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Franck Gabriel

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# COMBINATORIAL THEORY OF PERMUTATION-INVARIANT RANDOM MATRICES I: PARTITIONS, GEOMETRY AND RENORMALIZATION.

*by*

Franck Gabriel

E-mail: [franck.gabriel@normalesup.org](mailto:franck.gabriel@normalesup.org)

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**Abstract.** — In this article, we define and study a geometry on the set of partitions of an even number of objects. One of the definitions involves the partition algebra, a structure of algebra on the set of such partitions depending on an integer parameter  $N$ . Then we emulate the theory of random matrices in a combinatorial framework: for any parameter  $N$ , we introduce a family of linear forms on the partition algebras which allows us to define a notion of weak convergence similar to the convergence in moments in random matrices theory.

A renormalization of the partition algebras allows us to consider the weak convergence as a simple convergence in a fixed space. This leads us to the definition of a deformed partition algebra for any integer parameter  $N$  and to the definition of two transforms: the cumulants transform and the exclusive moments transform. Using an improved triangular inequality for the distance defined on partitions, we prove that the deformed partition algebras, endowed with a deformation of the linear forms converge as  $N$  go to infinity. This result allows us to prove combinatorial properties about geodesics and a convergence theorem for semi-groups of functions on partitions.

At the end we study a sub-algebra of functions on infinite partitions with finite support : a new addition operation and a notion of  $\mathcal{R}$ -transform are defined. We introduce the set of multiplicative functions which becomes a Lie group for the new addition and multiplication operations. For each of them, the Lie algebra is studied.

The appropriate tools are developed in order to understand the algebraic fluctuations of the moments and cumulants for converging sequences. This allows us to extend all the results we got for the zero order of fluctuations to any order.

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## 1. Introduction

This article is the first of a self-contained set of three articles [8], [9] and [10] on a combinatorial method in random matrices theory based on a geometry on partitions and a new point of view on usual/free cumulants based on dualities between groups and sub-algebras of partitions. This general method allows to work with random matrices which are invariant by conjugation by the symmetric group instead of the unitary or orthogonal group, besides, no more assumption about the factorization of moments is needed. The first article is about the combinatorial framework based on the partition algebra. In the second article we will apply this framework to random matrices, and the third one will put the emphasis on the random walks on the symmetric group and the link with the  $\mathfrak{S}_\infty$ -Yang-Mills measure.

This set of articles has to be considered as the continuation of what could be called the Gauge Theory School in random matrices. The article of F. Xu [18] is one of the pioneer work about this point of view on random matrices. Later, this point of view was developed by A. Sengupta [17], then highly improved by T. Lévy [12], [13], then it was used by two students of T. Lévy: A. Dahlqvist in [7] and G. Cébron [5], [4].

We wrote these articles as a lesson for graduate students with the intention that no special requirement is needed to understand them. The reader will find a new presentation and introduction to the random matrices theory. To achieve this, we only used the Gauge Theory School's papers, the seminal article for partition algebras [11], and the book [16] which, in some sense, we tried partially to generalize. Another point of view on random matrices which are invariant by conjugation by the symmetric group was given first by C. Male in his paper on traffics [14]. Yet, the goal was to develop the ideas of the Gauge Theory School and thus we did not use this article. In the forthcoming article [6], the author and his coauthor build connections between the notions developed here and the notions developed in [14]. In some sense, these articles can be seen also a bridge to go from the book [16] to the traffic interpretation of [14], traffics which have shown their importance in the study of random graphs [15]. At the moment the author was finishing these articles, he was informed of M. Capitaine and M. Casalis's work, [3], on their Schur-Weyl's interpretation of non-commutative free cumulants for unitary and orthogonal invariant random matrices.

The point of view developed in the three articles [8], [9] and [10] allows us to recover in a very simple way some famous theorems. The reader will also find in these articles a simple tool box in order to prove new convergences of random matrices (for example random walks on the symmetric group). He will also find the tools in order to understand the algebraic fluctuations of moments of random matrices. Besides, this point of view allows to define a general notion of freeness for matrices which are invariant by conjugation by the symmetric group and we construct the first non-commutative multiplicative Lévy processes for this notion of freeness. We will formulate two equivalent

definitions of this freeness: one based on cumulants, and the other on moments. This freeness notion is linked with a new  $\mathcal{R}$ -transform which generalizes the old known  $\mathcal{R}$ -transform. A Kreweras complement is defined for partitions: this generalizes the notion already set for permutations. Amongst others, we will state a matricial Wick's theorem, which allows to recover the Wick law for Gaussian Hermitian or symmetric matrices. We will also recover theorems about convergence of Hermitian Lévy processes proved in [2], [1] and unitary Lévy processes proved in [4]: we extend them to the symmetric and the orthogonal case. A new central limit theorem will be stated, which generalizes the non-commutative and the commutative central limit theorem. In the article [9], convergences of random walks on the symmetric group will be proved, and will be used in order to show that the Wilson loops of the  $\mathfrak{S}_N$ -Yang-Mills measure converge in probability when  $N$  goes to infinity. This will imply the first result known about convergence of ramified coverings on the disk. We will also see how to inject the usual theory of probabilities in this framework. This last assertion shows that one could, in this framework, study the probabilistic fluctuations.

**1.1. Renormalization and a physical point of view.** — In this article, we emulate the theory of random matrices in a combinatorial framework. Given a partition  $p$  of a number of points, and an integer  $N$ , we consider  $(p, N)$  as a physical system involving  $N$  particles. When the number of points is even, by polarizing the points in two sets, we can consider  $(p, N)$  as a discrete time transformation operation. A partition  $p$  can be seen as an elementary evolution of a system of size  $N$ : we can define the composition of two partitions. Later in the paper, we consider these discrete-time transformations also as the Hamiltonian of continuous time transformations.

An evolution of a system of size  $N$  is a linear combination of elementary evolutions of size  $N$ . Thus, every transformation is uniquely characterized by a size  $N$  and by a finite number of coefficients which, as we will see in the article, are bare quantities. Two questions arise: how to describe a system of infinite size and how to renormalize the bare quantities. As one does for perturbative renormalization, the important idea is to consider observables: we define some observables, one for each partition. In Theorem 4.1, we show how the bare coefficient must be renormalized in order to have finite observables at the limit  $N = \infty$ .

Then, we show that, by using the same renormalization, the composition of two evolutions converge also: this is proved in Theorems 6.1 and 7.1. In Theorem 7.2, we consider continuous-time evolution transformations: we show that if the Hamiltonian is renormalized as we did for discrete time transformations, then the evolution converges. In Theorem 10.2, we characterize the Hamiltonian so that the factorization property of large  $N$  holds.

We study also the development in power of  $1/N$  of systems of size  $N$  which converge to a continuous system.

The main novelty is to show that, even if one knows how to renormalize the bare constants, it does not seem interesting to define a vector space of infinite systems since all systems considered are defined in the same vector space whose basis is the set of partitions of  $2k$  elements. In order to have an interesting space of infinite systems, one has to consider a renormalization of the algebras in which are defined the  $N$ -dimensional

systems: the limit defines a non-trivial algebra in which one can study continuous evolutions of continuous systems.

Let us remark that a consequence of our results is that, in our toy-model, given a continuous system, one has canonically a sequence of approximations by systems involving  $N$  particles.

**1.2. Layout of the article.** — Using the set  $\mathcal{P}_k$  of partitions of  $2k$  elements as basis, one can define an algebra known as the partition algebra which definition depends on a parameter  $N \in \mathbb{N}$ : the partition algebra  $\mathbb{C}[\mathcal{P}_k(N)]$ . For a comprehensive study of this algebra, we recommend the article [11]. The main definitions are set in Section 2.

In Section 3, we define a geometry on the set of partitions of  $2k$  elements which generalizes a well-known geometry on the symmetric group  $\mathfrak{S}_k$ . Using this new geometry, in Section 4 we define two notions of convergence of sequences which are shown to be equivalent. We define the notion of coordinate numbers, normalized moments, exclusive coordinate numbers and exclusive normalized moments. One of the results that we prove is that exclusive coordinate numbers and exclusive normalized moments are equal. In Section 5, a new deformed partition algebra is defined:  $\mathbb{C}[\mathcal{P}_k(N, N)]$ . These algebras are shown to converge to a new algebra: this is obtained by an improvement of the triangle inequality proved in Section 6 for the distance defined on the set of partitions of  $2k$  elements. Let us remark that we define in the same section a Kreweras complement for partitions which generalizes the notion for permutations. We use these results in Section 7 in order to show that the multiplication is continuous for the notion of convergence of elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ . We also study the convergence of semi-groups in  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ . In Section 8, using the convergence of sequences defined in Section 4, we show how one can prove combinatorial results, for exemple, a new proof of the improved triangle inequality is given.

In Section 9, we develop the notion of algebraic fluctuations, and extend the results already proved for the zero order of fluctuations to any order.

In Section 10, we construct an algebra  $\mathfrak{E}[\mathcal{P}]$  which elements are functions on  $\cup_{k \in \mathbb{N}} \mathcal{P}_k$ . This algebra can be endowed with two special laws:  $\boxplus$  and  $\boxtimes$ . We study two subgroups of  $\mathfrak{E}[\mathcal{P}]$  associated with the operations  $\boxplus$  and  $\boxtimes$ , the group of multiplicative invertible elements. These groups are Lie groups, the Lie algebras of these groups are studied. We also define the  $\mathcal{R}_A$ -transform, which generalizes the usual  $\mathcal{R}$ -transform and we define two others transformations linked with the notion of exclusive moments. To finish the article, we extend these definitions to the setting of higher order fluctuations.

**1.3. Acknowledgements.** — This work has been realized during my PhD at the LPMA which offered me the necessary liberty to complete this article. Many thanks to the researchers and administrative staff of the LPMA. I am grateful to my PhD advisor Thierry Lévy for supporting this research, for his helpful comments and corrections which led to improvements in this manuscript and for the useful discussions about mathematics and other subjects. I would like to express my special gratitude to Terence Tao, particularly for his blog which, during a period of doubts, made me enjoy maths again. This project began when I wanted to understand the link between the work of T. Lévy and the formulation given at the Pims summer school by David Brydges of Wick's theorem, I am really thankful to him for this. Also many thanks are due to Antoine

Dahlqvist who explained me the duality between permutations and partitions. I would like to thank Tom Halverson and Arun Ram for answering my questions about partitions. I am also grateful to Guillaume Cébron since I learned a lot thanks to his papers. Thanks to Patrick Gabriel for his interest in my work and the discussions related to this work that we had together, Marie-Françoise Gabriel and Catherine Lam for trying to correct the English in this manuscript. At last, a thought to all the people which are supporting researchers and whose names never appear in the acknowledgements.

## 2. Partition algebra

**2.1. First definitions.** — Let  $k$  and  $N$  be two integers. We will consider three different algebras  $\mathbb{C}[\mathfrak{S}_k]$ ,  $\mathbb{C}[\mathcal{B}_k(N)]$ ,  $\mathbb{C}[\mathcal{P}_k(N)]$ : respectively the symmetric algebra, the Brauer algebra, and the partition algebra. These algebras satisfy the inclusions:

$$\mathbb{C}[\mathfrak{S}_k] \subset \mathbb{C}[\mathcal{B}_k(N)] \subset \mathbb{C}[\mathcal{P}_k(N)].$$

Thus, we will first construct  $\mathbb{C}[\mathcal{P}_k(N)]$  and we will see the two others algebras as sub-algebras of  $\mathbb{C}[\mathcal{P}_k(N)]$ . The reference article for the partition algebra is the article [11] of T.Halverson and A.Ram.

Let us consider  $2k$  elements which we denote by:  $1, \dots, k$  and  $1', \dots, k'$ . We define  $\mathcal{P}_k$  as the set of set partitions of  $\{1, \dots, k\} \cup \{1', \dots, k'\}$ . If  $k = 0$ , we consider  $\mathcal{P}_k = \{\emptyset\}$ . Let  $p$  be an element of  $\mathcal{P}_k$ . We will denote by  $p^1, \dots, p^r$  the blocks in  $p$ . The number of connected components  $\text{nc}(p)$ , the propagating number  $\text{pn}(p)$  and the length  $l(p)$  of  $p$  are defined respectively by:

$$\text{nc}(p) = r,$$

$$\text{pn}(p) = \# \{i, p^i \text{ contains both an element of } \{1, \dots, k\} \text{ and one of } \{1', \dots, k'\}\},$$

$$l(p) = k.$$

Any partition  $p \in \mathcal{P}_k$  can be represented by a graph. For this we consider two rows:  $k$  vertices are in the top row, labeled by 1 to  $k$  from left to right and  $k$  vertices are in the bottom row, labeled from  $1'$  to  $k'$  from left to right. Any edge between two vertices means that the labels of the two vertices are in the same block of the partition  $p$ . Examples are given in Figure 1 and 2.

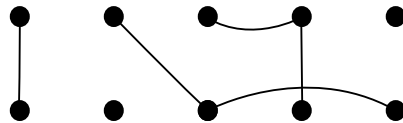
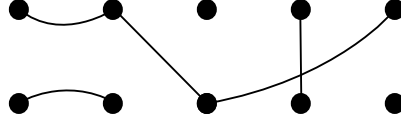
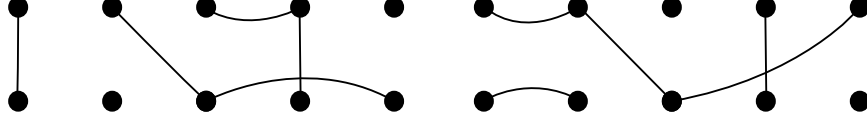


FIGURE 1. Partition  $p_1 = \{\{1', 1\}\{2'\}\{2, 3', 5'\}\{3, 4, 4'\}\{5\}\}$ .

The notion of tensor product of partitions will be also very useful.

**Definition 2.1.** — Let  $k$  and  $l$  be two integers. Let  $p$  be an element of  $\mathcal{P}_k$  and let  $p'$  be an element of  $\mathcal{P}_l$ . Let us consider two diagrams: one associated with  $p$ , another with  $p'$ . Let  $p \otimes p'$  be the partition in  $\mathcal{P}_{k+l}$  associated with the diagram where one has put the diagram associated with  $p$  on the left of the diagram associated with  $p'$ .

FIGURE 2. Partition  $p_2 = \{\{1', 2'\}\{1, 2, 3', 5\}\{3\}\{4', 4\}\{5'\}\}$ .FIGURE 3. Partition  $p_1 \otimes p_2$ .

Let  $k$  be an integer. Let  $p_1$  and  $p_2$  be two elements of  $\mathcal{P}_k$ . We say that  $p_1$  is *coarser* than  $p_2$  if any two elements which are in the same block of  $p_2$  are also in the same block of  $p_1$ . This order is directed: for any partitions  $p_1$  and  $p_2$  in  $\mathcal{P}_k$  there exists a third partition  $p_3$  which is coarser than  $p_1$  and  $p_2$ : for example, one can consider the partition  $p_1 \vee p_2$  defined as follows.

**Definition 2.2.** — *Let  $k$  be an integer. Let  $p_1$  and  $p_2$  be two elements of  $\mathcal{P}_k$ . We define  $p_1 \vee p_2$  as the partition in  $\mathcal{P}_k$  such that for any  $i, j \in \{1, \dots, k\} \cup \{1', \dots, k'\}$ ,  $i$  and  $j$  are in the same block of  $p_1 \vee p_2$  if and only if there exists  $i = x_0, x_1, \dots, x_l = j$  with  $x_n \in \{1, \dots, k\} \cup \{1', \dots, k'\}$  such that  $x_n$  and  $x_{n+1}$  are in the same block of  $p_1$  or  $p_2$  for any  $n \in \{0, \dots, l-1\}$ .*

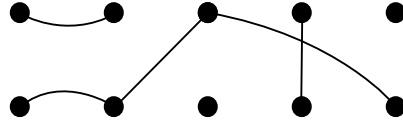
It is always interesting to have a graphical representation for the operations defined on partitions. One can recover a diagram representing  $p_1 \vee p_2$  by putting a diagram representing  $p_2$  over one representing  $p_1$ .

FIGURE 4. Two diagrams which represent  $p_1 \vee p_2 = \{\{1, 1', 2, 2', 3', 5, 5'\}, \{3, 4, 4'\}\}$ .

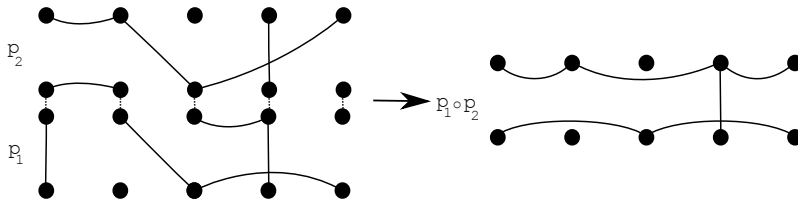
Let us play a little with the graphical representation of  $p_1$  and  $p_2$  in order to define other natural operations on the set of partitions.

We will use later the transposition of a partition: it is the partition obtained by permuting the role of  $\{1, \dots, k\}$  and  $\{1', \dots, k'\}$ . For example if  $k = 3$ , let  $p = \{\{1, 1', 3'\}, \{2, 3\}, \{2'\}\}$ , then  ${}^t p = \{\{1', 1, 3\}, \{2', 3'\}, \{2\}\}$ . For every diagram associated with  $p$ , the diagram obtained by flipping it according to a horizontal axis is a diagram associated with  ${}^t p$ . One can find an example in Figure 5

Another thing we can do is to put one diagram representing  $p_2$  above one diagram representing  $p_1$ . Let us identify the lower vertices of  $p_2$  with the upper vertices of  $p_1$ . We obtain a graph with vertices on three levels, then erase the vertices in the middle row, keeping the edges obtained by concatenation of edges passing through the deleted

FIGURE 5. Partition  ${}^t p_2$ 

vertices. Any connected component entirely included in the middle row is then removed. Let us denote by  $\kappa(p_1, p_2)$  the number of such connected components. We obtain an other diagram associated with a partition denoted by  $p_1 \circ p_2$ . For any elements  $p_1$  and  $p_2$  of  $\mathcal{P}_k$ , the partition  $p_1 \circ p_2$  does not depend on the choice of diagrams representing the partitions  $p_1$  and  $p_2$ .

FIGURE 6. Partition  $p_1 \circ p_2$ .

The set of Brauer elements and the set of partitions will be stable by this operation of concatenation.

**Definition 2.3.** — *The set of Brauer elements  $\mathcal{B}_k$  is the set of pair partitions in  $\mathcal{P}_k$ . The set of permutation  $\mathfrak{S}_k$  is the set of pair partitions in  $\mathcal{P}_k$  whose propagating number is equal to  $k$ .*

For any  $p_1$  and  $p_2$  in  $\mathcal{B}_k$  (resp.  $\mathfrak{S}_k$ ),  $p_1 \circ p_2 \in \mathcal{B}_k$  (resp.  $\mathfrak{S}_k$ ). Let us define the three algebras  $\mathbb{C}[\mathfrak{S}_k]$ ,  $\mathbb{C}[\mathcal{B}_k(N)]$  and  $\mathbb{C}[\mathcal{P}_k(N)]$ .

**Definition 2.4.** — *Let  $k$  and  $N$  be two integers. The partition algebra  $\mathbb{C}[\mathcal{P}_k(N)]$  is the associative algebra over  $\mathbb{C}$  with basis  $\mathcal{P}_k$  endowed with the multiplication defined by:*

$$\forall p_1, p_2 \in \mathcal{P}_k, p_1 p_2 = N^{\kappa(p_1, p_2)} (p_1 \circ p_2).$$

*The Brauer algebra  $\mathbb{C}[\mathcal{B}_k(N)]$  (resp. symmetric algebra  $\mathbb{C}[\mathfrak{S}_k]$ ) is the sub-algebra of  $\mathbb{C}[\mathcal{P}_k(N)]$  generated by the elements of  $\mathcal{B}_k$  (resp. the elements of  $\mathfrak{S}_k$ ).*

**Notation 2.1.** — *In all the paper,  $A_k$  will stand either for  $\mathcal{P}_k$  or  $\mathcal{B}_k$  or  $\mathfrak{S}_k$ . Thus for any  $N \in \mathbb{N}$ ,  $\mathbb{C}[A_k(N)]$  will stand for  $\mathbb{C}[\mathcal{P}_k(N)]$ ,  $\mathbb{C}[\mathcal{B}_k(N)]$  or  $\mathbb{C}[\mathfrak{S}_k(N)]$ .*

Let us remark that actually, the algebra  $\mathbb{C}[\mathfrak{S}_k(N)]$  does not depend on  $N$ . We can see any permutation  $\sigma \in \mathfrak{S}_k$  as a bijection from  $\{1, \dots, k\}$  to itself: for any  $i \in \{1, \dots, k\}$  there exists a unique  $j \in \{1', \dots, k'\}$  such that  $\{i, j'\} \in \sigma$ , we set  $\sigma(i) = j$ . For any permutations  $\sigma_1$  and  $\sigma_2$ , the bijection associated with  $\sigma_1 \sigma_2$  is the composition of the two bijections associated with  $\sigma_1$  and  $\sigma_2$ .

We can extend the operations of transposition, tensor product and multiplication on the partition algebra, by linearity or bi-linearity.



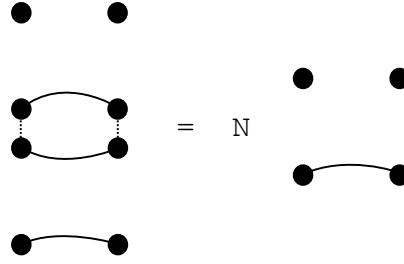


FIGURE 7. Example of a product which involves the counting of loops.

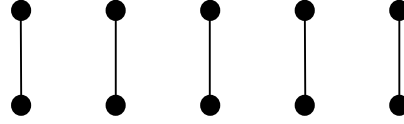
The sub-algebra  $\mathbb{C}[\mathfrak{S}_k]$  is not only stable for the  $\circ$  operation. It also satisfies the following property which can be proved by looking at the propagating number.

**Lemma 2.1.** — *Let  $p, p' \in \mathcal{P}_k$ , if  $p \circ p' \in \mathfrak{S}_k$  then  $p$  and  $p'$  are in  $\mathfrak{S}_k$ .*

Besides, for any partition  $\sigma \in \mathfrak{S}_k$  and any  $p \in A_k$ ,  $\kappa(\sigma, p) = \kappa(p, \sigma) = 0$ . Let us remark that, for any integer  $N$ , the algebras  $\mathbb{C}[A_k(N)]$  have the same neutral element, denoted by  $id_k$  or  $id$ , for the product operation:

$$id_k = \{\{i, i'\}, i \in \{1, \dots, k\}\},$$

whose diagram for  $k = 5$  is drawn in figure 8. A consequence of Lemma 2.1 is that, as  $id_k \in \mathfrak{S}_k$ , the only invertible elements of  $A_k(N)$ , for the multiplication operation, are the permutations. The inverse of a permutation  $\sigma$  is  $\sigma^{-1} = {}^t\sigma$ .

FIGURE 8. The neutral element  $id_5$ .

We will later need some special permutations.

**Definition 2.5.** — *Let  $k$  be an integer. Let  $I \subset \{1, \dots, k\}$ :  $I = \{i_1, \dots, i_l\}$  with  $i_1 < \dots < i_l$ . We define  $\sigma_I$  the permutation which sends  $i_j$  on  $j$  for any  $j \in \{1, \dots, l\}$  and  $i \notin I$  on  $l + i - \#\{n, i_n < i\}$ . This is the partition:*

$$\sigma_I = \{\{i_j, j'\}, j \in \{1, \dots, l\}\} \cup \{\{i, (l + i - \#\{n, i_n < i\})'\}, i \notin I\}.$$

**Definition 2.6.** — *The transposition  $(1, 2)$  is the partition  $\sigma_{\{2\}}$  in  $\mathcal{P}_2$  defined by:*

$$(1, 2) = \{\{1, 2'\}, \{2, 1'\}\}.$$

*The Weyl contraction is the Brauer element in  $\mathcal{P}_2$  defined by:*

$$[1, 2] = \{\{1, 2\}\{1', 2'\}\}.$$

*These partitions are drawn in Figure 9.*

FIGURE 9. The transposition  $(1, 2)$  and the Weyl contraction  $[1, 2]$ .

**Definition 2.7.** — Let  $k$  be an integer. Let  $i, j$  be two distinct integers in  $\{1, \dots, k\}$ . The transposition  $(i, j)$  in  $\mathfrak{S}_k$  is:

$$(i, j) = \sigma_{\{i, j\}}^{-1}((1, 2) \otimes Id_{k-2})\sigma_{\{i, j\}} = \{\{i', j\}, \{i, j'\}\} \cup \{\{l, l'\}, l \notin \{i, j\}\}.$$

The set of transpositions on  $k$  elements is:

$$\mathcal{T}_k = \{(i, j), i, j \in \{1, \dots, k\}, i \neq j\}.$$

The Weyl contraction  $[i, j]$  in  $\mathcal{B}_k$  is:

$$[i, j] = \sigma_{\{i, j\}}^{-1}([1, 2] \otimes Id_{k-2})\sigma_{\{i, j\}} = \{\{i, j\}, \{i', j'\}\} \cup \{\{l, l'\}, l \notin \{i, j\}\}.$$

Due to the remark we made after Lemma 2.1, the product does not depend on which  $\mathbb{C}[\mathcal{B}_k(N)]$  one considers to define the product. We denote by  $\mathcal{W}_k$  the set of Weyl contractions in  $\mathcal{B}_k$ :

$$\mathcal{W}_k = \{[i, j], i, j \in \{1, \dots, k\}, i \neq j\}.$$

A notion linked with the tensor operation, which will be central in the asymptotic freeness results in the article [9], is the notion of irreducibility of partitions. Let  $k$  be an integer. Let  $p \in \mathcal{P}_k$ .

**Definition 2.8.** — A cycle of  $p$  is a block of  $p \vee id$ . The set of cycles of  $p$  is denoted by  $\mathcal{C}(p)$ . The number of cycles of  $p$  is denoted by  $c(p)$ . The partition  $p$  is composed if  $c(p) > 1$ . The partition  $p$  is irreducible if it is not composed. By convention, the empty partition is irreducible.

Let us consider the set of irreducible partitions.

**Definition 2.9.** — For any integer  $k$ , we will denote by  $A_k^{(i)}$  the set of irreducible partitions of  $A_k$ .

It has to be noted that for any integer  $k$ :

$$\mathfrak{S}_k^{(i)} = \{\sigma(1, \dots, k)\sigma^{-1}, \sigma \in \mathfrak{S}_k\}.$$

The partition  $p \in \mathcal{P}_k$  is composed if and only if there exist  $p_1$  and  $p_2$  two partitions non equal to the empty partition, and  $I$  a subset of  $\{1, \dots, k\}$  such that  $\#I = l(p_1)$ ,  $l(p_2) = k - \#I$  and:

$$\sigma_I^{-1}(p_1 \otimes p_2)\sigma_I = p.$$

Let us define the decomposition of  $p$  into two partitions.

**Definition 2.10.** — *The set of decompositions of  $p$  into two partitions is:*

$$\mathfrak{F}_2(p) = \{(p_1, p_2, I), \sigma_I^{-1}(p_1 \otimes p_2)\sigma_I = p\}.$$

Let us remark that for any partition, even the irreducible partitions,  $\mathfrak{F}_2(p) \neq \emptyset$ . For example, if  $p$  is irreducible:

$$\mathfrak{F}_2(p) = \{(p, \emptyset, \{1, \dots, k\}), (\emptyset, p, \emptyset)\}.$$

Let also remark that  $\mathfrak{F}_2(\emptyset) = \{(\emptyset, \emptyset, \emptyset)\}$ .

We will need a notion of weak irreducibility later: this is based on the notion of extraction and restriction. For any partition  $p$  we have a lot of choice in order to represent  $p$  as a graph: the complete graph which represents  $p$  is the graph such that  $i$  and  $j$ , two elements of  $\{1, \dots, k\} \cup \{1', \dots, k'\}$  are linked if and only if  $i$  and  $j$  are in the same block of  $p$ .

**Definition 2.11.** — *Let  $k$  be an integer, let  $p$  be in  $\mathcal{P}_k$ . Let  $J$  be a subset of  $\{1, \dots, k\} \cup \{1', \dots, k'\}$ . Let us denote by  $J^s$  the symmetrization of  $J$ :*

$$J^s = J \cup \{j \in \{1', \dots, k'\}, \exists i \in J \cap \{1, \dots, k\}, j = i'\} \cup \{i \in \{1, \dots, k\}, i' \in J\}.$$

We define:

- *The extraction of  $p$  to  $J$ , denoted  $p_J$ . Let us take the complete graph which represents  $p$ , let us erase all the vertices which are not in  $J^s$  and all the edges which are not from two vertices in  $J^s$  and at last let us label, from left to right the vertices. This is the graph of  $p_J$ .*
- *The restriction of  $p$  to  $J$ , denoted  $p|_J$ . Let us take the complete graph which represents  $p$ , let us erase all the edges which are not from two vertices in  $J$  and let us connect each  $i \notin J$  with  $i'$ . This is the graph of  $p|_J$ .*

By convention, if  $J^s = \{1, \dots, k\} \cup \{1', \dots, k'\}$ , then  $p_J = \emptyset$  and  $p|_J = id$ .

**Definition 2.12.** — *The support of  $p$  is:*

$$S(p) = \{1, \dots, k\} \setminus \{i \in \{1, \dots, k\}, \{i, i'\} \in p\}.$$

*The partition  $p$  is weakly irreducible if  $p_{S(p)}$  is irreducible. In particular  $Id_k$  is weakly irreducible.*

**2.2. Partitions and representation.** — In this section, we define a natural action of the partition algebra (and by restriction of the Brauer and of the symmetric algebra) on  $(\mathbb{C}^N)^{\otimes k}$ . This action will be useful in order to translate combinatorial properties into linear algebraic properties.

Let  $N$  and  $k$  be two integers.

**Definition 2.13.** — *For any  $p \in \mathcal{P}_k$  and any  $k$ -uples  $(i_1, \dots, i_k)$  and  $(i_{1'}, \dots, i_{k'})$  of elements of  $\{1, \dots, N\}$ , we set:*

$$p_{i_{1'}, \dots, i_{k'}}^{i_1, \dots, i_k} = \begin{cases} 1, & \text{if for any two elements } r \text{ and } s \in \{1, \dots, k\} \cup \{1', \dots, k'\} \text{ which} \\ & \text{are in the same block of } p, \text{ one has } i_r = i_s, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $k$  and  $N$  be any integers. We can now define the action of the partition algebra  $\mathbb{C}[\mathcal{P}_k(N)]$  on  $(\mathbb{C}^N)^{\otimes k}$ . Let  $(e_1, \dots, e_N)$  be the canonical basis of  $\mathbb{C}^N$ .

**Definition 2.14.** — *The action of the partition algebra  $\mathbb{C}[\mathcal{P}_k(N)]$  on  $(\mathbb{C}^N)^{\otimes k}$  is defined by the fact that for any  $p \in \mathcal{P}_k$ , for any  $(i_1, \dots, i_k) \in \{1, \dots, N\}^k$ :*

$$p.(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_{(i_{1'}, \dots, i_{k'}) \in \{1, \dots, N\}^k} p_{i_{1'}, \dots, i_{k'}}^{i_1, \dots, i_k} e_{i_{1'}} \otimes \dots \otimes e_{i_{k'}}.$$

This action defines a representation of the partition algebra  $\mathbb{C}[\mathcal{P}_k(N)]$  on  $(\mathbb{C}^N)^{\otimes k}$  which we denote by  $\rho_N^{\mathcal{P}_k}$ :

$$\rho_N^{\mathcal{P}_k} : \mathbb{C}[\mathcal{P}_k(N)] \mapsto \text{End} \left( (\mathbb{C}^N)^{\otimes k} \right).$$

Let us define  $E_i^j$  be the matrix which sends  $e_j$  on  $e_i$ . Let  $p$  be a partition in  $\mathcal{P}_k$ . We can write the matrix of  $\rho_N^{\mathcal{P}_k}(p)$  in the basis  $(e_{i_1} \otimes \dots \otimes e_{i_k})_{(i_l)_{l=1}^k \in \{1, \dots, N\}^k}$ :

$$(1) \quad \rho_N^{\mathcal{P}_k}(p) = \sum_{(i_1, \dots, i_k, i_{1'}, \dots, i_{k'}) / p_{i_{1'}, \dots, i_{k'}}^{i_1, \dots, i_k} = 1} E_{i_1}^{i_{1'}} \otimes \dots \otimes E_{i_k}^{i_{k'}}.$$

For example, if  $p$  is the transposition  $(1, 2)$ , then:

$$\rho_N^{\mathcal{P}_2}((1, 2)) = \sum_{a, b=1}^N E_a^b \otimes E_b^a.$$

We think that this presentation allows to understand, in an easier way, the representation  $\rho_N^{\mathcal{P}_k}$ . We illustrate in Figure 10, how to find the partition which representation is given by a sum of the form (1). The partition  $p_1$  used in Figure 10 is the partition drawn in Figure 1.

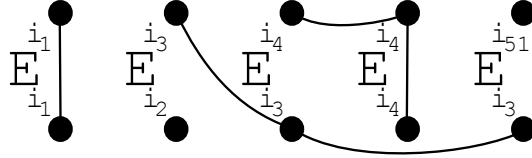


FIGURE 10.  $\sum_{i_1, i_2, i_3, i_4, i_5} E_{i_1}^{i_1} \otimes E_{i_2}^{i_3} \otimes E_{i_3}^{i_4} \otimes E_{i_4}^{i_4} \otimes E_{i_5}^{i_3} = \rho_N^{\mathcal{P}_5}(p_1)$ .

Let us suppose that  $N \geq 2k$ . Using the Theorem 3.6 in [11], the application  $\rho_N^{\mathcal{P}_k}$  is injective. Actually, if one considers only its restriction to the symmetric algebra or the Brauer algebra, it is enough to ask for  $N \geq k$ . For  $N = k - 1$  this result does not hold, this is a consequence of the Mandelstam's identity which asserts that:

$$\sum_{\sigma \in \mathfrak{S}_k} (-1)^{\epsilon(\sigma)} \rho_{k-1}^{\mathcal{P}_k}(\sigma) = 0,$$

where  $\epsilon(\sigma)$  is the signature of  $\sigma$ .

Let us remark that the natural action of  $\mathbb{C}[\mathcal{P}_k(N)]$  on  $(\mathbb{C}^N)^{\otimes k}$  behaves well under the operation of product tensor.

**Lemma 2.2.** — *Let  $k$  and  $k'$  be two integers. Let  $p \in \mathbb{C}[\mathcal{P}_k]$  and  $p' \in \mathbb{C}[\mathcal{P}_{k'}]$ . We have for any integer  $N$ :*

$$\rho_N^{\mathcal{P}_{k+k'}}(p \otimes p') = \rho_N^{\mathcal{P}_k}(p) \otimes \rho_N^{\mathcal{P}_{k'}}(p').$$

**2.3. The exclusive basis of  $\mathbb{C}[\mathcal{P}_k]$ .** — The basis used to define the partition algebra is quite natural, yet, it is not always very easy to work with. Indeed, if we look at the representation  $\rho_N^{\mathcal{P}_k}$  of a partition, we see that the condition we used to define the delta function is not exclusive. It means that we did not use the following exclusive delta function:

$$(p_{i_1, \dots, i_k}^{i_1, \dots, i_k})^{ex} = \begin{cases} 1, & \text{if for any two elements } r \text{ and } s \in \{1, \dots, k\} \cup \{1', \dots, k'\}, \\ & i_r = i_s \text{ if and only if } r \text{ and } s \text{ are in the same block of } p, \\ 0, & \text{otherwise.} \end{cases}$$

By changing in Definition 2.14 the delta function defined in Definition 2.13 by this new exclusive delta function, we define a new function:

$$\tilde{\rho}_N^k : \mathbb{C}[\mathcal{P}_k(N)] \rightarrow \text{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right).$$

Does it exist, for any partition  $p \in \mathcal{P}_k$  an element  $p^c \in \mathbb{C}[\mathcal{P}_k]$  such that for any integer  $N$ ,  $\rho_N^{\mathcal{P}_k}(p^c) = \tilde{\rho}_N^k(p)$ ? The answer is given by the following definition, as explained by Equation (2.3) of [11].

**Definition 2.15.** — *For any  $k \in \mathbb{N}$ . We define the family  $(p^c)_{p \in \mathcal{P}_k}$  as the only family of elements in  $\mathbb{C}[\mathcal{P}_k]$  defined by the relation:*

$$p = \sum_{p' \text{ coarser than } p} p'^c.$$

The notion of being coarser defines a partial order on  $\mathcal{P}_k$ : the relation can be inverted. The family  $(p^c)_{p \in \mathcal{P}_k}$  is well defined and it is a basis of the partition algebra  $\mathbb{C}[\mathcal{P}_k]$ . We will call  $(p^c)_{p \in \mathcal{P}_k}$  the exclusive partition basis, it satisfies the following proposition.

**Proposition 2.1.** — *For any integers  $k$  and  $N$ , for any partition  $p \in \mathcal{P}_k$ ,*

$$\rho_N^{\mathcal{P}_k}(p^c) = \tilde{\rho}_N^k(p).$$

### 3. Geometry on the set of partitions

Let  $k$  be an integer. In this section, we define a new geometry on the set of partitions  $\mathcal{P}_k$  which generalizes some well-known geometry on the symmetric group. We will see three ways to construct a distance on  $\mathcal{P}_k$ : one will allow us to work with linear algebra, another to compute the distance in a combinatorial way, and the last one will use a graph which we will consider as the generalized Cayley graph of  $\mathcal{P}_k$ .

Depending on the context, we will consider a partition either as an element of  $\mathcal{P}_k$  or as an element of  $\text{End}\left(\left(\mathbb{C}^N\right)^{\otimes k}\right)$  via the action defined in Definition 2.14. We remind the reader that  $(e_1, \dots, e_N)$  is the canonical base of  $\mathbb{C}^N$ . The family  $\{e_{i_1} \otimes \dots \otimes e_{i_k}, (i_1, \dots, i_k) \in \{1, \dots, N\}^k\}$  is a basis of  $\left(\mathbb{C}^N\right)^{\otimes k}$ : let  $Tr^k$  be the trace with respect

to this canonical basis. We do not renormalize it, thus  $Tr^k \left( Id_{(\mathbb{C}^N)^{\otimes k}} \right) = N^k$ . We can define the trace of a partition.

**Definition 3.1.** — Let  $k$  and  $N$  be two integers, let  $p$  be a partition in  $\mathcal{P}_k$ . We define:

$$Tr_N(p) = Tr^k \left( \rho_N^{\mathcal{P}_k}(p) \right).$$

For any integer  $N$ , we extend  $Tr_N$  by linearity to  $\mathbb{C}[\mathcal{P}_k(N)]$ .

Let us remark that, if one does not want to use the representation  $\rho_N^{\mathcal{P}_k}$ , one could have also define the trace by defining for any partition  $p \in \mathcal{P}_k$ ,

$$(2) \quad Tr_N(p) = N^{\text{nc}(p \vee id)}.$$

We can now define a distance on  $\mathcal{P}_k$ .

**Proposition 3.1.** — Let  $k$  and  $N$  be two integers, let  $p$  and  $p'$  be two elements of  $\mathcal{P}_k$ . The number:

$$d(p, p') = -\log_N \left( \frac{Tr_N({}^t p p')}{\sqrt{Tr_N({}^t p p) Tr_N({}^t p' p')}} \right)$$

does not depend on  $N$ : it is called the distance between  $p$  and  $p'$ .

The fact that  $d(p, p')$  does not depend on  $N$  is a consequence of Lemma 3.1. Actually we have not prove yet that it is a distance, even if it is fairly easy to see that it satisfies the strict positivity property: it is a consequence of the Cauchy-Schwarz's inequality.

The easiest way to prove that  $d(p, p')$  does not depend on  $N$  is to show that it is a combinatorial object.

**Lemma 3.1.** — Let  $k$  and  $N$  be two integers, for any  $p$  and  $p'$  in  $\mathcal{P}_k$ :

$$d(p, p') = \frac{1}{2} (\text{nc}(p) + \text{nc}(p')) - \text{nc}(p \vee p').$$

*Proof.* — This is a consequence of the following equality which holds for any  $p$  and  $p'$  in  $\mathcal{P}_k$  and any positive integer  $N$ :

$$(3) \quad Tr_N({}^t p p') = N^{\text{nc}({}^t p \circ p' \vee id) + \kappa({}^t p, p')} = N^{\text{nc}(p \vee p')},$$

which is a consequence of Equality 2 and the combinatorial equality:

$$\text{nc}({}^t p \circ p' \vee id) + \kappa({}^t p, p') = \text{nc}(p \vee p').$$

This latter equality can be understood by flipping the diagram of  ${}^t p$  over the one of  $p'$ : the flip transposes  ${}^t p$  thus we get the two diagrams of  $p$  and  $p'$  one over the other. By definition, the diagram constructed by putting a diagram representing  $p'$  over one representing  $p$  is associated with  $p \vee p'$ .  $\square$

It remains to show that  $d$  satisfies the triangular inequality on the set of partitions  $\mathcal{P}_k$ . For that we will show that it is a geodesic distance on a graph.

**Definition 3.2.** — Let  $k$  be an integer. We define the weighted graph  $\mathbb{G}_k = (\mathbb{V}_k, \mathbb{E}_k, w_k)$  such that:

- the set of vertices  $\mathbb{V}_k$  is  $\mathcal{P}_k$ ,

- there exists an edge  $e$  in  $\mathbb{E}_k$  between  $p$  and  $p'$  two elements of  $\mathcal{P}_k$  if and only if:
  - one can go from one to the other by gluing two blocks. Let us suppose that we can go from  $p$  to  $p'$ . If  $p$  is the partition  $\{p^1, \dots, p^r\}$  then there exist  $i$  and  $j$ , distinct, such that  $p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{p^i \cup p^j\}$ . The weight of the edge  $e$  is set to 0.5:  $w_k(e) = 0.5$ .
  - one can go from one to the other by permuting two elements of  $\{1, \dots, k\} \cup \{1', \dots, k'\}$  which are in distinct blocks. Let us suppose that we can go from  $p$  to  $p'$  by permuting two elements. In this case, if  $p$  is the partition  $\{p^1, \dots, p^r\}$ , there exist  $s, t \in \{1, \dots, k, 1', \dots, k'\}$  distinct and  $i, j \in \{1, \dots, r\}$  distinct, such that  $s \in p^i$ ,  $t \in p^j$  and  $p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{(p^i \setminus \{s\}) \cup \{t\}, (p^j \setminus \{t\}) \cup \{s\}\}$ . The weight of the edge  $e$  is set to 1:  $w_k(e) = 1$ .

**Remark 3.1.** — The graph  $\mathbb{G}_k$  plays the role of the Cayley graph of  $\mathcal{P}_k$ . Actually, if one considers the subgraph  $\mathbb{S}_k$  obtained by restraining it to the vertices which are permutations, one really obtains the Cayley graph of the symmetric group  $\mathfrak{S}_k$ . The Cayley graph  $\mathbb{B}_k$  of  $\mathcal{B}_k$  is defined as the restriction of  $\mathbb{G}_k$  to the vertices which are in  $\mathcal{B}_k$ .

We gave this definition so that the reader can understand easily why this graph is a generalization of the usual Cayley graph. Yet, there is an other graph which will be used in Proposition 3.2. Let us define  $\mathbb{G}'_k = (\mathbb{V}'_k, \mathbb{E}'_k, w'_k)$  such that:

- the set of vertices  $\mathbb{V}'_k$  is  $\mathcal{P}_k$ ,
- there exists an edge in  $\mathbb{E}'_k$  between  $p$  and  $p'$  two elements of  $\mathcal{P}_k$  if and only if one can go from one to the other by gluing two blocks,
- the weight function  $w'_k$  is constant equal to 1/2.

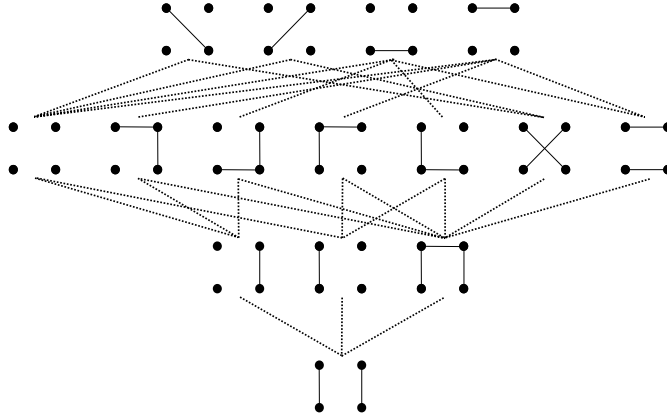


FIGURE 11. The graph  $\mathbb{G}'_2$ .

The graphs  $\mathbb{G}_k$  and  $\mathbb{G}'_k$  are interesting as they allow us to better understand the distance  $d$ .

**Proposition 3.2.** — Let  $k$  be an integer. Let  $p$  and  $p'$  be two elements of  $\mathcal{P}_k$ . Let us define  $C_{\mathbb{G}_k}(p, p')$  (resp.  $C_{\mathbb{G}'_k}(p, p')$ ) the set of paths  $\pi$  in  $\mathbb{G}_k$  (resp.  $\mathbb{G}'_k$ ) which begin in  $p$

and finish in  $p'$ . Let us define the geodesic distance on  $\mathbb{G}_k$  and on  $\mathbb{G}'_k$  between  $p$  and  $p'$  by:

$$d_{\mathbb{G}_k}(p, p') = \min_{\pi \in C_{\mathbb{G}_k}(p, p'), \pi = e_1 \dots e_l} w(e_1) + \dots + w(e_l),$$

$$d_{\mathbb{G}'_k}(p, p') = \min_{\pi \in C_{\mathbb{G}'_k}(p, p'), \pi = e_1 \dots e_l} w(e_1) + \dots + w(e_l).$$

We have the equalities:

$$d(p, p') = d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p').$$

*Proof.* — Let  $p$  and  $p'$  be two elements of  $\mathcal{P}_k$ . It is enough to prove that  $d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p')$  and  $d(p, p') = d_{\mathbb{G}'_k}(p, p')$ .

First, let us show that  $d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p')$ . This assertion comes from the fact that one can permute two elements of  $\{1, \dots, k\} \cup \{1', \dots, k'\}$  in the partition  $p$  by gluing two blocks of  $p$  and then splitting one block of the resulting partition. Indeed, let us suppose that  $p = \{p^1, \dots, p^r\}$ . Let  $s, t \in \{1, \dots, k, 1', \dots, k'\}$ , distinct, and let  $i, j \in \{1, \dots, r\}$ , distinct, such that  $s \in p^i$  and  $t \in p^j$ . Then

$$p' = \{p^s, s \in \{1, \dots, r\} \setminus \{i, j\}\} \cup \{(p^i \setminus \{s\}) \cup \{t\}, (p^i \setminus \{t\}) \cup \{s\}\}$$

can be obtained by:

1. gluing  $p_i$  and  $p_j$ ,
2. splitting  $p_i \cup p_j$  in two:  $(p^i \setminus \{s\}) \cup \{t\}$  and  $(p^i \setminus \{t\}) \cup \{s\}$ .

The weight of this path is equal to  $0.5 + 0.5 = 1$ . Thus, to compute the distance  $d_{\mathbb{G}_k}(p, p')$ , it is enough to look only at paths in  $\mathbb{G}'_k$ :  $d_{\mathbb{G}_k}(p, p') = d_{\mathbb{G}'_k}(p, p')$ .

Then, let us show that  $d(p, p') = d_{\mathbb{G}'_k}(p, p')$ . For this, let us see what happens to the distance  $d(p, p')$  between  $p$  and  $p'$  when one moves from  $p'$  to one neighborhood of  $p'$  in  $\mathbb{G}'_k$ . Suppose first that we glue two blocks of  $p'$ , then  $\text{nc}(p)$  is constant,  $\text{nc}(p')$  decreases by 1 and  $\text{nc}(p \vee p')$  stays constant or decreases by 1. In this case  $d(p, p')$  will increase or decrease by 0.5. If we cut one block of  $p'$ , then  $\text{nc}(p)$  is constant,  $\text{nc}(p')$  increases by 1 and  $\text{nc}(p \vee p')$  stays constant or increases by 1. In this case  $d(p, p')$  will also increase or decrease by 0.5.

Thus a gluing/cutting can at most increase the value of  $d(p, p')$  by 0.5. It implies that  $d(p, p') \leq d_{\mathbb{G}'_k}(p, p')$ . We have to show now that  $d_{\mathbb{G}'_k}(p, p') \leq d(p, p')$ . Let us remark that  $p \vee p'$  is coarser than  $p$ : we can go from  $p$  to  $p \vee p'$  by doing  $\text{nc}(p) - \text{nc}(p \vee p')$  gluing of blocks of  $p$ . The same holds for  $p'$ : we can go from  $p'$  to  $p \vee p'$  by doing  $\text{nc}(p') - \text{nc}(p \vee p')$  gluing of blocks of  $p'$ . Thus one can go from  $p$  to  $p \vee p'$  and then from  $p \vee p'$  to  $p'$  in  $\text{nc}(p) + \text{nc}(p') - 2\text{nc}(p \vee p')$  steps in  $\mathbb{G}'_k$ . Thus  $d_{\mathbb{G}'_k}(p, p') \leq \frac{1}{2}[\text{nc}(p) + \text{nc}(p') - 2\text{nc}(p \vee p')] = d(p, p')$ .  $\square$

The function  $d_{\mathbb{G}'_k}$  is a geodesic distance on a graph: it is thus a distance. As we have just shown that  $d = d_{\mathbb{G}'_k}$ , the next corollary is immediately proved.

**Corollary 3.1.** — *The function  $d : \mathcal{P}_k \times \mathcal{P}_k \rightarrow \mathbb{R}^+$  is a distance.*

**Lemma 3.2.** — *The restriction of  $d$  to the permutation group is quite usual:*

$$d(\sigma, \sigma') = k - \text{nc}(\sigma^{-1}\sigma'),$$



for any  $\sigma, \sigma' \in \mathfrak{S}_k$ . This distance is in fact the geodesic distance on the Cayley graph  $\mathcal{S}_k$  of  $\mathfrak{S}_k$ .

By Lemma 6.26 of [12], the restriction of the distance  $d$  to  $\mathcal{B}_k$  is also the geodesic distance on the Cayley graph  $\mathbb{B}_k$  of  $\mathcal{B}_k$ .

Using this distance, we can define a notion of set-geodesic for any of the three sets of partitions we are interested in. We remind the reader that the notation  $A_k$  was settled in Notation 2.1.

**Definition 3.3.** — Let  $p \in A_k$ , the set-geodesic  $[id, p]_{A_k}$  is defined by:

$$[id, p]_{A_k} = \{p' \in A_k, d(id, p) = d(id, p') + d(p', p)\}.$$

A geodesic in a graph between two vertices  $p$  and  $p'$  is a path in this graph which length is equal to the geodesic distance. Using Proposition 3.2 and Lemma 3.2, one shows that for any  $p \in A_k$ , the set-geodesic  $[id, p]_{A_k}$  is the union of the geodesics between  $id$  and  $p$  in the Cayley graph of  $A_k$ .

The distance on  $A_k$  allows to define a new partial order on  $A_k$ .

**Definition 3.4.** — Let  $p$  and  $p'$  be elements of  $A_k$ , we write that  $p \leq p'$  if and only if  $d(id, p) = d(id, p') + d(p, p')$ .

This is a partial order as the restriction of  $d$  to  $A_k \times A_k$  is a distance. In the following lemma, we show that the geodesic in the Cayley graph of  $\mathcal{P}_k$  between two permutations either stay in the set of permutations or intersect  $\mathcal{P}_k \setminus \mathcal{B}_k$ . Using the fact that  $[id, p]_{A_k}$  is the union of the geodesics between  $id$  and  $p$  in the Cayley graph of  $A_k$ , we get an equality between  $[id, \sigma]_{\mathcal{B}_k}$  and  $[id, \sigma]_{\mathfrak{S}_k}$ .

**Lemma 3.3.** — Let  $k$  be an integer. Let  $\sigma \in \mathfrak{S}_k$ , then:

$$[id, \sigma]_{\mathcal{B}_k} = [id, \sigma]_{\mathfrak{S}_k}.$$

*Proof.* — Let  $k$  be an integer. We will do a proof by contradiction. Let  $S \subset \mathfrak{S}_k$  be the set of permutations such that:

$$[id, \sigma]_{\mathcal{B}_k} \neq [id, \sigma]_{\mathfrak{S}_k}.$$

Let  $\sigma \in S$  be a permutation such that  $d(id, \sigma) = \min_{\sigma' \in S} d(id, \sigma')$ .

Let  $b$  be an element of  $\mathcal{B}_k \setminus \mathfrak{S}_k$  such that  $b \in [id, \sigma]_{\mathcal{B}_k}$ . There exists a geodesic in  $\mathcal{B}_k$  which goes through  $b$  and goes from  $id$  to  $\sigma$ . Thus, there exists  $b' \in \mathcal{B}_k$  such that  $d(id, b') = 1$  and  $b' \in [id, \sigma]_{\mathcal{B}_k}$ . The element  $b'$  can not be a permutation. Indeed, if  $b'$  was a permutation, then  $[b', \sigma]_{\mathcal{B}_k} \neq [b', \sigma]_{\mathfrak{S}_k}$  and thus,  $[id, b'^{-1}\sigma]_{\mathcal{B}_k} \neq [id, b'^{-1}\sigma]_{\mathfrak{S}_k}$ . Yet  $d(id, b'^{-1}\sigma) = d(b', \sigma) = d(id, \sigma) - 1$ . This would contradict the fact that  $d(id, \sigma) = \min_{\sigma' \in S} d(id, \sigma')$ .

Thus  $b'$  must be an element of  $\mathcal{B}_k \setminus \mathfrak{S}_k$ . As  $d(id, b') = 1$ , there exist  $i$  and  $j$  in  $\{0, \dots, k\}$  such that:

$$b' = [i, j],$$

where  $[i, j]$  is the Weyl contraction in  $\mathcal{B}_k$ . Thus there exist  $i$  and  $j$  in  $\{0, \dots, k\}$  such that  $[i, j] \in [id, \sigma]_{\mathcal{B}_k}$ . Recall that  $c(\sigma)$  is the number of cycles of  $\sigma$ . We have:

$$\begin{aligned} & d(id, [i, j]) + d([i, j], \sigma) - d(id, \sigma) \\ &= \text{nc}([i, j]) + \text{nc}(id \vee \sigma) - \text{nc}([i, j] \vee id) - \text{nc}([i, j] \vee \sigma) \\ &= k + c(\sigma) - (k - 1) - \text{nc}([i, j] \vee \sigma) \\ &= 1 + c(\sigma) - c(\sigma) + \delta_{i \text{ and } j \text{ not in the same cycle of } \sigma} \\ &= 1 + \delta_{i \text{ and } j \text{ not in the same cycle of } \sigma} > 0. \end{aligned}$$

Thus  $[i, j] \notin [id, \sigma]_{\mathcal{B}_k}$ : this yields the contradiction.  $\square$

This lemma is the key point which will allow us to explain in the second article [9] why processes on  $U(N)$  and  $O(N)$  have the same limit when one only considers usual moments.

The last property, known in the  $\mathfrak{S}_k$  and  $\mathcal{B}_k$  case, is still true for  $\mathcal{P}_k$ : a geodesic between  $id$  and  $p_1 \otimes p_2$  must be the tensor product of the geodesic between  $id$  and  $p_1$  and the geodesic between  $id$  and  $p_2$ .

**Lemma 3.4.** — *Let  $p \in A_k$ , we have:*

$$[id, p]_{A_k} \simeq \prod_{C \in \mathcal{C}(p)} \left[ id_{\frac{\#C}{2}}, p_C \right]_{A_{\frac{\#C}{2}}}.$$

*In particular if  $p_1$  and  $p_2$  are two partitions,  $p_1 \in \mathcal{A}_{k_1}$  and  $p_2 \in \mathcal{A}_{k_2}$ , then  $p' \in [id, p_1 \otimes p_2]_{A_{k_1+k_2}}$  if and only if there exist  $p'_1 \in \mathcal{A}_{k_1}$  and  $p'_2 \in \mathcal{A}_{k_2}$  such that  $p' = p'_1 \otimes p'_2$ .*

Let us finish this section with two propositions on geodesics. Let us define a notion of default in order to simplify the proofs.

**Definition 3.5.** — *Let  $p$  and  $p'$  be two elements of  $A_k$ . We define the default of  $p'$  not being on the geodesic  $[id, p]_{A_k}$  by:*

$$\text{df}(p', p) = d(id, p') + d(p', p) - d(id, p).$$

A simple but useful lemma is the following.

**Lemma 3.5.** — *Let  $k$  be an integer, let  $p \in \mathcal{P}_k$  and  $p' \in \mathcal{P}_k$  such that  $p$  is coarser than  $p'$ . Then:*

$$\text{df}(p', p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p) + \text{nc}(p \vee id).$$

*Proof.* — This is a simple calculation, where one has to use the fact that  $\text{nc}(p \vee p') = \text{nc}(p)$  since  $p$  is coarser than  $p'$ .  $\square$

**Proposition 3.3.** — *Let  $p$  and  $p'$  be two partitions in  $\mathcal{P}_k$ . Then  $p' \in [id, p]_{A_k}$  if and only if  $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$  and  $p' \in [id, p \vee p']_{A_k}$ .*

One can see this proposition as a direct consequence of the forthcoming Theorem 10.4, by considering the element of  $\mathfrak{E}[A]$  which is equal to  $p$ , with  $p \in \mathcal{P}_k$ . Yet, we give a direct proof: the proof is simple yet, without Theorem 10.4, it would have been trickier to guess the following proposition.

*Proof of Proposition 3.3.* — Let  $p$  and  $p'$  be two partitions in  $\mathcal{P}_k$ . Using the Lemma 3.4, we see that  $p' \in [id, p]_{\mathcal{P}_k}$  if and only if  $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$  and  $p' \in [id, p]_{\mathcal{P}_k}$ , thus, if and only if  $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$  and:

$$\text{df}(p', p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p \vee p') + \text{nc}(p \vee id) = 0,$$

which is equivalent to  $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$  and:

$$\text{df}(p', p \vee p') = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p \vee p') + \text{nc}(p \vee p' \vee id) = 0,$$

which is again equivalent to  $\text{nc}(p \vee p' \vee id) = \text{nc}(p \vee id)$  and  $p' \in [id, p \vee p']_{\mathcal{P}_k}$ .  $\square$

For the last geometric proposition, we need to define the left and right parts of a partition  $p$ .

**Definition 3.6.** — Let  $k$  and  $l$  be two integers. Let  $p \in A_{k+l}$ , we denote by  $p_k^g$  the extraction of  $p$  to  $\{1, \dots, k\}$  and  $p_k^d$  the extraction of  $p$  to  $\{k+1, \dots, k+l\}$ . The partition  $p_k^g$  is in  $\mathcal{P}_k$  and  $p_k^d$  is in  $\mathcal{P}_l$ .

**Proposition 3.4.** — Let  $k_1$  and  $k_2$  be two positive integers and let  $k = k_1 + k_2$ . Let  $p$  be an element of  $\mathcal{P}_k$ . Let  $p_1$  and  $p_2$  be respectively in  $\mathcal{P}_{k_1}$  and  $\mathcal{P}_{k_2}$ . We have equivalence between:

1.  $p$  is coarser than  $p_1 \otimes p_2$  and  $p_1 \otimes p_2 \in [id, p]_{\mathcal{P}_k}$ ,
2.  $p_{k_1}^g$  is coarser than  $p_1$ ,  $p_1$  is in  $[id, p_{k_1}^g]_{\mathcal{P}_{k_1}}$ ,  $p_{k_1}^d$  is coarser than  $p_2$ ,  $p_2$  is in  $[id, p_{k_2}^d]_{\mathcal{P}_{k_2}}$  and  $p_{k_1}^g \otimes p_{k_1}^d \in [id, p]_{\mathcal{P}_k}$ .

*Proof.* — Let  $k_1$  and  $k_2$  be two positive integers and let  $k = k_1 + k_2$ . Let  $p$  be an element of  $\mathcal{P}_k$ . Let  $p_1$  and  $p_2$  be respectively in  $\mathcal{P}_{k_1}$  and  $\mathcal{P}_{k_2}$ .

First of all, it is easy to see that  $p$  is coarser than  $p_1 \otimes p_2$  if and only if  $p_{k_1}^g$  is coarser than  $p_1$  and  $p_{k_1}^d$  is coarser than  $p_2$ .

Let us suppose that  $p$  is coarser than  $p_1 \otimes p_2$ , let us show that  $p_1 \otimes p_2 \in [id, p]_{A_k}$  if and only if  $p_1 \in [id, p_{k_1}^g]_{\mathcal{P}_{k_1}}$ ,  $p_2 \in [id, p_{k_2}^d]_{\mathcal{P}_{k_2}}$  and  $p_{k_1}^g \otimes p_{k_1}^d \in [id, p]_{A_k}$ . Since for any partitions the default between two partitions is always positive, this is equivalent to show that:

$$\text{df}(p_1 \otimes p_2, p) = \text{df}(p_1, p_{k_1}^g) + \text{df}(p_2, p_{k_2}^d) + \text{df}(p_{k_1}^g \otimes p_{k_1}^d, p).$$

Yet, using Lemme 3.5:

$$\begin{aligned} & \text{df}(p_1 \otimes p_2, p) - \text{df}(p_1, p_{k_1}^g) - \text{df}(p_2, p_{k_2}^d) - \text{df}(p_{k_1}^g \otimes p_{k_1}^d, p) \\ &= \text{nc}(p_1 \otimes p_2) - \text{nc}((p_1 \otimes p_2) \vee id) - \text{nc}(p) + \text{nc}(p \vee id) \\ & \quad - \text{nc}(p_1) + \text{nc}(p_1 \vee id) + \text{nc}(p_{k_1}^l) - \text{nc}(p_{k_1}^g \vee id) \\ & \quad - \text{nc}(p_2) + \text{nc}(p_2 \vee id) + \text{nc}(p_{k_2}^r) - \text{nc}(p_{k_2}^d \vee id) \\ & \quad - \text{nc}(p_{k_1}^g \otimes p_{k_1}^d) + \text{nc}((p_{k_1}^g \otimes p_{k_1}^d) \vee id) + \text{nc}(p) - \text{nc}(p \vee id) \\ &= 0, \end{aligned}$$

since:

$$\begin{aligned} \text{nc}(p_1 \otimes p_2) &= \text{nc}(p_1) + \text{nc}(p_2), \\ \text{nc}((p_1 \otimes p_2) \vee id) &= \text{nc}(p_1 \vee id) + \text{nc}(p_2 \vee id), \\ \text{nc}(p_k^g \otimes p_k^d) &= \text{nc}(p_k^g) + \text{nc}(p_k^d), \\ \text{nc}((p_k^g \otimes p_k^d) \vee id) &= \text{nc}(p_k^g \vee id) + \text{nc}(p_k^d \vee id) \end{aligned}$$

This ends the proof.  $\square$

#### 4. Convergence of elements of $\prod_{N \in \mathbb{N}} \mathbb{C}[P_k(N)]$

##### 4.1. Coordinate numbers and moments. —

*4.1.1. Definitions.* — Let  $k$  be an integer, recall the notation  $A_k$  defined in Notation 2.1. For each integer  $N$ , we have defined an algebra  $\mathbb{C}[A_k(N)]$ . Let  $(E_N)_{N \in \mathbb{N}}$  be a sequence such that for any integer  $N$ ,  $E_N \in \mathbb{C}[A_k(N)]$ . For each integer  $N$ , the algebra  $\mathbb{C}[A_k(N)]$ , seen as a vector space has the same basis  $A_k$ . Thus, we could study the convergence of  $(E_N)_{N \in \mathbb{N}}$  only from the vector space point of view by saying that the sequence  $(E_N)_{N \in \mathbb{N}}$  converges if and only if the coordinates of  $E_N$  in the basis  $A_k$  converge. Actually, this convergence forgets the fact that  $\mathbb{C}[A_k(N)]$  is an algebra which depends on an integer  $N$ . In order to define a better definition of convergence, we have to define the coordinate numbers of  $E$  in  $\mathbb{C}[A_k(N)]$ .

**Definition 4.1.** — *Let  $N$  be an integer. Let  $E$  be an element of  $\mathbb{C}[A_k(N)]$ . We define the numbers  $(\kappa^p(E))_{p \in A_k}$  as the only numbers such that:*

$$E = \sum_{p \in A_k} \frac{\kappa^p(E)}{N^{\frac{-k + \text{nc}(p)}{2} + d(id, p)}} p.$$

*The family  $(\kappa^p(E))_{p \in A_k(N)}$  is called the coordinate numbers of  $E$ .*

After Definition 4.4, we will explain how we get this definition, and why this definition is in fact the most natural thing one can do. We will need to use the following equality: for any integer  $k$ , for any  $p \in A_k$ ,

$$(4) \quad \frac{-k + \text{nc}(p)}{2} + d(id, p) = \text{nc}(p) - \text{nc}(p \vee id).$$

This implies the following remark.

**Remark 4.1.** — *For any integer  $k$ , any integer  $N$ , for any  $E \in \mathbb{C}[P_k(N)]$ :*

$$E = \sum_{p \in A_k} \frac{\kappa^p(E)}{N^{\text{nc}(p) - \text{nc}(p \vee id)}} p.$$

We will consider the coordinate numbers as linear applications from  $\mathbb{C}[A_k(N)]$  to  $\mathbb{R}$ :

$$\begin{aligned} \kappa^p : \mathbb{C}[A_k(N)] &\rightarrow \mathbb{R} \\ E &\mapsto \kappa^p(E). \end{aligned}$$

The notion of coordinate numbers allows us to define a strong convergence for any sequence  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Definition 4.2.** — Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The sequence  $(E_N)_{N \in \mathbb{N}}$  converges strongly if the coordinate numbers of  $E_N$  converges when  $N$  goes to infinity: for any  $p \in A_k$ ,  $\kappa^p(E_N)$  converges when  $N$  goes to infinity.

The goal now is to give a dual definition of convergence. We have seen in Definition 2.14 that any element of  $\mathbb{C}[A_k(N)]$  can be seen as an element of  $\text{End}\left((\mathbb{C}^N)^{\otimes k}\right)$  and we defined in Definition 3.1 the trace of any element  $\mathbb{C}[A_k(N)]$ . Using this trace and the structure of algebra of  $\mathbb{C}[A_k(N)]$ , we define, for any element of  $\mathbb{C}[A_k(N)]$  and any element  $p \in A_k$ , the  $p$ -normalized moment of  $E$ .

**Definition 4.3.** — Let  $N \in \mathbb{N}$ , let  $p \in A_k$  and  $E \in \mathbb{C}[A_k(N)]$ . The  $p$ -normalized moment of  $E$  is:

$$m_p(E) = \frac{1}{\text{Tr}_N(p)} \text{Tr}_N(E^t p).$$

Using these normalized moments, we can define a weak notion of convergence for any sequence  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Definition 4.4.** — The sequence  $(E_N)_{N \in \mathbb{N}}$  converges in moments if the normalized moments of  $E_N$  converges when  $N$  goes to infinity: for any  $p \in A_k$ ,  $m_p(E_N)$  converges when  $N$  goes to infinity.

*4.1.2. Coordinate numbers-moments transformation.* — We can now explain how we ended up with Definition 4.1 and we had the idea to define the distance on the set of partitions. The idea behind these definitions is that we want to know, given a sequence of  $E_N \in \mathbb{C}[A_k(N)]$ , how the usual coordinates of  $E_N$  in the basis  $A_k$  must scale so that for any  $p \in A_k$ ,  $m_p(E_N)$  converges when  $N$  goes to infinity. Let  $N$  be an integer, we have  $E_N = \sum_{p \in A_k} a_N^p p$ . Thus

$$m_{p_0}(E_N) = \sum_{p \in A_k} \frac{\text{Tr}_N(p^t p_0)}{\text{Tr}_N(p_0)} a_N^p.$$

Thus the vector  $m_N = (m_{p_0}(E_N))_{p_0}$  and  $a_N = (a_N^p)_p$  are linked by the relation  $m_N = M_N a_N$  where  $M_N = \left( \frac{\text{Tr}_N(p^t p_0)}{\text{Tr}_N(p_0)} \right)_{p_0, p}$ .

There are then two possible possibilities: to invert  $M_N$  for  $N$  big enough. This is the usual way, which leads to the Weingarten function. Or, one can make the following Ansatz: if we write the system, we see that for any  $p$ ,  $(a_N)_p$  is going to be multiplied by  $(M_N)_{p_0, p}$  for any  $p_0 \in A_k$ . Thus we make the assumption that  $(m_N)_p$  must grow as the inverse of the maximum of  $(M_N)_{p_0, p}$  over  $p_0$ . That is  $a_N^p \sim a^p N^{-\eta(p)}$ , where  $\eta_p$  is given by:

$$\eta_p = \sup_{p_0} \lim_{N \rightarrow \infty} \log_N \left( \frac{\text{Tr}_N(p^t p_0)}{\text{Tr}_N(p_0)} \right).$$

The goal now is to know in which  $p_0$  the supremum is obtained. It is more than tempting, seeing the scalar product  $Tr_N(p_0 \ ^t p)$  to write what is inside the  $\log_N$  as:

$$\begin{aligned} \frac{Tr_N(p \ ^t p_0)}{Tr_N(p_0)} &= \frac{Tr_N(p \ ^t p_0)}{\sqrt{Tr_N(p \ ^t p)Tr_N(p_0 \ ^t p_0)}} \frac{\sqrt{Tr_N(p_0 \ ^t p_0)Tr_N(p \ ^t p)}}{Tr_N(p_0)} \\ &= \frac{Tr_N(p \ ^t p_0)}{\sqrt{Tr_N(p \ ^t p)Tr_N(p_0 \ ^t p_0)}} \frac{\sqrt{Tr_N(p_0 \ ^t p_0)Tr_N(id_k \ ^t id_k)}}{Tr_N(p_0 \ ^t id_k)} \sqrt{\frac{Tr_N(p \ ^t p)}{Tr_N(id_k \ ^t id_k)}}. \end{aligned}$$

We recognize thus the distance that we defined. In fact the intuition that is should be a distance comes from the fact that one can write:

$$\eta_p = \sup_{p_0} [-d(p, p_0) + d(p_0, id_k)] + \frac{1}{2}(-k + nc(p)).$$

If  $d$  was a distance, then by the triangle inequality, for any  $p_0$ ,

$$d(p_0, id_k) - d(p_0, p) \leq d(p, id_k).$$

This shows that the supremum is obtained at  $p_0 = p$ , and thus the Ansatz tells us that:

$$a_N^p \sim a^p N^{-[\frac{1}{2}(-k+nc(p))+d(id,p)]},$$

to be compared with the Definition 4.1.

The first main result is given by Theorem 4.1 which shows the equivalence between strong and weak convergence.

**Theorem 4.1.** — *Let  $(E_N)_{N \in \mathbb{N}}$  be a sequence such that for any  $N \in \mathbb{N}$ ,*

$$E_N \in \mathbb{C}[A_k(N)].$$

*It converges strongly if and only if it converges in moments. Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges in moments or strongly, for any  $p \in A_k$ :*

$$(5) \quad \lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id,p]_{A_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

*Proof.* — Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ , let  $p \in A_k$  and let  $N$  be an integer. Using the coordinate numbers of  $E_N$ , we can calculate the  $p$ -normalized moments of  $E$ :

$$\begin{aligned} m_p(E_N) &= \frac{1}{Tr_N(p)} Tr_N(E_N \ ^t p) = \frac{1}{Tr_N(p)} Tr_N \left( \sum_{p' \in A_k(N)} \frac{\kappa^{p'}(E_N)}{N^{-\frac{-k+nc(p')}{2}+d(id,p')}} p' \ ^t p \right) \\ &= \sum_{p' \in A_k} \kappa^{p'}(E_N) \frac{Tr_N(p' \ ^t p)}{Tr_N(p) N^{-\frac{-k+nc(p')}{2}+d(id,p')}}. \end{aligned}$$

Using the definition of the distance, in Proposition 3.1, one has:

$$\begin{aligned} Tr_N(p' \ ^t p) &= N^{-d(p,p') + \frac{nc(p)+nc(p')}{2}}, \\ Tr_N(p) &= N^{-d(id,p) + \frac{nc(p)+k}{2}}. \end{aligned}$$

Thus:

$$m_p(E_N) = \sum_{p' \in A_k} \kappa^{p'}(E_N) N^{-d(p,p') + \frac{nc(p)+nc(p')}{2} + d(id,p) - \frac{nc(p)+k}{2} + \frac{k-nc(p')}{2} - d(id,p')}.$$

Hence:

$$(6) \quad m_p(E_N) = \sum_{p' \in A_k} \kappa^{p'}(E_N) N^{d(id,p) - d(id,p') - d(p,p')}.$$

Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges strongly. The triangular inequality for  $d$  shows that for any  $p \in A_k(N)$  converges when  $N$  goes to infinity. Besides, it allows us to write that for any  $p \in A_k(N)$ :

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id,p]_{A_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

Now, let us suppose that it converges in moments. We can write (6) as:

$$m^N = G_N \kappa^N,$$

where:

$$\begin{aligned} m^N &= (m_p(E_N))_{p \in A_k(N)}, \\ \kappa^N &= (\kappa^p(E_N))_{p \in A_k(N)}, \\ G_N &= \left( N^{d(id,p) - d(id,p') - d(p,p')} \right)_{p,p' \in A_k(N)}. \end{aligned}$$

Thus the sequence  $(G_N)_{N \in \mathbb{N}}$  converges to the matrix of the partial order  $\leq$  defined in Definition 3.4:

$$\lim_{N \rightarrow \infty} G_N = G,$$

where  $G_{p,p'} = \delta_{p \leq p'}$ . This last matrix is invertible, thus  $\kappa^N = G_N^{-1} m^N$  converges to  $G^{-1} m$  where  $m = (\lim_{N \rightarrow \infty} m_p(E_N))_{p \in A_k}$ .  $\square$

Let us take some notations in order to simplify our up-coming discussions.

**Notation 4.1.** — Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . From now on, we will say that  $(E_N)_{N \in \mathbb{N}}$  converges if and only if it converges either strongly or in moments. Besides, let suppose that  $(E_N)_{N \in \mathbb{N}}$  converges, then we will set, for any partition  $p \in A_k$  and any  $P \subset A_k$ :

$$\begin{aligned} m_p(E) &= \lim_{N \rightarrow \infty} m_p(E_N), \\ \kappa^p(E) &= \lim_{N \rightarrow \infty} \kappa^p(E_N) \\ \kappa^P(E) &= \sum_{p \in P} \kappa^p(E). \end{aligned}$$

**4.2. Consequences of Theorem 4.1.**— We have already an interesting corollary of Theorem 4.1.

**Theorem 4.2.** — For this theorem, let us suppose that  $A$  is equal either to  $\mathfrak{S}$  or  $\mathcal{B}$ . Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  which converges in moments, then for any  $p \in \mathcal{P}_k$ , the limit of  $m_p(E_N)$  exists. Besides, for any  $p \in \mathcal{P}_k$ , the following equality holds:

$$m_p(E) = \sum_{p' \in A_k, p' \in [id, p]_{\mathcal{P}_k}} \kappa^{p'}(E).$$

In the case where  $A = \mathcal{B}$ , one can also prove that, under some hypothesis, the convergence of the  $\mathfrak{S}$ -moments is equivalent to the convergence of the  $\mathfrak{S}$ -coordinate numbers.

**Theorem 4.3.** — Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$  and let us suppose that for any  $p \in \mathcal{B}_k$ ,  $(m_p(E_N))_{N \in \mathbb{N}}$  is bounded.

The following assertions are equivalent:

- for any  $\sigma \in \mathfrak{S}_k$ ,  $\kappa^\sigma(E_N)$  converges when  $N$  goes to infinity,
- for any  $\sigma \in \mathfrak{S}_k$ ,  $m_\sigma(E_N)$  converges when  $N$  goes to infinity.

and if one of the condition is satisfied, then for any  $\sigma \in \mathfrak{S}_N$ ,

$$m_\sigma(E) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{\sigma'}(E).$$

*Proof.* — Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$  which satisfies the hypothesis of the theorem. First of all, using the same notations of the proof of Theorem 4.1, we know that, for  $N$  big enough  $\kappa^N = G_N^{-1} m^N$ . As the sequence  $(m^N)_{N \in \mathbb{N}}$  is bounded and as  $G_N^{-1}$  converges to  $G^{-1}$  when  $N$  goes to infinity, we deduce that  $(\kappa^N)_{N \in \mathbb{N}}$  is also bounded.

Let  $\sigma \in \mathfrak{S}_k$ . Using the Equation (6), for any integer  $N$ ,

$$m_\sigma(E_N) = \sum_{p' \in \mathcal{B}_k} \kappa^{p'}(E_N) N^{d(id, \sigma) - d(id, p') - d(\sigma, p')}.$$

Yet, if  $p' \in \mathcal{B}_k \setminus \mathfrak{S}_k$ , using Lemma 3.3,  $d(id, \sigma) - d(id, p') - d(\sigma, p') < 0$ .

Let us suppose that for any  $\sigma' \in \mathfrak{S}_k$ ,  $\kappa^{\sigma'}(E_N)$  converges, then  $m_\sigma(E_N)$  converges as  $N$  goes to infinity, and:

$$\lim_{N \rightarrow \infty} m_\sigma(E_N) = \sum_{p' \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{p'}(E_N).$$

Let us suppose now that for any  $\sigma \in \mathfrak{S}_k$ ,  $m_\sigma(E_N)$  converges when  $N$  goes to infinity, then for any increasing sequence  $(i_N)_{N \in \mathbb{N}}$  of integers such that for any  $\sigma' \in \mathfrak{S}_k$ ,  $\kappa^{\sigma'}(E_{i_N})$  converges, we have:

$$\lim_{N \rightarrow \infty} m_\sigma(E_N) = \sum_{p' \in [id, \sigma]_{\mathfrak{S}_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_{i_N}).$$



Hence, for any  $p' \in \mathfrak{S}_k$ ,  $\lim_{N \rightarrow \infty} \kappa^{p'}(E_{i_N})$  does not depend on the sequence  $(i_N)_{N \in \mathbb{N}}$ : this shows that for any  $\sigma' \in \mathfrak{S}_k$ ,  $\kappa^{\sigma'}(E_N)$  converges when  $N$  goes to infinity. Again we get also:

$$\lim_{N \rightarrow \infty} m_\sigma(E_N) = \sum_{p' \in [id, \sigma]_{\mathfrak{S}_k}} \kappa^{p'}(E_N).$$

This finishes the proof.  $\square$

### 4.3. Exclusive coordinate numbers and exclusive moments. —

*4.3.1. Exclusive coordinate numbers.* — In Section 2.3, we defined an other basis of  $\mathbb{C}[\mathcal{P}_k]$ , namely the exclusive basis. In the case we are working with an element  $E \in \mathbb{C}[A_k(N)]$  we can also define the exclusive coordinate numbers.

**Definition 4.5.** — *Let  $k$  and  $N$  be two integers. Let  $E$  be an element of  $\mathbb{C}[A_k(N)]$ . We define the numbers  $(\kappa_c^p(E))_{p \in \mathcal{P}_k}$  as the only numbers such that:*

$$E = \sum_{p \in \mathcal{P}_k} \frac{\kappa_c^p(E)}{N^{d(id,p) + \frac{-k+nc(p)}{2}}} p^c = \sum_{p \in \mathcal{P}_k} \frac{\kappa_c^p(E)}{N^{nc(p) - nc(p \vee id)}} p^c.$$

The family  $(\kappa_c^p(E))_{p \in \mathcal{P}_k}$  is called the exclusive coordinate numbers of  $E$ .

The next proposition shows that one can choose to work either with the exclusive basis or with the usual basis of  $\mathbb{C}[\mathcal{P}_k]$  in order to study the convergence of  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Theorem 4.4.** — *Let  $k$  be an integer. Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .*

*The exclusive coordinate numbers  $(\kappa_c^p(E_N))_{p \in A_k}$  converge as  $N$  goes to infinity if and only if  $(E_N)_{N \in \mathbb{N}}$  converges. Besides, if  $(E_N)_{N \in \mathbb{N}}$  converges then for any  $p \in \mathcal{P}_k$ ,  $\kappa_c^p(E_N)$  converges as  $N$  goes to infinity, and for any  $p \in \mathcal{P}_k$ :*

$$\lim_{N \rightarrow \infty} \kappa_c^p(E_N) = \sum_{p' \in A_k, p' \text{ finer than } p, p' \in [id, p]_{\mathcal{P}_k}} \lim_{N \rightarrow \infty} \kappa_{p'}(E_N)$$

*Proof.* — Let  $k$  be an integer, let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Then for any integer  $N$ :

$$\begin{aligned} E_N &= \sum_{p \in A_k} \frac{\kappa^p(E_N)}{N^{nc(p) - nc(p \vee id)}} p^c = \sum_{p \in A_k} \frac{\kappa^p(E_N)}{N^{nc(p) - nc(p \vee id)}} \sum_{p' \text{ coarser than } p, p' \in \mathcal{P}_k} p'^c \\ &= \sum_{p \in A_k, p' \text{ coarser than } p, p' \in \mathcal{P}_k} \kappa^p(E_N) N^{-nc(p) + nc(p \vee id) + nc(p') - nc(p' \vee id)} \frac{p'^c}{N^{nc(p') - nc(p' \vee id)}}, \end{aligned}$$

and using Lemma 3.5:

$$E_N = \sum_{p' \in \mathcal{P}_k} \left( \sum_{p \in A_k, p \text{ finer than } p'} \kappa^p(E_N) N^{-df(p, p')} \right) \frac{p'^c}{N^{nc(p') - nc(p' \vee id)}}.$$

Thus, for any integer  $N$ , for any  $p' \in \mathcal{P}_k$

$$(7) \quad \kappa_c^{p'}(E_N) = \sum_{p \in A_k, p \text{ finer than } p'} \kappa^p(E_N) N^{-\text{df}(p, p')}.$$

The result follows from this equality, and the usual arguments already explained in Theorem 4.1.  $\square$

Let us remark that, using the Equality (7), one has the following proposition.

**Proposition 4.1.** — *Let  $A$  be either  $\mathfrak{S}$  or  $\mathfrak{B}$ . Let  $N$  be an integer, let  $E \in \mathbb{C}[A_k(N)]$ , for any  $p \in A_k$ :*

$$\kappa_c^p(E) = \kappa^p(E).$$

*Proof.* — This is a consequence of Equality (7) and the fact that  $p'$  in  $\mathcal{P}_k$  is finer than  $p' \in A_k$  implies that  $p' \notin A_k$ .  $\square$

**4.3.2. Exclusive moments.** — As we did for the coordinate numbers, one can define exclusive normalized moments.

**Definition 4.6.** — *Let  $N \in \mathbb{N}$ , let  $p \in \mathcal{P}_k$  and  $E \in \mathbb{C}[A_k(N)]$ . The  $p$ -exclusive normalized moment of  $E$  is:*

$$m_{p^c}(E) = \frac{1}{\text{Tr}_N(p)} \text{Tr}_N(E \text{ }^t(p^c)).$$

One can also give a combinatorial definition of the  $p$ -exclusive normalized moment.

**Lemma 4.1.** — *Let  $p$  and  $p'$  be in  $\mathcal{P}_k$ , then:*

$$\text{Tr}_N(p \text{ }^t(p'^c)) = \delta_{p' \text{ coarser than } p} \frac{N!}{(N - \text{nc}(p'))!}.$$

The easiest way to prove this lemma is to do it graphically: we see that  $p'$  must be coarser than  $p$ , if not the trace is equal to zero, and if  $p'$  is coarser than  $p$ , it is equal to  $\frac{N!}{(N - \text{nc}(p'))!}$ .

**Definition 4.7.** — *Let  $p$  and  $p'$  be in  $\mathcal{P}_k$ . We say that  $p'$  is coarser-compatible than  $p$  if and only if  $p'$  is coarser than  $p$  and  $\text{nc}(p \vee id) = \text{nc}(p' \vee id)$  and  $p'$  is coarser than  $p$ .*

The condition  $p'$  coarser compatible with  $p$  just means that one can glue only blocks of  $p$  which are in the same cycle in order to get  $p'$ . Similarly to what we proved for coordinate numbers, we prove the following proposition. Let us consider  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Proposition 4.2.** — *The sequence  $(E_N)_{N \in \mathbb{N}}$  converges in exclusive normalized moments if and only if for any  $p \in \mathcal{P}_k$ ,  $(m_{p^c}(E_N))_{N \in \mathbb{N}}$  converges. Besides, if  $(E_N)_{N \in \mathbb{N}}$  converges in normalized moments then for any  $p \in \mathcal{P}_k$ :*

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in \mathcal{P}_k, p' \text{ coarser-compatible than } p} \lim_{N \rightarrow \infty} m_{p'^c}(E_N).$$

*Proof.* — It is enough to consider  $(E_N)_{N \in \mathbb{N}}$  an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$ . By computation:

$$m_p(E_N) = \sum_{p' \text{ coarser than } p} N^{\text{nc}(p' \vee \text{id}) - \text{nc}(p \vee \text{id})} m_{p'^c}(E_N).$$

We are in the same setting as for the proof of Theorem 4.1: we can write this equality as:

$$m_N = G_N m_{c,N},$$

where  $(m_N)_p = m_p(E_N)$ ,  $(m_{c,N})_p = m_{p^c}(E_N)$  and  $G_N$  converges to the matrix of the partial order of being coarser-compatible. With the same arguments than in the proof of Theorem 4.1, we get that  $m_N$  converges to infinity if and only if  $m_{c,N}$  converges to infinity: the sequence  $(E_N)_{N \in \mathbb{N}}$  converges in  $\mathcal{P}_k$ -exclusive normalized moments if and only if it converges in normalized moments and in that case:

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \text{ coarser-compatible than } p} \lim_{N \rightarrow \infty} m_{p'^c}(E_N).$$

This finishes the proof.  $\square$

*4.3.3. In the exclusive world, coordinate numbers and moments are equal.* — We will prove that the limit of exclusive normalized moments are in fact equal to the limit of the exclusive coordinate numbers. Let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Theorem 4.5.** — *Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges in normalized moments. Then, for any  $p \in \mathcal{P}_k$ ,*

$$\lim_{N \rightarrow \infty} m_{p^c}(E_N) = \lim_{N \rightarrow \infty} \kappa_c^p(E_N).$$

*Proof.* — We will prove that for any integer  $N$ , any  $p \in A_k$ , seen as an element of  $\mathbb{C}[A_k(N)]$ , for any  $p' \in \mathcal{P}_k$ ,

$$\kappa_c^{p'}(p) = \left( \prod_{i=0}^{\text{nc}(p')-1} \binom{N}{N-k} \right) m_{p'^c}(p).$$

Indeed by the Equality 7, we get that for any  $p' \in \mathcal{P}_k$ :

$$(8) \quad \kappa_c^{p'}(p) = \delta_{p' \text{ coarser than } p} N^{\text{nc}(p') - \text{nc}(p' \vee \text{id})}.$$

Let  $p' \in \mathcal{P}_k$ , by Lemma 4.1:

$$m_{p'^c}(p) = \frac{1}{N^{\text{nc}(p' \vee \text{id})}} \text{Tr}_N(p^t(p'^c)) = \delta_{p' \text{ coarser than } p} \frac{N!}{(N - \text{nc}(p'))!} N^{-\text{nc}(p' \vee \text{id})}.$$

The theorem is now a simple consequence of a linearity argument and taking  $N$  going to infinity.  $\square$

Let us remark that one can prove Theorem 4.5 also by a purely combinatorial argument using Proposition 3.3, we give the proof below.

*Combinatorial proof of 4.5.* — It is enough to show that  $\lim_{N \rightarrow \infty} \kappa^p(E_N)$  satisfies the Equality in Proposition 4.2: for any  $p \in \mathcal{P}_k$ ,

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in \mathcal{P}_k, p' \text{ coarser-compatible than } p} \lim_{N \rightarrow \infty} \kappa_c^{p'}(E_N).$$

Using the fact that for any  $p \in \mathcal{P}_k$ :

$$\lim_{N \rightarrow \infty} \kappa_c^p(E_N) = \sum_{p' \in A_k, p' \text{ finer than } p, p' \in [id, p]_{\mathcal{P}_k}} \lim_{N \rightarrow \infty} \kappa_{p'}(E_N),$$

we only have to prove that for any  $p \in \mathcal{P}_k$ :

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in \mathcal{P}_k, p'' \in A_k, p' \text{ coarser-compatible than } p, p'' \text{ finer than } p', p'' \in [id, p']_{\mathcal{P}_k}} \lim_{N \rightarrow \infty} \kappa^{p''}(E_N).$$

Using a slight modification of Proposition 3.3, there exists  $p' \in \mathcal{P}_k$  coarser than  $p \vee p''$ , coarser compatible than  $p$  and such that  $p'' \in [id, p']_{\mathcal{P}_k}$  if and only if  $p'' \in [id, p]_{\mathcal{P}_k}$ . Thus we only have to prove that for any  $p \in \mathcal{P}_k$ :

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p'' \in A_k, p'' \in [id, p]_{\mathcal{P}_k}} \lim_{N \rightarrow \infty} \kappa^{p''}(E_N).$$

Using the Theorem 4.2, we can conclude.  $\square$

Using Theorem 4.5, Theorem 4.4 and Proposition 4.1, one can give an expression of the exclusive moments which involves the coordinate numbers, and one can link the exclusive normalized moments with the coordinate numbers.

**Theorem 4.6.** — *Let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges in normalized moments. Then, for any  $p \in \mathcal{P}_k$ ,*

$$\lim_{N \rightarrow \infty} m_{p^c}(E_N) = \sum_{p' \in A_k, p' \text{ finer than } p, p' \in [id, p]_{\mathcal{P}_k}} \lim_{N \rightarrow \infty} \kappa_{p'}(E_N).$$

Besides, let us suppose until the end of the theorem that  $A$  is equal either to  $\mathfrak{S}$  or  $\mathfrak{B}$ , then for any  $N \in \mathbb{N}$  and any  $p \in A_k$ :

$$\kappa^p(E_N) = \left( \prod_{i=0}^{\text{nc}(p')-1} \binom{N}{N-k} \right) m_{p^c}(E_N).$$

In particular, for any  $p \in A_k$ :

$$\lim_{N \rightarrow \infty} \kappa^p(E_N) = \lim_{N \rightarrow \infty} m_{p^c}(E_N).$$

At the beginning of this section, we have argued that the simplest notion of convergence of elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  was not interesting as it did not take into account the fact that  $\mathbb{C}[A_k(N)]$  is an algebra which depends on the parameter  $N$ . In the following section, we will slightly modify the product defined on  $\mathbb{C}[A_k(N)]$  in order to define a new algebra  $\mathbb{C}[A_k(N, N)]$ . In this new algebra the strong convergence will be the usual notion of convergence in vector spaces.

## 5. The deformed partition algebra

Let us define a deformation of the partition algebra by modifying the multiplication which was set in Definition 2.4.

**Definition 5.1.** — *Let  $k$  and  $N$  be two integers. We define the application:*

$$M_k^N : A_k \rightarrow A_k$$

$$p \mapsto \frac{1}{N^{d(id,p) + \frac{-k+nc(p)}{2}}} p.$$

*This application can be extended as an isomorphism of vector spaces from  $\mathbb{C}[A_k]$  to itself.*

Let  $k$  and  $N$  be two integers. Seen as a vector space, the algebra  $\mathbb{C}[A_k(N)]$  is isomorphic to  $\mathbb{C}[A_k]$ . Thus, we can see  $M_k^N$  as an isomorphism of vector spaces from  $\mathbb{C}[A_k]$  to  $\mathbb{C}[A_k(N)]$ . Let us endow  $\mathbb{C}[A_k]$  with a structure of associative algebra by taking the pullback of the structure of algebra of  $\mathbb{C}[A_k(N)]$  by  $M_k^N$ : for any  $p_1, p_2$  in  $A_k$  the new product of  $p_1$  with  $p_2$  is given by:

$$p_1 \cdot_N p_2 = (M_k^N)^{-1} [M_k^N(p_1)M_k^N(p_2)].$$

This is the deformed algebra  $\mathbb{C}[A_k(N, N)]$ . Using the definition of  $M_k^N$ , one gets the following proposition.

**Proposition 5.1.** — *Let  $N$  be an integer. The deformed algebra  $\mathbb{C}[A_k(N, N)]$  is the associative algebra over  $\mathbb{C}$  with basis  $\mathcal{P}_k$ , endowed with the multiplication defined by the fact that for any  $p_1, p_2 \in A_k$ :*

$$p_1 \cdot_N p_2 = N^{\kappa(p_1, p_2)} N^{d(id, p_1 \circ p_2) - d(id, p_1) - d(id, p_2) + \frac{k+nc(p_1 \circ p_2) - nc(p_1) - nc(p_2)}{2}} (p_1 \circ p_2).$$

One can write the exponent in an other form so that it looks like a triangle inequality.

**Lemma 5.1.** — *Let  $p$  and  $p'$  in  $A_k$ , let  $N$  be an integer. We have the equality:*

$$d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p')$$

$$= d({}^t p', p) - d(id, p) - d(id, p') + k + nc(p \circ p') - nc(p) - nc(p') + 2\kappa(p, p').$$

*Proof.* — For this, we consider  $N$  to the power to the r.h.s and the l.h.s. and we use the following equality:

$$N^{-d({}^t p', p)} = \frac{Tr(pp')}{N^{\frac{nc(p)+nc(p')}{2}}} = N^{\kappa(p, p')} \frac{Tr(p \circ p')}{N^{\frac{nc(p)+nc(p')}{2}}} = N^{\kappa(p, p')} \frac{N^{-d(id, p \circ p') + \frac{k+nc(p \circ p')}{2}}}{N^{\frac{nc(p)+nc(p')}{2}}}.$$

This allows to prove Lemma 5.1. □

Using the definition of the deformed algebra  $\mathbb{C}[A_k(N, N)]$ , we have the straightforward proposition.

**Proposition 5.2.** — *Let  $k$  and  $N$  be two integers. The application  $M_k^N$  can be extended as an isomorphism of algebra from  $\mathbb{C}[A_k(N, N)]$  to  $\mathbb{C}[A_k(N)]$ . Its extension will be also denoted by  $M_k^N$ .*

*For any integer  $N$ , the deformed algebra  $\mathbb{C}[A_k(N, N)]$  is isomorphic to  $\mathbb{C}[A_k(N)]$ .*

Actually, the application  $M_k^N$  is not only compatible with the multiplication, but also with the  $\otimes$  operation defined in Definition 2.1.

**Lemma 5.2.** — *Let  $k, k'$  and  $N$  be any integers. Let  $p \in A_k$  and  $p' \in A_{k'}$ . The following equality holds:*

$$(9) \quad M_{k+k'}^N(p \otimes p') = M_k^N(p) \otimes M_{k'}^N(p').$$

The definition of the morphism  $M_k^N$  was not chosen randomly: it was set so that the following lemma holds.

**Lemma 5.3.** — *Let  $E \in \mathbb{C}[A_k(N)]$ , we have:*

$$(M_k^N)^{-1}(E) = \sum_{p \in A_k} \kappa^p(E)p.$$

Thus, one can see that we will be able to formulate the strong convergence in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  by using the morphisms  $(M_k^N)_{N \in \mathbb{N}}$  and the usual notion of convergence in vector spaces. Indeed, for any integers  $N$  and  $k$ , any element in  $\mathbb{C}[A_k(N, N)]$  can be considered as an element of  $\mathbb{C}[A_k]$ . This allows to state the following lemma.

**Lemma 5.4.** — *Let  $(E_N)_{N \in \mathbb{N}}$  be an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The sequence  $(E_N)_{N \in \mathbb{N}}$  converges strongly if and only if:*

$$(M_k^N)^{-1}(E_N)$$

*converges when  $N$  goes to infinity in  $\mathbb{C}[A_k]$  for the usual convergence in finite dimensional vector spaces.*

## 6. Refined geometry of the partition algebra

In the last section, we defined the deformed algebra  $\mathbb{C}[A_k(N, N)]$  and we explained that the strong convergence can be seen as the natural notion of convergence in finite dimensional vector space as soon as one works in the deformed algebra. In this section, we will study the convergence of the algebras  $\mathbb{C}[A_k(N, N)]$ .

The core of Section 3 was to prove the triangular inequality for the function  $d$  defined on  $A_k$  in Definition 3.1. The study of the convergence of the algebras  $\mathbb{C}[A_k(N, N)]$  will use intensively the following improved triangular inequality for  $A_k$ .

**Proposition 6.1.** — *Let  $k$  be an integer. Let  $p$  and  $p'$  be two elements of  $\mathcal{P}_k$ . We have the following improved triangular inequality:*

$$d(p', p) \leq d(p', id) + d(p, id) - k - \text{nc}(p \circ {}^t p') + \text{nc}(p) + \text{nc}(p') - 2\kappa(p, {}^t p').$$

The restriction of the improved triangle inequality to the permutations is obvious as it is a consequence of the usual triangle inequality. Indeed, for any permutations  $\sigma$  and  $\sigma'$ ,  $\text{nc}(\sigma) = 0$  and  $\kappa(p, p') = 0$ . Yet, this is indeed an improved triangular inequality as soon as one considers elements on  $\mathcal{B}_k$ : let us suppose that  $p$  and  $p'$  are equal to the Weyl

contraction  $[1, 2]$ . The triangular inequality asserts that  $0 \leq 2$ , since  $d(id, [1, 2]) = 1$ . Yet, in this case:

$$d(p', id) + d(p, id) - k - \text{nc}(p \circ {}^t p') + \text{nc}(p) + \text{nc}(p') - 2\kappa(p, {}^t p') = 0.$$

The improved triangular inequality asserts thus the stronger fact that  $0 \leq 0$ .

In fact, we can see this improved triangular inequality as a consequence of the usual triangular inequality and an inequality between  $d(p, p \circ p')$  and  $d(id, p')$ . If we consider  $p$  and  $p'$  in the symmetric group, then we know that  $d(p, p \circ p') = d(p, pp') = d(id, p')$ . Yet, this equality does not hold any more in the general case, we only get the following inequality.

**Proposition 6.2.** — *Let  $k$  be an integer. Let  $p$  and  $p'$  in  $\mathcal{P}_k$ . We have the following inequality:*

$$d(p, p \circ p') \leq d(id, p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

*Proof.* — Let  $k$  be an integer. Let  $p$  and  $p'$  in  $\mathcal{P}_k$ . Let us define  $\tau \in \mathfrak{S}_k$ :

$$\tau_k = (1, k+1)(2, k+2) \dots (k, 2k).$$

Let us apply the triangular inequality:

$$(10) \quad d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau) \leq d(p \otimes id_k, p \otimes p') + d(p \otimes p', ((p \circ p') \otimes id_k) \tau).$$

The goal is to understand each of these three terms. The term  $d(p \otimes id_k, p \otimes p')$  is simple:

$$d(p \otimes id_k, p \otimes p') = d(id, p').$$

Let us study  $d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau)$ . Using the definition of the distance in Proposition 3.1, and the Equality 3:

$$N^{-d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau)} = \frac{\text{Tr}_N [(p \otimes id_k) {}^t(((p \circ p') \otimes id_k) \tau)]}{N^{\frac{\text{nc}(p)+k}{2}} N^{\frac{\text{nc}(p \circ p')+k}{2}}},$$

since  $\text{nc}(p \otimes id_k) = \text{nc}(p) + k$  and  $\text{nc}(((p \circ p') \otimes id_k) \tau) = \text{nc}((p \circ p') \otimes id_k) = \text{nc}(p \circ p') + k$ . Yet:

$$\text{Tr}_N [(p \otimes id_k) {}^t(((p \circ p') \otimes id_k) \tau)] = \text{Tr}_N [p {}^t(p \circ p')].$$

Thus, using again Proposition 3.1:

$$d(p \otimes id_k, ((p \circ p') \otimes id_k) \tau) = d(p, p \circ p') + k.$$

Let us consider  $d(p \otimes p', ((p \circ p') \otimes id_k) \tau)$ . Using the same arguments:

$$N^{-d(p \otimes p', ((p \circ p') \otimes id_k) \tau)} = \frac{\text{Tr}_N (pp' {}^t(p \circ p'))}{N^{\frac{\text{nc}(p)+\text{nc}(p')}{2}} N^{\frac{\text{nc}(p \circ p')+k}{2}}}.$$

Using the definition of  $\kappa(p, p')$  and the Equality 3:

$$\begin{aligned} N^{-d(p \otimes p', ((p \circ p') \otimes id_k) \tau)} &= N^{\kappa(p, p')} \frac{\text{Tr}_N (p \circ p' {}^t(p \circ p'))}{N^{\frac{\text{nc}(p)+\text{nc}(p')}{2}} N^{\frac{\text{nc}(p \circ p')+k}{2}}} \\ &= N^{\kappa(p, p') + \frac{1}{2}[\text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') - k]}. \end{aligned}$$

Thus:

$$d(p \otimes p', ((p \circ p') \otimes id_k) \tau) = -\kappa(p, p') - \frac{1}{2}[\text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') - k].$$

Let us come back to the triangular inequality 10. This shows that:

$$d(p, p \circ p') + k \leq d(id, p') - \kappa(p, p') - \frac{1}{2}[\text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') - k],$$

and thus:

$$d(p, p \circ p') \leq d(id, p') - \frac{\text{nc}(p \circ p') + k - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

This is the inequality we wanted to prove.  $\square$

*Proof of Proposition 6.1.* — Let  $k$  be an integer. Let  $p$  and  $p'$  be two elements of  $A_k$ . Using the triangular inequality:

$$d(id, p \circ p') \leq d(id, p) + d(p, p \circ p').$$

And an application of Proposition 6.2 implies that:

$$(11) \quad d(id, p \circ p') \leq d(id, p) + d(id, p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

And using Lemma 5.1:

$$d({}^t p', p) \leq d(id, p) - d(id, p') + k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') + 2\kappa(p, p').$$

The result follows then from the fact that  $\text{nc}({}^t p') = \text{nc}(p')$ .  $\square$

We can generalize the inequality (11) to a  $n$ -uple of elements of  $A_k$ .

**Lemma 6.1.** — *Let  $k$  be an integer. For any integer  $n$ , for any  $(p_i)_{i=1}^n \in A_k^n$ :*

$$d(id, \circ_{i=1}^n p_i) \leq \sum_{i=1}^n d(id, p_i) - \frac{1}{2} \left[ (n-1)k + \text{nc}(\circ_{i=1}^n p_i) - \sum_{i=1}^n \text{nc}(p_i) \right] - \sum_{i=1}^{n-1} \kappa(p_i, p_{i+1}),$$

where we have used the notation  $\circ_{i=1}^n p_i = p_1 \circ \dots \circ p_n$ .

In fact, the best way to understand the improved triangular inequality is to work with the equivalent inequality (11). This formulation of the improved triangular inequality leads us to the next notion.

**Definition 6.1.** — *Let  $p$  and  $p'$  be two elements of  $A_k$ . We will say that  $p \prec p \circ p'$  if and only if:*

$$d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} + \kappa(p, p') = 0.$$

Let  $p_0 \in A_k$ . We will write that  $p \prec p_0$  if there exists  $p' \in A_k$  such that  $p_0 = p \circ p'$  and  $p \prec p \circ p'$ .

**Definition 6.2.** — *Let us suppose that  $p \prec p_0$ . We define for any  $p \prec p_0$ :*

$$K_{p_0}(p) = \{p' \in A_k, p \circ p' = p_0\}.$$



Let us suppose that  $p \prec p \circ p'$ . We recall that:

$$\begin{aligned} d(id, p \circ p') &\leq d(id, p) + d(p, p \circ p') \\ &\leq d(id, p) + d(id, p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p'). \end{aligned}$$

Thus, if the first term and the third one are equal, then  $p \in [id, p \circ p']_{A_k}$ . We have shown the following lemma.

**Lemma 6.2.** — *Let  $k$  be an integer. Let  $p$  and  $p_0$  in  $A_k$ . If  $p \prec p_0$  then there exists  $p' \in A_k$  such that  $p_0 = p \circ p'$  and*

$$p \in [id, p_0]_{A_k}.$$

Let us remark that  $\{\sigma' \in \mathfrak{S}_k, \sigma' \prec \sigma\} = [id, \sigma]_{\mathfrak{S}_k}$ . This is due to the fact that  $\kappa(\sigma, \sigma') = 0$  for any couple of permutations, the fact that  $\text{nc}$  is constant on the set of permutations and the fact that any permutation is invertible. Using Lemma 2.1 one can have the better result.

**Lemma 6.3.** — *Let  $k$  be an integer. Let  $\sigma \in \mathfrak{S}_k$ , then:*

$$\{p \in \mathcal{P}_k, p \prec \sigma\} = [id, \sigma]_{\mathfrak{S}_k}.$$

Let us state a consequence of Lemma 6.2: the factorization property for  $\prec$ .

**Lemma 6.4.** — *Let  $k$  and  $l$  be two integers. Let  $a \in \mathcal{P}_k$  and  $b \in \mathcal{P}_l$ . For any  $p \in \mathcal{P}_{k+l}$  such that  $p \prec a \otimes b$ , there exist  $p_1 \prec a$  and  $p_2 \prec b$  such that  $p = p_1 \otimes p_2$ .*

This lemma is a consequence of Lemma 6.2 and the factorization property for the geodesics stated in Lemma 3.4.

Let  $p$  and  $p_0$  in  $A_k$  such that  $p \prec p_0$ . Let us have a little discussion on  $K_{p_0}(p)$ : by definition this is not empty but it is not reduced to a unique partition. For example, one can show that if  $p = \{\{1, 2, 1', 2'\}\}$  and  $p_0 = \{\{1', 2'\}, \{1\}, \{2\}\}$  then:

$$K_{p_0}(p) = \{\{\{1\}, \{2\}, \{1'\}, \{2'\}\}, \{\{1\}, \{2\}, \{1', 2'\}\}\}.$$

Let  $k$  be an integer. Let  $(1, \dots, k)$  be the  $k$ -cycle in  $\mathfrak{S}_k$ . It is well-known that the set of non-crossing partition over  $\{1, \dots, k\}$  is isomorphic to  $[id, (1, \dots, k)]_{\mathfrak{S}_k}$ . From now on, we will consider any non-crossing partition over  $\{1, \dots, k\}$  as an element of  $[id, (1, \dots, k)]_{\mathfrak{S}_k}$ . The following lemma is straightforward.

**Lemma 6.5.** — *The notion of  $K_{p_0}(p)$  generalizes the notion of Kreweras complement for non-crossing partitions over  $\{1, \dots, k\}$  and  $p_0 = (1, \dots, k)$ .*

We are going now to see one of the main results of the paper, namely the fact that the improved triangle inequality implies the convergence of the deformed algebras  $(\mathbb{C}[A_k(N, N)])_{N \in \mathbb{N}}$  stated in the forthcoming Theorem 6.1. Before doing so, we need to define the notion of convergence of algebras.

**Definition 6.3.** — *Let  $C$  be a finite set of elements. For any  $N \in \mathbb{N} \cup \{\infty\}$ , let  $L_N$  be an algebra such that  $C$  is a linear basis of  $L_N$ . For any elements  $x$  and  $y$  of  $C$ , for each  $N \in \mathbb{N} \cup \{\infty\}$ , we denote the product of  $x$  with  $y$  in  $L_N$  by  $x.Ny$ .*

We say that  $L_N$  converges to the algebra  $L_\infty$  when  $N$  goes to infinity if for any  $x$  and  $y$  in  $C$ ,

$$x \cdot_N y \xrightarrow{N \rightarrow \infty} x \cdot_\infty y \text{ in } \mathbb{C}[C],$$

for the usual notion of convergence in finite dimensional linear spaces.

Let us state the convergence of the deformed algebras  $(\mathbb{C}[A_k(N, N)])_{N \in \mathbb{N}}$ .

**Theorem 6.1.** — *As  $N$  goes to infinity, the deformed algebra  $\mathbb{C}[A_k(N, N)]$  converges to the deformed algebra  $\mathbb{C}[A_k(\infty, \infty)]$  which is the associative algebra over  $\mathbb{C}$  with basis  $A_k$  endowed with the multiplication defined by:*

$$\forall p, p' \in \mathcal{P}_k, pp' = \delta_{p \prec p'} p \circ p'.$$

*Proof.* — Let  $k$  be an integer. For any  $N \in \mathbb{N} \cup \{\infty\}$ ,  $A_k$  is a linear basis of  $\mathbb{C}[A_k(N, N)]$ . By bi-linearity of the product, it is enough to prove that for any  $p$  and  $p'$  in  $A_k$ ,  $p \cdot_N p'$  converges to  $\delta_{p \prec p'} p \circ p'$ .

Let  $p$  and  $p'$  be two elements of  $\mathcal{P}$ . We have:

$$p \cdot_N p' = N^{d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p') (p \circ p').$$

By the version of the improved triangle inequality stated in Proposition 6.1 or in the inequality (11), we have:

$$d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p') \leq 0.$$

According to Definition 6.1, we have  $p \cdot_N p' \xrightarrow{N \rightarrow \infty} \delta_{p \prec p'} p \circ p'$ .  $\square$

To conclude this section, let us remark that for any integer  $k$ , we have the inclusion of algebras:

$$\mathbb{C}[\mathfrak{S}_k(\infty, \infty)] \subset \mathbb{C}[\mathcal{B}_k(\infty, \infty)] \subset \mathbb{C}[\mathcal{P}_k(\infty, \infty)].$$

## 7. Consequences of the convergence of the deformed algebras.

**7.1. Convergence of a product.** — Let  $k$  be an integer. As usual, let  $A_k$  be  $\mathfrak{S}_k$ ,  $\mathcal{B}_k$  or  $\mathcal{P}_k$ . Let us give the first consequence of Theorem 6.1 for the product of two elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Recall the Notation 4.1.

**Theorem 7.1.** — *Let  $(E_N)_{N \in \mathbb{N}}, (F_N)_{N \in \mathbb{N}}$  be elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  converge, then the sequence  $(E_N F_N)_{N \in \mathbb{N}}$  converges. Besides, for any  $p_0 \in A_k$ :*

$$(12) \quad \kappa^{p_0}(EF) = \sum_{p \in A_k, p \prec p_0} \kappa^p(E) \kappa^{K_{p_0}(p)}(F),$$

$$(13) \quad m_{p_0}(EF) = \sum_{p \in A_k, p \in [id, p_0]_{A_k}} \kappa^p(E) m_{p \circ p_0}(F).$$

*Proof.* — Let  $(E_N)_{N \in \mathbb{N}}, (F_N)_{N \in \mathbb{N}}$  elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  converge. We have by definition:

$$(M_k^N)^{-1}(E_N F_N) = (M_k^N)^{-1}(E_N) \cdot_N (M_k^N)^{-1}(F_N).$$

We know, by Lemma 5.4, that  $(M_k^N)^{-1}(E_N)$  and  $(M_k^N)^{-1}(F_N)$ , seen as elements of  $\mathbb{C}[A_k]$ , converge when  $N \rightarrow \infty$ . Besides, the algebra  $\mathbb{C}[A_k(N, N)]$  converges to  $\mathbb{C}[A_k(\infty, \infty)]$ , as it was proved in Theorem 6.1. Thus  $(M_k^N)^{-1}(E_N F_N)$  converges when  $N$  goes to infinity. Again, by Lemma 5.4 and Theorem 4.1, this shows that  $(E_N F_N)_{N \in \mathbb{N}}$  converges.

Besides, using Lemma 5.3, we have:

$$\begin{aligned} (M_k^N)^{-1}(E_N F_N) &= \sum_{p \in A_k} \kappa^{p_0}(E_N F_N) p_0, \\ (M_k^N)^{-1}(E_N) \cdot_N (M_k^N)^{-1}(F_N) &= \sum_{p \in A_k, p' \in A_k} \kappa^p(E_N) \kappa^{p'}(F_N) p \cdot_N p'. \end{aligned}$$

Using the formula for the limit of  $\cdot_N$  shown in Theorem 6.1, for any  $p_0 \in \mathcal{P}_k$ :

$$\kappa^{p_0}(EF) = \sum_{p \in A_k, p \prec p_0} \kappa^p(E) \kappa^{K_{p_0}(p)}(F).$$

For the second equality, one could use the link, between  $A_k$ -moments and coordinate numbers when  $N \rightarrow \infty$  given by Equality (5). Yet, this happens to be more difficult than a direct proof. Indeed, by bi-linearity, we have only to show that the equality (13) holds when, for any integer  $N$ :

$$E_N = \frac{1}{N^{\frac{-k+nc(p)}{2}+d(id,p)}} p.$$

Let  $N$  be an integer, let us suppose that  $E_N$  is of this form. Let  $p_0 \in A_k$ , we have:

$$\begin{aligned} m_{p_0}(E_N F_N) &= \frac{1}{N^{\frac{-k+nc(p)}{2}+d(id,p)}} \frac{Tr(F_N {}^t p_0 p)}{Tr(p_0)} \\ &= \frac{N^{\kappa({}^t p_0, p)}}{N^{\frac{-k+nc(p)}{2}+d(id,p)}} \frac{Tr(F_N {}^t p_0 \circ p)}{Tr(p_0)} \\ &= \frac{N^{\kappa({}^t p_0, p)}}{N^{\frac{-k+nc(p)}{2}+d(id,p)}} \frac{Tr({}^t p \circ p_0) m_{{}^t p \circ p_0}(F_N)}{Tr(p_0)} \\ &= \frac{1}{N^{\frac{-k+nc(p)}{2}+d(id,p)}} \frac{Tr({}^t p p_0)}{Tr(p_0)} m_{{}^t p \circ p_0}(F_N). \end{aligned}$$

We remind the reader that, for any  $p$  and  $p_0$  in  $A_k$ :

$$\frac{1}{N^{\frac{-k+nc(p)}{2}+d(id,p)}} \frac{Tr({}^t p p_0)}{Tr(p_0)} = N^{d(id, p_0) - d(id, p) - d(p, p_0)}.$$

This equality allows to finish the proof, as:

$$m_{p_0}(E_N F_N) = N^{d(id, p_0) - d(id, p) - d(p, p_0)} m_{{}^t p \circ p_0}(F_N).$$

Using the triangular inequality, one gets finally that  $m_{p_0}(E_N F_N)$  converges when  $N$  goes to infinity to  $\delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}}(F_N)$ .  $\square$

**Remark 7.1.** — We can show the similar result that, under the same assumptions:

$$(14) \quad m_{p_0}(EF) = \sum_{p \in A_k, p \in [id, p_0]_{A_k}} m_{p_0 \circ^t p}(E) \kappa^p(F).$$

**7.2. Semi-groups.** — Let  $k$  be an integer. In this subsection, we will study convergence of sequences of semi-groups in  $\mathbb{C}[A_k(N)]$ . Semi-groups in different algebras will appear in the paper: for this paper, a family  $(a_t)_{t \geq 0}$  is a semi-group if there exists  $h$ , called the generator, such that for any  $t_0 \geq 0$ :

$$\frac{d}{dt} \Big|_{t=t_0} a_t = h a_{t_0}.$$

If we consider the algebra  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ , we are led to the next definition.

**Definition 7.1.** — The family  $((E_t^N)_N)_{t \geq 0}$  is a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  if there exists  $(H_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ , called the generator, such that for any  $t \geq 0$ , for any integer  $N$ :

$$\frac{d}{dt} \Big|_{t=t_0} E_t^N = H_N E_{t_0}^N.$$

From now on, let us suppose that  $((E_t^N)_N)_{t \geq 0}$  is a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  whose generator is  $(H_N)_{N \in \mathbb{N}}$ . Let us define the convergence for semi-groups in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Definition 7.2.** — The semi-group  $((E_t^N)_N)_{t \geq 0}$  converges if and only if for any  $t \geq 0$ ,  $E_t^N$  converges as  $N$  goes to infinity.

The next theorem, one of the main theorems of the paper, shows that a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  converges if the initial condition and the generator converge. Recall the Notation 4.1.

**Theorem 7.2.** — The semi-group  $((E_t^N)_N)_{t \geq 0}$  converges if the sequences  $(E_0^N)_{N \in \mathbb{N}}$  and  $(H_N)_{N \in \mathbb{N}}$  converge as  $N$  goes to infinity.

Besides, we have the two differential systems of equations:

$$(15) \quad \forall p \in A_k, \forall t_0 \geq 0, \quad \frac{d}{dt} \Big|_{t=t_0} \kappa^p(E_t) = \sum_{p_1 \in A_k, p_1 < p} \kappa^{p_1}(H) \kappa^{K_p(p_1)}(E_{t_0}).$$

$$(16) \quad \forall p \in A_k, \forall t_0 \geq 0, \quad \frac{d}{dt} \Big|_{t=t_0} m_p(E_t) = \sum_{p_1 \in [id, p]_{A_k}} \kappa^{p_1}(H) m_{t_{p_1 \circ p}}(E_{t_0}).$$

*Proof.* — Let us suppose that  $(H_N)_{N \in \mathbb{N}}$  converges. For any integer  $N$  and any  $t \geq 0$ , we define:

$$\begin{aligned} \tilde{E}_t^N &= (M_k^N)^{-1}(E_t^N), \\ \tilde{H}_N &= (M_k^N)^{-1}(H_N). \end{aligned}$$

As for any integer  $N$ ,  $M_k^N$  is a morphism of algebra, the family  $((\tilde{E}_t^N)_{N \in \mathbb{N}})_{t \geq 0}$  is a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N, N)]$  and its generator is  $(\tilde{H}_N)_{N \in \mathbb{N}}$ . An application of Lemma 5.3 allows us to write the condition of semi-group in the basis  $A_k$ ; for any  $t_0 \geq 0$ :

$$\frac{d}{dt} \Big|_{t=t_0} \sum_{p_0 \in A_k} \kappa^{p_0}(E_{t_0}^N) p_0 = \left( \sum_{p \in A_k} \kappa^p(H_N) p \right) \cdot_N \left( \sum_{p' \in A_k} \kappa^{p'}(E_{t_0}^N) p' \right).$$

Then the following equality must hold for any integer  $N$ , any  $t_0 \geq 0$  and any  $p_0 \in A_k$ :

$$\begin{aligned} & \frac{d}{dt} \Big|_{t=t_0} \kappa^{p_0}(E_t^N) \\ &= \sum_{p, p' \in A_k, p \circ p' = p_0} \kappa^p(H_N) \kappa^{p'}(E_{t_0}^N) N^{d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p')}. \end{aligned}$$

Let us take  $N$  going to infinity. Because of the hypothesis and because of the improved triangular inequality, this differential system converges:  $\kappa^p(E_t^N)$  must converge for any  $p \in A_k$  and any real  $t \geq 0$ . Besides, we get for any  $t_0 \geq 0$ :

$$\forall p \in A_k, \frac{d}{dt} \Big|_{t=t_0} \kappa^p(E_t) = \sum_{p_1 \in A_k, p_1 \prec p} \kappa^{p_1}(H) \kappa^{K_p(p_1)}(E_{t_0}).$$

Since the semi-group converges, using the usual notations, we can write that for any  $p \in A_k$  and any  $t_0 \geq 0$ :

$$\frac{d}{dt} \Big|_{t=t_0} m_p(E_t) = m_p(H E_{t_0}),$$

and using equality (13), one has:

$$\lim_{N \rightarrow \infty} m_p(H_N E_{t_0}^N) = \sum_{p_1 \in [id, p]_{A_k}} \kappa^{p_1}(H) m_{t_{p_1 \circ p}}(E_{t_0}).$$

Hence we recover the second system of differential equations.  $\square$

Of course one also has, by using equality (14) instead of (13), that for any  $p \in \mathcal{P}_k$  and any  $t_0 \geq 0$ :

$$\frac{d}{dt} \Big|_{t=t_0} m_{p_0}(E_t) = \sum_{p \in [id, p_0]_{A_k}} m_{p_0 \circ^t p}(H) \kappa^p(E_{t_0}).$$

Moreover, Theorem 7.2 can be very easily generalized for any semi-group with time dependent generator. In order to finish the section, let us prove a consequence of Lemma 3.3.

**Theorem 7.3.** — *Let  $((E_t^N)_{N \in \mathbb{N}})_{t \geq 0}$  be a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$ . Let us suppose that the sequence  $(E_0^N)_{N \in \mathbb{N}}$  converges as  $N$  goes to infinity. Let us suppose that for any  $\sigma \in \mathfrak{S}_k$ ,  $\kappa^\sigma(H_N)$  converges when  $N$  goes to infinity. Then for any  $\sigma \in \mathfrak{S}_k$ , for any*

positive real  $t$ ,  $\kappa^\sigma(E_t^N)$  converges as  $N$  goes to infinity. Besides for any  $\sigma \in \mathfrak{S}_k$  and any  $t_0 \geq 0$ :

$$(17) \quad \frac{d}{dt}|_{t=t_0} \kappa^\sigma(E_t) = \sum_{\sigma \in \mathfrak{S}_k, \sigma_1 \prec \sigma} \kappa^{\sigma_1}(H) \kappa^{K_\sigma(\sigma_1)}(E_{t_0}).$$

*Proof.* — Let  $((E_t^N)_N)_{t \geq 0}$  be a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{B}_k(N)]$  which satisfies the hypothesis of the theorem. Let  $\sigma \in \mathfrak{S}_k$  and let  $N$  be an integer. We have seen in the last proof that for any  $t_0 \geq 0$ :

$$\begin{aligned} & \frac{d}{dt}|_{t=t_0} \kappa^\sigma(E_t^N) \\ &= \sum_{p, p' \in \mathcal{B}_k, p \circ p' = \sigma} \kappa^p(H_N) \kappa^{p'}(E_{t_0}^N) N^{d(id, p \circ p') - d(id, p) - d(id, p') + \frac{k + nc(p \circ p') - nc(p) - nc(p')}{2} + \kappa(p, p')}. \end{aligned}$$

Yet, by Lemma 2.1, if  $p \circ p' = \sigma$ , then  $p$  and  $p'$  are in  $\mathfrak{S}_k$ . Thus,

$$\frac{d}{dt}|_{t=t_0} \kappa^\sigma(E_t^N) = \sum_{p, p' \in \mathfrak{S}_k, p \circ p' = \sigma} \kappa^p(H_N) \kappa^{p'}(E_{t_0}^N) N^{d(id, p \circ p') - d(id, p) - d(id, p')}.$$

Thus, we see that  $((\kappa^\sigma(E_t^N))_{\sigma \in \mathfrak{S}_k})_{t \geq 0}$  satisfies a linear differential system whose coefficients converge by hypothesis. Thus, for any  $\sigma \in \mathfrak{S}_k$ , for any positive real  $t$ ,  $\kappa^\sigma(E_t^N)$  converges as  $N$  goes to infinity. The Equation 17 is obtained by taking  $N$  going to infinity in the last equation.  $\square$

## 8. A new way to get combinatorial properties

In Section 6, we showed new inequalities on the set of partitions  $\mathcal{P}_k$ . The proofs were quite combinatorial, and used only the notion of distance. In this section, we want to show that one can prove new inequalities or equalities, by using Theorem 4.1 as a black box.

**8.1. Geometric consequences of Theorem 4.1.** — First, let us give a new proof of the improved triangular inequality.

*Proof of Proposition 6.1.* — Let  $k$  be an integer. Let  $p$  and  $p'$  be two elements of  $A_k$ . Let us consider  $(\mathbf{p}_N)_{N \in \mathbb{N}}$  and  $(\mathbf{p}'_N)_{N \in \mathbb{N}}$  such that for any integer  $N$ :

$$\begin{aligned} \mathbf{p}_N &= M_k^N(p), \\ \mathbf{p}'_N &= M_k^N(p'). \end{aligned}$$

Using Lemma 5.4,  $(\mathbf{p}_N)_{N \in \mathbb{N}}$  and  $(\mathbf{p}'_N)_{N \in \mathbb{N}}$  converge strongly. Let  $N$  be an integer. Applying the equality (9), we have:

$$\mathbf{p}_N \otimes \mathbf{p}'_N = M_{2k}^N(p \otimes p').$$

Thus, using Lemma 5.4,  $\mathbf{p}_N \otimes \mathbf{p}'_N$  converges strongly when  $N$  goes to infinity. An application of Theorem 4.1 shows that it converges in moments: for any  $\tilde{p} \in A_{2k}$ ,

$$m_{\tilde{p}}(\mathbf{p}_N \otimes \mathbf{p}'_N) \text{ converges when } N \rightarrow \infty.$$

For any partition  $\tilde{p} \in A_k$ , we define  $P(\tilde{p})$  be the partition in  $A_{2k}$ :

$$P(\tilde{p}) = (\tilde{p} \otimes id_k)(1, k+1)(2, k+2) \dots (k, 2k).$$

Then for any  $E \in \mathbb{C}[A_k(N)]$  and  $F \in \mathbb{C}[A_k(N)]$ , and any  $p_0 \in A_k$ , we have:

$$m_{P(p_0)}(E \otimes F) = m_{p_0}(EF).$$

Thus for any  $p_0 \in A_k$ ,  $m_{p_0}(\mathbf{p}_N \mathbf{p}'_N)$  which is equal to  $m_{P(p_0)}(\mathbf{p}_N \otimes \mathbf{p}'_N)$  converges as  $N$  goes to infinity. Using again the Theorem 4.1, we have that  $\mathbf{p}_N \mathbf{p}'_N$  converges strongly as  $N$  goes to infinity. It implies, because of Lemma 5.4 that  $(M_k^N)^{-1}(\mathbf{p}_N \mathbf{p}'_N)$  converges in  $\mathbb{C}[A_k]$  when  $N$  goes of infinity. We can calculate this last expression:

$$\begin{aligned} (M_k^N)^{-1}(\mathbf{p}_N \mathbf{p}'_N) &= (M_k^N)^{-1}(M_k^N(p)M_k^N(p')) \\ &= p \cdot_N p' \\ &= N^{d({}^t p', p) - d(id, p) - d(id, p') + k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p') + 2\kappa(p, p')} (p \circ p'), \end{aligned}$$

where we used Lemma 5.1. Thus we must have that for any  $p$  and  $p'$  in  $A_k$ :

$$d({}^t p', p) \leq d(id, p) + d(id, p') - k - \text{nc}(p \circ p') + \text{nc}(p) + \text{nc}(p') - 2\kappa(p, p').$$

The improved inequality is just a consequence of the last inequality as soon as we see that for any  $p \in A_k$ ,  $\text{nc}({}^t p) = \text{nc}(p)$ , and  $d(id, p) = d(id, {}^t p)$ .  $\square$

Again, using the same ideas, one can show the following interesting property.

**Proposition 8.1.** — *Let  $p_0, p_1$  and  $p_2$  be three partitions in  $A_k$ . Let  $\tau$  be the partition in  $A_{2k}$  defined by:*

$$\tau = (1, k+1)(2, k+2) \dots (k, 2k).$$

We have:

$$\delta_{p_1 \otimes p_2 \in [id, (p_0 \otimes id_k) \tau]_{A_{2k}}} = \delta_{p_1 \circ p_2 \in [id, p_0]_{A_k}} \delta_{p_1 \prec p_1 \circ p_2}.$$

*Proof.* — Let  $p_0, p_1$  and  $p_2$  be three partitions in  $A_k$ . Let us consider  $(\mathbf{p}_N^1)_{N \in \mathbb{N}}$  and  $(\mathbf{p}_N^2)_{N \in \mathbb{N}}$  such that for any integer  $N$ ,  $\mathbf{p}_N^1 = M_k^N(p_1)$  and  $\mathbf{p}_N^2 = M_k^N(p_2)$ .

Using Lemma 5.4,  $(\mathbf{p}_N^1)_{N \in \mathbb{N}}$  and  $(\mathbf{p}_N^2)_{N \in \mathbb{N}}$  converge strongly. Thus,  $(\mathbf{p}_N^1 \otimes \mathbf{p}_N^2)_{N \in \mathbb{N}}$  converges strongly, and by Theorem 4.1 it converges in moments.

Let us calculate, using two ways, the limit of  $m_{(p_0 \otimes id_k) \tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2]$ , where  $\tau = (1, k+1)(2, k+2) \dots (k, 2k)$ .

First, using Theorem 4.1 and the Equation (5), we get that:

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k) \tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \sum_{p \in [(p_0 \otimes id_k) \tau]_{A_{2k}}} \lim_{N \rightarrow \infty} \kappa^p [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2].$$

Yet, for any  $p \in A_{2k}$ ,  $\kappa^p [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \delta_{p=p_1 \otimes p_2}$ , thus:

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k) \tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \delta_{p_1 \otimes p_2 \in [id, (p_0 \otimes id_k) \tau]_{A_{2k}}}.$$

Then, using the fact that  $m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = m_{p_0} [\mathbf{p}_N^1 \mathbf{p}_N^2]$ , and using again Theorem 4.1 and the Equation (5):

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \sum_{p \in [id, p_0]_{A_k}} \lim_{N \rightarrow \infty} \kappa^p [\mathbf{p}_N^1 \mathbf{p}_N^2].$$

Let  $p \in A_k$ ,  $\kappa^p [\mathbf{p}_N^1 \mathbf{p}_N^2]$  is the coefficient of  $p$  in the expression  $(M_k^N)^{-1} (\mathbf{p}_N^1 \mathbf{p}_N^2)$ . Let us remark that  $(M_k^N)^{-1} (\mathbf{p}_N^1 \mathbf{p}_N^2) = (M_k^N)^{-1} (M_k^N(p_1) M_k^N(p_2)) = p_1 \cdot_N p_2$  which converges in  $\mathbb{C}[A_k]$  to  $\delta_{p_1 \prec p_1 \circ p_2} p_1 \circ p_2$ . Thus,

$$\lim_{N \rightarrow \infty} \kappa^p [\mathbf{p}_N^1 \mathbf{p}_N^2] = \delta_{p_1 \prec p_1 \circ p_2} \delta_{p = p_1 \circ p_2}.$$

This implies that:

$$\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2] = \delta_{p_1 \prec p_1 \circ p_2} \delta_{p_1 \circ p_2 \in [id, p_0]_{A_k}}.$$

Using the two ways to compute  $\lim_{N \rightarrow \infty} m_{(p_0 \otimes id_k)\tau} [\mathbf{p}_N^1 \otimes \mathbf{p}_N^2]$ , we get:

$$\delta_{p_1 \otimes p_2 \in [id, (p_0 \otimes id_k)\tau]_{A_{2k}}} = \delta_{p_1 \circ p_2 \in [id, p_0]_{A_k}} \delta_{p_1 \prec p_1 \circ p_2}.$$

which was the desired equality.  $\square$

In fact, one can always prove the results by a combinatorial argument: the ideas we present are more an automatic way to get combinatorial results that one can prove after by combinatorial means. For exemple, let us consider Definition 9.1. Using Proposition 8.1, one can now expect that  $\text{df}(p_1 \otimes p_2, (p \otimes id_k)\tau) = \text{df}(p_1 \circ p_2, p) + \eta(p_1, p_2)$ . Indeed, we have the following proposition.

**Proposition 8.2.** — *Let  $p_0, p_1$  and  $p_2$  be three partitions in  $A_k$ . Let  $\tau$  be the partition in  $A_{2k}$  defined by:*

$$\tau = (1, k+1)(2, k+2) \dots (k, 2k).$$

*We have:*

$$\text{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) = \text{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2).$$

*Proof.* — The proof is only based on calculations. Let  $p$  and  $p'$  be two partitions in  $A_k$ , then:

$$\text{df}(p', p) = \text{nc}(p') - \text{nc}(p' \vee id) - \text{nc}(p' \vee p) + \text{nc}(p \vee id),$$

and  $\eta(p, p')$  is equal to:

$$\text{nc}(p) + \text{nc}(p') - \text{nc}(p \circ p') - \text{nc}(p \vee id) - \text{nc}(p' \vee id) + \text{nc}(p \circ p' \vee id) - \kappa(p, p').$$

Thus:

$$\begin{aligned} & \text{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \text{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) \\ &= \text{nc}(p_1 \circ p_2) - \text{nc}((p_1 \circ p_2) \vee id) - \text{nc}((p_1 \circ p_2) \vee p_0) + \text{nc}(p_0 \vee id) + \text{nc}(p_1) + \text{nc}(p_2) \\ & \quad - \text{nc}(p_1 \circ p_2) - \text{nc}(p_1 \vee id) - \text{nc}(p_2 \vee id) + \text{nc}((p_1 \circ p_2) \vee id) - \kappa(p_1, p_2) \\ & \quad - \text{nc}(p_1 \otimes p_2) + \text{nc}((p_1 \otimes p_2) \vee id) + \text{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]) \\ & \quad - \text{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}). \end{aligned}$$



Using the following equalities:

$$\begin{aligned} \text{nc}(p_1 \otimes p_2) &= \text{nc}(p_1) + \text{nc}(p_2), \\ \text{nc}([p_1 \otimes p_2] \vee id) &= \text{nc}(p_1 \vee id) + \text{nc}(p_2 \vee id), \end{aligned}$$

we get:

$$\begin{aligned} &\text{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \text{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) \\ &= -\text{nc}((p_1 \circ p_2) \vee p_0) + \text{nc}(p_0 \vee id) - \kappa(p_1, p_2) \\ &+ \text{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]) - \text{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}). \end{aligned}$$

The equalities:

$$\begin{aligned} \text{Tr}_N(p_0) &= \text{Tr}_N((p_0 \otimes id_k)\tau), \\ N^{\kappa(p_1, p_2)} \text{Tr}_N((p_1 \circ p_2)^t p_0) &= \text{Tr}_N((p_1 p_2)^t p_0) = \text{Tr}_N((p_1 \otimes p_2)^t [(p_0 \otimes id_k)\tau]), \end{aligned}$$

allow us to prove, as an application of Equations (2) and (3), that:

$$\begin{aligned} \text{nc}(p_0 \vee id) &= \text{nc}([(p_0 \otimes id_k)\tau] \vee id_{2k}), \\ \text{nc}((p_1 \circ p_2) \vee p_0) + \kappa(p_1, p_2) &= \text{nc}([(p_0 \otimes id_k)\tau] \vee [p_1 \otimes p_2]). \end{aligned}$$

Thus  $\text{df}(p_1 \circ p_2, p_0) + \eta(p_1, p_2) - \text{df}(p_1 \otimes p_2, (p_0 \otimes id_k)\tau) = 0$ .  $\square$

**8.2. Combinatorial consequences of Theorem 4.1.** — Let us remark the following important, yet simple theorem.

**Theorem 8.1.** — *Let  $(m_p)_{p \in A_k}$  be a family of complex numbers. There exists a sequence  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  which converges and such that:*

$$\lim_{N \rightarrow \infty} m_p(E_N) = m_p.$$

*Proof.* — Let us consider  $(m_p)_{p \in A_k}$  a family of complex numbers. Let us consider  $(\kappa^p)_{p \in A_k}$ , the unique family of real such that for any  $p \in A_k$ :

$$m_p = \sum_{p' \in [id, p]_{A_k}} \kappa^{p'}.$$

Let us consider then:

$$E_N = M_k^N \left( \sum_{p \in A_k} \kappa^p p \right).$$

According to Lemma 5.4,  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  converges strongly. Thus, by Theorem 4.1 it converges in moments and for any  $p \in A_k$ :

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id, p]_{A_k}} \lim_{N \rightarrow \infty} \kappa^{p'}(E_N).$$

Yet, using Lemma 5.3,  $\kappa^{p'}(E_N)$  is equal to  $\kappa^{p'}$ . Thus:

$$\lim_{N \rightarrow \infty} m_p(E_N) = \sum_{p' \in [id, p]_{A_k}} \kappa^{p'} = m_p.$$

This concludes the proof.  $\square$

This theorem is very important, as actually, it shows that, in order to understand the transformation between moments and coordinate numbers, we have an approximation setting in which one can work with: the space of convergent sequences in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us show some examples of propositions that one can get using this point of view. For this, we need the notion of cumulants and exclusive moments. Let us consider  $(m_p)_{p \in A_k}$  a family of complex numbers.

**Definition 8.1.** — *The cumulants of  $(m_p)_{p \in A_k}$  is the unique family of complex numbers  $(\kappa^p)_{p \in A_k}$  such that for any  $p \in A_k$ :*

$$m_p = \sum_{p' \in [id, p]_{A_k}} \kappa^{p'}.$$

*The exclusive moments of  $(m_p)_{p \in A_k}$  is the only family  $(m_{p^c})_{p \in \mathcal{P}_k}$  of complex numbers such that:*

$$m_p = \sum_{p' \in \mathcal{P}_k, p' \text{ coarser-compatible than } p} m_{p'^c}.$$

Let us consider the cumulants  $(\kappa^p)_{p \in A_k}$  and the exclusive moments  $(m_{p^c})_{p \in \mathcal{P}_k}$  of  $(m_p)_{p \in A_k}$ .

**Proposition 8.3.** — *Let  $p$  and  $p_0$  be two elements of  $A_k$ . Then:*

$$\delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}} = \sum_{p' \in [id, p_0]_{A_k}} \delta_{p \prec p'} \kappa^{K_{p'}(p)}.$$

where for any  $P \subset A_k$ ,  $\kappa^P = \sum_{p \in P} \kappa^p$ .

By specifying  $p = id$  in Proposition 8.4, we get back the Equation (5). Besides, one can get a similar formula for  $m_{p_0 \circ t_p}(E)$  by using the Equation (14). Using Theorem 8.1, the last proposition is a consequence of Proposition 8.4.

**Proposition 8.4.** — *For any integer  $N$ , let us consider  $E_N$  an element of  $\mathbb{C}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges. Let  $p$  and  $p_0$  be two elements of  $A_k$ . Then:*

$$\delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}}(E) = \sum_{p' \in [id, p_0]_{A_k}} \delta_{p \prec p'} \kappa^{K_{p'}(p)}(E).$$

*Proof.* — Let  $p$  and  $p_0$  be two elements of  $A_k$ . Let us consider for any  $N$ ,  $M_k^N(p) \in \mathbb{C}[A_k(N)]$ . The sequence  $(M_k^N(p))_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  converges by Lemma 5.4. Let us apply the Theorem 7.1 to the product  $M_k^N(p)E_N$ . We remind the reader that  $\kappa^{p'}(M_k^N(p)) = \delta_{p=p'}$ .

From Theorem 7.1, Equation (12), we know that

$$\lim_{N \rightarrow \infty} \kappa^{p'}(M_k^N(p)E_N) = \delta_{p \prec p'} \kappa^{K_{p'}(p)}(E).$$

Let us use the Equation (5):

$$\lim_{N \rightarrow \infty} m_{p_0}(M_k^N(p)E_N) = \sum_{p' \in [id, p_0]_{A_k}} \delta_{p \prec p'} \kappa^{K_{p'}(p)}(E).$$

Yet, according to Equation (13),

$$\lim_{N \rightarrow \infty} m_{p_0}(M_k^N(p)E_N) = \delta_{p \in [id, p_0]_{A_k}} m_{t_{p \circ p_0}}(E),$$

hence the equality stated in Proposition 8.4.  $\square$

Let us show one more exemple. Using Theorem 8.1, one can translate Theorem 4.6.

**Theorem 8.2.** — *For any  $p \in A_k$ :*

$$m_{p^c} = \sum_{p' \text{ finer than } p, p' \in [id, p]_{A_k}} \kappa_{p'}.$$

**8.3. Convergence of the modified observables.** — In Section 5, we have defined a deformed partition algebra, by deforming the multiplication. Yet, we have not defined any deformed linear form  $m_p$  on the algebra  $\mathbb{C}[A_k(N, N)]$ . In fact, on  $\mathbb{C}[A_k(N, N)]$ , for any  $p \in \mathcal{P}_k$  we define:

$$\begin{aligned} m_p^N : \mathbb{C}[A_k(N, N)] &\rightarrow \mathbb{C} \\ E &\mapsto m_p(M_k^N(E)). \end{aligned}$$

A consequence of Theorem 4.1 is that for any  $E \in A_k$ , for any  $p \in \mathcal{P}_k$ ,  $m_p^N(E)$  converges as  $N$  goes to infinity: let us denote the limit by  $m_p^\infty(E)$ . We already know that the algebra  $\mathbb{C}[A_k(N, N)]$  converges to  $\mathbb{C}[A_k(\infty, \infty)]$  when  $N$  goes to infinity. Thus, we have that:

**Theorem 8.3.** — *For any integer  $k$ ,  $(\mathbb{C}[A_k(N, N)], (m_p^N)_{p \in \mathcal{P}_k})$  converges to*

$$(\mathbb{C}[A_k(\infty, \infty)], (m_p^\infty)_{p \in \mathcal{P}_k})$$

*as  $N$  goes to infinity. This means that:*

1. *the algebra  $\mathbb{C}[A_k(N, N)]$  converges to  $\mathbb{C}[A_k(\infty, \infty)]$  as  $N$  goes to infinity,*
2. *for any  $E \in \mathbb{C}[A_k(N, N)]$ , for any  $p \in \mathcal{P}_k$ ,  $m_p^N(E)$  converges to  $m_p^\infty(E)$  as  $N$  goes to infinity, where  $m_p^\infty(E)$  is defined below.*

*Besides, let  $E = \sum_{p \in A_k} E_p p$  and  $F = \sum_{p \in A_k} F_p p$  in  $\mathbb{C}[A_k(\infty, \infty)]$ , then*

$$EF = \sum_{p_1, p_2 \in A_k} E_{p_1} F_{p_2} \delta_{p_1 \prec p_1 \circ p_2} p_1 \circ p_2.$$

*And if  $p_0 \in A_k$ :*

$$m_{p_0}^\infty(E) = \sum_{p \in A_k} \delta_{p \in [id, p_0]_{A_k}} E_p.$$

In fact, this theorem has to be read in the other way: given the algebra with linear forms  $(\mathbb{C}[A_k(\infty, \infty)], (m_p^\infty)_{p \in \mathcal{P}_k})$ , one can find an approximation given by  $(\mathbb{C}[A_k(N, N)], (m_p^N)_{p \in \mathcal{P}_k})$ .

### 9. Algebraic fluctuations

In this section, we generalize Sections 4, 5 and 7 in order to study the asymptotic developments of the coordinate numbers and normalized moments. The proofs will be either omitted or simplified as they will use the same arguments as we have seen in Sections 4, 5 and 7.

In order to study the asymptotic developments, we need to introduce two notions of default. One already seen is linked with the triangular inequality and the other to the improved triangular inequality. Let  $k$  be an integer.

**Definition 9.1.** — *Let  $p$  and  $p'$  be two elements of  $A_k$ . We define the default of  $p'$  not being on the geodesic  $[id, p]_{A_k}$  by:*

$$\text{df}(p', p) = d(id, p') + d(p', p) - d(id, p).$$

We define also the default  $\eta(p, p')$  that  $p \prec p \circ p'$  is not satisfied by:

$$d(id, p) + d(id, p') - d(id, p \circ p') - \frac{k + \text{nc}(p \circ p') - \text{nc}(p) - \text{nc}(p')}{2} - \kappa(p, p').$$

We warn the reader that, in general:

$$\text{df}(p, p \circ p') \neq \eta(p, p'),$$

even if this equality is true when one considers  $p, p' \in \mathfrak{S}_k$ . Let us remark that if  $p$  and  $p_0$  are elements of  $A_k$ ,  $p \prec p_0$  holds if and only if there exists  $p'$  such that  $p_0 = p \circ p'$  and  $\eta(p, p') = 0$ .

Let us define the  $N$ -development algebra of order  $m$  of  $A_k$ . This algebra is the good setting in order to study fluctuations of the coordinate numbers and moments.

**Definition 9.2.** — *Let  $N, k$  and  $m$  be integers, let  $X$  be a formal variable. The  $N$ -development algebra of order  $m$  of  $A_k$ ,  $\mathbb{C}_{(m)}[A_k(N)]$ , is the associative algebra generated by the elements of the form:*

$$\frac{p}{X^i},$$

where  $p \in A_k$  and  $i \in \{0, \dots, m\}$ . The product is defined such that, for any  $p$  and  $p'$  in  $A_k$ , and any  $i$  and  $j$  in  $\{0, \dots, m\}$ :

$$\frac{p}{X^i} \cdot \frac{p'}{X^j} = \frac{1}{N^{\max(i+j+\eta(p,p')-m, 0)}} \frac{p \circ p'}{X^{\min(i+j+\eta(p,p'), m)}}.$$

This product is well defined: indeed the improved triangle inequality, Proposition 6.1 or Lemma 6.1, assert that for any  $p, p' \in A_k$ ,  $\eta(p, p') \geq 0$ , thus, for any  $i, j \in \{0, \dots, m\}$ , any  $p, p' \in A_k$ , we have  $\min(i + j + \eta(p, p'), m) \geq 0$ . This implies that:

$$\frac{p \circ p'}{X^{\min(i+j+\eta(p,p'), m)}}$$

is an element of the canonical basis of the  $N$ -development algebra of order  $m$  of  $A_k$ .

There is an important remark to be done: once one has defined the  $N$ -development algebra of order  $m$  of  $A_k$ , we will not have any energy to spend in order to get interesting results.

Let us also remark that for any integers  $k, N$  and  $m$  in  $\mathbb{N}$ ,  $\mathbb{C}_{(m)}[\mathfrak{S}_k(N)] \subset \mathbb{C}_{(m)}[\mathcal{B}_k(N)] \subset \mathbb{C}_{(m)}[\mathcal{P}_k(N)]$ , where these inclusions are inclusions of algebras.

**9.1. Coordinate numbers.** — Let us remark that for any integer  $N$ , the  $N$ -development algebra of order 0 of  $A_k$  is canonically isomorphic to  $\mathbb{C}[A_k(N, N)]$ .

**Lemma 9.1.** — *Let  $N$  be an integer, the application:*

$$\begin{aligned} L_k^N : A_k &\rightarrow \mathbb{C}_{(0)}[A_k(N)] \\ p &\mapsto \frac{p}{X^0} \end{aligned}$$

*can be extended as an isomorphism of algebra between  $\mathbb{C}[A_k(N, N)]$  and  $\mathbb{C}_{(0)}[A_k(N)]$ .*

*Proof.* — Let us show that for any  $p, p'$  in  $A_k$ ,  $L_k^N(p \cdot_N p') = L_k^N(p)L_k^N(p')$ . As for any  $p, p' \in A_k$ ,  $\eta(p, p') \geq 0$ ,  $L_k^N(p)L_k^N(p')$  is equal to:

$$\frac{p}{X^0} \frac{p'}{X^0} = \frac{1}{N^{\eta(p, p')}} \frac{p \circ p'}{X^0} = \frac{1}{N^{\eta(p, p')}} L_k^N(p \circ p').$$

Yet, looking at the definition of  $\eta(p, p')$  given in Definition 9.1, for any integer  $N$  the following equation holds in  $\mathbb{C}[A_k(N, N)]$ :

$$p \cdot_N p' = \frac{1}{N^{\eta(p, p')}} p \circ p'.$$

This allows to conclude. □

Using this remark, we define, for any  $i \leq m$ , the coordinate numbers of order  $i$  of any element of  $\mathbb{C}_{(m)}[A_k(N)]$  as following.

**Definition 9.3.** — *Let  $N$  and  $m$  be two integers. Let  $E \in \mathbb{C}_{(m)}[A_k(N)]$ . The coordinate numbers of  $E$  up to the order  $m$  are the elements  $(\kappa_i^p(E))_{i \in \{0, \dots, m\}, p \in \mathcal{P}_k}$  such that:*

$$E = \sum_{p \in A_k} \sum_{i=0}^m \kappa_i^p(E) \frac{p}{X^i}.$$

*Let  $p \in A_k$  and  $i \leq m$ . The number  $\kappa_i^p(E)$  is the coordinate number of  $E$  on  $p$  of order  $i$ .*

We define also a notion of convergence for  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ . In order to do so, we must not forget that, when  $m = 0$ ,  $\mathbb{C}_{(m)}[A_k(N)]$  is isomorphic to the deformed algebra  $\mathbb{C}[A_k(N, N)]$  and not the algebra  $\mathbb{C}[A_k(N)]$ .

**Definition 9.4.** — *Let  $m \in \mathbb{N}$ . The sequence  $(E_N)_{N \in \mathbb{N}}$  converges if and only if for any  $i \in \{0, \dots, m-1\}$ , and any  $p \in A_k$ ,  $\kappa_i^p(E_N)$  is independent of  $N$ , and for any  $p \in A_k$ ,  $\kappa_m^p(E_N)$  converges when  $N$  goes to infinity.*

**Notation 9.1.** — *Let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges as  $N$  goes to infinity. We denote, for any  $i \in \{0, \dots, m\}$ , and any  $p \in A_k$ :*

$$\kappa_i^p(E) = \lim_{N \rightarrow \infty} \kappa_i^p(E_N).$$

**9.2. Convergences:  $\mathbb{C}_{(m)}[A_k(N)]$  and multiplication.** — Using Lemma 9.1, Theorem 6.1, as the algebra  $\mathbb{C}[A_k(N, N)]$  is isomorphic to  $\mathbb{C}_{(0)}[A_k(N)]$  by an isomorphism which sends the canonical base of the first algebra on the canonical base of the second algebra, we know that the algebra  $\mathbb{C}_{(0)}[A_k(N)]$  converges as  $N$  tends to infinity. In fact, the result holds for any  $m \in \mathbb{N}$ .

**Definition 9.5.** — Let  $N, k$  and  $m$  be three integers. Let  $X$  be a formal variable. The  $\infty$ -development algebra of order  $m$  of  $A_k$ , denoted by  $\mathbb{C}_{(m)}[A_k(\infty)]$  is the associative algebra generated by the elements of the form:

$$\frac{p}{X^i},$$

where  $p \in A_k$  and  $i \in \{0, \dots, m\}$ . The product is defined such that, for any  $p$  and  $p'$  in  $A_k$ , and any  $i$  and  $j$  in  $\{0, \dots, m\}$ ,

$$\frac{p}{X^i} \frac{p'}{X^j} = \delta_{i+j+\eta(p,p') \leq m} \frac{p \circ p'}{X^{i+j+\eta(p,p')}}.$$

Let us recall Definition 6.3, where we defined the convergence of algebras. We then have the following proposition.

**Proposition 9.1.** — Let  $k$  and  $m$  be two integers. When  $N$  goes to infinity, the  $N$ -development algebra of order  $m$  of  $A_k$ ,  $\mathbb{C}_{(m)}[A_k(N)]$  converges to the  $\infty$ -development algebra of order  $m$  of  $A_k$ , namely  $\mathbb{C}_{(m)}[A_k(\infty)]$ .

*Proof.* — Let  $k$  be an integer. The algebras  $\mathbb{C}_{(m)}[A_k(N)]$  have, for any integer  $N$ , the same linear basis  $\left(\frac{p}{X^i}\right)_{i \in \{0, \dots, m\}, p \in A_k}$ . Since for any  $p, p' \in A_k$ , any  $i, j \in \mathbb{N}$ :

$$\frac{p}{X^i} \frac{p'}{X^j} = \frac{1}{N^{\max(i+j+\eta(p,p')-m, 0)}} \frac{p \circ p'}{X^{\min(i+j+\eta(p,p'), m)}} \xrightarrow{N \rightarrow \infty} \delta_{i+j+\eta(p,p') \leq m} \frac{p \circ p'}{X^{i+j+\eta(p,p')}},$$

where the first product is seen in  $\mathbb{C}_{(m)}[A_k(N)]$ , the algebra  $\mathbb{C}_{(m)}[A_k(N)]$  converges to  $\mathbb{C}_{(m)}[A_k(\infty)]$  as  $N$  goes to infinity.  $\square$

Let us write the first easiest consequence of the Proposition 9.1, which can be proved by using a bi-linearity argument, Proposition 9.1 and Definition 9.4.

**Proposition 9.2.** — Let  $m$  be an integer, let  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  be elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ . Let us suppose that the two sequences  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  converge. The sequence  $(E_N F_N)_{N \in \mathbb{N}}$  converges and, using Notations 9.1, for any  $i_0 \in \{0, \dots, m\}$  and for any  $p_0 \in A_k$ :

$$\kappa_{i_0}^{p_0}(EF) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(E) \kappa_{i_0 - \eta(p, p') - i}^{p'}(F).$$

As for Section 7.2, the good behavior of the product, given by Proposition 9.2, implies a criteria for the convergence of semi-groups in  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ .

**Definition 9.6.** — Let  $m$  and  $k$  be two integers. Let  $((E_t^N)_N)_{t \geq 0}$  be a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ . The semi-group  $((E_t^N)_N)_{t \geq 0}$  converges if and only if for any  $t \geq 0$ ,  $(E_t^N)_{N \in \mathbb{N}}$  converges.

We have the following proposition, whose proof relies on the ideas behind the proof of Proposition 9.2.

**Proposition 9.3.** — *Let  $m \in \mathbb{N}$ . Let us consider  $((E_t^N)_N)_{t \geq 0}$  a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$  which generator is denoted by  $(H_N)_{N \in \mathbb{N}}$ . It converges if the sequences  $(E_0^N)_{N \in \mathbb{N}}$  and  $(H_N)_{N \in \mathbb{N}}$  converge. Besides, using Notation 9.1, for any  $p \in A_k$ , for any  $t_0 \geq 0$  and any  $i \in \{0, \dots, m\}$ ,*

$$\frac{d}{dt} \Big|_{t=t_0} \kappa_{i_0}^{p_0}(E_t) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(H) \kappa_{i_0 - \eta(p, p') - i}^{p'}(E_{t_0}).$$

In order to finish this section, let us introduce the evaluation morphism: it is a morphism which allows to inject an element from  $\mathbb{C}_{(m)}[A_k(N)]$  in  $\mathbb{C}[A_k(N)]$ . Let  $N$  and  $m$  be two integers. The function  $eval_N$  is defined by:

$$\begin{aligned} eval_{(m)}^N : \mathbb{C}_{(m)}[A_k(N)] &\rightarrow \mathbb{C}[A_k(N)] \\ \sum_{p \in A_k} \sum_{i=0}^m \kappa_i^p(E) \frac{p}{X^i} &\mapsto \sum_{p \in A_k} \sum_{i=0}^m \kappa_i^p(E) \frac{1}{N^i} \frac{p}{N^{-\frac{k}{2} + \frac{nc(p)}{2} + d(id, p)}}. \end{aligned}$$

**Lemma 9.2.** — *For any integers  $N$  and  $m$ ,  $eval_{(m)}^N$  is a morphism of algebra.*

*Proof.* — Let  $N$  and  $m$  be two integers, let  $i, j \in \{0, \dots, m\}$  and  $p, p' \in A_k$ . Then:

$$\begin{aligned} eval_{(m)}^N \left( \frac{p}{X^i} \frac{p'}{X^j} \right) &= eval_{(m)}^N \left( \frac{1}{N^{\max(i+j+\eta(p, p')-m, 0)}} \frac{p \circ p'}{X^{\min(i+j+\eta(p, p'), m)}} \right) \\ &= \frac{1}{N^{i+j+\eta(p, p') - \frac{k}{2} + \frac{nc(p \circ p')}{2} + d(id, p \circ p')}} p \circ p' \\ &= \left( \frac{1}{N^i} \frac{p}{N^{-\frac{k}{2} + \frac{nc(p)}{2} + d(id, p)}} \right) \left( \frac{1}{N^j} \frac{p'}{N^{-\frac{k}{2} + \frac{nc(p')}{2} + d(id, p')}} \right) \\ &= eval_{(m)}^N \left( \frac{p}{X^i} \right) eval_{(m)}^N \left( \frac{p'}{X^j} \right). \end{aligned}$$

The other properties are easily verified.  $\square$

The function  $eval_{(m)}^N$  has an inverse if and only if  $m = 0$ . This will motivate us in order to define a notion of convergence up to order  $m$  of fluctuations for sequences in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Then, given a linear or multiplicative problem in  $\mathbb{C}[A_k(N)]$ , one can try to find a similar problem in  $\mathbb{C}_{(m)}[A_k(N)]$ , solve this last problem, and push by  $eval_{(m)}^N$  the solution on a solution of the first problem.

**9.3. Convergence at any order of fluctuations in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .** — We are interested in elements in  $\mathbb{C}[A_k(N)]$  and we want to define a notion of strong convergence up to the  $m^{\text{th}}$  order of fluctuations.

**Definition 9.7.** — *Let  $m$  be an integer, let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The sequence  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations if and only if there exist two families of real  $(\kappa_i^p)_{i \in \{0, \dots, m-1\}, p \in A_k}$  and  $(\kappa_{m, N}^p)_{p \in A_k, N \in \mathbb{N}}$  such that:*

- $\forall p \in A_k, \kappa^p(E_N) = \sum_{i=0}^{m-1} \frac{\kappa_i^p}{N^i} + \frac{\kappa_{m,N}^p}{N^m},$
- $\forall p \in A_k, \kappa_{m,N}^p$  converges as  $N$  goes to infinity.

The families  $(\kappa_i^p)_{i \in \{0, \dots, m-1\}, p \in A_k}$  and  $(\kappa_{m,N}^p)_{p \in A_k}$  are uniquely defined.

For any  $p \in A_k$ , any integer  $N$  and any  $i \in \{0, \dots, m-1\}$ ,  $\kappa_i^p$  is the coordinate number of  $E_N$  on  $p$  of order  $i$ , and  $\kappa_{m,N}^p$  is the coordinate number of  $E_N$  on  $p$  of order  $m$ .

**Notation 9.2.** — Let  $m$  be an integer. Let  $(E_N)_{N \in \mathbb{N}}$  in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  such that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations. From now on, the coordinate numbers of  $E_N$  on  $p$  of order  $i$  will be denoted by  $\kappa_i^p(E_N)$ . For any  $p \in A_k$  and any  $i \in \{0, \dots, m\}$ , we will define:

$$\kappa_i^p(E) = \lim_{N \rightarrow \infty} \kappa_i^p(E_N).$$

When one works in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ , one has to be aware that the coordinate numbers of higher order of fluctuations are only defined for a sequence  $(E_N)_{N \in \mathbb{N}}$  which converges strongly. Thus, one must not forget that the notation  $\kappa_i^p(E_N)$  means that we are looking at the coordinate numbers of  $E_N$  seen as an element of the sequence  $(E_N)_{N \geq 0}$ .

The Definition 9.7 might seem strange as it only uses once the notion of convergence. Yet, it is easy to see that an equivalent definition is the following one.

Let  $m$  be an integer, let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . It converges strongly up to the  $m^{\text{th}}$  order of fluctuations if and only if there exists a family  $(\kappa_i^p)_{i \in \{0, \dots, m\}, p \in A_k}$  of real numbers such that for any  $i \in \{0, \dots, m\}$ ,

$$N^i \left( \kappa^p(E_N) - \sum_{j=0}^{i-1} \frac{\kappa_j^p}{N^j} \right) \xrightarrow{N \rightarrow \infty} \kappa_i^p,$$

with the convention  $\sum_{j=0}^{-1} \frac{\kappa_j^p}{N^j} = 0$ . This definition explains why the families  $(\kappa_i^p)_{i \in \{0, \dots, m-1\}, p \in A_k}$  and  $(\kappa_{m,N}^p)_{N \in \mathbb{N}, p \in A_k}$  defined in Definition 9.7 are uniquely defined.

The next lemma makes a link between the convergence of elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$  and the convergence up to the  $m^{\text{th}}$  order of fluctuations of elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .

**Lemma 9.3.** — Let  $m \in \mathbb{N}$ . Let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges. Then  $\left( \text{eval}_{(m)}^N(E_N) \right)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations.

The notion of strong convergence to the  $m^{\text{th}}$  order of fluctuations allows to inject canonically an element of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  which converges strongly up to the  $m^{\text{th}}$  order of fluctuations into  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ .

**Definition 9.8.** — Let  $m$  be an integer, let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations. For any  $p \in A_k$ , any integer  $N$ , let  $(\kappa_i^p)_{i \in \{0, \dots, m-1\}}$  and  $\kappa_{m,N}^p$  be the coordinate numbers of  $E_N$



on  $p$ . We define the lift of the sequence  $(E_N)_{N \in \mathbb{N}}$  as  $(\tilde{E}_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$  defined by:

$$\tilde{E}_N = \sum_{p \in A_k} \left( \left( \sum_{i=0}^{m-1} \kappa_i^p \frac{p}{X^i} \right) + \kappa_{m,N}^p \frac{p}{X^m} \right).$$

The following lemma is then straightforward.

**Lemma 9.4.** — *Let  $m \in \mathbb{N}$ , let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  and let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations. Let  $(\tilde{E}_N)_{N \in \mathbb{N}}$  be its canonical lift in  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ . Then  $(\tilde{E}_N)_{N \in \mathbb{N}}$  converges as  $N$  goes to infinity and for any  $N \in \mathbb{N}$ , one has  $\text{eval}_{(m)}^N(\tilde{E}_N) = E_N$ .*

We are going to define a weak notion of convergence up to the  $m^{\text{th}}$  order of fluctuations and we will show that this notion is equivalent to the strong convergence notion we defined in Definition 9.7.

**Definition 9.9.** — *Let  $m$  be an integer, let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The sequence  $(E_N)_{N \in \mathbb{N}}$  converges in moments up to the  $m^{\text{th}}$  order of fluctuations if and only if there exist two families  $(m_p^i)_{i \in \{0, \dots, m-1\}, p \in A_k}$  and  $(m_{p,N}^m)_{N \in \mathbb{N}, p \in A_k}$  such that:*

- $\forall p \in A_k$ ,  $m_p(E_N) = \sum_{i=0}^{m-1} \frac{m^i}{N^i} + \frac{m_p^m}{N^m}$ ,
- $\forall p \in A_k$ ,  $m_p^{m,N}$  converges as  $N$  goes to infinity.

The families  $(m_p^i)_{i \in \{0, \dots, m-1\}, p \in A_k}$  and  $(m_{p,N}^m)_{N \in \mathbb{N}, p \in A_k}$  are uniquely defined.

For any  $p \in A_k$ , any integer  $N$ , and any  $i \in \{0, \dots, m-1\}$ ,  $m_p^i$  is the  $i^{\text{th}}$ -order fluctuations of the  $p$ -normalized moment of  $E_N$ , and  $m_{p,N}^m$  is the  $m^{\text{th}}$ -order fluctuations of the  $p$ -normalized moment of  $E_N$ .

**Notation 9.3.** — *Let  $m$  be an integer. Let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  such that  $(E_N)_{N \in \mathbb{N}}$  converges in moments up to the  $m^{\text{th}}$  order of fluctuations. From now on, the  $i^{\text{th}}$ -order fluctuations of the  $p$ -normalized moment of  $E_N$  will be denoted by  $m_p^i(E_N)$ . For any  $p \in A_k$  and any  $i \in \{0, \dots, m\}$ , we define:*

$$m_p^i(E) = \lim_{N \rightarrow \infty} m_p^i(E_N).$$

We can state a remark for the fluctuations of the  $p$ -normalized moments of  $E_N$  similar to the one explained just after Notation 9.2 about the coordinate numbers of  $E_N$  on  $p$  of order  $i$ .

The next theorem shows that the strong convergence up to the  $m^{\text{th}}$  order of fluctuations is equivalent to the convergence in moments up to the  $m^{\text{th}}$  order of fluctuations.

**Theorem 9.1.** — *Let  $m \in \mathbb{N}$ , let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The sequence  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations if and only if it converges in moments up to the  $m^{\text{th}}$  order of fluctuations. We will say that  $(E_N)_{N \in \mathbb{N}}$  converges up to the  $m^{\text{th}}$  order of fluctuations.*

Let us suppose that  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[\mathcal{P}_k(N)]$  converges up to the  $m^{\text{th}}$  order of fluctuations. Using Notations 9.2 and 9.3 and the Definition 9.1, we have that, for any  $i_0 \in \{0, \dots, m\}$  and any  $p \in A_k$ :

$$(18) \quad m_p^{i_0}(E) = \sum_{p' \in A_k, \text{df}(p', p) \leq i_0} \kappa_{i_0 - \text{df}(p', p)}^{p'}(E).$$

*Proof.* — Let  $m$  be an integer and let  $(E_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us consider  $p$  in  $A_k$ .

Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations. The coordinate numbers of  $E_N$  are defined up to order  $m$  of fluctuations and:

$$E_N = \sum_{p \in A_k} \sum_{i=0}^m \frac{\kappa_i^p(E_N)}{N^i} \frac{p}{N^{-\frac{k}{2} + \frac{nc(p)}{2} + d(id, p)}}.$$

Besides, for any  $p \in A_k$  and any  $i \leq m-1$ ,  $\kappa_i^p(E_N)$  does not depend on  $N$  and  $\kappa_m^p(E_N)$  converges when  $N$  goes to infinity.

We can compute the  $p$ -normalized moments of  $E_N$ , using the same arguments as for the proof of Theorem 4.1. For any  $N \in \mathbb{N}$  and any  $p \in A_k$ :

$$\begin{aligned} m_p(E_N) &= \frac{1}{\text{Tr}_N(p)} \text{Tr}_N(E_N) = \sum_{p' \in A_k} \sum_{i=0}^m \kappa_i^{p'}(E_N) \frac{1}{N^{i + \text{df}(p', p)}} \\ &= \sum_{j=0}^{m-1} \left( \sum_{(p', i) \in A_k \times \{0, \dots, m-1\}, i + \text{df}(p, p') = j} \kappa_i^{p'}(E_N) \right) \frac{1}{N^j} \\ &\quad + \left( \sum_{(p', i) \in A_k \times \{0, \dots, m\}, i + \text{df}(p, p') \geq m} \frac{\kappa_i^{p'}(E_N)}{N^{i + \text{df}(p, p') - m}} \right) \frac{1}{N^m}. \end{aligned}$$

Let us define for any  $N \in \mathbb{N}$ , any  $j \in \{0, \dots, m-1\}$  and any  $p \in A_k$ :

$$m_p^j(E_N) = \sum_{(p', i) \in A_k \times \{0, \dots, m-1\}, i + \text{df}(p, p') = j} \kappa_i^{p'}(E_N)$$

and

$$m_p^m(E_N) = \sum_{(p', i) \in A_k \times \{0, \dots, m\}, i + \text{df}(p, p') \geq m} \frac{\kappa_i^{p'}(E_N)}{N^{i + \text{df}(p, p') - m}},$$

so that, for any  $p \in A_k$  and any  $N \in \mathbb{N}$ :

$$m_p(E_N) = \sum_{j=0}^{m-1} \frac{m_p^j(E_N)}{N^j} + \frac{m_p^m(E_N)}{N^m}.$$

For any  $p \in A_k$  and any  $i \leq m-1$ ,  $m_p^i(E_N)$  does not depend on  $N$  and for any  $p \in A_k$ ,  $\kappa_m^p(E_N)$  converges when  $N$  goes to infinity. Thus  $m_p^m(E_N)$  converges when  $N$  goes to infinity to

$$\sum_{p' \in A_k, \text{df}(p', p) \leq m} \kappa_{m - \text{df}(p', p)}^{p'}(E).$$

By Definition 9.9, this shows that  $(E_N)_{N \in \mathbb{N}}$  converges in moments up to the  $m^{\text{th}}$  order of fluctuations and the Equation (18) holds.

Let us suppose now that  $(E_N)_{N \in \mathbb{N}}$  converges in moments up to the  $m^{\text{th}}$  order of fluctuations. Then, by Theorem 4.1, it converges strongly up to order 0 of fluctuation. Let us suppose that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to order  $l$  of fluctuations with  $l < m$ . Thus, the coordinate numbers of  $E_N$  up to order  $l$  of fluctuations are well defined and we can write:

$$E_N = \sum_{p \in \mathcal{P}_k} \left( \sum_{j=0}^{l-1} \frac{\kappa_j^p(E)}{N^j} + \frac{\kappa_l^p(E_N)}{N^l} \right) p,$$

where, for any  $p \in \mathcal{P}_k$ ,  $\kappa_l^p(E_N)$  is converging when  $N$  goes to infinity to a number  $\kappa_l^p(E)$ . We can use the computation, that we already did, of the normalized moments of  $E_N$ . For any partition  $p \in A_k$ :

$$\begin{aligned} m_p(E_N) &= \sum_{j=0}^{l-1} \left( \sum_{(p', i) \in A_k \times \{0, \dots, l-1\}, i + \text{df}(p, p') = j} \kappa_i^{p'}(E) \right) \frac{1}{N^j} \\ &\quad + \left( \sum_{(p', i) \in A_k \times \{0, \dots, l\}, i + \text{df}(p, p') \geq l} \frac{\kappa_i^{p'}(E_N)}{N^{i + \text{df}(p, p') - l}} \right) \frac{1}{N^l}. \end{aligned}$$

Thus, using the same notations than those used in the first part of the proof, we get:

$$\begin{aligned} m_p(E_N) &= \sum_{j=0}^l \frac{m_p^j(E)}{N^j} + \sum_{p' \in A_k, \text{df}(p, p') = 0} \frac{\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E)}{N^l} \\ &\quad + \sum_{(p', i) \in A_k \times \{0, \dots, l\}, i + \text{df}(p, p') - l = 1} \frac{\kappa_i^{p'}(E_N)}{N^{l+1}} + o\left(\frac{1}{N^{l+1}}\right). \end{aligned}$$

Let us use the fact that  $(E_N)_{N \in \mathbb{N}}$  converges in moments up to the order  $l + 1$  of fluctuations: for any  $p \in A_k$ ,

$$N^{l+1} \left( m_p(E_N) - \sum_{j=0}^l \frac{m_p^j(E)}{N^j} \right)$$

converges as  $N$  goes to infinity. This implies that for any  $p \in A_k$ ,

$$\sum_{p' \in [id, p]_{A_k}} N(\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E))$$

converges as  $N$  goes to infinity. We are thus in the same setting as for the order 0 of fluctuations: for any  $p \in A_k$ ,

$$N(\kappa_l^{p'}(E_N) - \kappa_l^{p'}(E))$$

converges as  $N$  goes to infinity: this is equivalent to say that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to order  $l + 1$  of fluctuations. This implies by recurrence that  $(E_N)_{N \in \mathbb{N}}$  converges strongly up to order  $m$  of fluctuations.  $\square$

**9.4. Multiplication and convergence of fluctuations in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ .** — The results in Section 9.3 were only algebraic: we will now give the similar results for elements in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The main ingredients used in order to do so are Lemma 9.2, Lemma 9.3 and Lemma 9.4 which respectively assert that  $eval_{(m)}^N$  is a morphism of algebra, compatible with the strong convergence notion and, in some sense, can be inverted.

**Theorem 9.2.** — *Let  $m \in \mathbb{N}$ . Let  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  be elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us suppose that the sequences  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  converge up to the  $m^{\text{th}}$  order of fluctuations. Then, the sequence  $(E_N F_N)_{N \in \mathbb{N}}$  converges up to the  $m^{\text{th}}$  order of fluctuations.*

*Besides, using the Notations 9.2 and 9.3, for any  $i_0 \in \{0, \dots, m\}$  and for any  $p_0 \in A_k$ :*

$$(19) \quad \kappa_{i_0}^{p_0}(EF) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i=0}^{i_0 - \eta(p, p')} \kappa_i^p(E) \kappa_{i_0 - \eta(p, p') - i}^{p'}(F).$$

$$(20) \quad m_{p_0}^{i_0}(EF) = \sum_{p_1 \in A_k} \sum_{i+j+\text{df}(p_1, p_0)=i_0} \kappa_i^{p_1}(E) m_{p_1 \circ p_0}^j(F).$$

*Proof.* — Let  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  be elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Let us suppose that the sequences  $(E_N)_{N \in \mathbb{N}}$  and  $(F_N)_{N \in \mathbb{N}}$  converge strongly or in moments up to the  $m^{\text{th}}$  order of fluctuations.

By Lemma 9.4, let us consider the canonical lifts of  $(E_N)_{N \in \mathbb{N}}$  (resp.  $(F_N)_{N \in \mathbb{N}}$ ) in  $\prod_{N \in \mathbb{N}} \mathbb{C}_{(m)}[A_k(N)]$ :  $(\tilde{E}_N)_{N \in \mathbb{N}}$  (resp.  $(\tilde{F}_N)_{N \in \mathbb{N}}$ ). The two sequences  $(\tilde{E}_N)_{N \in \mathbb{N}}$  and  $(\tilde{F}_N)_{N \in \mathbb{N}}$  converge. According to Proposition 9.2, the sequence  $(\tilde{E}_N \tilde{F}_N)_{N \in \mathbb{N}}$  converges. For any  $i_0 \in \{0, \dots, m\}$  and for any  $p_0 \in A_k$ :

$$(21) \quad \kappa_{i_0}^{p_0}(\tilde{E}\tilde{F}) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(\tilde{E}) \kappa_{i_0 - \eta(p, p') - i}^{p'}(\tilde{F}).$$

An application of Lemma 9.3 shows that the sequence  $(eval_{(m)}^N(\tilde{E}_N \tilde{F}_N))_{N \in \mathbb{N}}$  converges up to the  $m^{\text{th}}$  order of fluctuations. As  $eval_{(m)}^N$  is a morphism of algebra, Lemma 9.2, for any  $N \in \mathbb{N}$ ,

$$eval_{(m)}^N(\tilde{E}_N \tilde{F}_N) = eval_{(m)}^N(\tilde{E}_N) eval_{(m)}^N(\tilde{F}_N) = E_N F_N.$$

We deduce that  $(E_N F_N)_{N \in \mathbb{N}}$  converges strongly up to the  $m^{\text{th}}$  order of fluctuations. The equality (19) is deduced from (21).

In order to prove the equality (20), the best way is to come back to the definitions, and do a proof similar to the one for (13) in Theorem 7.1.  $\square$

Let us consider the implication of Proposition 9.3 for the semi-groups in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . From now on, let us suppose that  $((E_t^N)_N)_{t \geq 0}$  is a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  whose generator is  $(H_N)_{N \in \mathbb{N}}$ . We would like to state a theorem for the fluctuations of  $((E_t^N)_N)_{t \geq 0}$  similar to Theorem 7.2. For this, we need the following definition.

**Definition 9.10.** — Let  $m \in \mathbb{N}$ . The semi-group  $((E_t^N)_N)_{t \geq 0}$  converges to the  $m^{\text{th}}$  order of fluctuations if and only if for any  $t \geq 0$ ,  $(E_t^N)_{N \in \mathbb{N}}$  converges up to the  $m^{\text{th}}$  order of fluctuations.

We can now state the theorem about the convergence to the  $m^{\text{th}}$  order of fluctuations of a semi-group in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . The proof will not be given, as it is a direct consequence of Proposition 9.3 with a lift-argument as for the last proof.

**Theorem 9.3.** — Let  $m \in \mathbb{N}$ . The semi-group  $((E_t^N)_N)_{t \geq 0}$  converges to the  $m^{\text{th}}$  order of fluctuations if the sequences  $(E_0^N)_{N \in \mathbb{N}}$  and  $(H_N)_{N \in \mathbb{N}}$  converge up to the  $m^{\text{th}}$  order of fluctuations.

Besides, we have the two differential systems of equations:

$$\forall p_0 \in A_k, \forall t_0 \geq 0, \forall i_0 \in \{0, \dots, m\},$$

$$\frac{d}{dt} \Big|_{t=t_0} \kappa_{i_0}^{p_0}(E_t) = \sum_{p, p' \in A_k, \eta(p, p') \leq i_0, p \circ p' = p_0} \sum_{i \in \{0, \dots, i_0 - \eta(p, p')\}} \kappa_i^p(H) \kappa_{i_0 - \eta(p, p') - i}^{p'}(E_{t_0}).$$

$$\forall p_0 \in A_k, \forall t_0 \geq 0, \forall i \in \{0, \dots, m\},$$

$$\frac{d}{dt} \Big|_{t=t_0} m_{p_0}^{i_0}(E_t) = \sum_{p_1 \in A_k} \sum_{i+j+\text{df}(p_1, p_0)=i_0} \kappa_i^{p_1}(H_t) m_{p_1 \circ p_0}^j(E_{t_0}).$$

## 10. An introduction to the general $\mathcal{R}$ -transform

**10.1. The zero order.** — Up to now, we only worked with partitions which have a fixed length: we worked in  $A_k$  for a fixed integer  $k$ . Yet, we could have worked with  $A_\infty = \cup_{k \in \mathbb{N}} A_k$  endowed with the product:  $pp' = \delta_{|p|=|p'|} pp'$  where we recall that  $|p|$  is the length of  $p$ . With this definition, we see that all the results hold when one changes  $k$  by  $k = \infty$ . For example  $\mathbb{C}[A_\infty(N, N)]$  converges when  $N$  goes to infinity to an algebra  $\mathbb{C}[A_\infty(\infty, \infty)]$ . We could have studied this algebra, yet, in the theory of random matrices, we will see that the first elements  $E \in \mathbb{C}[A_\infty(\infty, \infty)]$  we naturally obtain are the elements which, seen as elements of  $\mathbb{C}[A_\infty]$ , are invariant and such that  $E_0 = 1$ . The invariance of  $E$  means that for any integer  $k$  and any  $\sigma \in \mathfrak{S}_k$ :

$$\sigma E_k \sigma^{-k} = E_k,$$

where  $E_k$  is the restriction of  $E$  on  $A_k$ . This definition allows to make the link with the usual theory of  $\mathcal{R}$ -transform. In fact, we will not use the invariance by  $\mathfrak{S}$  in most of what we will do. Yet, we preferred to write it like this since it is the setting in which one works when one consider the limit of one sequence of random matrices  $(A_N)_{N \in \mathbb{N}}$ . Yet, we will need similar results in [9] for the non-conjugation invariant case and we will use them by referring to the theorem proved in the conjugation invariant case.

10.1.1. *Order zero: general definitions and Lie algebras.* — Recall that  $A$  is either  $\mathfrak{S}$ ,  $\mathcal{B}$  or  $\mathcal{P}$ .

**Definition 10.1.** — Let us define the algebra:

$$\mathfrak{E}_g[A] = \prod_{k=0}^{\infty} \mathbb{C}^{\mathfrak{S}}[A_k(\infty, \infty)],$$

where, for any integer  $k$ ,  $\mathbb{C}^{\mathfrak{S}}[A_k(\infty, \infty)]$  is the algebra of elements of  $\mathbb{C}[A_k(\infty, \infty)]$  which, seen as elements of  $\mathbb{C}[A_k]$  are invariant by conjugation by any element of  $\mathfrak{S}_k$ . Actually, two subspaces of  $\mathfrak{E}_g[A]$  will be interesting for us:

$$\begin{aligned} \mathfrak{E}[A] &= \{E \in \mathfrak{E}_g[A], E_{\emptyset} = 1\}, \\ \mathfrak{e}[A] &= \{E \in \mathfrak{E}_g[A], E_{\emptyset} = 0\}. \end{aligned}$$

Any element  $E \in \mathfrak{E}_g[A]$  is of the form:

$$\left( \sum_{p \in A_k} (E_k)_p p \right)_{k \in \mathbb{N}}.$$

In order to simplify the notations, we will use the following convention: for any integer  $k$ , for any  $p \in A_k$ ,

$$E_p = E(p) = (E_k)_p,$$

and for any positive integer  $k$ :

$$E_k = \sum_{p \in A_k} E_p p.$$

The algebra  $\mathfrak{E}_g[A]$  is naturally endowed with a natural addition and multiplication given, for any  $E, F \in \mathfrak{E}_g[A]$  and any  $k \in \mathbb{N}^*$  by:

$$\begin{aligned} (E + F)_k &= E_k + F_k \\ (E \boxtimes F)_k &= E_k F_k. \end{aligned}$$

By convention  $(E \boxtimes F)_{\emptyset} = E_{\emptyset} F_{\emptyset}$ . Besides, one can construct an other law on  $\mathfrak{E}_g[A]$ .

**Definition 10.2.** — Let  $E$  and  $F$  be two elements of  $\mathfrak{E}_g[A]$ . We denote by  $E \boxplus F$  the element of  $\mathfrak{E}_g[A]$  such that for any  $p \in A_{|p)}$ :

$$(E \boxplus F)_p = \sum_{(p_1, p_2, I) \in \mathfrak{F}_2(p)} E(p_1) F(p_2),$$

where  $\mathfrak{F}_2(p)$  was defined in Definition 2.10.

In fact, the two operations  $\boxtimes$  and  $\boxplus$  are convolution operations.

**Remark 10.1.** — The sets  $\mathfrak{E}[A]$  and  $\mathfrak{e}[A]$  are stable by the  $\boxplus$  and  $\boxtimes$  operations. Besides,  $\mathfrak{E}[A]$  is an affine space whose underlying vector space is  $\mathfrak{e}[A]$ .

The operation  $\boxplus$  on  $\mathfrak{E}[A]$  is commutative, it defines a structure of group on  $\mathfrak{E}[A]$ . The neutral element  $0_{\mathfrak{E}}$  is the only element in  $\mathfrak{E}[A]$  such that for any positive integer  $k$ , any  $p \in A_k$ ,  $(0_{\mathfrak{E}})_p = 0$ .

The operation  $\boxtimes$  is not commutative and the set of invertible elements in  $\mathfrak{E}[A]$  is the set of elements  $E$  such that  $E_{id_k} \neq 0$  for any  $k \geq 1$ , we denote it by  $G\mathfrak{E}[A]$ . We denote by  $1_{\mathfrak{E}}$  the neutral element for  $\boxtimes$  which is the only element such that for any  $k \geq 1$ ,  $(1_{\mathfrak{E}})_k = id_k$ .

Let us consider an interesting sub-vector space of  $\mathfrak{E}_g[A]$ : the sub-vector space of irreducible partitions.

**Definition 10.3.** — Recall the notation  $A_k^{(i)}$  was defined in Definition 2.9. We denote by  $\mathbb{C}^{\mathfrak{S}}[A_k^{(i)}]$  the vector space of elements of  $\mathbb{C}[A_k^{(i)}]$  which, seen as elements of  $\mathbb{C}[A_k^{(i)}]$  are invariant by conjugation by any element of  $\mathfrak{S}_k$ . We define:

$$\mathfrak{E}_g^{(i)}[A] = \prod_{k=0}^{\infty} \mathbb{C}^{\mathfrak{S}}[A_k^{(i)}].$$

Actually, two subspaces of  $\mathfrak{E}_g[A]$  will be interesting for us:

$$\begin{aligned} \mathfrak{E}^{(i)}[A] &= \mathfrak{E}_g^{(i)}[A] \cap \mathfrak{E}[A], \\ \mathfrak{e}^{(i)}[A] &= \mathfrak{E}_g^{(i)}[A] \cap \mathfrak{e}[A]. \end{aligned}$$

When  $A = \mathfrak{S}$ , we have already seen after Definition 2.9 that:

$$A_k^{(i)} = \{\sigma(1, \dots, k)\sigma^{-1}, \sigma \in \mathfrak{S}_k\}.$$

**Proposition 10.1.** — The affine space  $\mathfrak{E}^{(i)}[\mathfrak{S}]$  can be identified, by the following isomorphism, with the affine space  $\mathbb{C}_1[[z]]$  of formal power series which constant term is equal to 1:

$$\begin{aligned} \mathfrak{E}^{(i)}[\mathfrak{S}] &\rightarrow \mathbb{C}_1[[z]] \\ E &\mapsto \sum_{k \in \mathbb{N}} E_{(1, \dots, k)} z^k. \end{aligned}$$

Any element  $E$  in  $\mathfrak{E}_g[A]$  can be restricted in order to obtain an element of  $\mathfrak{E}_g^{(i)}[A]$  that we denote by  $E|_{\mathfrak{E}^{(i)}[A]}$ . Conversely, given an element of  $\mathfrak{E}_g^{(i)}[A]$ , one can inject it non-trivially in  $\mathfrak{E}_g[A]$  in a natural way. Recall the definition of the extraction of  $p$  in Definition 2.11, and the definition of cycles given in Definition 2.8. We only consider the injection of an element of  $\mathfrak{E}^{(i)}[A]$  in  $\mathfrak{E}[A]$ .

**Definition 10.4.** — For any  $E \in \mathfrak{E}^{(i)}[A]$ , we denote by  $\mathfrak{M}(E)$  the unique element of  $\mathfrak{E}[A]$  such that for any integer  $k$ , any  $p \in A_k$ ,

$$(\mathfrak{M}(E))_p = \prod_{C \in \mathcal{C}(p)} E_{p_C}.$$

Any element of the image of the application:

$$\begin{aligned} \mathfrak{M} : \mathfrak{E}^{(i)}[A] &\rightarrow \mathfrak{E}[A] \\ E &\mapsto \mathfrak{M}(E) \end{aligned}$$

is called multiplicative and we denote  $\mathfrak{ME}[A] = \mathfrak{M}[\mathfrak{E}[A]]$ .

Let us remark that  $0_{\mathfrak{E}}$  and  $1_{\mathfrak{E}}$  are multiplicative elements. This is not the only property satisfied by  $\mathfrak{ME}[A]$ .

**Theorem 10.1.** — *The set  $\mathfrak{ME}[A]$  is stable by the operations  $\boxplus$  and  $\boxtimes$ .*

*Proof.* — Let  $E$  and  $F$  be two elements of  $\mathfrak{M}(E)$ . Let us show that  $E \boxplus F$  is multiplicative. Let  $p_1$  and  $p_2$  be two partitions, we have to show that:

$$(22) \quad (E \boxplus F)_{p_1 \otimes p_2} = (E \boxplus F)_{p_1} (E \boxplus F)_{p_2}.$$

Yet, by definition:

$$(E \boxplus F)_{p_1 \otimes p_2} = \sum_{(a_1, a_2, I) \in \mathfrak{F}_2(p_1 \otimes p_2)} E_{a_1} F_{a_2},$$

and:

$$(E \boxplus F)_{p_1} (E \boxplus F)_{p_2} = \sum_{(a_1^1, a_2^1, I^1) \in \mathfrak{F}_2(p_1), (a_1^2, a_2^2, I^2) \in \mathfrak{F}_2(p_2)} E_{a_1^1} E_{a_2^1} F_{a_1^2} F_{a_2^2}.$$

Using the fact that  $E$  and  $F$  are multiplicative, that  $E_{\emptyset} = 1 = F_{\emptyset}$  and using the fact that for any  $(a_1, a_2, I) \in \mathfrak{F}_2(p_1 \otimes p_2)$ ,  $a_1$  and  $a_2$  can be decomposed into two parts in order to get two 3-tuples  $(a_1^1, a_2^1, I^1) \in \mathfrak{F}_2(p_1)$  and  $(a_1^2, a_2^2, I^2) \in \mathfrak{F}_2(p_2)$ , one gets the equality (22).

Let us show that  $E \boxtimes F$  is multiplicative. Let  $p_1$  and  $p_2$  be two partitions, we have to show that:

$$(E \boxtimes F)_{p_1 \otimes p_2} = (E \boxtimes F)_{p_1} (E \boxtimes F)_{p_2}.$$

By definition:

$$(E \boxtimes F)_{p_1 \otimes p_2} = \sum_{a, b / a \circ b = p_1 \otimes p_2, a \prec p_1 \otimes p_2} E_a F_b.$$

Yet, using Lemma 6.4, any partition  $a$  such that  $a \prec p_1 \otimes p_2$  can be decomposed as  $a_1 \otimes a_2$  such that  $a_1 \prec p_1$  and  $a_2 \prec p_2$ . Then if  $b$  is a partition such that  $a_1 \otimes a_2 \circ b = p_1 \otimes p_2$ ,  $b$  can be also decomposed as  $b = b_1 \otimes b_2$  with  $a_1 \otimes b_1 = p_1$  and  $a_2 \otimes b_2 = p_2$ . Using the multiplicative property of  $E$  and  $F$ , one gets:

$$\begin{aligned} (E \boxtimes F)_{p_1 \otimes p_2} &= \sum_{a_1, a_2, b_1, b_2 / a_1 \circ b_1 = p_1, a_2 \circ b_2 = p_2, a_1 \prec p_1, a_2 \prec p_2} E_{a_1} E_{a_2} F_{b_1} F_{b_2} \\ &= \sum_{a_1, b_1 / a_1 \circ b_1 = p_1, a_1 \prec p_1} E_{a_1} F_{b_1} \sum_{a_2, b_2 / a_2 \circ b_2 = p_2, a_2 \prec p_2} E_{a_2} F_{b_2} \\ &= (E \boxtimes F)_{p_1} (E \boxtimes F)_{p_2}. \end{aligned}$$

This ends the proof. □



Let us justify our notation  $\boxplus$ . If we consider the pull-back of the  $\boxplus$  operation from  $\mathfrak{M}\mathfrak{E}[A]$  to  $\mathfrak{E}^{(i)}[A]$  and if one consider only the coefficients for the non-empty partitions, one simply obtains the usual additive law on  $\mathfrak{E}^{(i)}[A]$ . We will also see in the article [9] that  $\boxplus$  is the natural operation which appears when one is working with sum of free elements.

We believe that the inverse of a multiplicative element for the  $\boxplus$  and  $\boxtimes$  is still multiplicative, but we have not yet written the proof. It is natural to wonder, as we have two semi-groups  $(\mathfrak{M}\mathfrak{E}[A], \boxplus)$  and  $(\mathfrak{M}\mathfrak{E}[A] \cap G\mathfrak{E}_A, \boxtimes)$  on which one can define differentiable one-parameter semi-groups, what are the ‘‘Lie algebras’’ of these two semi-groups. Let us remark that  $\mathfrak{M}\mathfrak{E}[A] \cap G\mathfrak{E}_A$  is only the set of elements  $E$  of  $\mathfrak{M}\mathfrak{E}[A]$  such that  $E_{id_1} \neq 0$ . We need to define two ways to inject  $\mathfrak{e}^{(i)}[A]$  in  $\mathfrak{e}[A]$ , the first of which is the natural injection.

**Definition 10.5.** — For any  $E \in \mathfrak{e}^{(i)}[A]$ , we denote by  $\mathfrak{J}(E)$  the unique element of  $\mathfrak{e}[A]$  such that, for any positive integer  $k$ , any irreducible  $p \in A_k$ ,

$$(\mathfrak{J}(E))_p = E_p,$$

and for any non-irreducible  $p \in A_k$ ,  $(\mathfrak{J}(E))_p = 0$ . We define  $\mathfrak{m}\mathfrak{e}_{\boxplus}[A] = \mathfrak{J}(\mathfrak{E}^{(i)}[A])$ .

The second injection uses the notion of support of a partition and the notion of weakly irreducible partitions defined in Definition 2.12. Recall also the notion of extraction defined in Definition 2.11.

**Definition 10.6.** — For any  $E \in \mathfrak{e}^{(i)}[A]$ , we denote by  $\mathfrak{J}(E)$  the unique element of  $\mathfrak{e}[A]$  such that, for any integer  $k$ , any weakly irreducible  $p \neq id_k$  in  $A_k$ :

$$(\mathfrak{J}(E))_p = E(p_S(p)),$$

and  $(\mathfrak{J}(E))_{id_k} = k(\mathfrak{J}(E))_{id_1}$  and for any other  $p \in A_k$ ,  $(\mathfrak{J}(E))_p = 0$ . We define  $\mathfrak{m}\mathfrak{e}_{\boxtimes}[A] = \mathfrak{J}(\mathfrak{E}^{(i)}[A])$ .

This might look strange that we change the definition for  $(\mathfrak{J}(E))_{id_k}$ . It is easier to understand it by using an other equivalent definition of the weakly irreducible notion. The partition  $p$  is weakly irreducible if there exist  $p_0$  irreducible and  $I \subset \{1, \dots, k\}$  such that  $p = \sigma_I^{-1}(p_0 \otimes Id_{k-l(p_0)})\sigma_I$ . The partition  $p_0$  is unique if and only if  $p \neq id_k$ . If  $p_0$  is unique then  $(\mathfrak{J}(E))_p = E(p_0)$ . If  $p = id_k$ , then  $id_k = \sigma_{\{l\}}^{-1}(id_1 \otimes id_{k-l})\sigma_{\{l\}}$  for any integer  $l \in \{1, \dots, k\}$ . We do not choose and we prefer to sum all the values:  $(\mathfrak{J}(E))_{id_k} = kE_{id_1}$ .

Due to the definitions, it is obvious that the sets  $\mathfrak{m}\mathfrak{e}_{\boxtimes}[A]$  and  $\mathfrak{m}\mathfrak{e}_{\boxplus}[A]$  are vector spaces. Let us define the exponentiation of any element of  $\mathfrak{e}[A]$  associated with the operation  $\boxplus$ .

**Definition 10.7.** — Let  $E \in \mathfrak{e}[A]$ . The  $\boxplus$ -semi group associated with  $E$  is the family  $(e_{\boxplus}^{tE})_{t \geq 0}$  of elements of  $\mathfrak{E}[A]$  such that for any  $t_0 \geq 0$ :

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} e_{\boxplus}^{tE} &= E \boxplus e_{\boxplus}^{t_0E}, \\ e_{\boxplus}^{0E} &= 0_{\mathfrak{E}}. \end{aligned}$$

Due to the commutativity of  $\boxplus$ , one has that for any  $E, F \in \mathfrak{e}[A]$ ,  $e_{\boxplus}^E \boxplus e_{\boxplus}^F = e_{\boxplus}^{E \boxplus F}$ . Let us define the exponentiation associated with the operation  $\boxtimes$ .

**Definition 10.8.** — Let  $E \in \mathfrak{e}[A]$ . The  $\boxtimes$ -semi group associated with  $E$  is the family  $(e_{\boxtimes}^{tE})_{t \geq 0}$  of elements of  $\mathfrak{E}[A]$  such that for any  $t_0 \geq 0$ :

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} e_{\boxtimes}^{tE} &= E \boxtimes e_{\boxtimes}^{t_0 E}, \\ e_{\boxtimes}^{0E} &= 1_{\mathfrak{e}}. \end{aligned}$$

We defined  $e_{\boxplus}^{tE}$  and  $e_{\boxtimes}^{tE}$  as a one-parameter semi-group for two reasons: it will appear later in this formulation, and it allows to have a Lie group/Lie algebra formalism. An equivalent definition is given by the next proposition.

**Proposition 10.2.** — Let  $E \in \mathfrak{e}[A]$ . For any  $t \in \mathbb{R}^+$ ,

$$e_{\boxplus}^{tE} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{\boxplus n} \quad \text{and} \quad e_{\boxtimes}^{tE} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E^{\boxtimes n},$$

where  $E^{\boxplus 0} = 0_{\mathfrak{e}}$  and  $E^{\boxtimes 0} = 1_{\mathfrak{e}}$ .

Actually, we will use implicitly this fact when we will have to compute a element of the form  $e_{\boxplus}^{tE}$  in the article [9]. Besides, if one wants to make everything explicit, for example this implies that for any  $t \in \mathbb{R}^+$ , any positive integer  $k$ , any  $p \in A_k$  and any  $E \in \mathfrak{e}[A]$ ,

$$(e_{\boxtimes}^{tE})_p = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{(p_1, \dots, p_n) \in A_k, p_1 \prec p_1 \circ p_2 \prec \dots \prec p_1 \circ \dots \circ p_n / p_1 \circ \dots \circ p_n = p} E_{p_1} E_{p_2} \dots E_{p_n}.$$

The next theorem shows that  $\mathfrak{me}_{\boxplus}[A]$  and  $\mathfrak{me}_{\boxtimes}[A]$  are the Lie algebras of respectively  $(\mathfrak{ME}[A], \boxplus)$  and  $(\mathfrak{ME}[A] \cap G\mathfrak{E}_A, \boxtimes)$ .

**Theorem 10.2.** — Let  $E \in \mathfrak{me}_{\boxplus}[A]$ . For any  $t \geq 0$ ,

$$e_{\boxplus}^{tE} \in \mathfrak{ME}[A].$$

Besides for any differentiable one-parameter semi-group  $(E^t)_{t \geq 0}$  in  $(\mathfrak{ME}[A], \boxplus)$  such that  $E^0 = 0_{\mathfrak{e}}$ , there exists  $E \in \mathfrak{me}_{\boxplus}[A]$  such that for any  $t \geq 0$ ,

$$e_{\boxplus}^{tE} = E^t.$$

Let  $E \in \mathfrak{me}_{\boxtimes}[A]$ . For any  $t \geq 0$ ,

$$e_{\boxtimes}^{tE} \in \mathfrak{ME}[A].$$

Besides for any differentiable one-parameter semi-group  $(E^t)_{t \geq 0}$  in  $(\mathfrak{ME}[A], \boxtimes)$  such that  $E^0 = 1_{\mathfrak{e}}$ , there exists  $E \in \mathfrak{me}_{\boxtimes}[A]$  such that for any  $t \geq 0$ ,

$$e_{\boxtimes}^{tE} = E^t.$$

*Proof.* — Let  $(E^t)_{t \geq 0}$  be a differentiable family of elements of  $\mathfrak{E}[A]$  such that  $E^0 = 0_{\mathfrak{e}}$  or  $E^0 = 1_{\mathfrak{e}}$ . In order to prove that  $E^t \in \mathfrak{ME}[A]$  for any real  $t \geq 0$ , it is enough to show that  $(E^t)_{t \geq 0}$  and  $\left( \mathfrak{M} \left( E^t_{|\mathfrak{e}^{(i)}[A]} \right) \right)_{t \geq 0}$  satisfy the same differential linear equations.

Besides, let  $(E^t)_{t \geq 0}$  be a differentiable one-parameter semi-group for the  $\boxplus$  operation (resp.  $\boxtimes$  operation), which is in  $\mathfrak{ME}[A]$  and which starts at  $0_{\mathfrak{e}}$  (resp.  $1_{\mathfrak{e}}$ ). In order to

prove that there exists  $E \in \mathfrak{m}\mathfrak{E}_{\boxplus}[A]$  (resp.  $E \in \mathfrak{m}\mathfrak{E}_{\boxtimes}[A]$ ) such that for any  $t \geq 0$ ,  $e_{\boxplus}^{tE} = E^t$  (resp.  $e_{\boxtimes}^{tE} = E^t$ ), it is enough to show that:

$$\left( (E^t)_{t \geq 0} \text{ and } \left( e_{\boxplus}^{t\mathfrak{J}\left(\left(\frac{d}{dt}\Big|_{t=0} E_t\right)_{|\mathfrak{E}^{(i)}[A]}\right)} \right)_{t \geq 0} \right. \\ \left. \left( \text{resp. } (E^t)_{t \geq 0} \text{ and } \left( e_{\boxtimes}^{t\mathfrak{J}\left(\left(\frac{d}{dt}\Big|_{t=0} E_t\right)_{|\mathfrak{E}^{(i)}[A]}\right)} \right)_{t \geq 0} \right) \right)$$

satisfy the same differential linear equations.

Let  $E \in \mathfrak{m}\mathfrak{E}_{\boxplus}[A]$ . For any  $t \geq 0$  we consider  $E^t = e_{\boxplus}^{tE}$ . Let  $n$  be an integer and let us consider  $n$  irreducible partitions  $p_1, \dots, p_n$  in  $\cup_{k \in \mathbb{N}} A_k^{(i)}$ . For any real  $t_0 \geq 0$ , we have:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_0} E_{p_1 \otimes \dots \otimes p_n}^t &= (E \boxplus E^{t_0})_{p_1 \otimes \dots \otimes p_n} \\ &= \sum_{(p'_1, p'_2, I) \in \mathfrak{F}_2(p_1 \otimes \dots \otimes p_n)} E(p'_1) E^{t_0}(p'_2). \end{aligned}$$

Yet, we must not forget that  $E$  is in  $\mathfrak{m}\mathfrak{E}_{\boxplus}[A]$ : for any integer  $k$ , any  $p \in \mathcal{P}_k$ , if  $p$  is not irreducible or if  $p = \emptyset$ , then  $E(p) = 0$ . Thus the sum we are considering can be taken over the  $(p'_1, p'_2, I) \in \mathfrak{F}_2(p_1 \otimes \dots \otimes p_n)$  such that  $p'_1$  is irreducible and not equal to  $\emptyset$ : this means in particular that  $p'_{1S(p'_1)}$  is one of the  $(p_i)_{i=1}^n$ . Thus:

$$\frac{d}{dt}\Big|_{t=t_0} E_{p_1 \otimes \dots \otimes p_n}^t = \sum_{i=1}^n E(p_i) E_{p_1 \otimes \dots \otimes p_{i-1} \otimes p_{i+1} \otimes \dots \otimes p_n}^{t_0}.$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=t_0} (E_{p_1}^t \dots E_{p_n}^t) &= \sum_{i=1}^n \left( \frac{d}{dt}\Big|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \sum_{i=1}^n E(p_i) \prod_{j \neq i} E_{p_j}^{t_0}. \end{aligned}$$

This allows to conclude that  $E^t \in \mathfrak{M}\mathfrak{E}[A]$  for any real  $t \geq 0$ .

Let  $(E^t)_{t \geq 0}$  be a differentiable one-parameter semi-group for the  $\boxplus$  operation which is in  $\mathfrak{M}\mathfrak{E}[A]$  and such that  $E^0 = 0_{\mathfrak{E}}$ . Then using the same calculation that we did, for any integer  $n$  and any irreducible partitions  $p_1, \dots, p_n$  in  $\cup_{k \in \mathbb{N}} A_k^{(i)}$ , for any real  $t_0 \geq 0$ , we have:

$$\frac{d}{dt}\Big|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) = \frac{d}{dt}\Big|_{t=t_0} (E_{p_1}^t \dots E_{p_n}^t) = \sum_{i=1}^n \left( \frac{d}{dt}\Big|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0}.$$

Yet  $p_i$  is irreducible, thus  $\frac{d}{dt}|_{t=t_0} E_{p_i}^t = \left( \left( \frac{d}{dt}|_{t=0} E^t \right) \boxplus E^{t_0} \right)_{p_i} = \frac{d}{dt}|_{t=0} E_{p_i}^t$ , and thus:

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) &= \sum_{i=1}^n \left( \frac{d}{dt}|_{t=0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \left( \mathfrak{J} \left( \left( \frac{d}{dt}|_{t=0} E^t \right)_{|\mathfrak{E}^{(i)}[A]} \right) \boxplus E^{t_0} \right)_{p_1 \otimes \dots \otimes p_n} \end{aligned}$$

and thus there exists  $E \in \mathfrak{m}_{\boxplus}[A]$  such that for any  $t \geq 0$ ,  $e_{\boxplus}^{tE} = E^t$ .

Now, let  $E \in \mathfrak{m}_{\boxtimes}[A]$ . For any  $t \geq 0$  we consider  $E^t = e_{\boxtimes}^{tE}$ . Let  $n$  be an integer and let us consider  $n$  irreducible partitions  $p_1, \dots, p_n$  in  $\cup_{k \in \mathbb{N}} A_k^{(i)}$ . For any real  $t_0 \geq 0$ , we have:

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} E_{p_1 \otimes \dots \otimes p_n}^t &= (E \boxtimes E^{t_0})_{p_1 \otimes \dots \otimes p_n} \\ &= \sum_{a, b / a \circ b = p_1 \otimes \dots \otimes p_n, a \prec p_1 \otimes \dots \otimes p_n} E_a E_b^{t_0}. \end{aligned}$$

Yet, we must not forget that  $E$  is in  $\mathfrak{m}_{\boxtimes}[A]$ : for any integer  $k$ , any  $p \in \mathcal{P}_k$ , if  $p$  is not weakly irreducible then  $E(p) = 0$ . Thus the sum we are considering can be taken over the  $(a, b) \in \mathfrak{F}_2(p_1 \otimes \dots \otimes p_n)$  such that  $a$  is weakly irreducible. Besides,  $E_{id_l} = lE_{id_1}$  for any integer  $l$ . Thus:

$$E_{id_{\sum_{i=1}^n l(p_i)}} E_{p_1 \otimes \dots \otimes p_n}^{t_0} = \sum_{i=1}^n E_{id_{l(p_i)}} E_{p_1 \otimes \dots \otimes p_n}^{t_0}.$$

Thus, we get:

$$\frac{d}{dt}|_{t=t_0} E_{p_1 \otimes \dots \otimes p_n}^t = \sum_{i=1}^n \sum_{a, b / a \circ b = p_i, a \prec p_i} E_a E_{p_1 \otimes \dots \otimes p_{i-1} \otimes b \otimes p_{i+1} \otimes \dots \otimes p_n}^{t_0}.$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1}^t \dots E_{p_n}^t) &= \sum_{i=1}^n \left( \frac{d}{dt}|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \sum_{i=1}^n \sum_{a, b / a \circ b = p_i, a \prec p_i} E_a E_b^{t_0} \prod_{j \neq i} E_{p_j}^{t_0}. \end{aligned}$$

This allows to conclude that  $E^t \in \mathfrak{M}\mathfrak{E}[A]$  for any real  $t \geq 0$ .

Let  $(E^t)_{t \geq 0}$  be a differentiable one-parameter semi-group for the  $\boxtimes$  operation which is in  $\mathfrak{M}\mathfrak{E}[A]$  and such that  $E^0 = 1_{\mathfrak{E}}$ . Then using the same calculation that we did, for any integer  $n$  and any irreducible partitions  $p_1, \dots, p_n$  in  $\cup_{k \in \mathbb{N}} A_k^{(i)}$ , for any real  $t_0 \geq 0$ , we have:

$$\frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) = \frac{d}{dt}|_{t=t_0} (E_{p_1}^t \dots E_{p_n}^t) = \sum_{i=1}^n \left( \frac{d}{dt}|_{t=t_0} E_{p_i}^t \right) \prod_{j \neq i} E_{p_j}^{t_0}.$$

Yet  $p_i$  is irreducible, thus  $\frac{d}{dt}|_{t=t_0} E_{p_i}^t = \sum_{a,b/aob=p_i, a \prec p_i} \frac{d}{dt}|_{t=0} E_{p_i}^t E_b^{t_0}$ , and thus:

$$\begin{aligned} \frac{d}{dt}|_{t=t_0} (E_{p_1 \otimes \dots \otimes p_n}^t) &= \sum_{i=1}^n \left( \sum_{a,b/aob=p_i, a \prec p_i} \frac{d}{dt}|_{t=0} E_{p_i}^t E_b^{t_0} \right) \prod_{j \neq i} E_{p_j}^{t_0} \\ &= \left( \mathfrak{J} \left( \left( \frac{d}{dt}|_{t=0} E_t \right)_{|\mathfrak{e}^{(i)}[A]} \right) \boxtimes E^{t_0} \right)_{p_1 \otimes \dots \otimes p_n} \end{aligned}$$

and thus there exists  $E \in \mathfrak{me}_{\boxtimes}[A]$  such that for any  $t \geq 0$ ,  $e_{\boxtimes}^{tE} = E^t$ .  $\square$

**Remark 10.2.** — In fact,  $\mathfrak{e}[A]$  is endowed with two structures of Lie algebras. Indeed, it is a vector space for the addition and multiplication by a scalar, and we can define two Lie brackets on it, one named  $[\cdot, \cdot]_{\boxplus}$  which comes from the  $\boxplus$  operation and the other named  $[\cdot, \cdot]_{\boxtimes}$  which comes from the  $\boxtimes$  operation. In order to know which bracket is considered on  $\mathfrak{e}[A]$ , we will denote it either by  $\mathfrak{e}_{\boxplus}[A]$  or by  $\mathfrak{e}_{\boxtimes}[A]$ .

Since the operation  $\boxplus$  is commutative, the bracket  $[\cdot, \cdot]_{\boxplus}$  is trivial. Thus  $\mathfrak{me}_{\boxplus}$  is a sub-Lie algebra of  $\mathfrak{e}_{\boxplus}$ .

Since the operation  $\boxtimes$  is not commutative, the bracket  $[\cdot, \cdot]_{\boxtimes}$  is not trivial and for any  $E$  and  $F$  in  $\mathfrak{e}_{\boxtimes}[A]$ ,

$$[E, F]_{\boxtimes} = E \boxtimes F - F \boxtimes E.$$

Then, it is not difficult to see directly that  $\mathfrak{me}_{\boxtimes}[A]$  is a sub-Lie algebra of  $\mathfrak{e}_{\boxtimes}[A]$ .

**10.1.2. The  $\mathcal{R}_A$ -transform.** — We will define the notion of  $\mathcal{R}_A$ -transform. This application will be defined as the inverse of the  $\mathcal{M}_A$ -transform whose definition lies on the Equation (5).

**Definition 10.9.** — The  $\mathcal{M}_A$ -transform is the application:

$$\begin{aligned} \mathcal{M}_A : \mathfrak{E}[A] &\rightarrow \mathfrak{E}[A] \\ E &\rightarrow \mathcal{M}_A(E) \end{aligned}$$

such that for any  $E \in \mathfrak{E}[A]$ , for any integer  $k$ , any  $p \in A_k$ :

$$(\mathcal{M}_A(E))_p = \sum_{p' \in [id, p]_{A_k}} E_{p'}.$$

This application is a bijection. Thus we can consider its inverse.

**Definition 10.10.** — The  $\mathcal{R}_A$ -transform is the inverse of the  $\mathcal{M}_A$ -transform:

$$\mathcal{R}_A = \mathcal{M}_A^{-1}.$$

We will often forget about the indices  $A$  when we will work with the  $\mathcal{R}$ -transforms. One can show that the  $\mathcal{R}_A$ -transform is a bijection from  $\mathfrak{ME}[A]$  to itself.

**Proposition 10.3.** — The  $\mathcal{R}_A$ -transform is a bijection from  $\mathfrak{ME}[A]$  to itself.

*Proof.* — We recall that the  $\mathcal{R}_A$ -transform is, by definition, a bijection from  $\mathfrak{E}[A]$  to itself. Let  $E \in \mathfrak{ME}[A]$ , we have to show that  $\mathcal{M}_A[E] \in \mathfrak{ME}[A]$  and  $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$ . Let  $k$  and  $l$  be any integers. Let  $p_1 \in \mathcal{P}_k$  and  $p_2 \in \mathcal{P}_l$ .

Let us show that  $\mathcal{M}_A[E] \in \mathfrak{ME}[A]$ . Using Lemma 3.4 and the multiplicative property of  $E$ , we have:

$$\begin{aligned} (\mathcal{M}_A[E])_{p_1 \otimes p_2} &= \sum_{p' \in [id, p_1 \otimes p_2]_{A_k}} E_{p'} = \sum_{p'_1 \in [id, p_1]_{A_k}, p'_2 \in [id, p_2]_{A_k}} E_{p'_1 \otimes p'_2} \\ &= \sum_{p'_1 \in [id, p_1]_{A_k}, p'_2 \in [id, p_2]_{A_k}} E_{p'_1} E_{p'_2} \\ &= \sum_{p'_1 \in [id, p_1]_{A_k}} E_{p'_1} \sum_{p'_2 \in [id, p_2]_{A_k}} E_{p'_2} \\ &= (\mathcal{M}_A[E])_{p_1} (\mathcal{M}_A[E])_{p_2}. \end{aligned}$$

Now, let us show that  $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$ . Let us consider  $(\tilde{E}_p)_{p \in \bigcup_{k \in \mathbb{N}} A_k^{(i)}}$  such that for any integer  $k$ , any  $p \in A_k^{(i)}$ ,

$$E_p = \sum_{p' \in [id, p]_{A_k}} \prod_{c \in \mathbb{C}(p')} \tilde{E}_{p_c}.$$

Using the multiplicativity of  $E$ , and Lemma 3.4, we see that  $E$  being in  $\mathfrak{ME}[A]$ , the family  $(\tilde{E}_p)_{p \in \bigcup_{k \in \mathbb{N}} A_k^{(i)}}$  satisfies in fact that for any integer  $k$ , any  $p \in A_k$ :

$$E_p = \sum_{p' \in [id, p]_{A_k}} \prod_{c \in \mathbb{C}(p')} \tilde{E}_{p_c}.$$

Thus  $\prod_{c \in \mathbb{C}(p')} \tilde{E}_{p_c}$  is equal to  $(\mathcal{R}_A[E])_{p'}$  and thus  $\mathcal{R}_A[E] \in \mathfrak{ME}[A]$ .  $\square$

We can also translate the Lemma 3.3 in terms of  $\mathcal{R}$ -transform. For this, we need the restriction function defined for any integer  $k$  by:

$$\begin{aligned} r : \quad \mathbb{C}[\mathcal{B}_k] &\rightarrow \mathbb{C}[\mathfrak{S}_k] \\ \sum_{b \in \mathcal{B}_k} f(b)b &\mapsto \sum_{\sigma \in \mathfrak{S}_k} f(\sigma)\sigma. \end{aligned}$$

**Proposition 10.4.** — *The following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{B}] & \xrightarrow{\mathcal{R}_{\mathcal{B}}} & \mathfrak{E}[\mathcal{B}] \\ r \downarrow & & \downarrow r \\ \mathfrak{E}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

*Proof.* — It is only a consequence of the fact that:

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{B}] & \xleftarrow{\mathcal{M}_{\mathcal{B}}} & \mathfrak{E}[\mathcal{B}] \\ r \downarrow & & \downarrow r \\ \mathfrak{E}[\mathfrak{S}] & \xleftarrow{\mathcal{M}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

is commutative. Indeed, using Lemma 3.3, if  $E \in \mathfrak{E}[\mathcal{B}]$ , and if  $\sigma \in \mathfrak{S}_k$ :

$$r[\mathcal{M}_{\mathcal{B}}(E)](\sigma) = (\mathcal{M}_{\mathcal{B}}(E))(\sigma) = \sum_{p \in [id, \sigma]_{\mathcal{B}_k}} E_p = \sum_{p \in [id, \sigma]_{\mathfrak{S}_k}} E_p = [\mathcal{M}_{\mathfrak{S}} \circ r(E)](\sigma).$$

This concludes the proof.  $\square$

It is well-known in the literature that there exists a transformation on  $\mathbb{C}_1[[z]]$  which we will call the  $\mathcal{R}_u$ -transform. In order to finish this section, we make the link between our  $\mathcal{R}_A$ -transform and the  $\mathcal{R}_u$ -transform.

**Definition 10.11.** — *Let  $M(z)$  be a formal power serie in  $\mathbb{C}_1[[z]]$ , that is a formal power serie of the form:*

$$M(z) = 1 + \sum_{n=1}^{\infty} a_n z^n.$$

*Let  $C(z)$  be the formal power serie  $C(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$  such that  $C[zM(z)] = M(z)$ . The  $\mathcal{R}_u$ -transform of  $M$  is  $C$ .*

The  $\mathcal{R}_A$ -transform is a generalization of the usual  $\mathcal{R}_u$ -transform. Indeed, we have the following theorem.

**Theorem 10.3.** — *Using the identification  $\mathfrak{E}^{(i)}[\mathfrak{S}] \simeq \mathbb{C}_1[[z]]$  explained in Proposition 10.1, the following diagram is commutative:*

$$\begin{array}{ccc} \mathfrak{E}^{(i)}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_u} & \mathfrak{E}^{(i)}[\mathfrak{S}] \\ \mathfrak{M} \downarrow & & \downarrow \mathfrak{M} \\ \mathfrak{E}[\mathfrak{S}] & \xrightarrow{\mathcal{R}_{\mathfrak{S}}} & \mathfrak{E}[\mathfrak{S}] \end{array}$$

*Proof.* — Let  $E$  be an element of  $\mathfrak{E}^{(i)}[\mathfrak{S}] \simeq \mathbb{C}_1[[z]]$ . Using Theorem 2.7 of [16], and using the bijection between non-crossing partitions of  $k$  elements and the set  $[id, (1, \dots, k)]_{\mathfrak{S}_k}$ , we know that  $\mathcal{R}_u(E)$  is characterized by the fact that for any integer  $k > 0$ :

$$E_{(1, \dots, k)} = \sum_{p \in [id, (1, \dots, k)]_{\mathfrak{S}_k}} \prod_{c \text{ cycle of } p} \mathcal{R}_u(E)(1, \dots, \#c).$$

Or, with our notations:

$$E_{(1, \dots, k)} = \sum_{p \in [id, (1, \dots, k)]_{\mathfrak{S}_k}} \mathfrak{M}[\mathcal{R}_u(E)].$$

By the factorization property of the geodesics, Lemma 3.4, for any  $\sigma \in \mathfrak{S}_k$ :

$$[\mathfrak{M}(E)](\sigma) = \sum_{\sigma' \in [id, \sigma]_{\mathfrak{S}_k}} \mathfrak{M}[\mathcal{R}_u(E)](\sigma).$$

This is equivalent to the fact that  $\mathcal{R}_{\mathfrak{S}}[\mathfrak{M}(E)] = \mathfrak{M}[\mathcal{R}_u(E)]$ .  $\square$

10.1.3. *Transformations linked with the exclusive moments.* — We only consider the case where  $A = \mathcal{P}$ .

**Definition 10.12.** — *The  $\mathcal{M}^{c \rightarrow}$ -transform is the application:*

$$\begin{aligned} \mathcal{M}^{c \rightarrow} : \mathfrak{E}[\mathcal{P}] &\rightarrow \mathfrak{E}[\mathcal{P}] \\ E &\mapsto \mathcal{M}^{c \rightarrow}(E), \end{aligned}$$

such that for any  $E \in \mathfrak{E}[\mathcal{P}]$ , for any integer  $k$ , any  $p \in \mathcal{P}_k$ :

$$(\mathcal{M}^{c \rightarrow}(E))_p = \sum_{p' \text{ coarser-compatible than } p} E_{p'}.$$

This application is a bijection, it is the application which transforms exclusive moments in moments. Thus we can consider its inverse. The  $\mathcal{M}^c$ -transform is the inverse of the  $\mathcal{M}^{c \rightarrow}$ -transform:  $\mathcal{M}^c = (\mathcal{M}^{c \rightarrow})^{-1}$ . Using the same arguments than Proposition 10.3, one can proof that the  $\mathcal{M}^c$  is a bijection from  $\mathfrak{M}\mathfrak{E}[\mathcal{P}]$  to itself.

Let us remark that this last proposition holds since, if  $p'$  is coarser-compatible than  $p_1 \otimes p_2$  this means that there exists  $p'_1$  and  $p'_2$  such that  $p' = p'_1 \otimes p'_2$  and such that  $p'_1$  (resp.  $p'_2$ ) is coarser-compatible than  $p_1$  (resp.  $p_2$ ). Thus, if one has defined  $\mathcal{M}^{c \rightarrow}(E)$  by replacing the coarser-compatibility order by the coarser order then this proposition (and other good properties) would not have hold.

Let us define a last transformation on  $\mathfrak{E}[\mathcal{P}]$ .

**Definition 10.13.** — *The  $\mathcal{M}^{\rightarrow c}$ -transform is the application:*

$$\begin{aligned} \mathcal{M}^{\rightarrow c} : \mathfrak{E}[\mathcal{P}] &\rightarrow \mathfrak{E}[\mathcal{P}] \\ E &\mapsto \mathcal{M}^{\rightarrow c}(E), \end{aligned}$$

such that for any  $E \in \mathfrak{E}[\mathcal{P}]$ , for any integer  $k$ , any  $p \in \mathcal{P}_k$ :

$$(\mathcal{M}^{\rightarrow c}(E))_p = \sum_{p' \text{ finer than } p, p' \in [id, p]_{\mathcal{P}_k}} E_{p'}$$

This is again a bijection. The applications defined above are actually linked.

**Theorem 10.4.** — *The following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{E}[\mathcal{P}] & \xrightarrow{\mathcal{M}^{\rightarrow c}} & \mathfrak{E}[\mathcal{P}] \\ & \searrow \mathcal{M}_{\mathcal{P}} & \downarrow \mathcal{M}^{c \rightarrow} \\ & & \mathfrak{E}[\mathcal{P}] \end{array}$$

*Proof.* — This is a straightforward application of Proposition 8.2.  $\square$



**10.2. Higher order.** — In Definition 9.5, we defined the  $\infty$ -development algebra of order  $m$  of  $A_k$ . Thus, one can also define a higher order  $\mathcal{R}$ -transform: we will only give definitions in this section. Let  $m \in \mathbb{N}$  be the higher order of fluctuations which we are working with.

**Definition 10.14.** — *Let us define the algebra:*

$$\mathfrak{E}_{g,(m)}[A] = \prod_{k=0}^{\infty} \mathbb{C}_{(m)}^{\mathfrak{S}}[A_k(\infty)]$$

where, for any integer  $k$ ,  $\mathbb{C}_{(m)}^{\mathfrak{S}}[A_k(\infty)]$  is the algebra of elements of  $\mathbb{C}_{(m)}[A_k(\infty)]$  which, seen as elements of  $\mathbb{C}[A_k]$ , are invariant by conjugation by any element of  $\mathfrak{S}_k$ . We also consider the subspace of  $\mathfrak{E}_{g,(m)}[A]$  defined by:

$$\mathfrak{E}_{(m)}[A] = \{E \in \mathfrak{E}_{g,(m)}[A], E_{\emptyset,0} = 1, E_{\emptyset,i} = 0, \forall i \geq 1\}.$$

Let us remark that  $\mathfrak{E}_{(0)}[A] = \mathfrak{E}[A]$ . Any element  $E \in \mathfrak{E}[A]$  is of the form:

$$\left( \sum_{p \in A_k, i \in \{0, \dots, m\}} (E_k)_{p,i} \frac{p}{X^i} \right)_{k \in \mathbb{N}}.$$

Again, in order to simplify the notations, we will use the following convention: for any  $p \in \cup_{k=0}^{\infty} A_k$  and any  $i \in \{0, \dots, m\}$ :

$$E_{p,i} = E^i(p) = (E_{l(p)})_{p,i},$$

and for any integer  $k$ :

$$E_k = \sum_{p \in A_k, i \in \{0, \dots, m\}} E_{p,i} \frac{p}{X^i}.$$

As for  $\mathfrak{E}_g[A]$ , the algebra  $\mathfrak{E}_{g,(m)}[A]$  is naturally endowed with a natural addition and multiplication given, for any  $E, F \in \mathfrak{E}_{g,(m)}[A]$ , by:

$$\begin{aligned} (E + F)_k &= E_k + F_k, \\ (E \boxtimes F)_k &= E_k F_k. \end{aligned}$$

Besides, we can also construct an other law on  $\mathfrak{E}_{g,(m)}[A]$ .

**Definition 10.15.** — *Let  $E$  and  $F$  be two elements of  $\mathfrak{E}_{g,(m)}[A]$ . We denote by  $E \boxplus F$  the element of  $\mathfrak{E}_{g,(m)}[A]$  such that for any positive integer  $k$ , any  $p \in A_k$  and any  $i \in \{0, \dots, m\}$ :*

$$(E \boxplus F)^i(p) = \sum_{(p_1, p_2, l) \in \mathfrak{F}_2(p)} \sum_{i_1=1}^i E^{i_1}(p_1) F^{i-i_1}(p_2),$$

where  $\mathfrak{F}_2(p)$  was defined in Definition 2.10.

Again, the subset  $\mathfrak{E}_{(m)}[A]$  is stable by the  $\boxtimes$  and  $\boxplus$  operations. Besides,  $\mathfrak{E}_{(m)}[A]$  is an affine space.

The operation  $\boxplus$  is commutative, it defines a structure of group on  $\mathfrak{E}_{(m)}[A]$ . The neutral element is the element  $0_{\mathfrak{E}_{(m)}} \in \mathfrak{E}_{(m)}[A]$  such that, for any positive integer  $k$ , any  $p \in A_k$  and any  $i \in \{0, \dots, m\}$ ,  $(E)_{p,i} = 0$ .

The operation  $\boxtimes$  is not commutative and the set of invertible elements in  $\mathfrak{E}_{(m)}[A]$  is the set of elements  $E$  such that  $E_{id_k,0} \neq 0$  for any  $k \geq 1$ . We denote by  $1_{\mathfrak{E}_{(m)}}$  the neutral element for  $\boxtimes$  which is the only element in  $\mathfrak{E}_{(m)}[A]$  such that for any  $k \geq 1$ ,  $(E_k)_k = \frac{id_k}{X^0}$ .

We can also define a  $\mathcal{R}_A^{(m)}$ -transform. For this, we need to define the  $\mathcal{M}_A^{(m)}$ -transform whose definition lies on Equality (18).

**Definition 10.16.** — *The general  $\mathcal{M}_A^{(m)}$  transform is the application:*

$$\begin{aligned} \mathcal{M}_A^{(m)} : \mathfrak{E}_{(m)}[A] &\rightarrow \mathfrak{E}_{(m)}[A] \\ E &\rightarrow \mathcal{M}_A^{(m)}(E) \end{aligned}$$

such that for any  $E \in \mathfrak{E}_{(m)}[A]$ , for any positive integer  $k$ , any  $p \in A_k$  and any  $i \in \{0, \dots, m\}$ :

$$\left(\mathcal{M}_A^{(m)}(E)\right)_{p,i} = \sum_{p' \in A_k, \text{df}(p',p) \leq i} E_{p',i-\text{df}(p',p)}.$$

This application is a bijection: we can consider its inverse.

**Definition 10.17.** — *The  $\mathcal{R}_A^{(m)}$ -transform is the inverse of the  $\mathcal{M}_A$ -transform:*

$$\mathcal{R}_A^{(m)} = \left(\mathcal{M}_A^{(m)}\right)^{-1}.$$

## 11. Conclusion

We have defined a geometry on partitions, and new notions of convergence for elements of  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$ . Using Schur-Weyl's duality and similar results, we will link the study of random matrices with the study of elements in  $\prod_{N \in \mathbb{N}} \mathbb{C}[A_k(N)]$  and in  $\mathfrak{E}[A]$ . In the article [9], we apply the results proved in this article to the theory of random matrices invariant in law by conjugation by the symmetric group. We also study additive and multiplicative unitary or orthogonal invariant Lévy processes. In the article [10], we apply the results of the first two articles to the study of random walks on the symmetric group and the study of the  $\mathfrak{S}_\infty$ -Yang-Mills theory.

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