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HOLDERIAN WEAK INVARIANCE PRINCIPLE UNDER A HANNAN TYPE CONDITION

DAVIDE GIRAUDO

ABSTRACT. We investigate the invariance principle in Hölder spaces for strictly stationary martingale difference sequences. In particular, we show that the sufficient condition on the tail in the i.i.d. case does not extend to stationary ergodic martingale differences. We provide a sufficient condition on the conditional variance which guarantee the invariance principle in Hölder spaces. We then deduce a condition in the spirit of Hannan one.

1. INTRODUCTION

One of the main problems in probability theory is the understanding of the asymptotic behavior of Birkhoff sums $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, where $(\Omega, \mathcal{F}, \mu, T)$ is a dynamical system and f a map from Ω to the real line.

One can consider random functions constructed from the Birkhoff sums

$$(1.1) \quad S_n^{\text{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]+1}, \quad t \in [0, 1].$$

and investigate the asymptotic behaviour of the sequence $(S_n^{\text{pl}}(f, t))_{n \geq 1}$ seen as an element of a function space. Donsker showed (cf. [Don51]) that the sequence $(n^{-1/2}(\mathbb{E}(f^2))^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion W when the sequence $(f \circ T^i)_{i \geq 0}$ is i.i.d. and zero mean. Then an intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [MPU06] for the main theorems in this direction.

Our purpose is to investigate the weak convergence of the sequence $(n^{-1/2}S_n^{\text{pl}}(f))_{n \geq 1}$ in Hölder spaces when $(f \circ T^i)_{i \geq 0}$ is a strictly stationary sequence. A classical method for showing a limit theorem is to use a martingale approximation, which allows to deduce the corresponding result if it holds for martingale difference sequences provided that the approximation is good enough. To the best of our knowledge, no result about the invariance principle in Hölder space for stationary martingale difference sequences is known.

1.1. The Hölder spaces. It is well known that standard Brownian motion's paths are almost surely Hölder regular of exponent α for each $\alpha \in (0, 1/2)$, hence it is natural to consider the random function defined in (1.1) as an element of $\mathcal{H}_\alpha[0, 1]$ and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for $\alpha \in (0, 1)$ the Hölder space $\mathcal{H}_\alpha[0, 1]$ of functions $x: [0, 1] \rightarrow \mathbb{R}$ such that $\sup_{s \neq t} |x(s) - x(t)| / |s - t|^\alpha$ is

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finite. The analogue of the continuity modulus in $C[0, 1]$ is w_α , defined by

$$(1.2) \quad w_\alpha(x, \delta) = \sup_{0 < |t-s| < \delta} \frac{|x(t) - x(s)|}{|t - s|^\alpha}.$$

We then define $\mathcal{H}_\alpha^0[0, 1]$ by $\mathcal{H}_\alpha^0[0, 1] := \{x \in \mathcal{H}_\alpha[0, 1], \lim_{\delta \rightarrow 0} w_\alpha(x, \delta) = 0\}$. We shall essentially work with the space $\mathcal{H}_\alpha^o[0, 1]$ which, endowed with $\|x\|_\alpha := w_\alpha(x, 1) + |x(0)|$, is a separable Banach space (while $\mathcal{H}_\alpha[0, 1]$ is not separable). Since the canonical embedding $\iota: \mathcal{H}_\alpha^o[0, 1] \rightarrow \mathcal{H}_\alpha[0, 1]$ is continuous, each convergence in distribution in $\mathcal{H}_\alpha^o[0, 1]$ also takes place in $\mathcal{H}_\alpha[0, 1]$.

Let us denote by D_j the set of dyadic numbers in $[0, 1]$ of level j , that is,

$$(1.3) \quad D_0 := \{0, 1\}, \quad D_j := \{(2l - 1)2^{-j}; 1 \leq l \leq 2^{j-1}\}, j \geq 1.$$

If $r \in D_j$ for some $j \geq 0$, we define $r^+ := r + 2^{-j}$ and $r^- := r - 2^{-j}$. For $r \in D_j$, $j \geq 1$, let Λ_r be the function whose graph is the polygonal path joining the points $(0, 0)$, $(r^-, 0)$, $(r, 1)$, $(r^+, 0)$ and $(1, 0)$. We can decompose each $x \in C[0, 1]$ as

$$(1.4) \quad x = \sum_{r \in D} \lambda_r(x) \Lambda_r = \sum_{j=0}^{+\infty} \sum_{r \in D_j} \lambda_r(x) \Lambda_r,$$

and the convergence is uniform on $[0, 1]$. The coefficients $\lambda_r(x)$ are given by

$$(1.5) \quad \lambda_r(x) = x(r) - \frac{x(r^+) + x(r^-)}{2}, \quad r \in D_j, j \geq 1,$$

and $\lambda_0(x) = x(0)$, $\lambda_1(x) = x(1)$.

Ciesielski proved (cf. [Cie60]) that $\{\Lambda_r; r \in D\}$ is a Schauder basis of $\mathcal{H}_\alpha^o[0, 1]$ and the norms $\|\cdot\|_\alpha$ and the sequential norm defined by

$$(1.6) \quad \|x\|_\alpha^{\text{seq}} := \sup_{j \geq 0} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(x)|,$$

are equivalent.

Considering the sequential norm, we can show (see Theorem 3 in [Suq99]) that a sequence $(\xi_n)_{n \geq 1}$ of random elements of \mathcal{H}_α^o vanishing at 0 is tight if and only if for each positive ε ,

$$(1.7) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{j\alpha} \max_{r \in D_j} |\lambda_r(\xi_n)| > \varepsilon \right\} = 0.$$

Notation 1.1. In the sequel, we will denote $r_{k,j} := k2^{-j}$ and $u_{k,j} := [nr_{k,j}]$ (or r_k and u_k for short). Notice that $u_{k+1,j} - u_{k,j} = [nr_{k,j} + n2^{-j}] - u_{k,j} \leq 2n2^{-j}$ if $j \leq \log n$, where $\log n$ denotes the binary logarithm of n and for a real number x , $[x]$ is the unique integer for which $[x] \leq x < [x] + 1$.

Remark 1.2. Since for each $x \in \mathcal{H}_{1/2-1/p}[0, 1]$, each $j \geq 1$ and each $r \in D_j$,

$$(1.8) \quad |\lambda_r(x)| \leq \frac{|x(r^+) - x(r)|}{2} + \frac{|x(r) - x(r^-)|}{2} \leq \max \{|x(r^+) - x(r)|, |x(r) - x(r^-)|\},$$

for a function f , the sequential norm of $n^{-1/2} S_n^{\text{pl}}(f)$ does not exceed

$$(1.9) \quad \sup_{j \geq 1} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} \left| S_n^{\text{pl}}(f, r_{k+1,j}) - S_n^{\text{pl}}(f, r_{k,j}) \right|.$$

Now, we state the result obtained by Račkauskas and Suquet in [RS03].

Theorem 1.3. *Let $p > 2$ and let $(f \circ T^j)_{j \geq 0}$ be an i.i.d. centered sequence with unit variance. Then the condition*

$$(1.10) \quad \lim_{t \rightarrow \infty} t^p \mu \{|f| > t\} = 0$$

is equivalent to the weak convergence of the sequence $(n^{-1/2} S_n^{\text{pl}}(f))_{n \geq 1}$ to a standard Brownian motion in the space $\mathcal{H}_{1/2-1/p}^o[0, 1]$.

1.2. Some facts about the $\mathbb{L}^{p,\infty}$ spaces. In the rest of the paper, χ denotes the indicator function. Let $p > 2$. We define the $\mathbb{L}^{p,\infty}$ space as the collection of functions $f: \Omega \rightarrow \mathbb{R}$ such that the quantity

$$(1.11) \quad \|f\|_{p,\infty}^p := \sup_{t>0} t^p \mu \{|f| > t\} < \infty.$$

This quantity is denoted like a norm, while it is not a norm (the triangle inequality may fail, for example if $X = [0, 1]$ endowed with the Lebesgue measure, $f(x) := x^{-1/p}$ and $g(x) := f(1-x)$; in this case $\|f + g\|_{p,\infty} \geq 2^{1+1/p}$ but $\|f\|_{p,\infty} + \|g\|_{p,\infty} = 2$). However, there exists a constant κ_p such that for each f ,

$$(1.12) \quad \|f\|_{p,\infty} \leq \sup_{A:\mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \chi_A] \leq \kappa_p \|f\|_{p,\infty}$$

and $N_p(f) := \sup_{A:\mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \chi_A]$ defines a norm. The first inequality in (1.12) can be seen from the estimate $t \mu \{|f| > t\} \leq \mathbb{E}[|f| \chi \{|f| > t\}]$; for the second one, we write

$$(1.13) \quad \mathbb{E}[|f| \chi_A] = \int_0^{+\infty} \mu(\{|f| > t\} \cap A) dt \leq \int_0^{+\infty} \min\{\mu\{|f| > t\}, \mu(A)\} dt,$$

and we bound the integrand by $\min\{t^{-p} \|f\|_{p,\infty}^p, \mu(A)\}$.

A function f satisfies (1.10) if and only if it belongs to the closure of bounded functions with respect to N_p . Indeed, if f satisfies (1.10), then the sequence $(f \chi_{\{|f| < n\}})_{n \geq 1}$ converges to f in $\mathbb{L}^{p,\infty}$. If $N_p(f - g) < \varepsilon$ with g bounded, then

$$(1.14) \quad \limsup_{t \rightarrow \infty} t^p \mu \{|f| > t\} \leq \limsup_{t \rightarrow \infty} t^p \mu \{|f - g| > t/2\} \leq 2^p \varepsilon.$$

We now provide two technical lemmas about $\mathbb{L}^{p,\infty}$ spaces. The first one will be used in the proof of the weak invariance principle for martingales, since we will have to control the tail function of the random variables involved in the construction of the truncated martingale (cf.(3.111)). The second one will provides an estimation of the $\mathbb{L}^{p,\infty}$ norm of a simple function, which will be used in the proof of Theorem 2.1, since the function m is constructed as a series of simple functions.

Lemma 1.4. *If $\lim_{t \rightarrow \infty} t^p \mu \{|f| > t\} = 0$, then for each sub- σ -algebra \mathcal{A} , we have $\lim_{t \rightarrow \infty} t^p \mu \{\mathbb{E}[|f| | \mathcal{A}] > t\} = 0$.*

Proof. For simplicity, we assume that f is non-negative. For a fixed t , the set $\{\mathbb{E}[f | \mathcal{A}] > t\}$ belongs to the σ -algebra \mathcal{A} , hence

$$(1.15) \quad t \mu \{\mathbb{E}[f | \mathcal{A}] > t\} \leq \mathbb{E}[\mathbb{E}[f | \mathcal{A}] \chi \{\mathbb{E}[f | \mathcal{A}] > t\}] = \mathbb{E}[f \chi \{\mathbb{E}[f | \mathcal{A}] > t\}].$$

By definition of N_p ,

$$(1.16) \quad \mathbb{E}[f \chi \{\mathbb{E}[f | \mathcal{A}] > t\}] \leq N_p(f \chi \{\mathbb{E}[f | \mathcal{A}] > t\}) \mu \{\mathbb{E}[f | \mathcal{A}] > t\}^{1-1/p},$$

hence

$$(1.17) \quad t^p \mu \{\mathbb{E}[f | \mathcal{A}] > t\} \leq N_p(f \chi \{\mathbb{E}[f | \mathcal{A}] > t\})^p.$$

Notice that

$$(1.18) \quad \forall s > 0, \quad N_p(f\chi\{\mathbb{E}[f | \mathcal{A}] > t\}) \leq s\mu\{\mathbb{E}[f | \mathcal{A}] > t\}^{1/p} + N_p(f\chi\{f > s\}),$$

hence

$$(1.19) \quad \limsup_{t \rightarrow \infty} \mathbb{E}[f\chi\{\mathbb{E}[|f| | \mathcal{A}] > t\}] \leq N_p(f\chi\{f > s\}) \leq \kappa_p \sup_{x \geq s} x^p \mu\{f > x\}.$$

By the assumption on the function f , the right hand side goes to 0 as s goes to infinity, which concludes the proof of the lemma. \square

Lemma 1.5. *Let $f := \sum_{i=0}^N a_i \chi(A_i)$, where the family $(A_i)_{i=0}^N$ is pairwise disjoint and $0 \leq a_N < \dots < a_0$. Then*

$$(1.20) \quad \|f\|_{p,\infty}^p \leq \max_{0 \leq j \leq N} a_j^p \sum_{i=0}^j \mu(A_i).$$

Proof. We have the equality

$$(1.21) \quad \mu\{f > t\} = \sum_{j=0}^N \chi_{(a_{j+1}, a_j]}(t) \sum_{i=0}^j \mu(A_i),$$

where $a_{N+1} := 0$, therefore

$$(1.22) \quad t^p \mu\{f > t\} \leq \max_{0 \leq j \leq N} a_j^p \sum_{i=0}^j \mu(A_i).$$

\square

2. MAIN RESULTS

The goal of the paper is to give a sharp sufficient condition on the moments of a strictly stationary martingale difference sequence which guarantees the weak invariance principle in $\mathcal{H}_\alpha^o[0, 1]$ for a fixed α .

We first show that Theorem 1.3 does not extend to strictly stationary ergodic martingale difference sequences, that is, sequences of the form $(m \circ T^i)_{i \geq 0}$ such that m is \mathcal{M} measurable and $\mathbb{E}[m | T\mathcal{M}] = 0$ for some sub- σ -algebra \mathcal{M} satisfying $T\mathcal{M} \subset \mathcal{M}$.

An application of Kolmogorov's continuity criterion shows that if $(m \circ T^i)_{i \geq 0}$ is a martingale difference sequence such that $m \in \mathbb{L}^{p+\delta}$ for some positive δ and $p > 2$, then the partial sum process $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}^o[0, 1]$ (see [KR91]).

We provide a condition on the quadratic variance which improves the previous approach (since the previous condition can be replaced by $m \in \mathbb{L}^p$). Then using martingale approximation we can provide a Hannan type condition which guarantees the weak invariance principle in $\mathcal{H}_\alpha^o[0, 1]$.

Theorem 2.1. *Let $p > 2$ and $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system with positive entropy. There exists a function $m: \Omega \rightarrow \mathbb{R}$ and a σ -algebra \mathcal{M} for which $T\mathcal{M} \subset \mathcal{M}$ such that:*

- the sequence $(m \circ T^i)_{i \geq 0}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i \geq 0}$;
- the convergence $\lim_{t \rightarrow +\infty} t^p \mu\{|m| > t\} = 0$ takes place;
- the sequence $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}^o[0, 1]$.

Theorem 2.2. Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system, \mathcal{M} a sub- σ -algebra of \mathcal{F} such that $T\mathcal{M} \subset \mathcal{M}$ and \mathcal{I} the collection of sets $A \in \mathcal{F}$ such that $T^{-1} = A$.

Let $p > 2$ and let $(m \circ T^j, T^{-i}\mathcal{M})$ be a strictly stationary martingale difference sequence. Assume that $t^p \mu \{|m| > t\} \rightarrow 0$ and $\mathbb{E}[m^2 | T\mathcal{M}] \in \mathbb{L}^{p/2}$. Then

$$(2.1) \quad n^{-1/2} S_n^{\text{pl}}(m) \rightarrow \eta \cdot W \text{ in distribution in } \mathcal{H}_{1/2-1/p}^o[0, 1],$$

where the random variable η is given by

$$(2.2) \quad \eta = \lim_{n \rightarrow \infty} \mathbb{E}[S_n^2 | \mathcal{I}] / n \text{ in } \mathbb{L}^1$$

and η is independent of the process $(W_t)_{t \in [0, 1]}$.

In particular, (2.1) takes place if m belongs to \mathbb{L}^p .

The key point of the proof of Theorem 2.2 is an inequality in the spirit of Doob's one, which gives $n^{-1} \mathbb{E}[\max_{1 \leq j \leq n} S_j(m)^2] \leq 2\mathbb{E}[m^2]$. It is used in order to establish tightness of the sequence $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$ in the space $C[0, 1]$.

Proposition 2.3. Let $p > 2$. There exists a constant C_p depending only on p such that if $(m \circ T^i)_{i \geq 1}$ is a martingale difference sequence, then the following inequality holds:

$$(2.3) \quad \sup_{n \geq 1} \left\| \left\| n^{-1/2} S_n^{\text{pl}}(m) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty}^p \leq C_p \left(\|m\|_{p, \infty}^p + \mathbb{E}(\mathbb{E}[m^2 | T\mathcal{M}])^{p/2} \right).$$

Remark 2.4. As Theorem 2.1 shows, the condition $\lim_{t \rightarrow \infty} t^p \mu \{|m| > t\} = 0$ alone for martingale difference sequences is not sufficient to obtain the weak convergence of $n^{-1/2} S_n^{\text{pl}}(m)$ in $\mathcal{H}_\alpha^o[0, 1]$ for $\alpha = 1/2 - 1/p$. For the constructed m in Theorem 2.1, the quadratic variance is κm^2 for some constant κ and m does not belong to the \mathbb{L}^p space.

By Lemma A.2 in [MRS12], the Hölder norm of a polygonal line is reached at two vertices, hence, for a function g ,

$$(2.4) \quad \left\| n^{-1/2} S_n^{\text{pl}}(g - g \circ T) \right\|_{\mathcal{H}_{1/2-1/p}^o} = n^{-1/p} \max_{1 \leq i < j \leq n} \frac{|g \circ T^j - g \circ T^i|}{(j - i)^{1/2-1/p}}$$

$$(2.5) \quad \leq 2n^{-1/p} \max_{1 \leq j \leq n} |g \circ T^j|.$$

As a consequence, if g belongs to \mathbb{L}^p , then the sequence $\left(\left\| n^{-1/2} S_n^{\text{pl}}(g - g \circ T) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right)_{n \geq 1}$ converges to 0 in probability. Therefore, we can exploit a martingale-coboundary decomposition in \mathbb{L}^p .

Corollary 2.5. Let $p > 2$ and let f be an \mathcal{M} -measurable function which can be written as

$$(2.6) \quad f = m + g - g \circ T,$$

where $m, g \in \mathbb{L}^p$ and $(m \circ T^i)_{i \geq 0}$ is a martingale difference sequence for the filtration $(T^{-i}\mathcal{M})_{i \geq 0}$. Then $n^{-1/2} S_n^{\text{pl}}(f) \rightarrow \eta W$ in distribution in $\mathcal{H}_{1/2-1/p}^o[0, 1]$, where η is given by (2.2) and independent of W .

We define for a function h the operators $\mathbb{E}_k(h) := \mathbb{E}[h | T^k \mathcal{M}]$ and $P_i(h) := \mathbb{E}_i(h) - \mathbb{E}_{i+1}(h)$. The condition $\sum_{i=0}^{\infty} \|P_i(f)\|_2$ was introduced by Hannan in [Han73] in order to deduce a central limit theorem. It actually implies the weak invariance principle (see Corollary 2 in [DMV07]).

Theorem 2.6. *Let $p > 2$ and let f be an \mathcal{M} -measurable function such that*

$$(2.7) \quad \mathbb{E} \left[f \mid \bigcap_{i \in \mathbb{Z}} T^i \mathcal{M} \right] = 0 \text{ and}$$

$$(2.8) \quad \sum_{i \geq 0} \|P_i(f)\|_p < \infty.$$

Then $n^{-1/2} S_n^{\text{pl}}(m) \rightarrow \eta W$ in distribution in $\mathcal{H}_{1/2-1/p}^o[0, 1]$, where η is given by (2.2) and independent of W .

3. PROOFS

3.1. Proof of Theorem 2.1. We need a result about dynamical systems of positive entropy for the construction of a counter-example.

Lemma 3.1. *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic probability measure preserving system of positive entropy. There exists two T -invariant sub- σ -algebras \mathcal{B} and \mathcal{C} of \mathcal{A} and a function $g: \Omega \rightarrow \mathbb{R}$ such that:*

- *the σ -algebras \mathcal{B} and \mathcal{C} are independent;*
- *the function g is \mathcal{B} -measurable, takes the values $-1, 0$ and 1 , has zero mean and the process $(g \circ T^n)_{n \in \mathbb{Z}}$ is independent;*
- *the dynamical system $(\Omega, \mathcal{C}, \mu, T)$ is aperiodic.*

This is Lemma 3.8 from [LV01].

We consider the following four increasing sequences of positive integers $(I_l)_{l \geq 1}$, $(J_l)_{l \geq 1}$, $(n_l)_{l \geq 1}$ and $(L_l)_{l \geq 1}$. We define $k_l := 2^{I_l + J_l}$ and impose the conditions:

$$(3.1) \quad \sum_{l=1}^{\infty} \frac{1}{L_l} < \infty;$$

$$(3.2) \quad \lim_{l \rightarrow \infty} J_l \cdot \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} = 1;$$

$$(3.3) \quad \lim_{l \rightarrow \infty} J_l 2^{-I_l/2} = 0;$$

$$(3.4) \quad \lim_{l \rightarrow \infty} n_l \sum_{i > l} \frac{k_i}{n_i} = 0;$$

$$(3.5) \quad \text{for each } l, \quad \sum_{i=1}^{l-1} \frac{k_l}{L_i} \left(\frac{n_i}{2^{I_i}} \right)^{1/p} < \frac{n_l^{1/p}}{2}.$$

Here \mathcal{N} denotes a random variable whose distribution is standard normal. Such sequences can be constructed as follows: first pick a sequence $(L_l)_{l \geq 1}$ satisfying (3.1), for example $L_l = l^2$. Then construct $(J_l)_{l \geq 1}$ such that (3.2) holds. Once the sequence $(J_l)_{l \geq 1}$ is constructed, define $(I_l)_{l \geq 1}$ satisfying (3.3). Now the sequence $(k_l)_{l \geq 1}$ is completely determined. Noticing that (3.4) is satisfied if the series $\sum_l k_l n_{l-1}/n_l$ converges, we construct the sequence $(n_l)_{l \geq 1}$ by induction; once $n_i, i \leq l-1$ are defined, we choose n_l such that $n_l \geq l^2 k_l n_{l-1}$ and (3.5) holds.

Using Rokhlin's lemma, we can find for any integer $l \geq 1$ a measurable set $C_l \in \mathcal{C}$ such that the sets $T^{-i}C_l, i = 0, \dots, n_l - 1$ are pairwise disjoint and $\mu \left(\bigcup_{i=0}^{n_l-1} T^{-i}C_l \right) > 1/2$.

For a fixed l , we define

$$(3.6) \quad k_{l,j} := 2^{I_l+J_l-j}, \quad 0 \leq j \leq J_l,$$

$$(3.7) \quad k_{l,j} := 2^{I_l+J_l-j}, \quad 0 \leq j \leq J_l \text{ and}$$

$$f_l := \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left(\frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \chi \left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l \right) + \\ + \frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l}} \right)^{1/p} \chi \left(\bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right),$$

$$(3.8) \quad f := \sum_{l=1}^{+\infty} f_l, \quad m := g \cdot f,$$

where g is the function obtained by Lemma 3.1.

Proposition 3.2. *We have the estimate $\|f_l\|_{p,\infty} \leq \kappa'_p L_l^{-1}$ for some constant κ'_p depending only on p . As a consequence, $\lim_{t \rightarrow \infty} t^p \mu\{|m| > t\} = 0$.*

Proof. Notice that

$$(3.9) \quad \left\| \frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l}} \right)^{1/p} \chi \left(\bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right) \right\|_{p,\infty}^p = \frac{1}{L_l^p} \frac{n_l}{k_{l,J_l}} k_{l,J_l} \cdot \mu(C_l) \leq \frac{1}{L_l^p}.$$

Next, using Lemma 1.5 with $N := J_l - 1$, $a_j := \frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l-j}} \right)^{1/p}$ and $A_j := \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l$, we obtain

$$(3.10) \quad \left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left(\frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \chi \left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i} C_l \right) \right\|_{p,\infty}^p \leq \max_{0 \leq j \leq J_l-1} \left(\frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \right)^p \sum_{i=0}^j \mu(A_j)$$

$$(3.11) \quad \leq \frac{1}{L_l^p} \max_{0 \leq j \leq J_l-1} \frac{n_l}{k_{l,J_l-j}} \sum_{i=0}^j \frac{k_{l,J_l-i}}{n_l}$$

$$(3.12) \quad = \frac{1}{L_l^p} \max_{0 \leq j \leq J_l-1} \sum_{i=0}^j \frac{2^{I_l+i}}{2^{I_l+j}}$$

$$(3.13) \quad \leq \frac{2}{L_l^p},$$

hence by (1.12), (3.9) and (3.13),

$$(3.14) \quad \|f_l\|_{p,\infty} \leq N_p \left(\frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l}} \right)^{1/p} \chi \left(\bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right) \right) +$$

$$(3.15) \quad + N_p \left(\frac{1}{L_l} \sum_{j=0}^{J_l-1} \left(\frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \chi \left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}} T^{-i} C_l \right) \right)$$

$$(3.16) \quad \leq \kappa_p \left\| \frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l}} \right)^{1/p} \chi \left(\bigcup_{i=0}^{k_{l,J_l}-1} T^{-i} C_l \right) \right\|_{p,\infty} +$$

$$(3.17) \quad + \kappa_p \left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left(\frac{n_l}{k_{l,J_l-j}} \right)^{1/p} \chi \left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}} T^{-i} C_l \right) \right\|_{p,\infty}$$

$$(3.18) \quad \leq \frac{1}{L_l} \kappa_p \left(1 + 2^{1/p} \right)$$

We thus define $\kappa'_p := \kappa_p (1 + 2^{1/p})$.

We fix $\varepsilon > 0$; using (3.1), we can find an integer l_0 such that $\sum_{l>l_0} 1/L_l < \varepsilon$. Since the function $\sum_{l=1}^{l_0} g f_l$ is bounded, we have,

$$(3.19) \quad \limsup_{t \rightarrow \infty} t^p \mu \{ |m| > t \} \leq \limsup_{t \rightarrow \infty} t^p \mu \left\{ \left| \sum_{l=1}^{l_0} g f_l \right| > \frac{t}{2} \right\} + 2^p \left\| \sum_{l>l_0} g f_l \right\|_{p,\infty}^p$$

$$(3.20) \quad = 2^p \left\| \sum_{l>l_0} g f_l \right\|_{p,\infty}^p$$

$$(3.21) \quad \leq \left(2 \sum_{l>l_0} N_p(f_l) \right)^p$$

$$(3.22) \quad \leq \kappa'_p \left(\sum_{l>l_0} \frac{1}{L_l} \right)^p$$

$$(3.23) \quad \leq \kappa'_p \varepsilon^p,$$

where the second inequality comes from inequalities (1.12). Since ε is arbitrary, the proof of Lemma 3.2 is complete. \square

We denote by \mathcal{M} the σ -algebra generated by \mathcal{C} and the random variables $g \circ T^k$, $k \leq 0$. It satisfies $\mathcal{M} \subset T^{-1}\mathcal{M}$.

Proposition 3.3. *The sequence $(m \circ T^i)_{i \geq 0}$ is a (stationary) martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i \geq 0}$.*

Proof. We have to show that $\mathbb{E}[m \mid T\mathcal{M}] = 0$. Since the σ -algebra \mathcal{C} is T -invariant, we have $T\mathcal{M} = \sigma(\mathcal{C} \cup \sigma(g \circ T^k, k \leq -1))$. This implies

$$(3.24) \quad \mathbb{E}[m \mid T\mathcal{M}] = \mathbb{E}[g f \mid T\mathcal{M}] = f \cdot \mathbb{E}[g \mid T\mathcal{M}].$$

Since g is centered and independent of $T\mathcal{M}$, Proposition 3.3 is proved. \square

It remains to prove that the process $(n^{-1/2}S_n^{\text{pl}}(m))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}^o[0, 1]$.

Proposition 3.4. *Under conditions (3.2), (3.3) and (3.4), there exists an integer l_0 such that for $l \geq l_0$*

$$(3.25) \quad P_l := \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gfl) - S_u(gfl)|}{v^{1/2-1/p}} \geq 1 \right\} \geq \frac{1}{16}.$$

Proof. Let us fix an integer $l \geq 1$. Assume that $\omega \in T^{-s}C_l$, where $k_l \leq s \leq n_l - 1$. Since $T^u\omega$ belongs to $T^{-(s-u)}C_l$ we have for $s - n_l \leq u \leq s$

$$(3.26) \quad (f_l \circ T^u)(\omega) = \begin{cases} \frac{1}{L_l} \left(\frac{n_l}{k_{l,J_l}} \right)^{1/p}, & \text{if } s - k_{l,J_l} < u \leq s; \\ \frac{1}{L_l} \left(\frac{n_l}{k_{l,j}} \right)^{1/p}, & \text{if } s - k_{l,j-1} < u \leq s - k_{l,j}, \text{ and } 1 \leq j \leq J_l; \\ 0, & \text{if } s - n_l \leq u < s - k_l. \end{cases}$$

As a consequence,

$$(3.27) \quad T^{-s}C_l \cap \left\{ \frac{1}{n_l^{1/p}} \max_{1 \leq j \leq J_l} \frac{|S_{s-k_{l,j-1}+1}(gfl) - S_{s-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \geq 1 \right\} \\ = T^{-s}C_l \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{s-k_{l,j-1}+1}(g) - S_{s-k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}.$$

Since for $k_l + 1 \leq s \leq n_l - k_l$ and $1 \leq j \leq J_l$, we have $1 \leq s - k_{l,j} \leq n_l - k_l$ and $1 \leq k_{l,j-1} - 1 - k_{l,j} \leq k_l$, the inequality

$$(3.28) \quad \chi(T^{-s}C_l) \cdot \max_{1 \leq j \leq J_l} \frac{|S_{s-k_{l,j-1}+1}(gfl) - S_{s-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \\ \leq \chi(T^{-s}(C_l)) \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gfl) - S_u(gfl)|}{v^{1/2-1/p}}$$

takes place and since the sets $(T^{-s}C_l)_{s=0}^{n_k-1}$ are pairwise disjoint, we obtain the lower bound

$$(3.29) \quad P_l \geq \sum_{s=1}^{n_l-2k_l} \mu \left(T^{-(s+k_l)}(C_l) \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{s+k_l-k_{l,j-1}+1}(gfl) - S_{s+k_l-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \geq 1 \right\} \right).$$

Using the fact that T is measure-preserving, this becomes

$$(3.30) \quad P_l \geq (n_l - 2k_l) \cdot \mu \left(T^{-k_l}(C_l) \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_l-k_{l,j-1}-1}(gfl) - S_{k_l-k_{l,j}}(gfl)|}{(k_{l,j-1} - 1 - k_{l,j})^{1/2-1/p}} \geq 1 \right\} \right),$$

and plugging (3.27) in the previous estimate, we get

$$(3.31) \quad P_l \geq (n_l - 2k_l) \mu \left(T^{-k_l}(C_l) \cap \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_l-k_{l,j-1}-1}(g) - S_{k_l-k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\} \right).$$

The sets $\left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_l - k_{l,j-1} + 1}(g) - S_{k_l - k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2 - 1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}$ and $T^{-k_l} C_l$ belong to the independent sub- σ -algebras \mathcal{B} and \mathcal{C} respectively, hence using the fact that the sequences $(g \circ T^i)_{i \geq 0}$ and $(g \circ T^{-i})_{i \geq 0}$ are identically distributed, we obtain

$$(3.32) \quad P_l \geq (n_l - 2k_l) \mu(C_l) \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2 - 1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}.$$

By construction, we have $n_l \cdot \mu(C_l) = \mu \left(\bigcup_{i=0}^{n_l-1} T^{-i} C_l \right) > 1/2$, hence

$$(3.33) \quad P_l \geq \frac{1}{2} \left(1 - 2 \frac{k_l}{n_l} \right) \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2 - 1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}.$$

It remains to find a lower bound for

$$(3.34) \quad P'_l := \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2 - 1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}.$$

Let us define the set

$$(3.35) \quad E_j := \left\{ \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2 - 1/p} k_{l,j-1}^{1/p}} \geq L_l \right\}$$

Since the sequence $(g \circ T^i)_{i \in \mathbb{Z}}$ is independent, the family $(E_j)_{1 \leq j \leq J_l}$ is independent, hence

$$(3.36) \quad P'_l \geq 1 - \prod_{j=1}^{J_l} (1 - \mu(E_j)).$$

We define the quantity

$$(3.37) \quad c_j := \mu \left\{ |\mathcal{N}| \geq \frac{L_l}{\|g\|_2} \left(\frac{k_{l,j-1}}{k_{l,j} - 1} \right)^{1/p} \right\}$$

(we recall that \mathcal{N} denotes a standard normally distributed random variable). By the Berry-Esseen theorem, we have for each $j \in \{1, \dots, J_l\}$,

$$(3.38) \quad |\mu(E_j) - c_j| \leq \frac{1}{\|g\|_2^3} \frac{1}{(k_{l,j-1} - 1)^{1/2}} \leq \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}.$$

Plugging the estimate (3.38) into (3.36) and noticing that for an integer N and $(a_n)_{n=1}^N, (b_n)_{n=1}^N$ two families of numbers in the unit interval,

$$(3.39) \quad \left| \prod_{n=1}^N a_n - \prod_{n=1}^N b_n \right| \leq \sum_{n=1}^N |a_n - b_n|,$$

we obtain

$$(3.40) \quad P'_l \geq 1 - \prod_{j=1}^{J_l} (1 - \mu(E_j)) + \prod_{j=1}^{J_l} (1 - c_j) - \prod_{j=1}^{J_l} (1 - c_j)$$

$$(3.41) \quad \geq 1 - \prod_{j=1}^{J_l} (1 - c_j) - \sum_{j=1}^{J_l} |\mu(E_j) - c_j|$$

$$(3.42) \quad \geq 1 - \prod_{j=1}^{J_l} (1 - c_j) - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}.$$

Notice that

$$(3.43) \quad 1 - \prod_{j=1}^{J_l} (1 - c_j) \geq 1 - \max_{1 \leq j \leq J_l} (1 - c_j)^{J_l}$$

and since $(I_l)_{\geq}$ is increasing and $I_1 \geq 1$, we have

$$(3.44) \quad \frac{k_{l,j-1}}{k_{l,j} - 1} = \frac{2}{1 - k_{l,j}^{-1}} \leq \frac{2}{1 - 2^{-I_l}} \leq 4$$

it follows by (3.37) that $c_j \geq \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\}$ for $1 \leq j \leq J_l$. We thus have

$$(3.45) \quad P'_l \geq 1 - \left(1 - \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} \right)^{J_l} - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}.$$

Using the elementary inequality

$$(3.46) \quad 1 - (1 - t)^n \geq nt - \frac{n(n-1)}{2} t^2$$

valid for a positive integer n and $t \in [0, 1]$, we obtain

$$(3.47) \quad P'_l \geq J_l \mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} - \frac{J_l^2}{2} \left(\mu \left\{ |\mathcal{N}| \geq 4^{1/p} \frac{L_l}{\|g\|_2} \right\} \right)^2 - J_l \frac{\sqrt{2}}{\|g\|_2^3} 2^{-I_l/2}.$$

By conditions (3.3) and (3.2), there exists an integer l'_0 such that if $l \geq l'_0$, then

$$(3.48) \quad \mu \left\{ \max_{1 \leq j \leq J_l} \frac{|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)|}{(k_{l,j} - 1)^{1/2-1/p} k_{l,j-1}^{1/p}} \geq L_l \right\} \geq \frac{1}{4}.$$

Combining (3.33) with (3.48), we obtain for $l \geq l'_0$

$$(3.49) \quad P_l \geq \frac{1}{8} \left(1 - 2 \frac{k_l}{n_l} \right).$$

By condition (3.4), we thus get that $P_l \geq 1/16$ for $l \geq l_0$, where $l_0 \geq l'_0$ and $k_l/n_l \leq 1/4$ if $l \geq l_0$.

This concludes the proof of Proposition 3.4. \square

Proposition 3.5. *Under conditions (3.1), (3.2), (3.3), (3.4) and (3.5), we have for l large enough*

$$(3.50) \quad \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m) - S_u(m)|}{v^{1/2-1/p}} \geq \frac{1}{2} \right\} \geq \frac{1}{32}.$$

Since the Hölder modulus of continuity of a piecewise linear function is reached at vertices, we derive the following corollary.

Corollary 3.6. *If $l \geq l_0$, then*

$$(3.51) \quad \mu \left\{ \omega_{1/2-1/p} \left(\frac{1}{\sqrt{n_l}} S_{n_l}^{\text{pl}}(m), \frac{k_l}{n_l} \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32}.$$

Therefore, for each positive δ , we have

$$(3.52) \quad \limsup_{n \rightarrow \infty} \mu \left\{ \omega_{1/2-1/p} \left(\frac{1}{\sqrt{n}} S_n^{\text{pl}}(m), \delta \right) \geq \frac{1}{2} \right\} \geq \frac{1}{32},$$

and the process $(n^{-1/2} S_n^{\text{pl}}(m))_{n \geq 1}$ is not tight in $\mathcal{H}_{1/2-1/p}^o[0, 1]$.

Proof of Proposition 3.5. Let l_0 be the integer given by Proposition 3.4 and let $l \geq l_0$. We define $m'_l := \sum_{i=1}^{l-1} gf_i$ and $m''_l := \sum_{i=l+1}^{+\infty} gf_i$.

We define for $i \geq 1$,

$$(3.53) \quad M_{l,i} := \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gf_i) - S_u(gf_i)|}{v^{1/2-1/p}}.$$

Let i be an integer such that $i < l$. Notice that for $1 \leq u \leq n_l - k_l$ and $v \leq k_l$, we have

$$(3.54) \quad |S_{u+v}(gf_i) - S_u(gf_i)| = U^u(|S_v(gf_i)|),$$

where $U(h)(\omega) = h(T(\omega))$ and since

$$(3.55) \quad |S_v(gf_i)| \leq v \|gf_i\|_\infty \leq \frac{k_l}{L_i} \left(\frac{n_i}{2^{L_i}} \right)^{1/p},$$

the estimate

$$(3.56) \quad M_{l,i} \leq \frac{k_l}{L_i n_l^{1/p}} \left(\frac{n_i}{2^{L_i}} \right)^{1/p}$$

holds. Since

$$(3.57) \quad \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leq \sum_{i=1}^{l-1} M_{l,i},$$

we have by (3.56),

$$(3.58) \quad \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leq \sum_{i=1}^{l-1} \frac{k_l}{L_i n_l^{1/p}} \left(\frac{n_i}{2^{L_i}} \right)^{1/p}.$$

By (3.5), the following bound takes place:

$$(3.59) \quad \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leq \frac{1}{2}.$$

The following set inclusions hold

$$(3.60) \quad \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m''_l) - S_u(m''_l)|}{v^{1/2-1/p}} \neq 0 \right\} \subset \bigcup_{i>l} \{M_{l,i} \neq 0\}$$

$$(3.61) \quad \subset \bigcup_{i>l} \bigcup_{u=1}^{n_l} \{U^u(gf_i) \neq 0\}.$$

We thus have

$$(3.62) \quad \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m_l'') - S_u(m_l'')|}{v^{1/2-1/p}} \neq 0 \right\} \leq \sum_{i>l} n_l \cdot \mu \{gf_i \neq 0\}$$

$$(3.63) \quad \leq n_l \sum_{i>l} \mu \{f_i \neq 0\}$$

$$(3.64) \quad = n_l \sum_{i>l} (k_i + 1) \mu(C_i)$$

$$(3.65) \quad \leq 2n_l \sum_{i>l} \frac{k_i}{n_i}.$$

and by (3.4), it follows that

$$(3.66) \quad \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m_l'') - S_u(m_l'')|}{v^{1/2-1/p}} \neq 0 \right\} \leq \frac{1}{32}$$

Accounting (3.59), we thus have

$$(3.67) \quad \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m) - S_u(m)|}{v^{1/2-1/p}} \geq \frac{1}{2} \right\} \\ \geq \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gfl + m_l'') - S_u(gfl + m_l'')|}{v^{1/2-1/p}} \geq 1 \right\} \\ \geq \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(gfl) - S_u(gfl)|}{v^{1/2-1/p}} \geq 1 \right\} \\ - \mu \left\{ \frac{1}{n_l^{1/p}} \max_{\substack{1 \leq u \leq n_l - k_l \\ 1 \leq v \leq k_l}} \frac{|S_{u+v}(m_l'') - S_u(m_l'')|}{v^{1/2-1/p}} \neq 0 \right\},$$

hence combining Proposition 3.4 with (3.66), we obtain the conclusion of Proposition 3.5. \square

Theorem 2.1 follows from Corollary 3.6 and Propositions 3.2 and 3.3.

3.2. Proof of Theorem 2.2 and Proposition 2.3.

Proof of Proposition 2.3. Let us fix a positive t . Recall the equivalence between $\|x\|_\alpha$ and $\|x\|_\alpha^{\text{seq}}$ and Notation 1.1. By Remark 1.2, we have to show that for some constant C depending only on p and each integer $n \geq 1$,

$$(3.68) \quad P(n, t) := t^p \mu \left\{ \sup_{j \geq 1} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2j} \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j}) \right| > t \right\} \leq \\ \leq C \left(\|m\|_{p, \infty}^p + \mathbb{E}(\mathbb{E}[m^2 | \mathcal{TM}])^{p/2} \right)$$

In the proof, we shall denote by C_p a constant depending only on p which may change from line to line.

We define

(3.69)

$$P_1(n, t) := \mu \left\{ \sup_{1 \leq j \leq \log n} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j}) \right| > t \right\}, \text{ and}$$

$$(3.70) \quad P_2(n, t) := \mu \left\{ \sup_{j > \log n} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j}) \right| > t \right\},$$

hence

$$(3.71) \quad P(n, t) \leq t^p P_1(n, t/2) + t^p P_2(n, t/2).$$

We estimate $P_2(n, t)$. For $j > \log n$, we have the inequality

$$(3.72) \quad r_{k+1, j} - r_{k, j} = (k+1)2^{-j} - k2^{-j} = 2^{-j} < 1/n,$$

hence if $r_{k, j}$ belongs to the interval $[l/n, (l+1)/n]$ for some $l \in \{0, \dots, n-1\}$, then

- either $r_{k+1, j} \in [l/n, (l+1)/n]$, and in this case,

$$(3.73) \quad \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j}) \right| = \left| m \circ T^{l+1} \right| 2^{-j} n \leq 2^{-j} n \max_{1 \leq l \leq n} \left| U^l(m) \right|;$$

- or $r_{k+1, j}$ belongs to the interval $[(l+1)/n, (l+2)/n]$. The estimates

$$(3.74) \quad \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, r_{k, j}) \right| \leq \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_n^{\text{pl}}(m, (l+1)/n) \right| + \\ + \left| S_n^{\text{pl}}(m, (l+1)/n) - S_n^{\text{pl}}(m, r_{k, j}) \right| \leq 2^{1-j} n \max_{1 \leq l \leq n} \left| U^l(m) \right|$$

hold.

Considering these two cases, we obtain

$$(3.75) \quad P_2(n, t) \leq \mu \left\{ \sup_{j > \log n} 2^{\alpha j} n 2^{1-j} n^{-1/2} \max_{1 \leq l \leq n} \left| U^l(m) \right| > t \right\}$$

$$(3.76) \quad \leq \mu \left\{ 2n^{\alpha-1/2} \max_{1 \leq l \leq n} \left| U^l(m) \right| > t \right\}$$

$$(3.77) \quad \leq n\mu \left\{ 2n^{-1/p} |m| > t \right\}$$

$$(3.78) \quad \leq \frac{2^p}{t^p} \sup_{x>0} x^p \mu \{ |m| > x \}.$$

Therefore, establishing inequality (3.68) reduces to find a constant C depending only on p such that

$$(3.79) \quad \sup_n \sup_t t^p P_1(n, t) \leq C \left(\|m\|_{p, \infty}^p + \mathbb{E} \left(\mathbb{E}[m^2 \mid T\mathcal{M}] \right)^{p/2} \right)$$

We define $u_{k, j} := [nr_{k, j}]$ for $k < 2^j$ and $j \geq 1$ (see Notation 1.1).

Notice that the inequalities

$$(3.80) \quad \left| S_{u_{k, j}}(m) - S_n^{\text{pl}}(m, r_{k, j}) \right| \leq \left| U^{u_{k, j}+1}(m) \right| \quad \text{and}$$

$$(3.81) \quad \left| S_n^{\text{pl}}(m, r_{k+1, j}) - S_{u_{k+1, j}}(m) \right| \leq \left| U^{u_{k+1, j}+1}(m) \right|$$

take place because if $j \leq \log n$, then

$$(3.82) \quad u_{k, j} \leq nr_{k, j} \leq u_{k, j} + 1 \leq u_{k+1, j} \leq nr_{k+1, j} \leq u_{k+1, j} + 1.$$

Therefore, $P_1(n, t) \leq P_{1,1}(n, t) + P_{1,2}(n, t)$, where

$$(3.83) \quad P_{1,1}(n, t) := \mu \left\{ \max_{1 \leq j \leq \log n} 2^{\alpha j} n^{-1/2} \max_{0 \leq k < 2^j} |S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)| > t/2 \right\},$$

$$(3.84) \quad P_{1,2}(n, t) := \mu \left\{ \max_{1 \leq j \leq \log n} 2^{\alpha j} n^{-1/2} \max_{1 \leq l \leq n} |U^l(m)| > t/4 \right\}.$$

Notice that

$$(3.85) \quad P_{1,2}(n, t) \leq \mu \left\{ n^{\alpha-1/2} \max_{1 \leq l \leq n} |U^l(m)| > t/4 \right\}$$

$$(3.86) \quad \leq n\mu \left\{ |m| > n^{1/p} t/4 \right\}$$

$$(3.87) \quad \leq 4^p t^{-p} \sup_{x \geq 0} x^p \mu \{ |m| > x \},$$

hence (3.79) will follow from the existence of a constant C depending only on p such that

$$(3.88) \quad \sup_n \sup_t t^p P_{1,1}(n, t) \leq C \left(\|m\|_{p,\infty}^p + \mathbb{E} \left(\mathbb{E}[m^2 | T\mathcal{M}] \right)^{p/2} \right).$$

We estimate $P_{1,1}(n, t)$ in the following way:

$$(3.89) \quad P_{1,1}(n, t) \leq \sum_{j=1}^{\log n} 2^j \max_{0 \leq k < 2^j} \mu \left\{ |S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)| > tn^{1/2} 2^{-1-\alpha j} \right\}$$

We define for $1 \leq j \leq \log n$ and $0 \leq k < 2^j$ the quantity

$$(3.90) \quad P(n, j, k, t) := \mu \left\{ |S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)| > tn^{1/2} 2^{-1-\alpha j} \right\}.$$

If $(f \circ T^j)_{j \geq 0}$ is a strictly stationary sequence, we define

$$(3.91) \quad Q_{f,n}(u) := \mu \left\{ \max_{1 \leq j \leq n} |f \circ T^j| > u \right\} + \mu \left\{ \left(\sum_{i=1}^n U^i \mathbb{E}[f^2 | T\mathcal{M}] \right)^{1/2} > u \right\}.$$

The following inequality is Theorem 1 of [Nag03]. It allows us to express the tail function of a martingale by that of the increments and the quadratic variance.

Theorem 3.7. *Let m be an \mathcal{M} -measurable function such that $\mathbb{E}[m | T\mathcal{M}] = 0$. Then for each positive y and each integer n ,*

$$(3.92) \quad \mu \{ |S_n(m)| > y \} \leq c(q, \eta) \int_0^1 Q_{m,n}(\varepsilon_q u \cdot y) u^{q-1} du,$$

where $q > 0$, $\eta > 0$, $\varepsilon_q := \eta/q$ and $c(q, \eta) := q \exp(3\eta e^{\eta+1} - \eta - 1)/\eta$.

We shall use (3.92) with $q := p+1$, $\eta = 1$ and $y := n^{1/2} 2^{-1-\alpha j} t$ in order to estimate $P(n, j, k, t)$:

$$(3.93) \quad P(n, j, k, t) \leq C_p \int_0^1 \mu \left\{ \max_{1 \leq i \leq u_{k+1,j} - u_{k,j}} |U^i(m)| > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\} u^p du \\ + C_p \int_0^1 \mu \left\{ \left(\sum_{i=u_{k,j}+1}^{u_{k+1,j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\} u^p du.$$

Exploiting the inequality $u_{k+1,j} - u_{k,j} \leq 2n2^{-j}$, we get from the previous bound

$$(3.94) \quad P(n, j, k, t) \leq C_p \int_0^1 \mu \left\{ \max_{1 \leq i \leq 2n2^{-j}} |U^i(m)| > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\} u^p du \\ + C_p \int_0^1 \mu \left\{ \left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\} u^p du.$$

We define for $j \leq \log n$, $t \geq 0$ and $u \in (0, 1)$,

$$(3.95) \quad P'(n, j, t, u) := \mu \left\{ \max_{1 \leq i \leq 2n2^{-j}} |U^i(m)| > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\}, \quad \text{and}$$

$$(3.96) \quad P''(n, j, t, u) := \mu \left\{ \left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\}.$$

Using the fact that the random variables $U^i(m)$, $1 \leq i \leq 2n2^{-j}$ are identically distributed, we derive the bound

$$(3.97) \quad P'(n, j, t, u) \leq 2n2^{-j} \mu \left\{ |m| > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1} \right\},$$

hence

$$(3.98) \quad P'(n, j, t, u) \leq 2n2^{-j} (n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1})^{-p} \|m\|_{p,\infty}^p \\ = 2^{p+1} \varepsilon_{p+1}^{-p} n^{1-p/2} 2^{j(-1+p\alpha)} t^{-p} u^{-p} \|m\|_{p,\infty}^p.$$

Since α and p are linked by the relationship $1/2 - 1/p = \alpha$, we have $p\alpha = p/2 - 1$ hence

$$(3.99) \quad \int_0^1 P'(n, j, t, u) u^p du \leq C_p t^{-p} n^{1-p/2} 2^{j(p/2-2)} \|m\|_{p,\infty}^p.$$

Notice the following set equalities:

$$(3.100) \quad \left\{ \left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t \right\} \\ = \left\{ \frac{1}{2n2^{-j}} \sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\}$$

and that $n2^{-j} \geq 1$ (because $j \leq \log n$), hence

$$(3.101) \quad \left\{ \left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t \right\} \subseteq \\ \subseteq \bigcup_{N \geq 2} \left\{ \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\},$$

from which it follows

$$(3.102) \quad \mu \left\{ \left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t \right\} \leq \\ \leq \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\}.$$

Combining (3.99) and (3.102), we obtain

$$(3.103) \quad \max_{0 \leq k < 2^j} P(n, j, k, t) \leq C_p t^{-p} n^{1-p/2} 2^{j(p/2-2)} \|m\|_{p,\infty}^p \\ + C_p \int_0^1 \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\} u^p du,$$

hence by (3.89) and (3.90),

$$(3.104) \quad P_{1,1}(n, t) \leq C_p t^{-p} \|m\|_{p,\infty}^p \sum_{j=1}^{\log n} 2^j 2^{j(p/2-2)} n^{1-p/2} + \\ + C_p \int_0^1 \sum_{j=1}^{\log n} 2^j \mu \left\{ \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) > 2^{-3} \varepsilon_{p+1}^2 u^2 2^{2j/p} t^2 \right\} u^p du.$$

From the elementary bounds

$$(3.105) \quad \sum_{j=1}^{\log n} 2^{j(p/2-1)} n^{1-p/2} \leq (1 - 2^{1-p/2})^{-1}$$

$$(3.106) \quad \sum_{j \geq 1} 2^j \mu \left\{ |g| > 2^{2j/p} \right\} \leq 2 \mathbb{E} |g|^{p/2}, \quad \text{for any non-negative function } g,$$

with

$$(3.107) \quad g := 2^3 \varepsilon_{p+1}^{-2} u^{-2} \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]), \quad u \in (0, 1)$$

we obtain

$$(3.108) \quad P_{1,1}(n, t) \leq C_p t^{-p} \|m\|_{p,\infty}^p + C_p t^{-p} \left\| \sup_{N \geq 2} \frac{1}{N} \sum_{i=1}^N U^i(\mathbb{E}[m^2 | T\mathcal{M}]) \right\|_{p/2}^{p/2}.$$

As the Koopman operator U is an \mathbb{L}^1 - \mathbb{L}^∞ contraction, Theorem 1 of [Ste61] gives the existence of a constant A_p such that for each $h \in \mathbb{L}^{p/2}$,

$$(3.109) \quad \left\| \sup_{N \geq 1} \frac{1}{N} \sum_{j=1}^N U^j(h) \right\|_{p/2} \leq A_p \|h\|_{p/2}.$$

Applying (3.109) with $h := \mathbb{E}[m^2 | T\mathcal{M}]$, we get by (3.108)

$$(3.110) \quad P_{1,1}(n, t) \leq C_p t^{-p} \|m\|_{p,\infty}^p + C_p t^{-p} \mathbb{E} (\mathbb{E}[m^2 | T\mathcal{M}])^{p/2},$$

which establishes (3.79). This concludes the proof of Proposition 2.3. \square

Proof of Theorem 2.2. The convergence of finite dimensional distributions can be proved using Theorem of [Bil68]. Its proof works for filtrations of the form $(T^{-i}\mathcal{M})_{i \geq 0}$ where $T\mathcal{M} \subset \mathcal{M}$ and also in the non-ergodic setting by considering the ergodic components.

We deduce tightness in Theorem 2.2 from Proposition 2.3 by a truncation argument. For a fixed R , we define

$$(3.111) \quad m_R := m\chi\{|m| \leq R\} - \mathbb{E}[m\chi\{|m| \leq R\} | T\mathcal{M}] \quad \text{and}$$

$$(3.112) \quad m'_R := m\chi\{|m| > R\} - \mathbb{E}[m\chi\{|m| > R\} | T\mathcal{M}].$$

In this way, the sequences $(m_R \circ T^i)_{i \geq 0}$ and $(m'_R \circ T^i)_{i \geq 0}$ are martingale difference sequences and $m = m_R + m'_R$.

Since $|m_R| \leq 2R$ and $(m_R \circ T^i)_{i \geq 0}$ is a martingale difference sequence, the sequence $(n^{-1/2}S_n^{\text{pl}}(m_R))_{n \geq 1}$ is tight in $\mathcal{H}_{1/2-1/p}^o[0, 1]$. Consequently, for each positive ε , the following convergence takes place:

$$(3.113) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha_j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(m_R) \right) \right| > \varepsilon n^{1/2} \right\} = 0.$$

Using Proposition 2.3, we derive the following bound, valid for each ε and each R ,

$$(3.114) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha_j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(m) \right) \right| > \varepsilon n^{1/2} \right\} \leq \\ \leq C_p \varepsilon^{-p} \left(\sup_{t \geq 0} t^p \mu \{ |m| \chi\{|m| > R\} > t \} + \sup_{t \geq 0} t^p \mu \{ \mathbb{E}[|m| \chi\{|m| > R\} | T\mathcal{M}] > t \} \right) + \\ + \varepsilon^{-p} C_p \mathbb{E} \left((\mathbb{E}[m^2 \chi\{|m| > R\} | T\mathcal{M}])^{p/2} \right).$$

The first term is $\sup_{t \geq R} t^p \mu \{|m| > t\}$, which goes to 0 as R goes to infinity.

The second term can be bounded by $\sup_{t \geq R} t^p \mu \{\mathbb{E}[|m| | T\mathcal{M}] > t\}$. Indeed, if $t \geq R$, we use the inclusion

$$(3.115) \quad \{ \mathbb{E}[|m| \chi\{|m| > R\} | T\mathcal{M}] > t \} \subset \{ \mathbb{E}[|m| | T\mathcal{M}] > t \},$$

and if $t < R$, then accounting the fact that the random variable $\mathbb{E}[|m| \chi\{|m| > R\} | T\mathcal{M}]$ is greater than R , we get

$$(3.116) \quad \mathbb{E}[|m| \chi\{|m| > R\} | T\mathcal{M}] = \mathbb{E}[|m| \chi\{|m| > R\} | T\mathcal{M}] \chi\{ \mathbb{E}[|m| | T\mathcal{M}] > R \} \\ \leq \mathbb{E}[|m| | T\mathcal{M}] \chi\{ \mathbb{E}[|m| | T\mathcal{M}] > R \},$$

from which it follows that

$$(3.117) \quad t^p \mu \{ \mathbb{E}[|m| \chi\{|m| > R\} | T\mathcal{M}] > t \} \leq R^p \mu \{ \mathbb{E}[|m| | T\mathcal{M}] > R \}.$$

By Lemma 1.4, the convergence

$$(3.118) \quad \lim_{R \rightarrow \infty} \sup_{t \geq R} t^p \mu \{ \mathbb{E}[|m| | T\mathcal{M}] > t \} = 0$$

takes place.

The third term of (3.114) converges to 0 as R goes to infinity by monotone convergence.

This concludes the proof of tightness in Theorem 2.2. □

3.3. Proof of Theorem 2.6. By (2.7), the equality $f = \sum_{i \geq 0} P_i(f)$ holds almost surely. For a fixed integer K , we define $f_K := \sum_{i=0}^K P_i(f)$. Then f_K satisfies the conditions of Corollary 2.5.

Indeed, we have the equalities

$$(3.119) \quad \begin{aligned} P_i(f) - P_0(U^i f) &= \mathbb{E}[f \mid T^i \mathcal{M}] - \mathbb{E}[U^i f \mid \mathcal{M}] - \mathbb{E}[f \mid T^{i+1} \mathcal{M}] + \mathbb{E}[U^i f \mid T \mathcal{M}] \\ (3.120) \quad &= (I - U^i) \mathbb{E}[f \mid T^i \mathcal{M}] - (I - U^i) \mathbb{E}[f \mid T^{i+1} \mathcal{M}] \end{aligned}$$

and the later term can be expressed as a coboundary noticing that $(I - U^i) = (I - U) \sum_{k=0}^{i-1} U^k$. Since $P_i(f)$ belongs to the \mathbb{L}^p space, we may write $f_K - \sum_{i=0}^K P_0(U^i f)$ as $(I - U)g_K$ where g_K belongs to the \mathbb{L}^p space. Defining $m_K := \sum_{i=0}^K P_0(U^i(f))$, the sequence $(m_K \circ T^i)_{i \geq 0}$ is a martingale difference sequence hence for each positive ε ,

$$(3.121) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f_K) \right) \right| > \varepsilon n^{1/2} \right\} = 0.$$

Now, we have to show that the convergence in (3.121) holds if f_K is replaced by $f - f_K$. To this aim, we use the inclusion

$$(3.122) \quad \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f - f_K) \right) \right| > \varepsilon n^{1/2} \right\} \subseteq \left\{ \sup_{j \geq 1} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f - f_K) \right) \right| > \varepsilon n^{1/2} \right\},$$

hence

$$(3.123) \quad \begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f - f_K) \right) \right| > \varepsilon n^{1/2} \right\} &\leq \varepsilon^{-p} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}}(f - f_K) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty}^p \\ (3.124) \quad &= \varepsilon^{-p} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}} \left(\sum_{i \geq K+1} P_i(f) \right) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty}^p, \end{aligned}$$

from which it follows that

$$(3.125) \quad \begin{aligned} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f - f_K) \right) \right| > \varepsilon n^{1/2} \right\} \\ \leq \varepsilon^{-p} \left(\sum_{i \geq K+1} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}}(P_i(f)) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty} \right)^p. \end{aligned}$$

Notice that for a fixed i , the sequence $(U^l(P_i(f)))_{l \geq 1}$ is a martingale difference sequence (with respect to the filtration $(T^{-i-l} \mathcal{M})_{l \geq 0}$). Therefore, by Proposition 2.3, we obtain

$$(3.126) \quad \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\text{pl}}(P_i(f)) \right\|_{\mathcal{H}_{1/2-1/p}^o} \right\|_{p, \infty} \leq C_p \|P_i(f)\|_p.$$

Plugging this estimate into (3.125), we obtain that for some constant C depending only on p ,

$$(3.127) \quad \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f - f_K) \right) \right| > \varepsilon n^{1/2} \right\} \leq C \varepsilon^{-p} \left(\sum_{i \geq K+1} \|P_i(f)\|_p \right)^p.$$

Combining (3.121) and (3.127), we obtain for each K :

$$(3.128) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \mu \left\{ \sup_{j \geq J} 2^{\alpha j} \max_{r \in D_j} \left| \lambda_r \left(S_n^{\text{pl}}(f) \right) \right| > n^{1/2} \varepsilon \right\} \leq \\ \leq C \varepsilon^{-p} \left(\sum_{i \geq K+1} \|P_i(f)\|_p \right)^p.$$

Since K is arbitrary, we conclude the proof of Theorem 2.6 thanks to assumption (2.8).

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