C}[A_k(N)] \rightarrow \mathbb{R} \\ E &\mapsto \kappa^p(E). \end{aligned}$$

The notion of coordinate numbers allows us to define a strong convergence for any sequence $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 4.2. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. The sequence $(E_N)_{N \in \mathbb{N}}$ converges strongly if the coordinate numbers of E_N converges when N goes to infinity: for any $p \in A_k$, $\kappa^p(E_N)$ converges when N goes to infinity.

The goal now is to give a dual definition of convergence. We have seen in Definition 2.15 that any element of $\mathbb{C}[A_k(N)]$ can be seen as an element of $\text{End}\left((\mathbb{C}^N)^{\otimes k}\right)$ and we defined in Definition 3.1 the trace of any element $\mathbb{C}[A_k(N)]$. Using this trace and the structure of algebra of $\mathbb{C}[A_k(N)]$, we define, for any element of $\mathbb{C}[A_k(N)]$ and any element $p \in A_k$, the p -normalized moment of E .

Definition 4.3. — Let $N \in \mathbb{N}$, let $p \in A_k$ and $E \in \mathbb{C}[A_k(N)]$. The p -normalized moment of E is:

$$m_p(E) = \frac{1}{\text{Tr}_N(p)} \text{Tr}_N(E^t p).$$

Using these normalized moments, we can define a weak notion of convergence for any sequence $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 4.4. — The sequence $(E_N)_{N \in \mathbb{N}}$ converges in moments if the normalized moments of E_N converges when N goes to infinity: for any $p \in A_k$, $m_p(E_N)$ converges when N goes to infinity.

4.1.2. Coordinate numbers-moments transformation. — We can now explain how we ended up with Definition 4.1 and how we had the idea to define the distance on the set of partitions. The idea behind these definitions is that we want to know, given a sequence $(E_N) \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$, how the usual coordinates of E_N in the basis A_k must scale so that for any $p \in A_k$, $m_p(E_N)$ converges when N goes to infinity. Let N be an integer, we have $E_N = \sum_{p \in A_k} a_N^p p$. Thus

$$m_{p_0}(E_N) = \sum_{p \in A_k} \frac{\text{Tr}_N(p^t p_0)}{\text{Tr}_N(p_0)} a_N^p.$$

Thus the vector $m_N = (m_{p_0}(E_N))_{p_0}$ and $a_N = (a_N^p)_p$ are linked by the relation $m_N = M_N a_N$ where $M_N = \left(\frac{\text{Tr}_N(p^t p_0)}{\text{Tr}_N(p_0)} \right)_{p_0, p}$.

There are then two possible possibilities: to invert M_N for N big enough. This is the usual way, which leads to the Weingarten function. Or, one can make the following Ansatz: if we write the system, we see that for any p , $(a_N)_p$ is going to be multiplied by $(M_N)_{p_0,p}$ for any $p_0 \in A_k$. Thus we make the assumption that $(a_N)_p$ must decrease as the inverse of the maximum of $(M_N)_{p_0,p}$ over p_0 . That is $a_N^p \sim a^p N^{-\eta_p}$, where η_p is given by:

$$\eta_p = \sup_{p_0} \lim_{N \rightarrow \infty} \log_N \left(\frac{\text{Tr}_N(p \ ^t p_0)}{\text{Tr}_N(p_0)} \right).$$

The goal now is to know in which p_0 the supremum is obtained. It is more than tempting, seeing the scalar product $\text{Tr}_N(p_0 \ ^t p)$ to write what is inside the \log_N as:

$$\begin{aligned} \frac{\text{Tr}_N(p \ ^t p_0)}{\text{Tr}_N(p_0)} &= \frac{\text{Tr}_N(p \ ^t p_0)}{\sqrt{\text{Tr}_N(p \ ^t p) \text{Tr}_N(p_0 \ ^t p_0)}} \frac{\sqrt{\text{Tr}_N(p_0 \ ^t p_0) \text{Tr}_N(p \ ^t p)}}{\text{Tr}_N(p_0)} \\ &= \frac{\text{Tr}_N(p \ ^t p_0)}{\sqrt{\text{Tr}_N(p \ ^t p) \text{Tr}_N(p_0 \ ^t p_0)}} \frac{\sqrt{\text{Tr}_N(p_0 \ ^t p_0) \text{Tr}_N(id_k \ ^t id_k)}}{\text{Tr}_N(p_0 \ ^t id_k)} \sqrt{\frac{\text{Tr}_N(p \ ^t p)}{\text{Tr}_N(id_k \ ^t id_k)}}. \end{aligned}$$

We recognize thus the distance that we defined. In fact the intuition that is should be a distance comes from the fact that one can write:

$$\eta_p = \sup_{p_0} [-d(p, p_0) + d(p_0, id_k)] + \frac{1}{2}(-k + \text{nc}(p)).$$

If d was a distance, then by the triangle inequality, for any p_0 ,

$$d(p_0, id_k) - d(p_0, p) \leq d(p, id_k).$$

This shows that the supremum is obtained at $p_0 = p$, and thus the Ansatz tells us that:

$$a_N^p \sim a^p N^{-[\frac{1}{2}(-k + \text{nc}(p)) + d(id, p)]},$$

to be compared with the Definition 4.1.

The first main result is given by Theorem 4.1 which shows the equivalence between strong and weak convergence.

Theorem 4.1. — *Let $(E_N)_{N \in \mathbb{N}}$ be a sequence such that for any $N \in \mathbb{N}$, E_N is an element of $\mathbb{C}[A_k(N)]$. The sequence $(E_N)_{N \in \mathbb{N}}$ converges strongly if and only if it converges in moments. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges in moments or strongly, for any $p \in A_k$:*

$$(7) \quad \lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id, p]_{A_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

Proof. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$, let $p \in A_k$ and let N be a positive integer. Using the coordinate numbers of E_N , we can calculate the p -normalized

moment of E_N :

$$\begin{aligned} m_p(E_N) &= \frac{1}{Tr_N(p)} Tr_N(E_N {}^t p) = \frac{1}{Tr_N(p)} Tr_N \left(\sum_{p' \in A_k} \frac{\kappa^{p'}(E_N)}{N^{\text{nc}(p') - \text{nc}(p' \vee id)}} p' {}^t p \right) \\ &= \sum_{p' \in A_k} \kappa^{p'}(E_N) \frac{Tr_N(p' {}^t p)}{Tr_N(p) N^{\text{nc}(p') - \text{nc}(p' \vee id)}}. \end{aligned}$$

Using the Equality (3):

$$m_p(E_N) = \sum_{p' \in A_k} \kappa^{p'}(E_N) N^{\text{nc}(p \vee p') - \text{nc}(p \vee id) - \text{nc}(p') + \text{nc}(p' \vee id)}.$$

Hence, using Definition 3.5:

$$(8) \quad m_p(E_N) = \sum_{p' \in A_k} \kappa^{p'}(E_N) N^{-\text{df}(p', p)}.$$

Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly. The triangle inequality for d shows that for any $p \in A_k$, $m_p(E_N)$ converges when N goes to infinity and:

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id, p]_{A_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

Now, let us suppose that it converges in moments. We can write (8) as:

$$m^N = G_N \kappa^N,$$

where $m^N = (m_p(E_N))_{p \in A_k(N)}$, $\kappa^N = (\kappa^p(E_N))_{p \in A_k(N)}$, and $G_N = \left(N^{-\text{df}(p', p)} \right)_{p, p' \in A_k(N)}$. The sequence $(G_N)_{N \in \mathbb{N}}$ converges to the matrix G defined in Definition 3.11, and since G is invertible, $\kappa^N = G_N^{-1} m^N$ converges to $G^{-1} m$ where $m = \left(\lim_{N \rightarrow \infty} m_p(E_N) \right)_{p \in A_k}$. \square

Let us take some notations in order to simplify our up-coming discussions.

Notation 4.1. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. From now on, we will say that $(E_N)_{N \in \mathbb{N}}$ converges if and only if it converges either strongly or in moments. Besides, let suppose that $(E_N)_{N \in \mathbb{N}}$ converges, then we will set, for any partition $p \in A_k$ and any $P \subset A_k$:

$$\begin{aligned} m_p(E) &= \lim_{N \rightarrow \infty} m_p(E_N), \\ \kappa^p(E) &= \lim_{N \rightarrow \infty} \kappa^p(E_N) \\ \kappa^P(E) &= \sum_{p \in P} \kappa^p(E). \end{aligned}$$

4.2. Consequences of Theorem 4.1.— We have already an interesting corollary of Theorem 4.1.

Theorem 4.2. — For this theorem, let us suppose that A is equal either to \mathfrak{S} or \mathcal{B} . Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ which converges in moments, then for any $p \in \mathcal{P}_k$, the limit of $m_p(E_N)$ exists. Besides, for any $p \in \mathcal{P}_k$, the following equality holds:

$$m_p(E) = \sum_{p' \in A_k, p' \leq p} \kappa^{p'}(E).$$

Proof. — If $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ converges in moments then it converges strongly. Thus seen as an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ it converges also strongly and thus in moments. The Equation (7) allows to conclude. \square

In the case where $A = \mathcal{B}$, one can also prove that, under some hypotheses, the convergence of the \mathfrak{S} -moments is equivalent to the convergence of the \mathfrak{S} -coordinate numbers.

Theorem 4.3. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ and let us suppose that for any $p \in \mathcal{B}_k$, $(m_p(E_N))_{N \in \mathbb{N}}$ is bounded. The following assertions are equivalent:

- for any $\sigma \in \mathfrak{S}_k$, $\kappa^\sigma(E_N)$ converges when N goes to infinity,
- for any $\sigma \in \mathfrak{S}_k$, $m_\sigma(E_N)$ converges when N goes to infinity.

and if one of the condition is satisfied, then for any $\sigma \in \mathfrak{S}_N$,

$$m_\sigma(E) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E).$$

Proof. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ which satisfies the hypothesis of the theorem. First of all, using the same notations of the proof of Theorem 4.1, we know that, for N big enough $\kappa^N = G_N^{-1} m^N$. As the sequence $(m^N)_{N \in \mathbb{N}}$ is bounded and as G_N^{-1} converges to G^{-1} when N goes to infinity, we deduce that $(\kappa^N)_{N \in \mathbb{N}}$ is also bounded.

Let $\sigma \in \mathfrak{S}_k$, using Equation (8), for any integer N ,

$$m_\sigma(E_N) = \sum_{p' \in \mathcal{B}_k} \kappa^{p'}(E_N) N^{-\text{df}(p', \sigma)}.$$

Yet, if $p' \in \mathcal{B}_k \setminus \mathfrak{S}_k$, using Lemma 3.9, $\text{df}(p', \sigma) < 0$.

Let us suppose that for any $\sigma' \in \mathfrak{S}_k$, $\kappa^{\sigma'}(E_N)$ converges, then $m_\sigma(E_N)$ converges as N goes to infinity, and:

$$\lim_{N \rightarrow \infty} m_\sigma(E_N) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E_N).$$

Let us suppose now that for any $\sigma \in \mathfrak{S}_k$, $m_\sigma(E_N)$ converges when N goes to infinity, then for any increasing sequence $(i_N)_{N \in \mathbb{N}}$ of integers such that for any $\sigma' \in \mathfrak{S}_k$, $\kappa^{\sigma'}(E_{i_N})$

converges, we have:

$$\lim_{N \rightarrow \infty} m_\sigma(E_N) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \lim_{N \rightarrow \infty} \kappa^{\sigma'}(E_{i_N}).$$

Hence, for any $\sigma' \in \mathfrak{S}_k$, $\lim_{N \rightarrow \infty} \kappa^{\sigma'}(E_{i_N})$ does not depend on the sequence $(i_N)_{N \in \mathbb{N}}$: this shows that for any $\sigma' \in \mathfrak{S}_k$, $\kappa^{\sigma'}(E_N)$ converges when N goes to infinity. Again we get also:

$$\lim_{N \rightarrow \infty} m_\sigma(E_N) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E_N).$$

This finishes the proof. \square

4.3. Exclusive coordinate numbers and moments. —

4.3.1. Exclusive coordinate numbers. — In Section 2.3, we defined an other basis of $\mathbb{C}[\mathcal{P}_k]$, namely the exclusive basis. In the case we are working with an element $E \in \mathbb{C}[A_k(N)]$ we can also define the exclusive coordinate numbers.

Definition 4.5. — *Let k and N be two positive integers. Let E be an element of $\mathbb{C}[A_k(N)]$. We define the numbers $(\kappa_c^p(E))_{p \in \mathcal{P}_k}$ as the only numbers such that:*

$$E = \sum_{p \in \mathcal{P}_k} \frac{\kappa_c^p(E)}{N^{d(id, p) + \frac{-k + nc(p)}{2}}} p^c = \sum_{p \in \mathcal{P}_k} \frac{\kappa_c^p(E)}{N^{nc(p) - nc(p \vee id)}} p^c.$$

The family $(\kappa_c^p(E))_{p \in \mathcal{P}_k}$ is called the exclusive coordinate numbers of E .

The next proposition shows that one can choose to work either with the exclusive basis or with the usual basis of $\mathbb{C}[A_k]$ in order to study the convergence of $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Theorem 4.4. — *Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. The exclusive coordinate numbers $(\kappa_c^p(E_N))_{p \in \mathcal{P}_k}$ converge as N goes to infinity if and only if $(E_N)_{N \in \mathbb{N}}$ converges. Besides, if $(E_N)_{N \in \mathbb{N}}$ converges then for any $p \in \mathcal{P}_k$, $\kappa_c^p(E_N)$ converges as N goes to infinity, and for any $p \in \mathcal{P}_k$:*

$$\lim_{N \rightarrow \infty} \kappa_c^p(E_N) = \sum_{p' \in A_k, p' \in \text{Sp}(p)} \lim_{N \rightarrow \infty} \kappa_{p'}(E_N)$$

Proof. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Then for any positive integer N :

$$\begin{aligned} E_N &= \sum_{p \in A_k} \frac{\kappa^p(E_N)}{N^{nc(p) - nc(p \vee id)}} p^c = \sum_{p \in A_k} \frac{\kappa^p(E_N)}{N^{nc(p) - nc(p \vee id)}} \sum_{p' \in \mathcal{P}_k | p' \text{ coarser than } p} p'^c \\ &= \sum_{p \in A_k, p' \in \mathcal{P}_k | p' \text{ coarser than } p} \kappa^p(E_N) N^{-nc(p) + nc(p \vee id) + nc(p') - nc(p' \vee id)} \frac{p'^c}{N^{nc(p') - nc(p' \vee id)}}, \end{aligned}$$

and using Lemma 3.3:

$$E_N = \sum_{p' \in \mathcal{P}_k} \left(\sum_{p \in A_k, p \text{ finer than } p'} \kappa^p(E_N) N^{-\text{df}(p, p')} \right) \frac{p'^c}{N^{\text{nc}(p') - \text{nc}(p' \vee \text{id})}}.$$

Thus, for any integer N , for any $p' \in \mathcal{P}_k$

$$(9) \quad \kappa_c^{p'}(E_N) = \sum_{p \in A_k, p \text{ finer than } p'} \kappa^p(E_N) N^{-\text{df}(p, p')}.$$

The result follows from this equality, the usual arguments already explained in Theorem 4.1 and Lemma 3.4. \square

Let us remark that, using Equality (9), one has the following proposition.

Proposition 4.1. — *Let A be either \mathfrak{S} or \mathfrak{B} . Let N be an integer, let $E \in \mathbb{C}[A_k(N)]$, for any $p \in A_k$:*

$$\kappa_c^p(E) = \kappa^p(E).$$

Proof. — This is a consequence of Equality (9) and the fact that p' in \mathcal{P}_k is finer than $p' \in A_k$ implies that $p' \notin A_k$. \square

4.3.2. Exclusive moments. — As we did for the coordinate numbers, one can define exclusive normalized moments.

Definition 4.6. — *Let $N \in \mathbb{N}$, let $p \in \mathcal{P}_k$ and $E \in \mathbb{C}[A_k(N)]$. The p -exclusive normalized moment of E is:*

$$m_{p^c}(E) = \frac{1}{\text{Tr}_N(p)} \text{Tr}_N(E \text{ }^t(p^c)).$$

One can also give a combinatorial definition of the p -exclusive normalized moment.

Lemma 4.1. — *Let p and p' be in \mathcal{P}_k , then:*

$$\text{Tr}_N(p \text{ }^t(p'^c)) = \delta_{p' \text{ coarser than } p} \frac{N!}{(N - \text{nc}(p'))!}.$$

The easiest way to prove this lemma is to do it graphically: we see that p' must be coarser than p , if not the trace is equal to zero, and if p' is coarser than p , it is equal to $\frac{N!}{(N - \text{nc}(p'))!}$.

Recall the Definition 3.8. Similarly to what we proved for coordinate numbers, we prove the following proposition. Let us consider $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Proposition 4.2. — *The sequence $(E_N)_{N \in \mathbb{N}}$ converges in normalized moments if and only if for any $p \in \mathcal{P}_k$, $(m_{p^c}(E_N))_{N \in \mathbb{N}}$ converges. Besides, if $(E_N)_{N \in \mathbb{N}}$ converges in normalized moments then for any $p \in \mathcal{P}_k$:*

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in \mathcal{P}_k, p' \in \text{Gl}_c(p)} \lim_{N \rightarrow \infty} m_{p'^c}(E_N).$$

Proof. — Because of Theorem 4.2, it is enough to consider $(E_N)_{N \in \mathbb{N}}$ an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$. By computation:

$$m_p(E_N) = \sum_{p' \text{ coarser than } p} N^{\text{nc}(p' \vee \text{id}) - \text{nc}(p \vee \text{id})} m_{p'}(E_N).$$

We are in the same setting as for the proof of Theorem 4.1: we can write this equality as:

$$m_N = C_N m_{c,N},$$

where $(m_N)_p = m_p(E_N)$, $(m_{c,N})_p = m_{p^c}(E_N)$ and C_N converges to the matrix C defined in Definition 3.11. With the same arguments than in the proof of Theorem 4.1, we get that m_N converges to infinity if and only if $m_{c,N}$ converges to infinity: the sequence $(E_N)_{N \in \mathbb{N}}$ converges in \mathcal{P}_k -exclusive normalized moments if and only if it converges in normalized moments and in this case:

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in \text{Gl}_c(p)} \lim_{N \rightarrow \infty} m_{p'}(E_N).$$

This finishes the proof. \square

4.3.3. In the exclusive world, coordinate numbers and moments are equal. — We will prove that the limit of exclusive normalized moments are in fact equal to the limit of the exclusive coordinate numbers. Let $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges in normalized moments.

Theorem 4.5. — *For any $p \in \mathcal{P}_k$,*

$$\lim_{N \rightarrow \infty} m_{p^c}(E_N) = \lim_{N \rightarrow \infty} \kappa_c^p(E_N).$$

Proof. — We will prove that for any integer N , any $p \in A_k$, seen as an element of $\mathbb{C}[A_k(N)]$, for any $p' \in \mathcal{P}_k$,

$$\kappa_c^{p'}(p) = \left(\prod_{i=0}^{\text{nc}(p')-1} \binom{N}{N-k} \right) m_{p'}(p).$$

Indeed by the Equality (9), we get that for any $p' \in \mathcal{P}_k$:

$$(10) \quad \kappa_c^{p'}(p) = \delta_{p' \text{ coarser than } p} N^{\text{nc}(p') - \text{nc}(p \vee \text{id})}.$$

Let $p' \in \mathcal{P}_k$, by Lemma 4.1:

$$m_{p'}(p) = \frac{1}{N^{\text{nc}(p' \vee \text{id})}} \text{Tr}_N(p^t(p'^c)) = \delta_{p' \text{ coarser than } p} \frac{N!}{(N - \text{nc}(p'))!} N^{-\text{nc}(p' \vee \text{id})}.$$

The theorem is now a simple consequence of a linearity argument and taking N going to infinity. \square

Let us remark that one can prove Theorem 4.5 also by a purely combinatorial argument using Theorem 3.3. Indeed, using similar notations as the one explained in Notation 4.1, the Theorem 4.4 shows that:

$$(\kappa_c^p(E))_{p \in \mathcal{P}_k} = S(\kappa^p(E))_{p \in \mathcal{P}_k}.$$

The Equation (7) asserts that:

$$(m_p(E))_{p \in \mathcal{P}_k} = G(\kappa^p(E))_{p \in \mathcal{P}_k} = CS(\kappa^p(E))_{p \in \mathcal{P}_k},$$

where the last equation is a consequence of Theorem 3.3. The Proposition 4.2 proves that:

$$(m_{p^c}(E))_{p \in \mathcal{P}_k} = C^{-1}(m_p(E))_{p \in \mathcal{P}_k}.$$

Thus:

$$(m_{p^c}(E))_{p \in \mathcal{P}_k} = S(\kappa^p(E))_{p \in \mathcal{P}_k} = (\kappa_c^p(E))_{p \in \mathcal{P}_k}.$$

Using this discussion, we see that one can give an expression of the exclusive moments which involves the coordinate numbers.

Theorem 4.6. — *For any $p \in \mathcal{P}_k$,*

$$\lim_{N \rightarrow \infty} m_{p^c}(E_N) = \sum_{p' \in A_k, p' \in \text{Sp}(p)} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

Using Lemma 3.6, one gets the following corollary.

Corollary 4.1. — *Let p be a partition in \mathcal{P}_k which does not have any pivotal block, then:*

$$(11) \quad \lim_{N \rightarrow \infty} m_{p^c}(E_N) = \delta_{p \in A_k} \lim_{N \rightarrow \infty} \kappa^p(E_N).$$

In particular, for any $p \in \mathcal{B}_k$, the Equality (11) is valid.

Recall the Definition 3.9, using Proposition 3.3, one gets also the following corollary.

Corollary 4.2. — *Let us suppose that A_k is equal either to \mathfrak{S}_k or \mathcal{B}_k . For any $p \in \mathcal{P}_k$:*

$$\lim_{N \rightarrow \infty} m_{p^c}(E_N) = \delta_{p \in \overline{A_k}} \lim_{N \rightarrow \infty} \kappa^{\text{Mb}(p)}(E_N).$$

Let us remark that some simple equalities hold also for finite N , since, for example, as a consequence of Equation 9 and the proof of Theorem 4.5, if A_k is equal either to \mathfrak{S}_k or \mathcal{B}_k , for any $N \in \mathbb{N}$ and any $p \in A_k$:

$$\kappa^p(E_N) = \left(\prod_{i=0}^{\text{nc}(p)-1} \binom{N}{N-k} \right) m_{p^c}(E_N).$$

At the beginning of this section, we have argued that the simplest notion of convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ was not interesting as it did not take into account the fact that $\mathbb{C}[A_k(N)]$ is an algebra which depends on the parameter N . In the Section 5, we will slightly modify the product defined on $\mathbb{C}[A_k(N)]$ in order to define a new algebra $\mathbb{C}[A_k(N, N)]$. In this new algebra the strong convergence will be the usual notion of convergence in vector spaces.

4.4. Projections and conjugation. — Let A_k be either \mathcal{B}_k or \mathfrak{S}_k . Recall the Definition 3.11 where we defined the matrices G , C and S . There exist two natural projections that one can define on $\mathbb{C}[\mathcal{P}_k]$. In order to define them, we need to define the restriction and extension applications.

Definition 4.7. — *The restriction on A_k is defined by:*

$$\begin{aligned} R_{|A_k} : \quad \mathbb{C}[\mathcal{P}_k] &\rightarrow \mathbb{C}[A_k] \\ (x_p)_{p \in \mathcal{P}_k} &\mapsto (x_p)_{p \in A_k}. \end{aligned}$$

The extension from A_k , denoted by E^{A_k} , is the unique application such that $R_{|A_k} \circ E^{A_k} = \text{Id}_{A_k}$ and $(E^{A_k}((x_p)_{p \in A_k}))_{p_0} = 0$ for any $p_0 \notin A_k$.

We define the restriction of G to A_k as:

$$G_{|A_k} = R_{|A_k} G E^{A_k}.$$

This is the matrix of the order \leq restrained to A_k .

Definition 4.8. — *The cumulant-projection on A is defined as the application:*

$$\mathcal{C}_\kappa^A : \mathbb{C}[\mathcal{P}_k] \rightarrow \mathbb{C}[\mathcal{P}_k]$$

such that:

$$\mathcal{C}_\kappa^A = E^{A_k} \circ (G_{|A_k})^{-1} \circ R_{|A_k} \circ G.$$

One can remark that $R_{|A} \circ G \circ E^{A_k} \circ (G_{|A_k})^{-1} = \text{Id}_{A_k}$.

Lemma 4.2. — *The cumulant-projection is a projection and $\text{Im}(\mathcal{C}_\kappa^A) = E^{A_k}(\mathbb{C}[A_k])$.*

Proof. — The first assertion is a consequence of the following computation:

$$\begin{aligned} \mathcal{C}_\kappa^A \circ \mathcal{C}_\kappa^A &= E^{A_k} \circ (G_{|A_k})^{-1} \circ R_{|A_k} \circ G \circ E^{A_k} \circ (G_{|A_k})^{-1} \circ R_{|A_k} \circ G \\ &= E^{A_k} \circ (G_{|A_k})^{-1} \circ (R_{|A_k} \circ G \circ E^{A_k} \circ (G_{|A_k})^{-1}) \circ R_{|A_k} \circ G \\ &= E^{A_k} \circ (G_{|A_k})^{-1} \circ R_{|A_k} \circ G = \mathcal{C}_\kappa^A. \end{aligned}$$

Besides, if $x \in E^{A_k}(\mathbb{C}[A_k])$, there exists $y \in \mathbb{C}[A_k]$ such that $x = E^{A_k}(y)$. This implies that $\mathcal{C}_\kappa^A(x) = \mathcal{C}_\kappa^A(E^{A_k}(y)) = E^{A_k} \circ (G_{|A_k})^{-1} \circ R_{|A_k} \circ G \circ E^{A_k}(y) = E^{A_k} \circ (G_{|A_k})^{-1} \circ G_{|A_k}(y) = E^{A_k}(y) = x$. Thus $E^{A_k}(\mathbb{C}[A_k]) \subset \text{Im}(\mathcal{C}_\kappa^A)$. It is clear also that $\text{Im}(\mathcal{C}_\kappa^A) \subset E^{A_k}(\mathbb{C}[A_k])$, hence the equality between the r.h.s. and the l.h.s. \square

Definition 4.9. — *The moment-projection on A is defined as the application:*

$$\mathcal{C}_m^A : \mathbb{C}[\mathcal{P}_k] \rightarrow \mathbb{C}[\mathcal{P}_k]$$

such that:

$$\mathcal{C}_m^A = G \circ \mathcal{C}_\kappa^A \circ G^{-1}.$$

Since \mathcal{C}_κ^A is a projection, \mathcal{C}_m^A is also a projection which image is given by $G \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$. It can also be interesting to define $\mathcal{C}_{m^c}^A$ by $\mathcal{C}_{m^c}^A = S \circ \mathcal{C}_\kappa^A \circ S^{-1}$. This is again a projection which image is given by $S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$. In Lemma 4.4, we characterize the set $S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$.

By looking at the definition of the cumulant and moment-projections, one can see that one needs to compute the inverse of $G|_{A_k}$: this amounts to compute the Möbius function for (A_k, \leq) . If A is equal to \mathfrak{S} , this Möbius functions is well-known since one can use the fact that (A_k, \leq) is isomorphic to the poset of non-crossing partitions for which the Möbius function has been computed by Kreweras.

Definition 4.10. — Let $(m_p)_{p \in \mathcal{P}_k}$ be a element of $\mathbb{C}[\mathcal{P}_k]$. We say that $(m_p)_{p \in \mathcal{P}_k}$ is G -invariant if:

$$(m_p)_{p \in \mathcal{P}_k} = \mathcal{C}_m^A((m_p)_{p \in \mathcal{P}_k}).$$

Using the fact that \mathcal{C}_m^A is a projection and that $G = CS$, we get the following lemma.

Lemma 4.3. — Let $(m_p)_{p \in \mathcal{P}_k}$ be a element of $\mathbb{C}[\mathcal{P}_k]$. It is G -invariant if and only if one of the following conditions is satisfied:

- $(m_p)_{p \in \mathcal{P}_k} \in G \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$,
- $G^{-1}(m_p)_{p \in \mathcal{P}_k} \in \mathbf{E}^{A_k}(\mathbb{C}[A_k])$,
- $C^{-1}(m_p)_{p \in \mathcal{P}_k} \in S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$.

The three conditions have to be understood as conditions on respectively the moments, the coordinate numbers, and the exclusive moments: this is why we wrote the three conditions even if they are obviously equivalent. Let us also remark that the sets $\mathbf{E}^{A_k}(\mathbb{C}[A_k])$ and $S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$ are easy to understand. Recall Definition 3.9 where we defined $\text{Mb}(p)$.

Lemma 4.4. — The set $\mathbf{E}^{A_k}(\mathbb{C}[A_k])$ is the set of elements x of $\mathbb{C}[\mathcal{P}_k]$ such that for any $p \notin A_k$, $x_p = 0$.

The set $S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$ is the set of elements x in $\mathbb{C}[\mathcal{P}_k]$ such that for any $p \in \mathcal{P}_k$, $x_p = \delta_{p \in \overline{A_k}} x_{\text{Mb}(p)}$.

Proof. — The first assertion is straightforward. The second is a direct consequence of Proposition 3.3 and Lemma 3.6. Indeed, if $x \in S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$, there exists $y \in \mathbb{C}[A_k]$ such that for any $p \in \mathcal{P}_k$:

$$x_p = \sum_{p' \in A_k | p' \sqsupset p} y_{p'}.$$

If $p \in A_k$, using Lemma 3.6, $x_p = y_p$ and if $p \notin A_k$, using Proposition 3.3, $x_p = y_{\text{Mb}(p)} = x_{\text{Mb}(p)}$. Using the same arguments, one can show that if x in $\mathbb{C}[\mathcal{P}_k]$ satisfies that for any $p \in \mathcal{P}_k$, $x_p = \delta_{p \in \overline{A_k}} x_{\text{Mb}(p)}$, then $x \in S \circ \mathbf{E}^{A_k}(\mathbb{C}[A_k])$. \square

Let us consider (A, G) an element of $\{(\mathfrak{S}, U), (\mathcal{B}, O), (\mathcal{P}, \mathfrak{S})\}$. For any positive integer N , $U(N)$, $O(N)$ and $\mathfrak{S}(N)$ are respectively the unitary group of size N , the orthogonal group of size N and the group of permutation matrices of size N . The notion of G -invariant element of $\mathbb{C}[\mathcal{P}_k]$ in Definition 4.10 is motivated by the following proposition.

Proposition 4.3. — Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ which converges when N goes to infinity. For any positive integer N , we define

$$F_N = \int_{G(N)} g^{\otimes k} \rho_N^{\mathcal{P}_k}(E_N) (g^*)^{\otimes k} dg,$$

where dg is the Haar probability measure on $G(N)$. There exists a sequence $(\tilde{E}_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ such that for any positive integer N : $\rho_N^{\mathcal{P}_k}(\tilde{E}_N) = F_N$. The sequence $(\tilde{E}_N)_{N \in \mathbb{N}}$ converges as N goes to infinity and:

$$\begin{aligned} \left(\kappa^{\mathcal{P}}(\tilde{E}) \right)_{p \in \mathcal{P}_k} &= \mathcal{C}_\kappa^A \left((\kappa^{\mathcal{P}}(E))_{p \in \mathcal{P}_k} \right), \\ \left(m_p(\tilde{E}) \right)_{p \in \mathcal{P}_k} &= \mathcal{C}_m^A \left((m_p(E))_{p \in \mathcal{P}_k} \right), \\ \left(m_{p^c}(\tilde{E}) \right)_{p \in \mathcal{P}_k} &= \mathcal{C}_{m^c}^A \left((m_{p^c}(E))_{p \in \mathcal{P}_k} \right). \end{aligned}$$

Proof. — Let us consider $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ two sequences which satisfy the assumptions of the proposition. The existence of a sequence $(\tilde{E}_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ such that for any positive integer N : $\rho_N^{\mathcal{P}_k}(\tilde{E}_N) = F_N$ is a consequence of the Schur-Weyl-Jones dualities which are explained in the Section 3 of [9]. Let us prove that $(\tilde{E}_N)_{N \in \mathbb{N}}$ converges as N goes to infinity and that the two equalities stated in the proposition are satisfied.

In order to do so, it is enough to show that the normalized moments of $(\tilde{E}_N)_{N \in \mathbb{N}}$ converge as N goes to infinity. Let p be in A_k , we have to prove that $m_p(\tilde{E}_N)$ converges as N goes to infinity. Yet, for any positive integer N :

$$\begin{aligned} m_p(\tilde{E}_N) &= \frac{1}{N^{\text{nc}(p \vee \text{id})}} \text{Tr}^k \left(\rho_N^{\mathcal{P}_k}(\tilde{E}_N) \rho_N^{\mathcal{P}_k}(t_p) \right) \\ &= \frac{1}{N^{\text{nc}(p \vee \text{id})}} \text{Tr}^k \left(\left(\int_{G(N)} g^{\otimes k} \rho_N^{\mathcal{P}_k}(E_N) (g^*)^{\otimes k} dg \right) \rho_N^{\mathcal{P}_k}(t_p) \right) \\ &= \frac{1}{N^{\text{nc}(p \vee \text{id})}} \int_{G(N)} \text{Tr}^k \left(\rho_N^{\mathcal{P}_k}(E_N) (g^*)^{\otimes k} \rho_N^{\mathcal{P}_k}(t_p) g^{\otimes k} \right) dg \end{aligned}$$

By the Schur-Weyl-Jones dualities, for any $g \in G_N$, $(g^*)^{\otimes k} \rho_N^{\mathcal{P}_k}(t_p) g^{\otimes k} = p$, thus:

$$m_p(\tilde{E}_N) = \frac{1}{N^{\text{nc}(p \vee \text{id})}} \text{Tr}^k \left(\rho_N^{\mathcal{P}_k}(E_N) \rho_N^{\mathcal{P}_k}(t_p) \right) = m_p(E_N).$$

This implies that $m_p(\tilde{E}_N)$ converges as N goes to infinity, and:

$$\left(m_p(\tilde{E}) \right)_{p \in A_k} = \mathbf{R}_{|A_k} \left((m_p(E))_{p \in \mathcal{P}_k} \right).$$

Considering this last equality with the following equalities:

$$\begin{aligned} \left(\kappa^p(\tilde{E})\right)_{p \in \mathcal{P}_k} &= \mathbf{E}^{A_k} \left((\kappa^p(\tilde{E}))_{p \in A_k} \right), \\ \left(\kappa^p(\tilde{E})\right)_{p \in A_k} &= (G|_{A_k})^{-1} \left((m_p(\tilde{E}))_{p \in A_k} \right), \\ \left(m_p(E)\right)_{p \in \mathcal{P}_k} &= G \left((\kappa^p(E))_{p \in \mathcal{P}_k} \right), \end{aligned}$$

one gets:

$$\left(\kappa^p(\tilde{E})\right)_{p \in \mathcal{P}_k} = \mathbf{E}^{A_k} \circ (G|_{A_k})^{-1} \circ R|_{A_k} \circ G \left((\kappa^p(E))_{p \in \mathcal{P}_k} \right),$$

which is nothing but the first equality we had to prove. In order to prove the second one, one can do the following computations:

$$\begin{aligned} \left((m_p(\tilde{E}))_{p \in \mathcal{P}_k} \right) &= G \left((\kappa^p(\tilde{E}))_{p \in \mathcal{P}_k} \right) = G \circ \mathcal{C}_\kappa^A \left((\kappa^p(E))_{p \in \mathcal{P}_k} \right) \\ &= G \circ \mathcal{C}_\kappa^A \circ G^{-1} \left((m_p(E))_{p \in \mathcal{P}_k} \right) \\ &= \mathcal{C}_m^A \left((m_p(E))_{p \in \mathcal{P}_k} \right). \end{aligned}$$

The third one is a consequence of the fact that $\left((m_{p^c}(\tilde{E}))_{p \in \mathcal{P}_k} \right) = S \left((\kappa^p(\tilde{E}))_{p \in \mathcal{P}_k} \right)$. \square

5. The deformed partition algebra

Let us define a deformation of the partition algebra by modifying the multiplication which was set in Definition 2.5. Let k and N be two positive integers.

Definition 5.1. — *We define the application:*

$$\begin{aligned} M_k^N : A_k &\rightarrow A_k \\ p &\mapsto \frac{1}{N d(id, p) + \frac{-k + nc(p)}{2}} p = \frac{1}{N nc(p) - nc(p \vee id)} p. \end{aligned}$$

This application can be extended as an isomorphism of vector spaces from $\mathbb{C}[A_k]$ to itself.

Seen as a vector space, the algebra $\mathbb{C}[A_k(N)]$ is isomorphic to $\mathbb{C}[A_k]$. Thus, we can see M_k^N as an isomorphism of vector spaces from $\mathbb{C}[A_k]$ to $\mathbb{C}[A_k(N)]$. Let us endow $\mathbb{C}[A_k]$ with a structure of associative algebra by taking the pullback of the structure of algebra of $\mathbb{C}[A_k(N)]$ by M_k^N : for any p_1, p_2 in A_k the new product of p_1 with p_2 is given by:

$$p_1 \cdot_N p_2 = (M_k^N)^{-1} [M_k^N(p_1) M_k^N(p_2)].$$

This is the deformed algebra $\mathbb{C}[A_k(N, N)]$. Using the definition of M_k^N , one gets the following proposition.

Proposition 5.1. — *The deformed algebra $\mathbb{C}[A_k(N, N)]$ is the associative algebra over \mathbb{C} with basis \mathcal{P}_k , endowed with the multiplication defined by the fact that for any $p_1, p_2 \in A_k$:*

$$p_1 \cdot_N p_2 = N^{\kappa(p_1, p_2)} N^{d(id, p_1 \circ p_2) - d(id, p_1) - d(id, p_2) + \frac{k + nc(p_1 \circ p_2) - nc(p_1) - nc(p_2)}{2}} (p_1 \circ p_2).$$

One can write the exponent in an other form so that it looks like a triangle inequality.

Lemma 5.1. — *Let p and p' in A_k . We have the equality:*

$$\begin{aligned} d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} + \kappa(p, p') \\ = d({}^t p', p) - d(id, p) - d(id, p') + k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') + 2\kappa(p, p'). \end{aligned}$$

Proof. — In order to prove this equality, for exemple one can consider N to the power to the r.h.s and the l.h.s. and one can use the following equality:

$$N^{-d({}^t p', p)} = \frac{\text{Tr}(pp')}{N^{\frac{\text{nc}(p) + \text{nc}(p')}{2}}} = N^{\kappa(p, p')} \frac{\text{Tr}(p \circ p')}{N^{\frac{\text{nc}(p) + \text{nc}(p')}{2}}} = N^{\kappa(p, p')} \frac{N^{-d(id, p \circ p') + \frac{k + \text{nc}(p \circ p')}{2}}}{N^{\frac{\text{nc}(p) + \text{nc}(p')}{2}}}.$$

This allows us to prove Lemma 5.1. \square

Using the definition of the deformed algebra $\mathbb{C}[A_k(N, N)]$, we have the straightforward proposition.

Proposition 5.2. — *The application M_k^N can be extended as an isomorphism of algebra from $\mathbb{C}[A_k(N, N)]$ to $\mathbb{C}[A_k(N)]$. Its extension will be also denoted by M_k^N .*

For any integer N , the deformed algebra $\mathbb{C}[A_k(N, N)]$ is isomorphic to $\mathbb{C}[A_k(N)]$.

Actually, the application M_k^N is not only compatible with the multiplication, but also with the \otimes operation defined in Definition 2.1.

Lemma 5.2. — *Let k and k' be two positive integers. Let $p \in A_k$ and $p' \in A_{k'}$. The following equality holds:*

$$(12) \quad M_{k+k'}^N(p \otimes p') = M_k^N(p) \otimes M_{k'}^N(p').$$

The definition of the morphism M_k^N was not chosen randomly: it was set so that the following lemma holds.

Lemma 5.3. — *Let $E \in \mathbb{C}[A_k(N)]$, we have:*

$$(M_k^N)^{-1}(E) = \sum_{p \in A_k} \kappa^p(E)p.$$

Thus, one can see that we will be able to formulate the strong convergence in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ by using the morphisms $(M_k^N)_{N \in \mathbb{N}}$ and the usual notion of convergence in vector spaces. Indeed, for any integers N and k , any element in $\mathbb{C}[A_k(N, N)]$ can be considered as an element of $\mathbb{C}[A_k]$. This allows us to state the following lemma.

Lemma 5.4. — *Let $(E_N)_{N \in \mathbb{N}}$ be an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. The sequence $(E_N)_{N \in \mathbb{N}}$ converges strongly if and only if:*

$$(M_k^N)^{-1}(E_N)$$

converges when N goes to infinity in $\mathbb{C}[A_k]$ for the usual convergence in finite dimensional vector spaces.

6. Refined geometry of the partition algebra

In the last section, we defined the deformed algebra $\mathbb{C}[A_k(N, N)]$ and we explained that the strong convergence can be seen as the natural notion of convergence in finite dimensional vector space as soon as one works in the deformed algebra. In this section, we will study the convergence of the algebras $\mathbb{C}[A_k(N, N)]$.

The core of Section 3 was to prove the triangle inequality for the function d defined on A_k in Definition 3.1. The study of the convergence of the algebras $\mathbb{C}[A_k(N, N)]$ will use intensively the following improved triangle inequality for A_k .

Proposition 6.1. — *Let p and p' be two elements of \mathcal{P}_k , the following improved triangle inequality holds:*

$$d(p', p) \leq d(p', id) + d(p, id) - k - \text{nc}(p \circ {}^t p') + \text{nc}(p) + \text{nc}(p') - 2\kappa(p, {}^t p').$$

The restriction of the improved triangle inequality to the permutations is obvious as it is a consequence of the usual triangle inequality. Indeed, for any permutations σ and σ' , $\text{nc}(\sigma) = 0$ and $\kappa(p, p') = 0$. Yet, this is indeed an improved triangle inequality as soon as one considers elements on \mathcal{B}_k : let us suppose that p and p' are equal to the Weyl contraction $[1, 2]$. The triangle inequality asserts that $0 \leq 2$, since $d(id, [1, 2]) = 1$. Yet, in this case:

$$d(p', id) + d(p, id) - k - \text{nc}(p \circ {}^t p') + \text{nc}(p) + \text{nc}(p') - 2\kappa(p, {}^t p') = 0.$$

The improved triangle inequality asserts thus the stronger fact that $0 \leq 0$.

In fact, we can see this improved triangle inequality as a consequence of the usual triangle inequality and an inequality between $d(p, p \circ p')$ and $d(id, p')$. If we consider p and p' in the symmetric group, then we know that $d(p, p \circ p') = d(p, pp') = d(id, p')$. Yet, this equality does not hold any more in the general case, we only get the following inequality.

Proposition 6.2. — *Let p and p' in \mathcal{P}_k , we have the following inequality:*

$$d(p, p \circ p') \leq d(id, p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

Proof. — Let p and p' in \mathcal{P}_k and let us define $\tau \in \mathfrak{S}_{2k}$:

$$\tau_k = (1, k+1)(2, k+2) \dots (k, 2k).$$

Let us apply the triangle inequality:

$$(13) \quad d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau) \leq d(p \otimes id_k, p \otimes p') + d(p \otimes p', ((p \circ p') \otimes id_k) \tau).$$

The goal is to understand each of these three terms. The term $d(p \otimes id_k, p \otimes p')$ is simple:

$$d(p \otimes id_k, p \otimes p') = d(id, p').$$

Let us study $d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau)$. Using the definition of the distance in Proposition 3.1, and the Equality (3):

$$N^{-d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau)} = \frac{\text{Tr}_N [(p \otimes id_k) {}^t((p \circ p') \otimes id_k) \tau]}{N^{\frac{\text{nc}(p)+k}{2}} N^{\frac{\text{nc}(p \circ p')+k}{2}}},$$

since $\text{nc}(p \otimes id_k) = \text{nc}(p) + k$ and $\text{nc}((p \circ p') \otimes id_k) \tau = \text{nc}((p \circ p') \otimes id_k) = \text{nc}(p \circ p') + k$. Yet:

$$\text{Tr}_N [(p \otimes id_k)^t ((p \circ p') \otimes id_k) \tau] = \text{Tr}_N [p^t (p \circ p')].$$

Thus, using again Proposition 3.1:

$$d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau) = d(p, p \circ p') + k.$$

Let us consider $d(p \otimes p', ((p \circ p') \otimes id_k) \tau)$. Using the same arguments:

$$N^{-d(p \otimes p', ((p \circ p') \otimes id_k) \tau)} = \frac{\text{Tr}_N (pp'^t (p \circ p'))}{N^{\frac{\text{nc}(p) + \text{nc}(p')}{2}} N^{\frac{\text{nc}(p \circ p') + k}{2}}}.$$

Using the definition of $\kappa(p, p')$ and the Equality (3):

$$\begin{aligned} N^{-d(p \otimes p', ((p \circ p') \otimes id_k) \tau)} &= N^{\kappa(p, p')} \frac{\text{Tr}_N (p \circ p'^t (p \circ p'))}{N^{\frac{\text{nc}(p) + \text{nc}(p')}{2}} N^{\frac{\text{nc}(p \circ p') + k}{2}}} \\ &= N^{\kappa(p, p') + \frac{1}{2}[\text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') - k]}. \end{aligned}$$

Thus:

$$d(p \otimes p', ((p \circ p') \otimes id_k) \tau) = -\kappa(p, p') - \frac{1}{2}[\text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') - k].$$

Let us come back to the triangle inequality (13). This shows that:

$$d(p, p \circ p') + k \leq d(id, p') - \kappa(p, p') - \frac{1}{2}[\text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') - k],$$

and thus:

$$d(p, p \circ p') \leq d(id, p') - \frac{\text{nc}(p \circ p') + k - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

This is the inequality we wanted to prove. \square

Proof of Proposition 6.1. — Let p and p' be two elements of A_k . Using the triangle inequality:

$$d(id, p \circ p') \leq d(id, p) + d(p, p \circ p').$$

And an application of Proposition 6.2 implies that:

$$(14) \quad d(id, p \circ p') \leq d(id, p) + d(id, p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

And using Lemma 5.1:

$$d({}^t p', p) \leq d(id, p) - d(id, p') + k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') + 2\kappa(p, p').$$

The result follows then from the fact that $\text{nc}({}^t p') = \text{nc}(p')$. \square

We can generalize the inequality (14) to a n -uple of elements of A_k .

Lemma 6.1. — *For any positive integer n , for any $(p_i)_{i=1}^n \in (A_k)^n$:*

$$d(id, \circ_{i=1}^n p_i) \leq \sum_{i=1}^n d(id, p_i) - \frac{1}{2} \left[(n-1)k + \text{nc}(\circ_{i=1}^n p_i) - \sum_{i=1}^n \text{nc}(p_i) \right] - \sum_{i=1}^{n-1} \kappa(p_i, p_{i+1}),$$

where we have used the notation $\circ_{i=1}^n p_i = p_1 \circ \dots \circ p_n$.

In fact, the best way to understand the improved triangle inequality is to work with the equivalent inequality (14). This formulation of the improved triangle inequality leads us to the next notion.

Definition 6.1. — Let p and p' be two elements of A_k . We will say that $p \prec p \circ p'$ if and only if:

$$d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} + \kappa(p, p') = 0.$$

Let $p_0 \in A_k$. We will write that $p \prec p_0$ if there exists $p' \in A_k$ such that $p_0 = p \circ p'$ and $p \prec p \circ p'$.

Definition 6.2. — Let us suppose that $p \prec p_0$. We define for any $p \prec p_0$:

$$K_{p_0}(p) = \{p' \in A_k, p \circ p' = p_0\}.$$

Let us suppose that $p \prec p \circ p'$. We recall that:

$$\begin{aligned} d(id, p \circ p') &\leq d(id, p) + d(p, p \circ p') \\ &\leq d(id, p) + d(id, p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p'). \end{aligned}$$

Thus, if the first term and the third term are equal, then $p \in [id, p \circ p']_{A_k}$. We have shown the following lemma.

Lemma 6.2. — Let p and p_0 in A_k , if $p \prec p_0$ then there exists $p' \in A_k$ such that $p_0 = p \circ p'$ and

$$p \in [id, p_0]_{A_k}.$$

Let us remark that for any $\sigma \in \mathfrak{S}_k$, $\{\sigma' \in \mathfrak{S}_k, \sigma' \prec \sigma\} = [id, \sigma]_{\mathfrak{S}_k}$. This is due to the fact that $\kappa(\sigma, \sigma') = 0$ for any couple of permutations, the fact that nc is constant on the set of permutations and the fact that any permutation is invertible. Using similar arguments and Lemma 2.1, one can have the better result.

Lemma 6.3. — Let k be an integer. Let $\sigma \in \mathfrak{S}_k$, then:

$$\{p \in \mathcal{P}_k, p \prec \sigma\} = [id, \sigma]_{\mathfrak{S}_k}.$$

Besides, for any $p \in \mathcal{P}_k$, $\{\sigma \in \mathfrak{S}_k, \sigma \prec p\} = [id, p]_{\mathcal{P}_k} \cap \mathfrak{S}_k$.

Let us state a consequence of Lemma 6.2: the factorization property for \prec .

Lemma 6.4. — Let k and l be two positive integers. Let $a \in \mathcal{P}_k$ and $b \in \mathcal{P}_l$. For any $p \in \mathcal{P}_{k+l}$ such that $p \prec a \otimes b$, there exist $p_1 \prec a$ and $p_2 \prec b$ such that $p = p_1 \otimes p_2$.

This lemma is a consequence of Lemma 6.2 and the factorization property for the geodesics stated in Lemma 3.8.

Let p and p_0 in A_k such that $p \prec p_0$. Let us have a little discussion on $K_{p_0}(p)$: by definition this is not empty but it is not reduced to a unique partition. For example, one can show that if $p = \{\{1, 2, 1', 2'\}\}$ and $p_0 = \{\{1', 2'\}, \{1\}, \{2\}\}$ then:

$$K_{p_0}(p) = \{\{\{1\}, \{2\}, \{1'\}, \{2'\}\}, \{\{1\}, \{2\}, \{1', 2'\}\}\}.$$

Let $(1, \dots, k)$ be the k -cycle in \mathfrak{S}_k that we already defined in Definition 2.10. It is well-known that the poset of non-crossing partition over $\{1, \dots, k\}$ is isomorphic to $([id, (1, \dots, k)]_{\mathfrak{S}_k}, \leq)$. From now on, we will consider any non-crossing partition over $\{1, \dots, k\}$ as an element of $[id, (1, \dots, k)]_{\mathfrak{S}_k}$. The following lemma is straightforward.

Lemma 6.5. — *If $p_0 = (1, \dots, k)$ and $p \in [id, (1, \dots, k)]_{\mathfrak{S}_k}$, then $K_{p_0}(p)$ is the Kreweras complement of the non-crossing partition corresponding to p .*

We are going now to see one of the main results of the paper, namely the fact that the improved triangle inequality implies the convergence of the deformed algebras $(\mathbb{C}[A_k(N, N)])_{N \in \mathbb{N}}$ stated in the forthcoming Theorem 6.1. Before doing so, we need to define the notion of convergence of algebras.

Definition 6.3. — *Let C be a finite set of elements. For any $N \in \mathbb{N} \cup \{\infty\}$, let L_N be an algebra such that C is a linear basis of L_N . For any elements x and y of C , for each $N \in \mathbb{N} \cup \{\infty\}$, we denote the product of x with y in L_N by $x \cdot_N y$.*

We say that L_N converges to the algebra L_∞ when N goes to infinity if for any x and y in C ,

$$x \cdot_N y \xrightarrow{N \rightarrow \infty} x \cdot_\infty y \text{ in } \mathbb{C}[C],$$

for the usual notion of convergence in finite dimensional linear spaces.

Let us state the convergence of the deformed algebras $(\mathbb{C}[A_k(N, N)])_{N \in \mathbb{N}}$.

Theorem 6.1. — *As N goes to infinity, the deformed algebra $\mathbb{C}[A_k(N, N)]$ converges to the deformed algebra $\mathbb{C}[A_k(\infty, \infty)]$ which is the associative algebra over \mathbb{C} with basis A_k endowed with the multiplication defined by:*

$$\forall p, p' \in \mathcal{P}_k, p \cdot_\infty p' = \delta_{p \prec p \circ p'} p \circ p'.$$

Proof. — For any $N \in \mathbb{N} \cup \{\infty\}$, A_k is a linear basis of $\mathbb{C}[A_k(N, N)]$. By bi-linearity of the product, it is enough to prove that for any p and p' in A_k , $p \cdot_N p'$ converges to $\delta_{p \prec p \circ p'} p \circ p'$. Let p and p' be two elements of \mathcal{P} . We have:

$$p \cdot_N p' = N^{d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} + \kappa(p, p')} (p \circ p').$$

By the version of the improved triangle inequality stated in Proposition 6.1 or in the inequality (14), we have:

$$d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} + \kappa(p, p') \leq 0.$$

According to Definition 6.1, we have $p \cdot_N p' \xrightarrow{N \rightarrow \infty} \delta_{p \prec p \circ p'} p \circ p'$. □

To conclude this section, let us remark that for any integer k , we have the inclusion of algebras:

$$\mathbb{C}[\mathfrak{S}_k(\infty, \infty)] \subset \mathbb{C}[\mathcal{B}_k(\infty, \infty)] \subset \mathbb{C}[\mathcal{P}_k(\infty, \infty)].$$

7. Consequences of the convergence of the deformed algebras.

7.1. Convergence of a product. — Let k be an integer. As usual, let A_k be \mathfrak{S}_k , \mathcal{B}_k or \mathcal{P}_k . Let us give the first consequence of Theorem 6.1 for the product of two elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Recall the Notation 4.1.

Theorem 7.1. — *Let $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ be two elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ converge, then the sequence $(E_N F_N)_{N \in \mathbb{N}}$ converges. Besides,*

– for any $p_0 \in A_k$:

$$(15) \quad \kappa^{p_0}(EF) = \sum_{p \in A_k, p \prec p_0} \kappa^p(E) \kappa^{K_{p_0}(p)}(F),$$

– for any $p_0 \in \mathcal{P}_k$:

$$(16) \quad m_{p_0}(EF) = \sum_{p \in A_k, p \leq p_0} \kappa^p(E) m_{p \circ p_0}(F).$$

Proof. — Let $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ converge. We have by definition:

$$(M_k^N)^{-1}(E_N F_N) = (M_k^N)^{-1}(E_N) \cdot_N (M_k^N)^{-1}(F_N).$$

We know, by Lemma 5.4, that $(M_k^N)^{-1}(E_N)$ and $(M_k^N)^{-1}(F_N)$, seen as elements of $\mathbb{C}[A_k]$, converge when N goes to infinity. Besides, the algebra $\mathbb{C}[A_k(N, N)]$ converges to $\mathbb{C}[A_k(\infty, \infty)]$, as it was proved in Theorem 6.1. Thus $(M_k^N)^{-1}(E_N F_N)$ converges when N goes to infinity. Again, by Lemma 5.4 and Theorem 4.1, this shows that $(E_N F_N)_{N \in \mathbb{N}}$ converges. Besides, using Lemma 5.3, we have:

$$\begin{aligned} (M_k^N)^{-1}(E_N F_N) &= \sum_{p \in A_k} \kappa^{p_0}(E_N F_N) p_0, \\ (M_k^N)^{-1}(E_N) \cdot_N (M_k^N)^{-1}(F_N) &= \sum_{p \in A_k, p' \in A_k} \kappa^p(E_N) \kappa^{p'}(F_N) p \cdot_N p'. \end{aligned}$$

Using the formula for the limit of \cdot_N shown in Theorem 6.1, for any $p_0 \in \mathcal{P}_k$:

$$\kappa^{p_0}(EF) = \sum_{p \in A_k, p \prec p_0} \kappa^p(E) \kappa^{K_{p_0}(p)}(F).$$

For the second equality, one could use the link, between A_k -moments and coordinate numbers when $N \rightarrow \infty$ given by Equality (7). Yet, this happens to be more difficult than a direct proof. Indeed, by bi-linearity, we have only to show that the equality (16) holds when, for any integer N :

$$E_N = \frac{1}{N^{\frac{-k+nc(p)}{2} + d(id,p)}} p,$$

with $p \in A_k$. Let N be an integer, let us suppose that E_N is of this form. Let $p_0 \in \mathcal{P}_k$, we have:

$$\begin{aligned} m_{p_0}(E_N F_N) &= \frac{1}{N^{\text{nc}(p) - \text{nc}(p \vee id)}} \frac{1}{N^{\text{nc}(p_0 \vee id)}} \text{Tr}(F_N {}^t p_0 p) \\ &= N^{-\text{nc}(p) + \text{nc}(p \vee id) - \text{nc}(p_0 \vee id)} N^{\kappa({}^t p_0, p)} \text{Tr}(F_N {}^t p_0 \circ p) \\ &= N^{-\text{nc}(p) + \text{nc}(p \vee id) - \text{nc}(p_0 \vee id)} N^{\kappa({}^t p_0, p) + \text{nc}({}^t p_0 \circ p \vee id)} m_{t_{p_0 p_0}}(F_N) \\ &= N^{-\text{nc}(p) + \text{nc}(p \vee id) - \text{nc}(p_0 \vee id)} N^{\kappa({}^t p_0, p) + \text{nc}({}^t p_0 \circ p \vee id)} m_{t_{p_0 p_0}}(F_N). \end{aligned}$$

Yet, we have seen in the proof of Lemma 3.1, that:

$$\text{nc}({}^t p_0 \circ p \vee id) + \kappa({}^t p_0, p) = \text{nc}(p_0 \vee p),$$

thus:

$$\begin{aligned} m_{p_0}(E_N F_N) &= N^{-\text{nc}(p) + \text{nc}(p \vee id) - \text{nc}(p_0 \vee id) + \text{nc}(p_0 \vee p)} m_{t_{p_0 p_0}}(F_N) \\ &= N^{-\text{df}(p, p_0)} m_{t_{p_0 p_0}}(F_N). \end{aligned}$$

One gets that $m_{p_0}(E_N F_N)$ converges when N goes to infinity to $\delta_{p \leq p_0} m_{t_{p_0 p_0}}(F_N)$. \square

Remark 7.1. — *Under the same assumptions, one can prove that for any $p_0 \in \mathcal{P}_k$:*

$$(17) \quad m_{p_0}(EF) = \sum_{p \in A_k, p \leq p_0} m_{p_0 \circ {}^t p}(E) \kappa^p(F).$$

7.2. Semi-groups. — In this subsection, we will study convergence of sequences of semi-groups in $\mathbb{C}[A_k(N)]$. Semi-groups in different algebras will appear in the paper: for this paper, a family $(a_t)_{t \geq 0}$ is a semi-group if there exists h , called the generator, such that for any $t_0 \geq 0$:

$$\frac{d}{dt} \Big|_{t=t_0} a_t = h a_{t_0}.$$

If we consider the algebra $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$, we are led to the next definition.

Definition 7.1. — *The family $((E_t^N)_N)_{t \geq 0}$ is a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ if there exists $(H_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$, called the generator, such that for any $t \geq 0$, for any integer N :*

$$\frac{d}{dt} \Big|_{t=t_0} E_t^N = H_N E_{t_0}^N.$$

From now on, let us suppose that $((E_t^N)_N)_{t \geq 0}$ is a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ whose generator is $(H_N)_{N \in \mathbb{N}}$. Let us define the convergence for semi-groups in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 7.2. — *The semi-group $((E_t^N)_N)_{t \geq 0}$ converges if and only if for any $t \geq 0$, E_t^N converges as N goes to infinity.*

The next theorem shows that a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ converges if the initial condition and the generator converge. Recall the Notation 4.1.

Theorem 7.2. — *The semi-group $((E_t^N)_N)_{t \geq 0}$ converges if the sequences $(E_0^N)_{N \in \mathbb{N}}$ and $(H_N)_{N \in \mathbb{N}}$ converge as N goes to infinity.*

Besides, we have the two differential systems of equations:

$$(18) \quad \forall p \in A_k, \forall t_0 \geq 0, \quad \frac{d}{dt}|_{t=t_0} \kappa^p(E_t) = \sum_{p_1 \in A_k, p_1 \prec p} \kappa^{p_1}(H) \kappa^{K_p(p_1)}(E_{t_0}).$$

$$(19) \quad \forall p \in \mathcal{P}_k, \forall t_0 \geq 0, \quad \frac{d}{dt}|_{t=t_0} m_p(E_t) = \sum_{p_1 \in A_k, p_1 \leq p} \kappa^{p_1}(H) m_{t_{p_1 \circ p}}(E_{t_0}).$$

Proof. — Let us suppose that $(H_N)_{N \in \mathbb{N}}$ converges. For any integer N and any $t \geq 0$, we define:

$$\begin{aligned} \tilde{E}_t^N &= (M_k^N)^{-1}(E_t^N), \\ \tilde{H}_N &= (M_k^N)^{-1}(H_N). \end{aligned}$$

As for any integer N , M_k^N is a morphism of algebra, the family $((\tilde{E}_t^N)_{N \in \mathbb{N}})_{t \geq 0}$ is a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N, N)]$ and its generator is $(\tilde{H}_N)_{N \in \mathbb{N}}$. An application of Lemma 5.3 allows us to write the condition of semi-group in the basis A_k ; for any $t_0 \geq 0$:

$$\frac{d}{dt}|_{t=t_0} \sum_{p_0 \in A_k} \kappa^{p_0}(E_{t_0}^N) p_0 = \left(\sum_{p \in A_k} \kappa^p(H_N) p \right) \cdot_N \left(\sum_{p' \in A_k} \kappa^{p'}(E_{t_0}^N) p' \right).$$

Then the following equality must hold for any positive integer N , any $t_0 \geq 0$ and any $p_0 \in A_k$:

$$\begin{aligned} &\frac{d}{dt}|_{t=t_0} \kappa^{p_0}(E_{t_0}^N) \\ &= \sum_{p, p' \in A_k, p \circ p' = p_0} \kappa^p(H_N) \kappa^{p'}(E_{t_0}^N) N^{d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p')}. \end{aligned}$$

Let us take N going to infinity. Because of the hypotheses and because of the improved triangle inequality, this differential system converges: $\kappa^p(E_t^N)$ must converge for any $p \in A_k$ and any real $t \geq 0$. Besides, we get for any $t_0 \geq 0$:

$$\forall p \in A_k, \frac{d}{dt}|_{t=t_0} \kappa^p(E_t) = \sum_{p_1 \in A_k, p_1 \prec p} \kappa^{p_1}(H) \kappa^{K_p(p_1)}(E_{t_0}).$$

Since the semi-group converges, using the usual notations, we can write that for any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}|_{t=t_0} m_p(E_t) = m_p(H E_{t_0}),$$

and using Equality (16), one has:

$$\lim_{N \rightarrow \infty} m_p(H_N E_{t_0}^N) = \sum_{p_1 \in A_k, p_1 \leq p} \kappa^{p_1}(H) m_{t_{p_1 \circ p}}(E_{t_0}).$$

Hence we recover the second system of differential equations. \square

Of course one also has, using Equality (17) instead of (16), that for any $p \in \mathcal{P}_k$ and any $t_0 \geq 0$:

$$\frac{d}{dt}|_{t=t_0} m_p(E_t) = \sum_{p_1 \in A_k, p_1 \leq p} m_{p \circ p_1}(H) \kappa^{p_1}(E_{t_0}).$$

Moreover, Theorem 7.2 can be very easily generalized for any semi-group with time dependent generator. In order to finish the section, let us prove a consequence of Lemma 2.1.

Theorem 7.3. — *Let $((E_t^N)_N)_{t \geq 0}$ be a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$. Let us suppose that the sequence $(E_0^N)_{N \in \mathbb{N}}$ converges as N goes to infinity. Let us suppose that for any $\sigma \in \mathfrak{S}_k$, $\kappa^\sigma(H_N)$ converges when N goes to infinity. Then for any $\sigma \in \mathfrak{S}_k$, for any positive real t , $\kappa^\sigma(E_t^N)$ converges as N goes to infinity. Besides for any $\sigma \in \mathfrak{S}_k$ and any $t_0 \geq 0$:*

$$(20) \quad \frac{d}{dt}|_{t=t_0} \kappa^\sigma(E_t) = \sum_{\sigma_1 \in \mathfrak{S}_k, \sigma_1 \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{\sigma_1}(H) \kappa^{\sigma_1^{-1}\sigma}(E_{t_0}).$$

Proof. — Let $((E_t^N)_N)_{t \geq 0}$ be a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ which satisfies the hypotheses of the theorem. Let $\sigma \in \mathfrak{S}_k$ and let N be a positive integer. We have seen in the last proof that for any $t_0 \geq 0$:

$$\begin{aligned} & \frac{d}{dt}|_{t=t_0} \kappa^\sigma(E_t^N) \\ &= \sum_{p, p' \in \mathcal{B}_k, p \circ p' = \sigma} \kappa^p(H_N) \kappa^{p'}(E_{t_0}^N) N^{d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p')}. \end{aligned}$$

Yet, by Lemma 2.1, if $p \circ p' = \sigma$, then p and p' are in \mathfrak{S}_k . Thus,

$$\frac{d}{dt}|_{t=t_0} \kappa^\sigma(E_t^N) = \sum_{p, p' \in \mathfrak{S}_k, p \circ p' = \sigma} \kappa^p(H_N) \kappa^{p'}(E_{t_0}^N) N^{-df(p, \sigma)}.$$

Thus, we see that $((\kappa^\sigma(E_t^N))_{\sigma \in \mathfrak{S}_k})_{t \geq 0}$ satisfies a linear differential system whose coefficients converge by hypothesis. Thus, for any $\sigma \in \mathfrak{S}_k$, for any positive real t , $\kappa^\sigma(E_t^N)$ converges as N goes to infinity. The Equation (20) is obtained by taking N going to infinity in the last equation. \square

8. Geometric and combinatorial consequences of Theorem 4.1

In Section 6, we showed new inequalities on the set of partitions \mathcal{P}_k . The proofs were quite combinatorial, and used only the notion of distance. In this section, we want to show that one can prove new inequalities or equalities, by using Theorem 4.1 as a black box.

8.1. Geometric consequences of Theorem 4.1. — First, let us give a new proof of the improved triangle inequality.

Proof of Proposition 6.1. — Let k be an integer. Let p and p' be two elements of A_k . Let us consider $(\mathbf{p}_N)_{N \in \mathbb{N}}$ and $(\mathbf{p}'_N)_{N \in \mathbb{N}}$ such that for any integer N :

$$\begin{aligned}\mathbf{p}_N &= M_k^N(p), \\ \mathbf{p}'_N &= M_k^N(p').\end{aligned}$$

Using Lemma 5.4, $(\mathbf{p}_N)_{N \in \mathbb{N}}$ and $(\mathbf{p}'_N)_{N \in \mathbb{N}}$ converge strongly. Let N be an integer. Applying the Equality (12), we have:

$$\mathbf{p}_N \otimes \mathbf{p}'_N = M_{2k}^N(p \otimes p').$$

Thus, using Lemma 5.4, $\mathbf{p}_N \otimes \mathbf{p}'_N$ converges strongly when N goes to infinity. An application of Theorem 4.1 shows that it converges in moments: for any $\tilde{p} \in A_{2k}$,

$$m_{\tilde{p}}(\mathbf{p}_N \otimes \mathbf{p}'_N) \text{ converges when } N \rightarrow \infty.$$

For any partition $\tilde{p} \in A_k$, we define $P(\tilde{p})$ be the partition in A_{2k} :

$$P(\tilde{p}) = (\tilde{p} \otimes id_k)(1, k+1)(2, k+2) \dots (k, 2k).$$

Then for any $E \in \mathbb{C}[A_k(N)]$ and $F \in \mathbb{C}[A_k(N)]$, and any $p_0 \in A_k$, we have:

$$m_{P(p_0)}(E \otimes F) = m_{p_0}(EF).$$

Thus for any $p_0 \in A_k$, $m_{p_0}(\mathbf{p}_N \mathbf{p}'_N)$ which is equal to $m_{P(p_0)}(\mathbf{p}_N \otimes \mathbf{p}'_N)$ converges as N goes to infinity. Using again the Theorem 4.1, we have that $\mathbf{p}_N \mathbf{p}'_N$ converges strongly as N goes to infinity. It implies, because of Lemma 5.4 that $(M_k^N)^{-1}(\mathbf{p}_N \mathbf{p}'_N)$ converges in $\mathbb{C}[A_k]$ when N goes of infinity. We can calculate this last expression:

$$\begin{aligned}(M_k^N)^{-1}(\mathbf{p}_N \mathbf{p}'_N) &= (M_k^N)^{-1}(M_k^N(p)M_k^N(p')) \\ &= p \cdot_N p' \\ &= N^{d({}^t p', p) - d(id, p) - d(id, p') + k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') + 2\kappa(p, p')} (p \circ p'),\end{aligned}$$

where we used Lemma 5.1. Thus we must have that for any p and p' in A_k :

$$d({}^t p', p) \leq d(id, p) + d(id, p') - k - \text{nc}(p \circ p') + \text{nc}(p) + \text{nc}(p') - 2\kappa(p, p').$$

The improved inequality is just a consequence of the last inequality as soon as we see that for any $p \in A_k$, $\text{nc}({}^t p) = \text{nc}(p)$, and $d(id, p) = d(id, {}^t p)$. \square

Again, using the same ideas, one can show the following interesting property, which will be important in order to compute in Theorem 7.6 in [9] the law of the product of two \mathcal{P} -free families.

Proposition 8.1. — *Let p_0, p_1 and p_2 be three partitions in A_k . Let τ be the partition in A_{2k} defined by:*

$$\tau = (1, k+1)(2, k+2) \dots (k, 2k).$$

We have:

$$\delta_{p_1 \otimes p_2 \in [id, (p_0 \otimes id_k) \tau]_{A_{2k}}} = \delta_{p_1 \circ p_2 \in [id, p_0]_{A_k}} \delta_{p_1 \prec p_1 \circ p_2}.$$

Proof. — Let p_0 , p_1 and p_2 be three partitions in A_k . Let us consider $(\mathbf{p}_N^1)_{N \in \mathbb{N}}$ and $(\mathbf{p}_N^2)_{N \in \mathbb{N}}$ such that for any integer N , $\mathbf{p}_N^1 = M_k^N(p_1)$ and $\mathbf{p}_N^2 = M_k^N(p_2)$. Using Lemma 5.4, $(\mathbf{p}_N^1)_{N \in \mathbb{N}}$ and $(\mathbf{p}_N^2)_{N \in \mathbb{N}}$ converge strongly. Thus, $(\mathbf{p}_N^1 \otimes \mathbf{p}_N^2)_{N \in \mathbb{N}}$ converges strongly, and by Theorem 4.1 it converges in moments. Let us calculate, using two ways, the limit of $m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2]$, where $\tau = (1, k+1)(2, k+2) \dots (k, 2k)$.

First, using Theorem 4.1 and the Equation (7), we get that:

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \sum_{p \in [(p_0 \otimes id_k)\tau]_{A_{2k}}} \lim_{N \rightarrow \infty} \kappa^p [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2].$$

Yet, for any $p \in A_{2k}$, $\kappa^p [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \delta_{p=p_1 \otimes p_2}$, thus:

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \delta_{p_1 \otimes p_2 \in [id, (p_0 \otimes id_k)\tau]_{A_{2k}}}.$$

Then, using the fact that $m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = m_{p_0} [\mathbf{p}_N^1 \mathbf{p}_N^2]$, and using again Theorem 4.1 and the Equation (7):

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \sum_{p \in [id, p_0]_{A_k}} \lim_{N \rightarrow \infty} \kappa^p [\mathbf{p}_N^1 \mathbf{p}_N^2].$$

Let $p \in A_k$, $\kappa^p [\mathbf{p}_N^1 \mathbf{p}_N^2]$ is the coefficient of p in the expression $(M_k^N)^{-1} (\mathbf{p}_N^1 \mathbf{p}_N^2)$. Let us remark that $(M_k^N)^{-1} (\mathbf{p}_N^1 \mathbf{p}_N^2) = (M_k^N)^{-1} (M_k^N(p_1) M_k^N(p_2)) = p_1 \cdot_N p_2$ which converges in $\mathbb{C}[A_k]$ to $\delta_{p_1 \prec p_1 \circ p_2} p_1 \circ p_2$. Thus,

$$\lim_{N \rightarrow \infty} \kappa^p [\mathbf{p}_N^1 \mathbf{p}_N^2] = \delta_{p_1 \prec p_1 \circ p_2} \delta_{p=p_1 \circ p_2}.$$

This implies that:

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \delta_{p_1 \prec p_1 \circ p_2} \delta_{p_1 \circ p_2 \in [id, p_0]_{A_k}}.$$

Using the two ways to compute $\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2]$, we get:

$$\delta_{p_1 \otimes p_2 \in [id, (p_0 \otimes id_k)\tau]_{A_{2k}}} = \delta_{p_1 \circ p_2 \in [id, p_0]_{A_k}} \delta_{p_1 \prec p_1 \circ p_2}.$$

which was the desired equality. \square

In fact, one can always prove the results by a combinatorial argument: the ideas we present are more an automatic way to get combinatorial results that one can prove after by combinatorial means. For example, let us consider Definition 9.1 ; using Proposition 8.1, one can now expect that $\mathbf{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) = \mathbf{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2)$. Indeed, we have the following proposition.

Proposition 8.2. — *Let p_0 , p_1 and p_2 be three partitions in A_k . Let τ be the partition in A_{2k} defined by:*

$$\tau = (1, k+1)(2, k+2) \dots (k, 2k).$$

We have:

$$\mathbf{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) = \mathbf{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2).$$

Proof. — The proof is only based on calculations. Let p and p' be two partitions in A_k , then:

$$\mathbf{df}(p', p) = \mathbf{nc}(p') - \mathbf{nc}(p' \vee id) - \mathbf{nc}(p' \vee p) + \mathbf{nc}(p \vee id),$$

and $\eta(p, p')$ is equal to:

$$\mathbf{nc}(p) + \mathbf{nc}(p') - \mathbf{nc}(p \circ p') - \mathbf{nc}(p \vee id) - \mathbf{nc}(p' \vee id) + \mathbf{nc}(p \circ p' \vee id) - \kappa(p, p').$$

Thus:

$$\begin{aligned} & \mathbf{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \mathbf{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) \\ &= \mathbf{nc}(p_1 \circ p_2) - \mathbf{nc}((p_1 \circ p_2) \vee id) - \mathbf{nc}((p_1 \circ p_2) \vee p_0) + \mathbf{nc}(p_0 \vee id) + \mathbf{nc}(p_1) + \mathbf{nc}(p_2) \\ & \quad - \mathbf{nc}(p_1 \circ p_2) - \mathbf{nc}(p_1 \vee id) - \mathbf{nc}(p_2 \vee id) + \mathbf{nc}((p_1 \circ p_2) \vee id) - \kappa(p_1, p_2) \\ & \quad - \mathbf{nc}(p_1 \otimes p_2) + \mathbf{nc}((p_1 \otimes p_2) \vee id) + \mathbf{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]) \\ & \quad - \mathbf{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}). \end{aligned}$$

Using the following equalities:

$$\begin{aligned} \mathbf{nc}(p_1 \otimes p_2) &= \mathbf{nc}(p_1) + \mathbf{nc}(p_2), \\ \mathbf{nc}([p_1 \otimes p_2] \vee id) &= \mathbf{nc}(p_1 \vee id) + \mathbf{nc}(p_2 \vee id), \end{aligned}$$

we get:

$$\begin{aligned} & \mathbf{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \mathbf{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) \\ &= -\mathbf{nc}((p_1 \circ p_2) \vee p_0) + \mathbf{nc}(p_0 \vee id) - \kappa(p_1, p_2) \\ & \quad + \mathbf{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]) - \mathbf{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}). \end{aligned}$$

The equalities:

$$\begin{aligned} Tr_N(p_0) &= Tr_N((p_0 \otimes id_k)\tau), \\ N^{\kappa(p_1, p_2)} Tr_N((p_1 \circ p_2)^t p_0) &= Tr_N((p_1 p_2)^t p_0) = Tr_N((p_1 \otimes p_2)^t [(p_0 \otimes id_k)\tau]), \end{aligned}$$

allow us to prove, as an application of Equations (2) and (3), that:

$$\begin{aligned} \mathbf{nc}(p_0 \vee id) &= \mathbf{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}), \\ \mathbf{nc}((p_1 \circ p_2) \vee p_0) + \kappa(p_1, p_2) &= \mathbf{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]). \end{aligned}$$

Thus $\mathbf{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \mathbf{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) = 0$. \square

8.2. Combinatorial consequences of Theorem 4.1. — Let us remark the following important, yet simple theorem.

Theorem 8.1. — *Let $(m_p)_{p \in A_k}$ be a family of complex numbers. There exists a sequence $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ which converges and such that:*

$$\lim_{N \rightarrow \infty} m_p(E_N) = m_p.$$

Proof. — Let us consider $(m_p)_{p \in A_k}$ a family of complex numbers. Let us consider $(\kappa^p)_{p \in A_k}$, the unique family of real such that for any $p \in A_k$:

$$m_p = \sum_{p' \in [id, p]_{A_k}} \kappa^{p'}.$$

Let us consider then:

$$E_N = M_k^N \left(\sum_{p \in A_k} \kappa^p p \right).$$

According to Lemma 5.4, $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ converges strongly. Thus, by Theorem 4.1 it converges in moments and for any $p \in A_k$:

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id, p]_{A_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

Yet, using Lemma 5.3, $\kappa^{p'}(E_N)$ is equal to $\kappa^{p'}$. Thus:

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id, p]_{A_k}} \kappa^{p'} = m_p.$$

This concludes the proof. \square

This theorem shows that, in order to understand the transformation between moments and coordinate numbers, we have an approximation setting in which one can work with: the space of convergent sequences in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us show some examples of propositions that one can get using this point of view. For this, we need the notion of cumulants and exclusive moments. Let us consider $(m_p)_{p \in A_k}$ a family of complex numbers.

Definition 8.1. — *The cumulants of $(m_p)_{p \in A_k}$ is the unique family of complex numbers $(\kappa^p)_{p \in A_k}$ such that for any $p \in A_k$:*

$$m_p = \sum_{p' \in [id, p]_{A_k}} \kappa^{p'}.$$

The exclusive moments of $(m_p)_{p \in A_k}$ is the only family $(m_{p^c})_{p \in \mathcal{P}_k}$ of complex numbers such that:

$$m_{p^c} = \sum_{p' \in \text{Sp}(p) \cap A_k} \kappa_{p'}.$$

Let us consider the cumulants $(\kappa^p)_{p \in A_k}$ and the exclusive moments $(m_{p^c})_{p \in \mathcal{P}_k}$ of $(m_p)_{p \in A_k}$.

Proposition 8.3. — *Let p and p_0 be two elements of A_k . Then:*

$$\delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}} = \sum_{p' \in [id, p_0]_{A_k}} \delta_{p \prec p'} \kappa^{K_{p'}(p)}.$$

where for any $P \subset A_k$, $\kappa^P = \sum_{p \in P} \kappa^p$.

By specifying $p = id$ in Proposition 8.4, we get back the Equation (7). Besides, one can get a similar formula for $m_{p_0 \circ t_p}(E)$ by using Equation (17). Using Theorem 8.1, the last proposition is a consequence of Proposition 8.4.

Proposition 8.4. — For any integer N , let us consider E_N an element of $\mathbb{C}[A_k(N)]$. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges. Let p and p_0 be two elements of A_k . Then:

$$\delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}}(E) = \sum_{p' \in [id, p_0]_{A_k}} \delta_{p \prec p'} \kappa^{K_{p'}(p)}(E).$$

Proof. — Let p and p_0 be two elements of A_k . Let us consider for any N , $M_k^N(p) \in \mathbb{C}[A_k(N)]$. The sequence $(M_k^N(p))_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ converges by Lemma 5.4. Let us apply the Theorem 7.1 to the product $M_k^N(p)E_N$. We remind the reader that $\kappa^{p'}(M_k^N(p)) = \delta_{p=p'}$ for any $p' \in A_k$. Using the Equation (15) in Theorem 7.1, we know that

$$\lim_{N \rightarrow \infty} \kappa^{p'}(M_k^N(p)E_N) = \delta_{p \prec p'} \kappa^{K_{p'}(p)}(E).$$

Let us use Equation (7):

$$\lim_{N \rightarrow \infty} m_{p_0}(M_k^N(p)E_N) = \sum_{p' \in [id, p_0]_{A_k}} \delta_{p \prec p'} \kappa^{K_{p'}(p)}(E).$$

Yet, according to Equation (16),

$$\lim_{N \rightarrow \infty} m_{p_0}(M_k^N(p)E_N) = \delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}}(E),$$

hence the equality stated in Proposition 8.4. \square

8.3. Convergence of the modified observables. — In Section 5, we have defined a deformed partition algebra, by deforming the multiplication. Yet, we have not defined any deformed linear form m_p on the algebra $\mathbb{C}[A_k(N, N)]$. In fact, on $\mathbb{C}[A_k(N, N)]$, for any $p \in \mathcal{P}_k$ we define:

$$m_p^N : \mathbb{C}[A_k(N, N)] \rightarrow \mathbb{C} \\ E \mapsto m_p(M_k^N(E)).$$

A consequence of Theorem 4.1 is that for any $E \in A_k$, for any $p \in \mathcal{P}_k$, $m_p^N(E)$ converges as N goes to infinity: let us denote the limit by $m_p^\infty(E)$. We already know that the algebra $\mathbb{C}[A_k(N, N)]$ converges to $\mathbb{C}[A_k(\infty, \infty)]$ when N goes to infinity. Thus, we have that:

Theorem 8.2. — For any integer k , $(\mathbb{C}[A_k(N, N)], (m_p^N)_{p \in \mathcal{P}_k})$ converges to

$$(\mathbb{C}[A_k(\infty, \infty)], (m_p^\infty)_{p \in \mathcal{P}_k})$$

as N goes to infinity. This means that:

1. the algebra $\mathbb{C}[A_k(N, N)]$ converges to $\mathbb{C}[A_k(\infty, \infty)]$ as N goes to infinity,
2. for any $E \in \mathbb{C}[A_k(N, N)]$, for any $p \in \mathcal{P}_k$, $m_p^N(E)$ converges to $m_p^\infty(E)$ as N goes to infinity, where $m_p^\infty(E)$ is defined below.

Besides, let $E = \sum_{p \in A_k} E_p p$ and $F = \sum_{p \in A_k} F_p p$ in $\mathbb{C}[A_k(\infty, \infty)]$, then

$$EF = \sum_{p_1, p_2 \in A_k} E_{p_1} F_{p_2} \delta_{p_1 \prec p_1 \circ p_2} p_1 \circ p_2.$$

And if $p_0 \in A_k$:

$$m_{p_0}^\infty(E) = \sum_{p \in A_k} \delta_{p \in [id, p_0]_{A_k}} E_p.$$

In fact, this theorem has to be read in the other way: given the algebra with linear forms $(\mathbb{C}[A_k(\infty, \infty)], (m_p^\infty)_{p \in \mathcal{P}_k})$, one can find an approximation given by $(\mathbb{C}[A_k(N, N)], (m_p^N)_{p \in \mathcal{P}_k})$.

9. Algebraic fluctuations

In this section, we generalize Sections 4, 5 and 7 in order to study the asymptotic developments of the coordinate numbers and normalized moments. The proofs will be either omitted or simplified as they will use the same arguments as we have seen in Sections 4, 5 and 7.

In order to study the asymptotic developments, we need to introduce two notions of defect. One already seen is linked with the triangle inequality and the other to the improved triangle inequality. Let k be a positive integer, let p and p' be two elements of A_k . We recall that we defined in Definition 3.5 the defect of p' from not being on the set-geodesic $[id, p]_{\mathcal{P}_k}$ by:

$$\text{df}(p', p) = d(id, p') + d(p', p) - d(id, p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p \vee p') + \text{nc}(p \vee id).$$

Definition 9.1. — *The defect $\eta(p, p')$ that $p \prec p \circ p'$ is not satisfied by:*

$$d(id, p) + d(id, p') - d(id, p \circ p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

We warn the reader that, in general:

$$\text{df}(p, p \circ p') \neq \eta(p, p'),$$

even if this equality is true when one considers $p, p' \in \mathfrak{S}_k$. Let us remark that if p and p_0 are elements of A_k , $p \prec p_0$ holds if and only if there exists p' such that $p_0 = p \circ p'$ and $\eta(p, p') = 0$.

Let us define the N -development algebra of order m of A_k . This algebra is the good setting in order to study fluctuations of the coordinate numbers and moments.

Definition 9.2. — *Let N, k and m be integers, let X be a formal variable. The N -development algebra of order m of A_k , $\mathbb{C}_{(m)}[A_k(N)]$, is the associative algebra generated by the elements of the form:*

$$\frac{p}{X^i},$$

where $p \in A_k$ and $i \in \{0, \dots, m\}$. The product is defined such that, for any p and p' in A_k , and any i and j in $\{0, \dots, m\}$:

$$\frac{p}{X^i} \cdot \frac{p'}{X^j} = \frac{1}{N^{\max(i+j+\eta(p, p')-m, 0)}} \frac{p \circ p'}{X^{\min(i+j+\eta(p, p'), m)}}.$$

This product is well defined: indeed the improved triangle inequality, Proposition 6.1 or Lemma 6.1, assert that for any $p, p' \in A_k$, $\eta(p, p') \geq 0$, thus, for any $i, j \in \{0, \dots, m\}$, any $p, p' \in A_k$, we have $\min(i + j + \eta(p, p'), m) \geq 0$. This implies that:

$$\frac{p \circ p'}{X^{\min(i+j+\eta(p,p'),m)}}$$

is an element of the canonical basis of the N -development algebra of order m of A_k . Let us remark that for any positive integers k, N and m , $\mathbb{C}_{(m)}[\mathfrak{S}_k(N)] \subset \mathbb{C}_{(m)}[\mathfrak{B}_k(N)] \subset \mathbb{C}_{(m)}[\mathcal{P}_k(N)]$, where these inclusions are inclusions of algebras.

9.1. Coordinate numbers. — Let N be a positive integer, the N -development algebra of order 0 of A_k is canonically isomorphic to $\mathbb{C}[A_k(N, N)]$.

Lemma 9.1. — *Let N be an integer, the application:*

$$\begin{aligned} L_k^N : A_k &\rightarrow \mathbb{C}_{(0)}[A_k(N)] \\ p &\mapsto \frac{p}{X^0} \end{aligned}$$

can be extended as an isomorphism of algebra between $\mathbb{C}[A_k(N, N)]$ and $\mathbb{C}_{(0)}[A_k(N)]$.

Proof. — Let us show that for any p, p' in A_k , $L_k^N(p \cdot_N p') = L_k^N(p)L_k^N(p')$. As for any $p, p' \in A_k$, $\eta(p, p') \geq 0$, $L_k^N(p)L_k^N(p')$ is equal to:

$$\frac{p}{X^0} \frac{p'}{X^0} = \frac{1}{N^{\eta(p,p')}} \frac{p \circ p'}{X^0} = \frac{1}{N^{\eta(p,p')}} L_k^N(p \circ p').$$

Yet, looking at the definition of $\eta(p, p')$ given in Definition 9.1, for any integer N the following equation holds in $\mathbb{C}[A_k(N, N)]$:

$$p \cdot_N p' = \frac{1}{N^{\eta(p,p')}} p \circ p'.$$

This allows us to conclude. □

Let m be a positive integer. We define, for any $i \leq m$, the coordinate numbers of order i of any element of $\mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.3. — *Let $E \in \mathbb{C}_{(m)}[A_k(N)]$. The coordinate numbers of E up to the order m are the elements $(\kappa_i^p(E))_{i \in \{0, \dots, m\}, p \in \mathcal{P}_k}$ such that:*

$$E = \sum_{p \in A_k} \sum_{i=0}^m \kappa_i^p(E) \frac{p}{X^i}.$$

Let $p \in A_k$ and $i \leq m$. The number $\kappa_i^p(E)$ is the coordinate number of E on p of order i .

We define also a notion of convergence. In order to do so, we must not forget that, when $m = 0$, $\mathbb{C}_{(m)}[A_k(N)]$ is isomorphic to the deformed algebra $\mathbb{C}[A_k(N, N)]$ and not the algebra $\mathbb{C}[A_k(N)]$. Let $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.4. — *The sequence $(E_N)_{N \in \mathbb{N}}$ converges if and only if for any $i \in \{0, \dots, m-1\}$, and any $p \in A_k$, $\kappa_i^p(E_N)$ is independent of N , and for any $p \in A_k$, $\kappa_m^p(E_N)$ converges when N goes to infinity.*

Notation 9.1. — Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges as N goes to infinity. We denote, for any $i \in \{0, \dots, m\}$ and any $p \in A_k$: $\kappa_i^p(E) = \lim_{N \rightarrow \infty} \kappa_i^p(E_N)$.

9.2. Convergences: $\mathbb{C}_{(m)}[A_k(N)]$ and multiplication. — Using Lemma 9.1 and Theorem 6.1, since the algebra $\mathbb{C}[A_k(N, N)]$ is isomorphic to $\mathbb{C}_{(0)}[A_k(N)]$ by an isomorphism which sends the canonical base of the first algebra on the canonical base of the second algebra, we know that the algebra $\mathbb{C}_{(0)}[A_k(N)]$ converges as N tends to infinity. In fact, the result holds for any $m \in \mathbb{N}$. Let k and m be two integers. Let X be a formal variable.

Definition 9.5. — The ∞ -development algebra of order m of A_k , denoted by $\mathbb{C}_{(m)}[A_k(\infty)]$ is the associative algebra generated by the elements of the form:

$$\frac{p}{X^i},$$

where $p \in A_k$ and $i \in \{0, \dots, m\}$. The product is defined such that, for any p and p' in A_k , and any i and j in $\{0, \dots, m\}$,

$$\frac{p}{X^i} \frac{p'}{X^j} = \delta_{i+j+\eta(p,p') \leq m} \frac{p \circ p'}{X^{i+j+\eta(p,p')}}.$$

Let us recall Definition 6.3, where we defined the convergence of algebras. We then have the following proposition.

Proposition 9.1. — When N goes to infinity, the N -development algebra of order m of A_k , $\mathbb{C}_{(m)}[A_k(N)]$ converges to the ∞ -development algebra of order m of A_k , namely $\mathbb{C}_{(m)}[A_k(\infty)]$.

Proof. — The algebras $\mathbb{C}_{(m)}[A_k(N)]$ have, for any integer N , the same linear basis $\left(\frac{p}{X^i}\right)_{i \in \{0, \dots, m\}, p \in A_k}$. Since for any $p, p' \in A_k$, any $i, j \in \mathbb{N}$:

$$\frac{p}{X^i} \frac{p'}{X^{i'}} = \frac{1}{N^{\max(i+j+\eta(p,p')-m, 0)}} \frac{p \circ p'}{X^{\min(i+j+\eta(p,p'), m)}} \xrightarrow{N \rightarrow \infty} \delta_{i+j+\eta(p,p') \leq m} \frac{p \circ p'}{X^{i+j+\eta(p,p')}},$$

where the first product is seen in $\mathbb{C}_{(m)}[A_k(N)]$, the algebra $\mathbb{C}_{(m)}[A_k(N)]$ converges to $\mathbb{C}_{(m)}[A_k(\infty)]$ as N goes to infinity. \square

Let us write the first easiest consequence of the Proposition 9.1, which can be proved by using a bi-linearity argument, Proposition 9.1 and Definition 9.4.

Proposition 9.2. — Let $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ be elements of $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$. Let us suppose that the two sequences $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ converge. The sequence $(E_N F_N)_{N \in \mathbb{N}}$ converges and, using Notations 9.1, for any $i_0 \in \{0, \dots, m\}$ and for any $p_0 \in A_k$:

$$\kappa_{i_0}^{p_0}(EF) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(E) \kappa_{i_0 - \eta(p, p') - i}^{p'}(F).$$

As for Section 7.2, the good behavior of the product, given by Proposition 9.2, implies a criterion for the convergence of semi-groups in $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.6. — Let $((E_t^N)_N)_{t \geq 0}$ be a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$. The semi-group $((E_t^N)_N)_{t \geq 0}$ converges if and only if for any $t \geq 0$, $(E_t^N)_{N \in \mathbb{N}}$ converges.

Proposition 9.3. — Let $m \in \mathbb{N}$. Let us consider $((E_t^N)_N)_{t \geq 0}$ a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ which generator is denoted by $(H_N)_{N \in \mathbb{N}}$. It converges if the sequences $(E_0^N)_{N \in \mathbb{N}}$ and $(H_N)_{N \in \mathbb{N}}$ converge. Besides, using Notation 9.1, for any $p \in A_k$, for any $t_0 \geq 0$ and any $i \in \{0, \dots, m\}$,

$$\frac{d}{dt} \Big|_{t=t_0} \kappa_{i_0}^{p_0}(E_t) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(H) \kappa_{i_0 - \eta(p, p') - i}^{p'}(E_{t_0}).$$

In order to finish this section, let us introduce the evaluation morphism: it is a morphism which allows us to inject an element from $\mathbb{C}_{(m)}[A_k(N)]$ in $\mathbb{C}[A_k(N)]$. Let N be a positive integer, the function $eval_N$ is defined by:

$$\begin{aligned} eval_{(m)}^N : \mathbb{C}_{(m)}[A_k(N)] &\rightarrow \mathbb{C}[A_k(N)] \\ \sum_{p \in A_k} \sum_{i=0}^m \kappa_i^p(E) \frac{p}{X^i} &\mapsto \sum_{p \in A_k} \sum_{i=0}^m \kappa_i^p(E) \frac{1}{N^i} \frac{p}{N^{-\frac{k}{2} + \frac{nc(p)}{2} + d(id, p)}}. \end{aligned}$$

Lemma 9.2. — For any positive integer N , $eval_{(m)}^N$ is a morphism of algebra.

Proof. — Let N be a positive integer, let $i, j \in \{0, \dots, m\}$ and $p, p' \in A_k$. Then:

$$\begin{aligned} eval_{(m)}^N \left(\frac{p}{X^i} \frac{p'}{X^j} \right) &= eval_{(m)}^N \left(\frac{1}{N^{\max(i+j+\eta(p, p')-m, 0)}} \frac{p \circ p'}{X^{\min(i+j+\eta(p, p'), m)}} \right) \\ &= \frac{1}{N^{i+j+\eta(p, p') - \frac{k}{2} + \frac{nc(p \circ p')}{2} + d(id, p \circ p')}} p \circ p' \\ &= \left(\frac{1}{N^i} \frac{p}{N^{-\frac{k}{2} + \frac{nc(p)}{2} + d(id, p)}} \right) \left(\frac{1}{N^j} \frac{p'}{N^{-\frac{k}{2} + \frac{nc(p')}{2} + d(id, p')}} \right) \\ &= eval_{(m)}^N \left(\frac{p}{X^i} \right) eval_{(m)}^N \left(\frac{p'}{X^j} \right). \end{aligned}$$

The other properties can be easily verified. \square

The function $eval_{(m)}^N$ has an inverse if and only if $m = 0$. This will motivate us in order to define a notion of convergence up to order m of fluctuations for sequences in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Then, given a linear or multiplicative problem in $\mathbb{C}[A_k(N)]$, one can try to find a similar problem in $\mathbb{C}_{(m)}[A_k(N)]$, solve this last problem, and push by $eval_{(m)}^N$ the solution on a solution of the first problem.

9.3. Convergence at any order of fluctuations in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. — We are interested in elements in $\mathbb{C}[A_k(N)]$ and we want to define a notion of strong convergence up to the m^{th} order of fluctuations. Let m be an integer, let $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Definition 9.7. — The sequence $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if there exist two families of real $(\kappa_i^p)_{i \in \{0, \dots, m-1\}, p \in A_k}$ and $(\kappa_{m, N}^p)_{p \in A_k, N \in \mathbb{N}}$ such that:

- $\forall p \in A_k, \kappa^p(E_N) = \sum_{i=0}^{m-1} \frac{\kappa_i^p}{N^i} + \frac{\kappa_{m,N}^p}{N^m},$
- $\forall p \in A_k, \kappa_{m,N}^p$ converges as N goes to infinity.

The families $(\kappa_i^p)_{i \in \{0, \dots, m-1\}, p \in A_k}$ and $(\kappa_{m,N}^p)_{p \in A_k}$ are uniquely defined.

For any $p \in A_k$, any integer N and any $i \in \{0, \dots, m-1\}$, κ_i^p is the coordinate number of E_N on p of order i , and $\kappa_{m,N}^p$ is the coordinate number of E_N on p of order m .

Notation 9.2. — Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. From now on, the coordinate numbers of E_N on p of order i will be denoted by $\kappa_i^p(E_N)$. For any $p \in A_k$ and any $i \in \{0, \dots, m\}$, we will define:

$$\kappa_i^p(E) = \lim_{N \rightarrow \infty} \kappa_i^p(E_N).$$

When one works in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$, one has to be aware that the coordinate numbers of higher order of fluctuations are only defined for a sequence $(E_N)_{N \in \mathbb{N}}$ which converges strongly. Thus, one must not forget that the notation $\kappa_i^p(E_N)$ means that we are looking at the coordinate numbers of E_N seen as an element of the sequence $(E_N)_{N \geq 0}$.

The Definition 9.7 might seem strange as it only uses once the notion of convergence. Yet, it is easy to see that an equivalent definition is the following one. The sequence $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if there exists a family $(\kappa_i^p)_{i \in \{0, \dots, m\}, p \in A_k}$ of real numbers such that for any $i \in \{0, \dots, m\}$,

$$N^i \left(\kappa^p(E_N) - \sum_{j=0}^{i-1} \frac{\kappa_j^p}{N^j} \right) \xrightarrow{N \rightarrow \infty} \kappa_i^p,$$

with the convention $\sum_{j=0}^{-1} \frac{\kappa_j^p}{N^j} = 0$. This equivalent definition explains why the families $(\kappa_i^p)_{i \in \{0, \dots, m-1\}, p \in A_k}$ and $(\kappa_{m,N}^p)_{N \in \mathbb{N}, p \in A_k}$ defined in Definition 9.7 are uniquely defined.

The next lemma allows to make a link between the convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ and the convergence up to the m^{th} order of fluctuations of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Lemma 9.3. — Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges. Then $\left(\text{eval}_{(m)}^N(E_N) \right)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations.

The notion of strong convergence to the m^{th} order of fluctuations allows us to inject canonically an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ which converges strongly up to the m^{th} order of fluctuations into $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$.

Definition 9.8. — Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. For any $p \in A_k$, any integer N , let $(\kappa_i^p)_{i \in \{0, \dots, m-1\}}$ and $\kappa_{m,N}^p$ be the coordinate numbers of E_N on p . We define the lift of the sequence $(E_N)_{N \in \mathbb{N}}$ as

$(\tilde{E}_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ defined by:

$$\tilde{E}_N = \sum_{p \in A_k} \left(\left(\sum_{i=0}^{m-1} \kappa_i^p \frac{p}{X^i} \right) + \kappa_{m,N}^p \frac{p}{X^m} \right).$$

The following lemma is then straightforward.

Lemma 9.4. — *Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. Let $(\tilde{E}_N)_{N \in \mathbb{N}}$ be its canonical lift in $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$. Then $(\tilde{E}_N)_{N \in \mathbb{N}}$ converges as N goes to infinity and for any $N \in \mathbb{N}$, one has $\text{eval}_{(m)}^N(\tilde{E}_N) = E_N$.*

We are going to define a weak notion of convergence up to the m^{th} order of fluctuations and we will show that this notion is equivalent to the strong convergence notion we defined in Definition 9.7.

Definition 9.9. — *The sequence $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations if and only if there exist two families $(m_p^i)_{i \in \{0, \dots, m-1\}, p \in A_k}$ and $(m_{p,N}^m)_{N \in \mathbb{N}, p \in A_k}$ such that:*

$$\begin{aligned} - \forall p \in A_k, m_p(E_N) &= \sum_{i=0}^{m-1} \frac{m_p^i}{N^i} + \frac{m_p^m}{N^m}, \\ - \forall p \in A_k, m_{p,N}^m &\text{ converges as } N \text{ goes to infinity.} \end{aligned}$$

The families $(m_p^i)_{i \in \{0, \dots, m-1\}, p \in A_k}$ and $(m_{p,N}^m)_{N \in \mathbb{N}, p \in A_k}$ are uniquely defined.

For any $p \in A_k$, any integer N , and any $i \in \{0, \dots, m-1\}$, m_p^i is the i^{th} -order fluctuations of the p -normalized moment of E_N , and $m_{p,N}^m$ is the m^{th} -order fluctuations of the p -normalized moment of E_N .

Notation 9.3. — *Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations. From now on, the i^{th} -order fluctuations of the p -normalized moment of E_N will be denoted by $m_p^i(E_N)$. For any $p \in A_k$ and any $i \in \{0, \dots, m\}$, we define:*

$$m_p^i(E) = \lim_{N \rightarrow \infty} m_p^i(E_N).$$

The same remark about the coordinate numbers of E_N on p of order i , explained just after Notation 9.2, can be made for the fluctuations of the p -normalized moments of E_N . The next theorem shows that the strong convergence up to the m^{th} order of fluctuations is equivalent to the convergence in moments up to the m^{th} order of fluctuations. We recall that $m \in \mathbb{N}$ and $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$.

Theorem 9.1. — *The sequence $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if it converges in moments up to the m^{th} order of fluctuations. We will say that $(E_N)_{N \in \mathbb{N}}$ converges up to the m^{th} order of fluctuations.*

Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges up to the m^{th} order of fluctuations, then, seen as an element of $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$, it converges also up to the m^{th} order of fluctuations and for any $i_0 \in \{0, \dots, m\}$ and any $p \in \mathcal{P}_k$:

$$(21) \quad \lim_{N \rightarrow \infty} m_p^{i_0}(E_N) = \sum_{p' \in A_k, \text{df}(p', p) \leq i_0} \kappa_{i_0 - \text{df}(p', p)}^{p'}(E).$$

Proof. — We have already seen the arguments in order to prove the second assertion. Let us prove that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations if and only if it converges in moments up to the m^{th} order of fluctuations. Let us consider p in A_k .

Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. The coordinate numbers of E_N are defined up to order m of fluctuations and:

$$E_N = \sum_{p \in A_k} \sum_{i=0}^m \frac{\kappa_i^p(E_N)}{N^i} \frac{p}{N^{\text{nc}(p) - \text{nc}(p \vee id)}}.$$

Besides, for any $p \in A_k$ and any $i \leq m-1$, $\kappa_i^p(E_N)$ does not depend on N and $\kappa_m^p(E_N)$ converges when N goes to infinity. We can compute the p -normalized moments of E_N , using the same arguments as for the proof of Theorem 4.1:

$$\begin{aligned} m_p(E_N) &= \frac{1}{\text{Tr}_N(p)} \text{Tr}_N(E_N \cdot p) = \sum_{p' \in A_k} \sum_{i=0}^m \kappa_i^{p'}(E_N) \frac{1}{N^{i + \text{df}(p', p)}} \\ &= \sum_{j=0}^{m-1} \left(\sum_{(p', i) \in A_k \times \{0, \dots, m-1\}, i + \text{df}(p', p) = j} \kappa_i^{p'}(E_N) \right) \frac{1}{N^j} \\ &\quad + \left(\sum_{(p', i) \in A_k \times \{0, \dots, m\}, i + \text{df}(p', p) \geq m} \frac{\kappa_i^{p'}(E_N)}{N^{i + \text{df}(p', p) - m}} \right) \frac{1}{N^m}. \end{aligned}$$

Let us define for any $N \in \mathbb{N}$, any $j \in \{0, \dots, m-1\}$ and any $p \in A_k$:

$$m_p^j(E_N) = \sum_{(p', i) \in A_k \times \{0, \dots, m-1\}, i + \text{df}(p', p) = j} \kappa_i^{p'}(E_N)$$

and

$$m_p^m(E_N) = \sum_{(p', i) \in A_k \times \{0, \dots, m\}, i + \text{df}(p', p) \geq m} \frac{\kappa_i^{p'}(E_N)}{N^{i + \text{df}(p', p) - m}},$$

so that, for any $p \in A_k$ and any $N \in \mathbb{N}$:

$$m_p(E_N) = \sum_{j=0}^{m-1} \frac{m_p^j(E_N)}{N^j} + \frac{m_p^m(E_N)}{N^m}.$$

For any $p \in A_k$ and any $i \leq m-1$, $m_p^i(E_N)$ does not depend on N and for any $p \in A_k$, $\kappa_m^p(E_N)$ converges when N goes to infinity. Thus $m_p^m(E_N)$ converges when N goes to infinity to

$$\sum_{p' \in A_k, \text{df}(p', p) \leq m} \kappa_{m - \text{df}(p', p)}^{p'}(E).$$

By Definition 9.9, this shows that $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations and the Equation (21) holds.

Let us suppose now that $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the m^{th} order of fluctuations. Then, by Theorem 4.1, it converges strongly up to order 0 of fluctuation. Let us suppose that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to order l of fluctuations with $l < m$.

Thus, the coordinate numbers of E_N up to order l of fluctuations are well defined and we can write:

$$E_N = \sum_{p \in \mathcal{P}_k} \left(\sum_{j=0}^{l-1} \frac{\kappa_j^p(E)}{N^j} + \frac{\kappa_l^p(E_N)}{N^l} \right) p,$$

where, for any $p \in \mathcal{P}_k$, $\kappa_l^p(E_N)$ is converging when N goes to infinity to a number $\kappa_l^p(E)$. We can use the computation of the normalized moments of E_N that we already did in order to show that for any partition $p \in A_k$:

$$\begin{aligned} m_p(E_N) &= \sum_{j=0}^{l-1} \left(\sum_{(p',i) \in A_k \times \{0, \dots, l-1\}, i + \text{df}(p',p) = j} \kappa_i^{p'}(E) \right) \frac{1}{N^j} \\ &\quad + \left(\sum_{(p',i) \in A_k \times \{0, \dots, l\}, i + \text{df}(p',p) \geq l} \frac{\kappa_i^{p'}(E_N)}{N^{i + \text{df}(p',p) - l}} \right) \frac{1}{N^l}. \end{aligned}$$

Thus, using the same notations than those used in the first part of the proof, we get:

$$\begin{aligned} m_p(E_N) &= \sum_{j=0}^l \frac{m_p^j(E)}{N^j} + \sum_{p' \in A_k, \text{df}(p',p) = 0} \frac{\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E)}{N^l} \\ &\quad + \sum_{(p',i) \in A_k \times \{0, \dots, l\}, i + \text{df}(p',p) - l = 1} \frac{\kappa_i^{p'}(E_N)}{N^{l+1}} + o\left(\frac{1}{N^{l+1}}\right). \end{aligned}$$

Let us use the fact that $(E_N)_{N \in \mathbb{N}}$ converges in moments up to the order $l + 1$ of fluctuations: for any $p' \in A_k$,

$$N^{l+1} \left(m_p(E_N) - \sum_{j=0}^l \frac{m_p^j(E)}{N^j} \right)$$

converges as N goes to infinity. This implies that for any $p \in A_k$,

$$\sum_{p' \in [id, p]_{A_k}} N(\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E))$$

converges as N goes to infinity. We are thus in the same setting as for the order 0 of fluctuations: for any $p \in A_k$,

$$N(\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E))$$

converges as N goes to infinity: this is equivalent to say that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to order $l + 1$ of fluctuations. This implies by recurrence that $(E_N)_{N \in \mathbb{N}}$ converges strongly up to order m of fluctuations. \square

9.4. Multiplication and convergence of fluctuations in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. — The results in Section 9.3 were only algebraic: we will now give the similar results for elements in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. The main ingredients used in order to do so are Lemma 9.2, Lemma 9.3 and Lemma 9.4 which respectively assert that $eval_{(m)}^N$ is a morphism of algebra, compatible with the strong convergence notion and, in some sense, can be inverted.

Theorem 9.2. — *Let $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ be elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that the sequences $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ converge up to the m^{th} order of fluctuations. Then, the sequence $(E_N F_N)_{N \in \mathbb{N}}$ converges up to the m^{th} order of fluctuations.*

Besides, using Notations 9.2 and 9.3, for any $i_0 \in \{0, \dots, m\}$:

– for any $p_0 \in A_k$:

$$(22) \quad \kappa_{i_0}^{p_0}(EF) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i=0}^{i_0 - \eta(p, p')} \kappa_i^p(E) \kappa_{i_0 - \eta(p, p') - i}^{p'}(F).$$

– for any $p_0 \in \mathcal{P}_k$:

$$(23) \quad m_{p_0}^{i_0}(EF) = \sum_{p_1 \in A_k} \sum_{i+j+\text{df}(p_1, p_0)=i_0} \kappa_i^{p_1}(E) m_{p_1 \circ p_0}^j(F).$$

Proof. — Let $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ be elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Let us suppose that the sequences $(E_N)_{N \in \mathbb{N}}$ and $(F_N)_{N \in \mathbb{N}}$ converge strongly or in moments up to the m^{th} order of fluctuations. By Lemma 9.4, let us consider the canonical lifts of $(E_N)_{N \in \mathbb{N}}$ (resp. $(F_N)_{N \in \mathbb{N}}$) in $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$: $(\tilde{E}_N)_{N \in \mathbb{N}}$ (resp. $(\tilde{F}_N)_{N \in \mathbb{N}}$). The two sequences $(\tilde{E}_N)_{N \in \mathbb{N}}$ and $(\tilde{F}_N)_{N \in \mathbb{N}}$ converge. According to Proposition 9.2, the sequence $(\tilde{E}_N \tilde{F}_N)_{N \in \mathbb{N}}$ converges. For any $i_0 \in \{0, \dots, m\}$ and for any $p_0 \in A_k$:

$$(24) \quad \kappa_{i_0}^{p_0}(\tilde{E}\tilde{F}) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(\tilde{E}) \kappa_{i_0 - \eta(p, p') - i}^{p'}(\tilde{F}).$$

An application of Lemma 9.3 shows that the sequence $\left(eval_{(m)}^N(\tilde{E}_N \tilde{F}_N) \right)_{N \in \mathbb{N}}$ converges up to the m^{th} order of fluctuations. As $eval_{(m)}^N$ is a morphism of algebra, Lemma 9.2, for any $N \in \mathbb{N}$,

$$eval_{(m)}^N(\tilde{E}_N \tilde{F}_N) = eval_{(m)}^N(\tilde{E}_N) eval_{(m)}^N(\tilde{F}_N) = E_N F_N.$$

We deduce that $(E_N F_N)_{N \in \mathbb{N}}$ converges strongly up to the m^{th} order of fluctuations. The equality (22) is deduced from (24).

In order to prove the equality (23), the best way is to come back to the definitions, and do a proof similar to the one for (16) in Theorem 7.1. \square

Let us consider the implication of Proposition 9.3 for semi-groups in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. From now on, let us suppose that $((E_t^N)_N)_{t \geq 0}$ is a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ whose generator is $(H_N)_{N \in \mathbb{N}}$. We would like to state a theorem for the fluctuations of $((E_t^N)_N)_{t \geq 0}$ similar to Theorem 7.2. For this, we need the following definition.

Definition 9.10. — The semi-group $((E_t^N)_N)_{t \geq 0}$ converges to the m^{th} order of fluctuations if and only if for any $t \geq 0$, $(E_t^N)_{N \in \mathbb{N}}$ converges up to the m^{th} order of fluctuations.

Now we can state the theorem about the convergence to the m^{th} order of fluctuations of a semi-group in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. The proof will not be given, as it is a direct consequence of Proposition 9.3 with a lift-argument as for the last proof.

Theorem 9.3. — The semi-group $((E_t^N)_N)_{t \geq 0}$ converges to the m^{th} order of fluctuations if the sequences $(E_0^N)_{N \in \mathbb{N}}$ and $(H_N)_{N \in \mathbb{N}}$ converge up to the m^{th} order of fluctuations. Besides, we have the two differential systems of equations:

– for any $p_0 \in A_k$, for any $t_0 \geq 0$, for any $i_0 \in \{0, \dots, m\}$:

$$\frac{d}{dt} \Big|_{t=t_0} \kappa_{i_0}^{p_0}(E_t) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(H) \kappa_{i_0 - \eta(p, p') - i}^{p'}(E_{t_0}).$$

– for any $p_0 \in \mathcal{P}_k$, for any $t_0 \geq 0$, for any $i \in \{0, \dots, m\}$:

$$\frac{d}{dt} \Big|_{t=t_0} m_{p_0}^{i_0}(E_t) = \sum_{p_1 \in A_k} \sum_{i+j+\text{df}(p_1, p_0)=i_0} \kappa_i^{p_1}(H_t) m_{t_{p_1 \circ p}}^j(E_{t_0}).$$

10. An introduction to the general \mathcal{R} -transform

10.1. The zero order. — Up to now, we only worked with partitions which have a fixed length: we worked in A_k for a fixed integer k . Yet, we could have worked with $A_\infty = \cup_{k \in \mathbb{N}} A_k$ endowed with the product: $pp' = \delta_{l(p)=l(p')} pp'$ where we recall that $l(p)$ is the length of p . With this definition, we see that all the results hold when one changes k by $k = \infty$. For example $\mathbb{C}[A_\infty(N, N)]$ converges when N goes to infinity to an algebra $\mathbb{C}[A_\infty(\infty, \infty)]$.

10.1.1. Order zero: general definitions and Lie algebras. — Recall that A is either \mathfrak{S} , \mathfrak{B} or \mathcal{P} .

Definition 10.1. — Let us define the algebra $\mathfrak{E}_g[A] = \prod_{k=0}^{\infty} \mathbb{C}[A_k(\infty, \infty)]$. Two subspaces of $\mathfrak{E}_g[A]$ will be interesting for us:

$$\mathfrak{E}[A] = \{E \in \mathfrak{E}_g[A], E_\emptyset = 1\} \text{ and } \mathfrak{e}[A] = \{E \in \mathfrak{E}_g[A], E_\emptyset = 0\}.$$

Any element $E \in \mathfrak{E}_g[A]$ is of the form $E = \left(\sum_{p \in A_k} (E_k)_{pp} \right)_{k \in \mathbb{N}}$. In order to simplify the notations, we will use the following conventions: for any integer k , for any $p \in A_k$,

$$E_p = E(p) = (E_k)_p,$$

and for any positive integer k :

$$E_k = \sum_{p \in A_k} E_p p.$$

The algebra $\mathfrak{E}_g[A]$ is naturally endowed with a natural addition and multiplication given, for any $E, F \in \mathfrak{E}_g[A]$ and any $k \in \mathbb{N}^*$ by:

$$\begin{aligned}(E + F)_k &= E_k + F_k \\ (E \boxtimes F)_k &= E_k F_k.\end{aligned}$$

By convention $(E \boxtimes F)_\emptyset = E_\emptyset F_\emptyset$. Besides, one can construct an other law on $\mathfrak{E}_g[A]$.

Definition 10.2. — *Let E and F be two elements of $\mathfrak{E}_g[A]$. We denote by $E \boxplus F$ the element of $\mathfrak{E}_g[A]$ such that for any $p \in A_{1(p)}$:*

$$(E \boxplus F)_p = \sum_{(p_1, p_2, I) \in \mathfrak{F}_2(p)} E(p_1)F(p_2),$$

where $\mathfrak{F}_2(p)$ was defined in Definition 2.11.

In fact, the two operations \boxtimes and \boxplus are convolution operations.

Remark 10.1. — *The sets $\mathfrak{E}[A]$ and $\mathfrak{e}[A]$ are stable by the \boxplus and \boxtimes operations. Besides, $\mathfrak{E}[A]$ is an affine space whose underlying vector space is $\mathfrak{e}[A]$.*

The operation \boxplus on $\mathfrak{E}[A]$ is commutative, it defines a structure of group on $\mathfrak{E}[A]$. The neutral element $0_{\mathfrak{E}}$ is the only element in $\mathfrak{E}[A]$ such that for any $k \in \mathbb{N}^*$, any $p \in A_k$, $(0_{\mathfrak{E}})_p = 0$.

The operation \boxtimes is not commutative and the set of invertible elements in $\mathfrak{E}[A]$ is the set of elements E such that $E_{id_k} \neq 0$ for any $k \geq 1$, we denote it by $G\mathfrak{E}[A]$. We denote by $1_{\mathfrak{E}}$ the neutral element for \boxtimes which is the only element such that for any $k \geq 1$, $(1_{\mathfrak{E}})_k = id_k$.

Let us consider an interesting sub-vector space of $\mathfrak{E}_g[A]$. Recall the notation $A_k^{(i)}$ defined in Definition 2.10.

Definition 10.3. — *We define $\mathfrak{E}_g^{(i)}[A] = \prod_{k=0}^{\infty} \mathbb{C} [A_k^{(i)}]$. Two subspaces of $\mathfrak{E}_g[A]$ will be interesting for us:*

$$\mathfrak{E}^{(i)}[A] = \mathfrak{E}_g^{(i)}[A] \cap \mathfrak{E}[A] \text{ and } \mathfrak{e}^{(i)}[A] = \mathfrak{E}_g^{(i)}[A] \cap \mathfrak{e}[A].$$

When $A = \mathfrak{S}$, we have already seen after Definition 2.10 that

$$A_k^{(i)} = \{\sigma^{-1}(1, \dots, k)\sigma, \sigma \in \mathfrak{S}_k\}.$$

Let us consider $(\mathfrak{E}^{(i)}[A])^{\mathfrak{S}}$ the set of elements of $\mathfrak{E}^{(i)}[A]$ which are invariant by conjugation by any permutation: this means that for any positive integer k and any $\sigma \in \mathfrak{S}_k$, the following equality holds in $\mathbb{C}[A_k(N)]$ for any positive integer N :

$$\sigma E_k \sigma^{-k} = E_k,$$

Proposition 10.1. — *The affine space $(\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}}$ can be identified with the affine space $\mathbb{C}_1[[z]]$ of formal power series which constant term is equal to 1 by the following*

isomorphism :

$$\begin{aligned} (\mathfrak{E}^{(i)}[\mathfrak{G}])^{\mathfrak{G}} &\rightarrow \mathbb{C}_1[[z]] \\ E &\mapsto \sum_{k \in \mathbb{N}} E_{(1, \dots, k)} z^k. \end{aligned}$$

Any element E in $\mathfrak{E}_g[A]$ can be restricted in order to obtain an element of $\mathfrak{E}_g^{(i)}[A]$ that we denote by $E|_{\mathfrak{E}^i[A]}$. Conversely, given an element of $\mathfrak{E}_g^{(i)}[A]$, one can inject it non-trivially in $\mathfrak{E}_g[A]$ in a natural way. Recall the definition of the extraction of p in Definition 2.12, and the definition of cycles given in Definition 2.9. We only consider the injection of an element of $\mathfrak{E}^{(i)}[A]$ in $\mathfrak{E}[A]$.

Definition 10.4. — For any $E \in \mathfrak{E}^{(i)}[A]$, we denote by $\mathfrak{M}(E)$ the unique element of $\mathfrak{E}[A]$ such that for any integer k , any $p \in A_k$,

$$(\mathfrak{M}(E))_p = \prod_{C \in \mathcal{C}(p)} E_{p_C}.$$

Any element of the image of the application:

$$\begin{aligned} \mathfrak{M} : \mathfrak{E}^{(i)}[A] &\rightarrow \mathfrak{E}[A] \\ E &\mapsto \mathfrak{M}(E) \end{aligned}$$

is called multiplicative and we denote $\mathfrak{M}\mathfrak{E}[A] = \mathfrak{M}[\mathfrak{E}[A]]$.

Let us remark that $0_{\mathfrak{E}}$ and $1_{\mathfrak{E}}$ are multiplicative elements. This is not the only property satisfied by $\mathfrak{M}\mathfrak{E}[A]$.

Theorem 10.1. — The set $\mathfrak{M}\mathfrak{E}[A]$ is stable by the operations \boxplus and \boxtimes .

Proof. — Let E and F be two elements of $\mathfrak{M}\mathfrak{E}[A]$. Let us show that $E \boxplus F$ is multiplicative. Let p_1 and p_2 be two partitions, we have to show that:

$$(25) \quad (E \boxplus F)_{p_1 \otimes p_2} = (E \boxplus F)_{p_1} (E \boxplus F)_{p_2}.$$

Yet, by definition:

$$(E \boxplus F)_{p_1 \otimes p_2} = \sum_{(a_1, a_2, I) \in \mathfrak{F}_2(p_1 \otimes p_2)} E_{a_1} F_{a_2},$$

and:

$$(E \boxplus F)_{p_1} (E \boxplus F)_{p_2} = \sum_{(a_1^1, a_2^1, I^1) \in \mathfrak{F}_2(p_1), (a_1^2, a_2^2, I^2) \in \mathfrak{F}_2(p_2)} E_{a_1^1} E_{a_1^2} F_{a_2^1} F_{a_2^2}.$$

Using the fact that E and F are multiplicative, that $E_{\emptyset} = 1 = F_{\emptyset}$ and using the fact that for any $(a_1, a_2, I) \in \mathfrak{F}_2(p_1 \otimes p_2)$, a_1 and a_2 can be decomposed uniquely into two parts in order to get two 3-tuples $(a_1^1, a_2^1, I^1) \in \mathfrak{F}_2(p_1)$ and $(a_1^2, a_2^2, I^2) \in \mathfrak{F}_2(p_2)$, one gets the Equality (25).

Let us show that $E \boxtimes F$ is multiplicative. Let p_1 and p_2 be two partitions, we have to show that:

$$(E \boxtimes F)_{p_1 \otimes p_2} = (E \boxtimes F)_{p_1} (E \boxtimes F)_{p_2}.$$

By definition:

$$(E \boxtimes F)_{p_1 \otimes p_2} = \sum_{a, b / a \circ b = p_1 \otimes p_2, a \prec p_1 \otimes p_2} E_a F_b.$$

Yet, using Lemma 6.4, any partition a such that $a \prec p_1 \otimes p_2$ can be decomposed uniquely as $a_1 \otimes a_2$ such that $a_1 \prec p_1$ and $a_2 \prec p_2$. Then if b is a partition such that $a_1 \otimes a_2 \circ b = p_1 \otimes p_2$, b can be also decomposed uniquely as $b = b_1 \otimes b_2$ with $a_1 \otimes b_1 = p_1$ and $a_2 \otimes b_2 = p_2$. Using the multiplicative property of E and F , one gets:

$$\begin{aligned} (E \boxtimes F)_{p_1 \otimes p_2} &= \sum_{a_1, a_2, b_1, b_2 / a_1 \circ b_1 = p_1, a_2 \circ b_2 = p_2, a_1 \prec p_1, a_2 \prec p_2} E_{a_1} E_{a_2} F_{b_1} F_{b_2} \\ &= \sum_{a_1, b_1 / a_1 \circ b_1 = p_1, a_1 \prec p_1} E_{a_1} F_{b_1} \sum_{a_2, b_2 / a_2 \circ b_2 = p_2, a_2 \prec p_2} E_{a_2} F_{b_2} \\ &= (E \boxtimes F)_{p_1} (E \boxtimes F)_{p_2}. \end{aligned}$$

This ends the proof. \square

Let us justify our notation \boxplus . If we consider the pull-back of the \boxplus operation from $\mathfrak{M}\mathfrak{E}[A]$ to $\mathfrak{E}^{(i)}[A]$ and if one consider only the coefficients for the non-empty partitions, one simply obtains the usual additive law on $\mathfrak{E}^{(i)}[A]$. We will also see in the article [9] that \boxplus is the natural operation which appears when one is working with sum of free elements.

We believe that the inverse of a multiplicative element for the \boxplus and \boxtimes is still multiplicative, but we have not yet written the proof. It is natural to wonder, as we have two semi-groups $(\mathfrak{M}\mathfrak{E}[A], \boxplus)$ and $(\mathfrak{M}\mathfrak{E}[A] \cap G\mathfrak{E}_A, \boxtimes)$ on which one can define differentiable one-parameter semi-groups, what are the ‘‘Lie algebras’’ of these two semi-groups. Let us remark that $\mathfrak{M}\mathfrak{E}[A] \cap G\mathfrak{E}_A$ is only the set of elements E of $\mathfrak{M}\mathfrak{E}[A]$ such that $E_{id_1} \neq 0$. We need to define two ways to inject $\mathfrak{e}^{(i)}[A]$ in $\mathfrak{e}[A]$, the first of which is the natural injection.

Definition 10.5. — For any $E \in \mathfrak{e}^{(i)}[A]$, we denote by $\mathfrak{J}(E)$ the unique element of $\mathfrak{e}[A]$ such that, for any positive integer k , any irreducible $p \in A_k$,

$$(\mathfrak{J}(E))_p = E_p,$$

and for any non-irreducible $p \in A_k$, $(\mathfrak{J}(E))_p = 0$. We define $\mathfrak{m}\mathfrak{e}_{\boxplus}[A] = \mathfrak{J}(\mathfrak{E}^{(i)}[A])$.

The second injection uses the notion of support of a partition and the notion of weakly irreducible partitions defined in Definition 2.13. Recall also the notion of extraction defined in Definition 2.12.

Definition 10.6. — For any $E \in \mathfrak{e}^{(i)}[A]$, we denote by $\mathfrak{J}(E)$ the unique element of $\mathfrak{e}[A]$ such that, for any integer k , any weakly irreducible $p \neq id_k$ in A_k :

$$(\mathfrak{J}(E))_p = E(pS(p)),$$

and $(\mathfrak{J}(E))_{id_k} = k (\mathfrak{J}(E))_{id_1}$ and for any other $p \in A_k$, $(\mathfrak{J}(E))_p = 0$. We define $\mathfrak{m}\mathfrak{e}_{\boxtimes}[A] = \mathfrak{J}(\mathfrak{E}^{(i)}[A])$.

Due to the definitions, it is obvious that the sets $\mathfrak{me}_{\boxtimes}[A]$ and $\mathfrak{me}_{\boxplus}[A]$ are vector spaces. Let us define the exponentiation of any element of $\mathfrak{e}[A]$ associated with the operation \boxplus .

Definition 10.7. — Let $E \in \mathfrak{e}[A]$. The \boxplus -semi group associated with E is the family $(e_{\boxplus}^{tE})_{t \geq 0}$ of elements of $\mathfrak{E}[A]$ such that for any $t_0 \geq 0$:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} e_{\boxplus}^{tE} &= E \boxplus e_{\boxplus}^{t_0 E}, \\ e_{\boxplus}^{0E} &= 0_{\mathfrak{e}}. \end{aligned}$$

Due to the commutativity of \boxplus , one has that for any $E, F \in \mathfrak{e}[A]$, $e_{\boxplus}^E \boxplus e_{\boxplus}^F = e_{\boxplus}^{E \boxplus F}$. Let us define the exponentiation associated with the operation \boxtimes .

Definition 10.8. — Let $E \in \mathfrak{e}[A]$. The \boxtimes -semi group associated with E is the family $(e_{\boxtimes}^{tE})_{t \geq 0}$ of elements of $\mathfrak{E}[A]$ such that for any $t_0 \geq 0$:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} e_{\boxtimes}^{tE} &= E \boxtimes e_{\boxtimes}^{t_0 E}, \\ e_{\boxtimes}^{0E} &= 1_{\mathfrak{e}}. \end{aligned}$$

We defined e_{\boxplus}^{tE} and e_{\boxtimes}^{tE} as a one-parameter semi-group for two reasons: it will appear later in this formulation, and it allows us to have a Lie group/Lie algebra formalism. As noticed by G. Cébron, an equivalent definition is given by the next proposition.

Proposition 10.2. — Let $E \in \mathfrak{e}[A]$, for any $t \in \mathbb{R}^+$,

$$e_{\boxplus}^{tE} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{\boxplus n} \quad \text{and} \quad e_{\boxtimes}^{tE} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{\boxtimes n},$$

where $E^{\boxplus 0} = 0_{\mathfrak{e}}$ and $E^{\boxtimes 0} = 1_{\mathfrak{e}}$.

Actually, we will use implicitly this fact when we will have to compute a element of the form e_{\boxplus}^{tE} in the article [9]. Besides, if one wants to make everything explicit, for example this implies that for any $t \in \mathbb{R}^+$, any positive integer k , any $p \in A_k$ and any $E \in \mathfrak{e}[A]$,

$$(e_{\boxtimes}^{tE})_p = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{(p_1, \dots, p_n) \in A_k, p_1 \prec p_1 \circ p_2 \prec \dots \prec p_1 \circ \dots \circ p_n / p_1 \circ \dots \circ p_n = p} E_{p_1} E_{p_2} \dots E_{p_n}.$$

The next theorem shows that $\mathfrak{me}_{\boxplus}[A]$ and $\mathfrak{me}_{\boxtimes}[A]$ are the Lie algebras of respectively $(\mathfrak{ME}[A], \boxplus)$ and $(\mathfrak{ME}[A] \cap G\mathfrak{E}_A, \boxtimes)$.

Theorem 10.2. — Let $E \in \mathfrak{me}_{\boxplus}[A]$, for any $t \geq 0$,

$$e_{\boxplus}^{tE} \in \mathfrak{ME}[A].$$

Besides for any differentiable one-parameter semi-group $(E^t)_{t \geq 0}$ in $(\mathfrak{ME}[A], \boxplus)$ such that $E^0 = 0_{\mathfrak{e}}$, there exists $E \in \mathfrak{me}_{\boxplus}[A]$ such that for any $t \geq 0$,

$$e_{\boxplus}^{tE} = E^t.$$

Let $E \in \mathfrak{m}\mathfrak{e}_{\boxtimes}[A]$. For any $t \geq 0$,

$$e_{\boxtimes}^{tE} \in \mathfrak{M}\mathfrak{E}[A].$$

Besides for any differentiable one-parameter semi-group $(E^t)_{t \geq 0}$ in $(\mathfrak{M}\mathfrak{E}[A], \boxtimes)$ such that $E^0 = 1_{\mathfrak{e}}$, there exists $E \in \mathfrak{m}\mathfrak{e}_{\boxtimes}[A]$ such that for any $t \geq 0$,

$$e_{\boxtimes}^{tE} = E^t.$$

Proof. — Before doing the proof, let us give the two general ideas that we will use.

1. Let $(E^t)_{t \geq 0}$ be a differentiable family of elements of $\mathfrak{E}[A]$ such that $E^0 = 0_{\mathfrak{e}}$ or $E^0 = 1_{\mathfrak{e}}$. In order to prove that $E^t \in \mathfrak{M}\mathfrak{E}[A]$ for any real $t \geq 0$, it is enough to show that $(E^t)_{t \geq 0}$ and $\left(\mathfrak{M}\left(E^t_{|\mathfrak{e}^{(i)}[A]}\right)\right)_{t \geq 0}$ satisfy the same differential linear equations.
2. Let $(E^t)_{t \geq 0}$ be a differentiable one-parameter semi-group for the \boxplus operation (resp. \boxtimes operation), which is in $\mathfrak{M}\mathfrak{E}[A]$ and which starts at $0_{\mathfrak{e}}$ (resp. $1_{\mathfrak{e}}$). In order to prove that there exists $E \in \mathfrak{m}\mathfrak{e}_{\boxplus}[A]$ (resp. $E \in \mathfrak{m}\mathfrak{e}_{\boxtimes}[A]$) such that for any $t \geq 0$, $e_{\boxplus}^{tE} = E^t$ (resp. $e_{\boxtimes}^{tE} = E^t$), it is enough to show that:

$$\left((E^t)_{t \geq 0} \text{ and } \left(e_{\boxplus}^{t\mathfrak{J}}\left(\left(\frac{d}{dt}\Big|_{t=0} E^t\right)_{|\mathfrak{e}^{(i)}[A]}\right) \right)_{t \geq 0} \right. \\ \left. \left(\text{resp. } (E^t)_{t \geq 0} \text{ and } \left(e_{\boxtimes}^{t\mathfrak{J}}\left(\left(\frac{d}{dt}\Big|_{t=0} E^t\right)_{|\mathfrak{e}^{(i)}[A]}\right) \right)_{t \geq 0} \right) \right)$$

satisfy the same differential linear equations.

Let us prove Theorem 10.2. Let $E \in \mathfrak{m}\mathfrak{e}_{\boxplus}[A]$. For any $t \geq 0$ we consider $E^t = e_{\boxplus}^{tE}$. Let n be a positive integer and let us consider n irreducible partitions p_1, \dots, p_n in $\cup_{k \in \mathbb{N}^*} A_k^{(i)}$. For any real $t_0 \geq 0$, we have:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_0} E^t_{p_1 \otimes \dots \otimes p_n} &= (E \boxplus E^{t_0})_{p_1 \otimes \dots \otimes p_n} \\ &= \sum_{(p'_1, p'_2, I) \in \mathfrak{F}_2(p_1 \otimes \dots \otimes p_n)} E(p'_1) E^{t_0}(p'_2). \end{aligned}$$

Yet, we must not forget that E is in $\mathfrak{m}\mathfrak{e}_{\boxplus}[A]$: for any integer k , any $p \in \mathcal{P}_k$, if p is not irreducible or if $p = \emptyset$, then $E(p) = 0$. Thus the sum we are considering can be taken over the $(p'_1, p'_2, I) \in \mathfrak{F}_2(p_1 \otimes \dots \otimes p_n)$ such that p'_1 is irreducible and not equal to \emptyset : this means in particular that $p'_{1S(p'_1)}$ is one of the $(p_i)_{i=1}^n$. Thus:

$$\frac{d}{dt}\Big|_{t=t_0} E^t_{p_1 \otimes \dots \otimes p_n} = \sum_{i=1}^n E(p_i) E^{t_0}_{p_1 \otimes \dots \otimes p_{i-1} \otimes p_{i+1} \otimes \dots \otimes p_n}.$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1}^t \cdots E_{p_n}^t) &= \sum_{i=1}^n \left(\frac{d}{dt}|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \sum_{i=1}^n E(p_i) \prod_{j \neq i} E_{p_j}^{t_0}. \end{aligned}$$

This allows us to conclude that $E^t \in \mathfrak{ME}[A]$ for any real $t \geq 0$.

Let $(E^t)_{t \geq 0}$ be a differentiable one-parameter semi-group for the \boxplus operation which is in $\mathfrak{ME}[A]$ and such that $E^0 = 0_{\mathfrak{E}}$. Then using the same calculation that we did, for any integer n and any irreducible partitions p_1, \dots, p_n in $\cup_{k \in \mathbb{N}^*} A_k^{(i)}$, for any real $t_0 \geq 0$, we have:

$$\frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) = \frac{d}{dt}|_{t=t_0} (E_{p_1}^t \cdots E_{p_n}^t) = \sum_{i=1}^n \left(\frac{d}{dt}|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0}.$$

Yet p_i is irreducible, thus using the fact that $\frac{d}{dt}|_{t=0} E_{\emptyset}^t = 0$, we get that $\frac{d}{dt}|_{t=t_0} E_{p_i}^t = \left(\left(\frac{d}{dt}|_{t=0} E^t \right) \boxplus E^{t_0} \right)_{p_i} = \frac{d}{dt}|_{t=0} E_{p_i}^t$, and thus:

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) &= \sum_{i=1}^n \left(\frac{d}{dt}|_{t=0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \left(\mathfrak{J} \left(\left(\frac{d}{dt}|_{t=0} E^t \right)_{|\mathfrak{E}^{(i)}[A]} \right) \boxplus E^{t_0} \right)_{p_1 \otimes \dots \otimes p_n} \end{aligned}$$

and thus there exists $E \in \mathfrak{me}_{\boxplus}[A]$ such that for any $t \geq 0$, $e_{\boxplus}^{tE} = E^t$.

Now, let $E \in \mathfrak{me}_{\boxtimes}[A]$. For any $t \geq 0$ we consider $E^t = e_{\boxtimes}^{tE}$. Let n be an integer and let us consider n irreducible partitions p_1, \dots, p_n in $\cup_{k \in \mathbb{N}^*} A_k^{(i)}$. For any real $t_0 \geq 0$, we have:

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} E_{p_1 \otimes \dots \otimes p_n}^t &= (E \boxtimes E^{t_0})_{p_1 \otimes \dots \otimes p_n} \\ &= \sum_{a, b / a \circ b = p_1 \otimes \dots \otimes p_n, a \prec p_1 \otimes \dots \otimes p_n} E_a E_b^{t_0}. \end{aligned}$$

Yet, we must not forget that E is in $\mathfrak{me}_{\boxtimes}[A]$: for any integer k , any $p \in \mathcal{P}_k$, if p is not weakly irreducible then $E(p) = 0$. Thus the sum we are considering can be taken over the couples (a, b) such that a is weakly irreducible. Besides, $E_{id_l} = lE_{id_1}$ for any integer l . Thus:

$$E_{id_{\sum_{i=1}^n l(p_i)}} E_{p_1 \otimes \dots \otimes p_n}^{t_0} = \sum_{i=1}^n E_{id_{l(p_i)}} E_{p_1 \otimes \dots \otimes p_n}^{t_0}.$$

Thus, we get:

$$\frac{d}{dt}|_{t=t_0} E_{p_1 \otimes \dots \otimes p_n}^t = \sum_{i=1}^n \sum_{a, b / a \circ b = p_i, a \prec p_i} E_a E_{p_1 \otimes \dots \otimes p_{i-1} \otimes b \otimes p_{i+1} \otimes \dots \otimes p_n}^{t_0}.$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1}^t \cdots E_{p_n}^t) &= \sum_{i=1}^n \left(\frac{d}{dt}|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \sum_{i=1}^n \sum_{a,b|a \circ b = p_i, a \prec p_i} E_a E_b^{t_0} \prod_{j \neq i} E_{p_j}^{t_0}. \end{aligned}$$

This allows us to conclude that $E^t \in \mathfrak{ME}[A]$ for any real $t \geq 0$.

Let $(E^t)_{t \geq 0}$ be a differentiable one-parameter semi-group for the \boxtimes operation which is in $\mathfrak{ME}[A]$ and such that $E^0 = 1_{\mathfrak{e}}$. Then using the same calculation that we did, for any integer n and any irreducible partitions p_1, \dots, p_n in $\cup_{k \in \mathbb{N}^*} A_k^{(i)}$, for any real $t_0 \geq 0$, we have:

$$\frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) = \frac{d}{dt}|_{t=t_0} (E_{p_1}^t \cdots E_{p_n}^t) = \sum_{i=1}^n \left(\frac{d}{dt}|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0}.$$

Yet for any $i \in \{1, \dots, n\}$, $\frac{d}{dt}|_{t=t_0} E_{p_i}^t = \sum_{a,b|a \circ b = p_i, a \prec p_i} \frac{d}{dt}|_{t=0} E_{p_i}^t E_b^{t_0}$, and thus:

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) &= \sum_{i=1}^n \left(\sum_{a,b|a \circ b = p_i, a \prec p_i} \frac{d}{dt}|_{t=0} E_{p_i}^t E_b^{t_0} \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \left(\mathfrak{J} \left(\left(\frac{d}{dt}|_{t=0} E_t \right) |_{\mathfrak{e}^{(i)}[A]} \right) \boxtimes E^{t_0} \right)_{p_1 \otimes \dots \otimes p_n} \end{aligned}$$

and thus there exists $E \in \mathfrak{me}_{\boxtimes}[A]$ such that for any $t \geq 0$, $e_{\boxtimes}^{tE} = E^t$. \square

Remark 10.2. — In fact, $\mathfrak{e}[A]$ is endowed with two structures of Lie algebras. Indeed, it is a vector space for the addition and multiplication by a scalar, and we can define two Lie brackets on it, one named $[\cdot, \cdot]_{\boxplus}$ which comes from the \boxplus operation and the other named $[\cdot, \cdot]_{\boxtimes}$ which comes from the \boxtimes operation. In order to know which bracket is considered on $\mathfrak{e}[A]$, we will denote it either by $\mathfrak{e}_{\boxplus}[A]$ or by $\mathfrak{e}_{\boxtimes}[A]$.

Since the operation \boxplus is commutative, the bracket $[\cdot, \cdot]_{\boxplus}$ is trivial. Thus \mathfrak{me}_{\boxplus} is a sub-Lie algebra of \mathfrak{e}_{\boxplus} . Since the operation \boxtimes is not commutative, the bracket $[\cdot, \cdot]_{\boxtimes}$ is not trivial and for any E and F in $\mathfrak{e}_{\boxtimes}[A]$,

$$[E, F]_{\boxtimes} = E \boxtimes F - F \boxtimes E.$$

Then, it is not difficult to see directly that $\mathfrak{me}_{\boxtimes}[A]$ is a sub-Lie algebra of $\mathfrak{e}_{\boxtimes}[A]$.

An element E belongs to $\mathfrak{me}_{\boxplus}[A]$ or $\mathfrak{me}_{\boxtimes}[A]$ if some conditions on its coordinates are satisfied. It would be interesting to know that are the conditions on the moments of E which allows us to know if E is in $\mathfrak{me}_{\boxplus}[A]$ or $\mathfrak{me}_{\boxtimes}[A]$. In the Section 10.2 of [9], we prove the following theorem.

Theorem 10.3. — Let E be in $\mathfrak{e}[A]$. Following the notations of Theorem 8.2, for any positive integer k , any $p \in \mathcal{P}_k$, we define:

$$m_p^\infty(E) = \sum_{p' \in A_k} \delta_{p' \in [id, p_0]_{A_k}} E_{p'}.$$

We have the following characterization of $\mathfrak{me}_{\boxplus}[A]$ and $\mathfrak{me}_{\boxtimes}[A]$.

1. $E \in \mathfrak{me}_{\boxplus}[A]$ if and only if for any positive integer k and l , for any $p_1 \in A_k$ and any $p_2 \in A_l$:

$$m_{p_1 \otimes p_2}^\infty(E) = 0.$$

2. $E \in \mathfrak{me}_{\boxtimes}[A]$ if and only if for any positive integer k and l , for any $p_1 \in A_k$ and any $p_2 \in A_l$:

$$m_{p_1 \otimes p_2}^\infty(E) = m_{p_1}^\infty(E) + m_{p_2}^\infty(E).$$

10.1.2. *The \mathcal{R}_A -transform.* — We will define the notion of \mathcal{R}_A -transform. This application will be defined as the inverse of the \mathcal{M}_A -transform whose definition lies on the Equation (7).

Definition 10.9. — The \mathcal{M}_A -transform is the application:

$$\begin{aligned} \mathcal{M}_A : \mathfrak{E}[A] &\rightarrow \mathfrak{E}[A] \\ E &\rightarrow \mathcal{M}_A(E) \end{aligned}$$

such that for any $E \in \mathfrak{E}[A]$, for any integer k , any $p \in A_k$:

$$(\mathcal{M}_A(E))_p = \sum_{p' \in [id, p]_{A_k}} E_{p'}.$$

This application is a bijection. Thus we can consider its inverse.

Definition 10.10. — The \mathcal{R}_A -transform is the inverse of the \mathcal{M}_A -transform:

$$\mathcal{R}_A = \mathcal{M}_A^{-1}.$$

We will often forget about the indices A when we will work with the \mathcal{R} -transforms. One can show that the \mathcal{R}_A -transform is a bijection from $\mathfrak{ME}[A]$ to itself.

Proposition 10.3. — The \mathcal{R}_A -transform is a bijection from $\mathfrak{ME}[A]$ to itself.

Proof. — We recall that the \mathcal{R}_A -transform is, by definition, a bijection from $\mathfrak{E}[A]$ to itself. Let $E \in \mathfrak{ME}[A]$, we have to show that $\mathcal{M}_A[E] \in \mathfrak{ME}[A]$ and $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$. Let k and l be two positive integers. Let $p_1 \in \mathcal{P}_k$ and $p_2 \in \mathcal{P}_l$.

Let us show that $\mathcal{M}_A[E] \in \mathfrak{ME}[A]$. Using Lemma 3.8 and the multiplicative property of E , we have:

$$\begin{aligned} (\mathcal{M}_A[E])_{p_1 \otimes p_2} &= \sum_{p' \in [id, p_1 \otimes p_2]_{A_k}} E_{p'} = \sum_{p'_1 \in [id, p_1]_{A_k}, p'_2 \in [id, p_2]_{A_k}} E_{p'_1 \otimes p'_2} \\ &= \sum_{p'_1 \in [id, p_1]_{A_k}, p'_2 \in [id, p_2]_{A_k}} E_{p'_1} E_{p'_2} \\ &= \sum_{p'_1 \in [id, p_1]_{A_k}} E_{p'_1} \sum_{p'_2 \in [id, p_2]_{A_k}} E_{p'_2} \\ &= (\mathcal{M}_A[E])_{p_1} (\mathcal{M}_A[E])_{p_2}. \end{aligned}$$

Now, let us show that $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$. Let us consider $(\tilde{E}_p)_{p \in \bigcup_{k \in \mathbb{N}^*} A_k^{(i)}}$ such that for any positive integer k , any $p \in A_k^{(i)}$,

$$E_p = \sum_{p' \in [id, p]_{A_k}} \prod_{c \in \mathcal{C}(p')} \tilde{E}_{p_c}.$$

Using the multiplicativity of E , and Lemma 3.8, we see that E being in $\mathfrak{ME}[A]$, the family $(\tilde{E}_p)_{p \in \bigcup_{k \in \mathbb{N}} A_k^{(i)}}$ satisfies in fact that for any integer k , any $p \in A_k$:

$$E_p = \sum_{p' \in [id, p]_{A_k}} \prod_{c \in \mathcal{C}(p')} \tilde{E}_{p_c}.$$

Thus $\prod_{c \in \mathcal{C}(p')} \tilde{E}_{p_c}$ is equal to $(\mathcal{R}_A[E])_{p'}$ and thus $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$. \square

We can also translate the Lemma 3.9 in terms of \mathcal{R} -transform. Recall Definition 4.7. We denote by $\mathcal{R}_{|\mathfrak{S}}$ the restriction function from $\mathfrak{E}[\mathcal{B}]$ to $\mathfrak{E}[\mathfrak{S}]$ such that for any $E \in \mathfrak{E}[\mathcal{B}]$, for any $k \in \mathbb{N}$,

$$(\mathcal{R}_{|\mathfrak{S}}(E))_k = (\mathcal{R}_{|\mathfrak{S}_k} \circ E^{\mathcal{B}_k})(E_k).$$

Proposition 10.4. — *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{B}] & \xrightarrow{\mathcal{R}_{\mathcal{B}}} & \mathfrak{E}[\mathcal{B}] \\ \mathcal{R}_{|\mathfrak{S}} \downarrow & & \downarrow \mathcal{R}_{|\mathfrak{S}} \\ \mathfrak{E}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

Proof. — It is only a consequence of the fact that:

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{B}] & \xleftarrow{\mathcal{M}_{\mathcal{B}}} & \mathfrak{E}[\mathcal{B}] \\ \mathcal{R}_{|\mathfrak{S}} \downarrow & & \downarrow \mathcal{R}_{|\mathfrak{S}} \\ \mathfrak{E}[\mathfrak{S}] & \xleftarrow{\mathcal{M}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

is commutative. Indeed, using Lemma 3.9, if $E \in \mathfrak{E}[\mathcal{B}]$, and if $\sigma \in \mathfrak{S}_k$:

$$\mathcal{R}_{|\mathfrak{S}}[\mathcal{M}_{\mathcal{B}}(E)](\sigma) = (\mathcal{M}_{\mathcal{B}}(E))(\sigma) = \sum_{p \in [id, \sigma]_{\mathcal{B}_k}} E_p = \sum_{p \in [id, \sigma]_{\mathfrak{S}_k}} E_p = [\mathcal{M}_{\mathfrak{S}} \circ \mathcal{R}_{|\mathfrak{S}}(E)](\sigma).$$

This concludes the proof. \square

It is well-known in the literature that there exists a notion of \mathcal{R} -transform on $\mathbb{C}_1[[z]]$ which we will call the \mathcal{R}_u -transform. In order to finish this section, we make the link between the \mathcal{R}_A -transform and the \mathcal{R}_u -transform.

Definition 10.11. — *Let $M(z)$ be a formal power serie in $\mathbb{C}_1[[z]]$, that is a formal power serie of the form:*

$$M(z) = 1 + \sum_{n=1}^{\infty} a_n z^n.$$

Let $C(z)$ be the formal power serie $C(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$ such that $C[zM(z)] = M(z)$. The \mathcal{R}_u -transform of M is C .

The \mathcal{R}_A -transform is a generalization of the usual \mathcal{R}_u -transform. Indeed, we have the following theorem.

Theorem 10.4. — *Using the identification $(\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} \simeq \mathbb{C}_1[[z]]$ explained in Proposition 10.1, the following diagram is commutative:*

$$\begin{array}{ccc} (\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} & \xrightarrow{\mathcal{R}_u} & (\mathfrak{E}^{(i)}[\mathfrak{S}])^{\mathfrak{S}} \\ \mathfrak{M} \downarrow & & \downarrow \mathfrak{M} \\ \mathfrak{E}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

Proof. — Let E be an element of $\mathfrak{E}^{(i)}[\mathfrak{S}] \simeq \mathbb{C}_1[[z]]$. Using Theorem 2.7 of [16], and using the bijection between non-crossing partitions of k elements and the set $[id, (1, \dots, k)]_{\mathfrak{S}_k}$, we know that $\mathcal{R}_u(E)$ is characterized by the fact that for any integer $k > 0$:

$$E_{(1, \dots, k)} = \sum_{p \in [id, (1, \dots, k)]_{\mathfrak{S}_k}} \prod_{c \text{ cycle of } p} \mathcal{R}_u(E)(1, \dots, \#c).$$

Or, with our notations:

$$E_{(1, \dots, k)} = \sum_{p \in [id, (1, \dots, k)]_{\mathfrak{S}_k}} \mathfrak{M}[\mathcal{R}_u(E)].$$

By the factorization property of the geodesics, Lemma 3.8, for any $\sigma \in \mathfrak{S}_k$:

$$[\mathfrak{M}(E)](\sigma) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \mathfrak{M}[\mathcal{R}_u(E)](\sigma).$$

This is equivalent to the fact that $\mathcal{R}_{\mathfrak{S}}[\mathfrak{M}(E)] = \mathfrak{M}[\mathcal{R}_u(E)]$. \square

10.1.3. *Transformations linked with the exclusive moments.* — For the sake of clarity, we only consider the case where $A = \mathcal{P}$.

Definition 10.12. — *The $\mathcal{M}^{c \rightarrow}$ -transform is the application:*

$$\begin{aligned} \mathcal{M}^{c \rightarrow} : \mathfrak{E}[\mathcal{P}] &\rightarrow \mathfrak{E}[\mathcal{P}] \\ E &\mapsto \mathcal{M}^{c \rightarrow}(E), \end{aligned}$$

such that for any $E \in \mathfrak{E}[\mathcal{P}]$, for any integer k , any $p \in \mathcal{P}_k$:

$$(\mathcal{M}^{c \rightarrow}(E))_p = \sum_{p' \in \text{Gl}_c(p)} E_{p'}.$$

This application is a bijection, it is the application which transforms exclusive moments in moments. Thus we can consider its inverse. The \mathcal{M}^c -transform is the inverse of the $\mathcal{M}^{c \rightarrow}$ -transform: $\mathcal{M}^c = (\mathcal{M}^{c \rightarrow})^{-1}$. Using the same arguments than Proposition 10.3, one can proof that the \mathcal{M}^c is a bijection from $\mathfrak{ME}[\mathcal{P}]$ to itself.

Let us remark that this last proposition holds since, if p' is coarser-compatible than $p_1 \otimes p_2$ this means that there exists p'_1 and p'_2 such that $p' = p'_1 \otimes p'_2$ and such that p'_1 (resp. p'_2) is coarser-compatible than p_1 (resp. p_2). Thus, if one has defined $\mathcal{M}^{c \rightarrow}(E)$ by replacing the coarser-compatibility order by the coarser order then this proposition (and other good properties) would not have hold.

Let us define a last transformation on $\mathfrak{E}[\mathcal{P}]$.

Definition 10.13. — *The $\mathcal{M}^{\rightarrow c}$ -transform is the application:*

$$\begin{aligned} \mathcal{M}^{\rightarrow c} : \mathfrak{E}[\mathcal{P}] &\rightarrow \mathfrak{E}[\mathcal{P}] \\ E &\mapsto \mathcal{M}^{\rightarrow c}(E), \end{aligned}$$

such that for any $E \in \mathfrak{E}[\mathcal{P}]$, for any integer k , any $p \in \mathcal{P}_k$:

$$(\mathcal{M}^{\rightarrow c}(E))_p = \sum_{p' \in \text{Sp}(p)} E_{p'}$$

This is again a bijection. The applications defined above are actually linked.

Theorem 10.5. — *The following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{P}] & \xrightarrow{\mathcal{M}^{\rightarrow c}} & \mathfrak{E}[\mathcal{P}] \\ & \searrow \mathcal{M}_{\mathcal{P}} & \downarrow \mathcal{M}^{c \rightarrow} \\ & & \mathfrak{E}[\mathcal{P}] \end{array}$$

Proof. — This is a straightforward application of Theorem 3.3. □

10.2. Higher order. — In Definition 9.5, we defined the ∞ -development algebra of order m of A_k . Thus, one can also define a higher order \mathcal{R} -transform: we will only give definitions in this section. Let $m \in \mathbb{N}$ be the higher order of fluctuations which we are working with.

Definition 10.14. — Let us define the algebra $\mathfrak{E}_{g,(m)}[A] = \prod_{k=0}^{\infty} \mathbb{C}_{(m)}[A_k(\infty)]$. We also consider the subspace of $\mathfrak{E}_{g,(m)}[A]$ defined by:

$$\mathfrak{E}_{(m)}[A] = \{E \in \mathfrak{E}_{g,(m)}[A], E_{\emptyset,0} = 1, E_{\emptyset,i} = 0, \forall i \geq 1\}.$$

Let us remark that $\mathfrak{E}_{(0)}[A] = \mathfrak{E}[A]$. Any element $E \in \mathfrak{E}[A]$ is of the form:

$$\left(\sum_{p \in A_k, i \in \{0, \dots, m\}} (E_k)_{p,i} \frac{p}{X^i} \right)_{k \in \mathbb{N}}.$$

Again, in order to simplify the notations, we will use the following conventions: for any $p \in \cup_{k=0}^{\infty} A_k$ and any $i \in \{0, \dots, m\}$:

$$E_{p,i} = E^i(p) = (E_{\mathbb{1}(p)})_{p,i},$$

and for any integer k :

$$E_k = \sum_{p \in A_k, i \in \{0, \dots, m\}} E_{p,i} \frac{p}{X^i}.$$

As for $\mathfrak{E}_g[A]$, the algebra $\mathfrak{E}_{g,(m)}[A]$ is naturally endowed with a natural addition and multiplication given, for any $E, F \in \mathfrak{E}_{g,(m)}[A]$, by:

$$\begin{aligned} (E + F)_k &= E_k + F_k, \\ (E \boxtimes F)_k &= E_k F_k. \end{aligned}$$

Besides, we can also construct an other law on $\mathfrak{E}_{g,(m)}[A]$.

Definition 10.15. — Let E and F be two elements of $\mathfrak{E}_{g,(m)}[A]$. We denote by $E \boxplus F$ the element of $\mathfrak{E}_{g,(m)}[A]$ such that for any positive integer k , any $p \in A_k$ and any $i \in \{0, \dots, m\}$:

$$(E \boxplus F)^i(p) = \sum_{(p_1, p_2, I) \in \mathfrak{F}_2(p)} \sum_{i_1=1}^i E^{i_1}(p_1) F^{i-i_1}(p_2),$$

where $\mathfrak{F}_2(p)$ was defined in Definition 2.11.

Again, the subset $\mathfrak{E}_{(m)}[A]$ is stable by the \boxtimes and \boxplus operations. Besides, $\mathfrak{E}_{(m)}[A]$ is an affine space.

The operation \boxplus is commutative, it defines a structure of group on $\mathfrak{E}_{(m)}[A]$. The neutral element is the element $0_{\mathfrak{E}_{(m)}} \in \mathfrak{E}_{(m)}[A]$ such that, for any positive integer k , any $p \in A_k$ and any $i \in \{0, \dots, m\}$, $(E)_{p,i} = 0$.

The operation \boxtimes is not commutative and the set of invertible elements in $\mathfrak{E}_{(m)}[A]$ is the set of elements E such that $E_{id_k,0} \neq 0$ for any $k \geq 1$. We denote by $1_{\mathfrak{E}_{(m)}}$ the neutral element for \boxtimes which is the only element in $\mathfrak{E}_{(m)}[A]$ such that for any $k \geq 1$, $(E_k)_k = \frac{id_k}{X^0}$.

We can also define a $\mathcal{R}_A^{(m)}$ -transform. For this, we need to define the $\mathcal{M}_A^{(m)}$ -transform whose definition lies on Equality (21).

Definition 10.16. — The general $\mathcal{M}_A^{(m)}$ transform is the application:

$$\begin{aligned} \mathcal{M}_A^{(m)} : \mathfrak{E}_{(m)}[A] &\rightarrow \mathfrak{E}_{(m)}[A] \\ E &\rightarrow \mathcal{M}_A^{(m)}(E) \end{aligned}$$

such that for any $E \in \mathfrak{E}_{(m)}[A]$, for any positive integer k , any $p \in A_k$ and any $i \in \{0, \dots, m\}$:

$$\left(\mathcal{M}_A^{(m)}(E)\right)_{p,i} = \sum_{p' \in A_k, \text{df}(p',p) \leq i} E_{p',i-\text{df}(p',p)}.$$

This application is a bijection: we can consider its inverse.

Definition 10.17. — The $\mathcal{R}_A^{(m)}$ -transform is the inverse of the \mathcal{M}_A -transform:

$$\mathcal{R}_A^{(m)} = \left(\mathcal{M}_A^{(m)}\right)^{-1}.$$

11. Conclusion

We have defined a geometry on partitions, and new notions of convergence for elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$. Using Schur-Weyl's duality and similar results, we will link the study of random matrices with the study of elements in $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ and in $\mathfrak{E}[A]$. In the article [9], we apply the results proved in this article to the theory of random matrices invariant in law by conjugation by the symmetric group. We also study additive and multiplicative unitary or orthogonal invariant Lévy processes. In the article [10], we apply the results of the first two articles to the study of random walks on the symmetric group and the study of the \mathfrak{S}_∞ -Yang-Mills theory.

References

- [1] F. Benaych-Georges. Classical and free infinitely divisible distributions and random matrices. *The Annals of Probability*, 33, no. 3:1134–1170, 2005.
- [2] T. Cabanal-Duvillard. A matrix representation of the Bercovici-Pata bijection. *Electronic Journal of Probability*, 10:632–661, 2005.
- [3] M. Capitaine and M. Casalis. Geometric interpretation of the cumulants for random matrices previously defined as convolutions on the symmetric group. *Lecture Notes in Mathematics, Séminaire de Probabilités XLI*, 1934:93–119, 2008.
- [4] G. Cébron. Matricial model for the free multiplicative convolution.
- [5] G. Cébron. Free convolution operators and free hall transform. *J. Funct. Anal.*, 265:2645–2708, 2013.
- [6] G. Cébron, A. Dahlqvist, and F. Gabriel. Traffic-Partitions correspondence. *In preparation*.
- [7] A. Dahlqvist. Integration formula for Brownian motion on classical compact Lie groups. *arXiv:1212.5107*, 2012.
- [8] F. Gabriel. Combinatorial theory of permutation-invariant random matrices I: Partitions, geometry and renormalization. *arxiv*, 2015.
- [9] F. Gabriel. Combinatorial theory of permutation-invariant random matrices II: Freeness and Lévy processes. *arxiv*, 2015.

- [10] F. Gabriel. Combinatorial theory of permutation-invariant random matrices III: Random walks on \mathfrak{S}_N , ramified coverings and the \mathfrak{S}_∞ Yang-Mills measure. *arxiv*, 2015.
- [11] T. Halverson and A. Ram. Partition algebras. *European Journal of Combinatorics*, 26:869–921, 2005.
- [12] T. Lévy. The master field on the plane. *arxiv.org/abs/1112.2452*.
- [13] T. Lévy. Schur–Weyl duality and the heat kernel measure on the unitary group. *Advances in Mathematics*, 218, no. 2:537–575, 2008.
- [14] C. Male. The distributions of traffics and their free product. *arxiv.org/abs/1111.4662*, 2011.
- [15] Camille Male and Sandrine Péché. Uniform regular weighted graphs with large degree: Wigner’s law, asymptotic freeness and graphons limit. <http://arxiv.org/abs/1410.8126>, 2014.
- [16] A. Nica and R. Speicher. *Lectures on the Combinatorics of Free Probability*. Number 335. Lecture Note Series, London Mathematical Society, Cambridge University Press, 2006.
- [17] A.N. Sengupta. Traces in Two-Dimensional QCD: The Large-N Limit. *Traces in geometry, number theory and quantum fields*, 218, no. 2:193–212, 2008.
- [18] F. Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190, no. 2:287–307, 1997.

FRANCK GABRIEL • *E-mail* : franck.gabriel@normalesup.org, Université Pierre et Marie Curie (Paris 6), Laboratoire de Probabilités et Modèles Aléatoires, 4, Place Jussieu, F-75252 Paris Cedex 05.