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To cite this version:

Davide Giraudo. Holderian weak invariance principle under a Hannan type condition. Stochastic Processes and their Applications, 2016, 126, 10.1016/j.spa.2015.09.001 . hal-01128232v2

HAL Id: hal-01128232

https://hal.science/hal-01128232v2

Submitted on 24 Dec 2015

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HOLDERIAN WEAK INVARIANCE PRINCIPLE UNDER A HANNAN TYPE CONDITION

DAVIDE GIRAUDO

Abstract. We investigate the invariance principle in Hölder spaces for strictly stationary martingale difference sequences. In particular, we show that the sufficient condition on the tail in the i.i.d. case does not extend to stationary ergodic martingale differences. We provide a sufficient condition on the conditional variance which guarantee the invariance principle in Hölder spaces. We then deduce a condition in the spirit of Hannan one.

1. Introduction

One of the main problems in probability theory is the understanding of the asymptotic behavior of Birkhoff sums $S_n(f) := \sum_{j=0}^{n-1} f \circ T^i$, where $(\Omega, \mathcal{F}, \mu, T)$ is a dynamical system and f a map from Ω to the real line.

One can consider random functions contructed from the Birkhoff sums

$$S_n^{\mathrm{pl}}(f,t) := S_{[nt]}(f) + (nt - [nt]) f \circ T^{[nt]+1}, \quad t \in [0,1]. \tag{1.1}$$

and investigate the asymptotic behaviour of the sequence $\left(S_n^{\mathrm{pl}}(f,t)\right)_{n \geqslant 1}$ seen as an element of a function space. Donsker showed (cf. [Don51]) that the sequence $(n^{-1/2}(\mathbb{E}(f^2))^{-1/2} S_n^{\mathrm{pl}}(f))_{n \geqslant 1}$ converges in distribution in the space of continuous functions on the unit interval to a standard Brownian motion W when the sequence $(f \circ T^i)_{i \geqslant 0}$ is i.i.d. and zero mean. Then an intensive research has then been performed to extend this result to stationary weakly dependent sequences. We refer the reader to [MPU06] for the main theorems in this direction.

Our purpose is to investigate the weak convergence of the sequence $(n^{-1/2} S_n^{\mathrm{pl}}(f))_{n \geqslant 1}$ in Hölder spaces when $(f \circ T^i)_{i \geqslant 0}$ is a strictly stationary sequence. A classical method for showing a limit theorem is to use a martingale approximation, which allows to deduce the corresponding result if it holds for martingale differences sequences provided that the approximation is good enough. To the best of our knowledge, no result about the invariance principle in Hölder space for stationary martingale difference sequences is known.

1.1. The Hölder spaces.

It is well known that standard Brownian motion's paths are almost surely Hölder regular of exponent α for each $\alpha \in (0, 1/2)$, hence it is natural to consider the random function defined in (1.1) as an element of $\mathcal{H}_\alpha[0,1]$ and try to establish its weak convergence to a standard Brownian motion in this function space.

Before stating the results in this direction, let us define for $\alpha \in (0,1)$ the Hölder space $\mathcal{H}_\alpha[0,1]$ of functions $x\colon [0,1] \to \mathbb{R}$ such that $\sup_{s \neq t} |x(s) - x(t)| / |s-t|^\alpha$ is

Date: December 24, 2015.

2010 *Mathematics Subject Classification.* 60F05; 60F17.

Key words and phrases. Invariance principle, martingales, strictly stationary process.

finite. The analogue of the continuity modulus in $C[0,1]$ is w_α, defined by

$$w_\alpha(x,\delta) = \sup_{0<|t-s|<\delta} \frac{|x(t)-x(s)|}{|t-s|^\alpha}. \tag{1.2}$$

We then define $\mathcal{H}^0_\alpha[0,1]$ by $\mathcal{H}^0_\alpha[0,1] := \{x \in \mathcal{H}_\alpha[0,1], \lim_{\delta\to 0} w_\alpha(x,\delta) = 0\}$. We shall essentially work with the space $\mathcal{H}^0_\alpha[0,1]$ which, endowed with $\|x\|_\alpha := w_\alpha(x,1) + |x(0)|$, is a separable Banach space (while $\mathcal{H}_\alpha[0,1]$ is not separable). Since the canonical embedding $\iota\colon \mathcal{H}^o_\alpha[0,1] \to \mathcal{H}_\alpha[0,1]$ is continuous, each convergence in distribution in $\mathcal{H}^o_\alpha[0,1]$ also takes place in $\mathcal{H}_\alpha[0,1]$.

Let us denote by D_j the set of dyadic numbers in $[0,1]$ of level j, that is,

$$D_0 := \{0,1\}, \quad D_j := \left\{(2l-1)2^{-j}; 1 \leqslant l \leqslant 2^{j-1}\right\}, j \geqslant 1. \tag{1.3}$$

If $r \in D_j$ for some $j \geqslant 0$, we define $r^+ := r + 2^{-j}$ and $r^- := r - 2^{-j}$. For $r \in D_j$, $j \geqslant 1$, let Λ_r be the function whose graph is the polygonal path joining the points $(0,0)$, $(r^-,0)$, $(r,1)$, $(r^+,0)$ and $(1,0)$. We can decompose each $x \in C[0,1]$ as

$$x = \sum_{r\in D} \lambda_r(x)\Lambda_r = \sum_{j=0}^{+\infty}\sum_{r\in D_j} \lambda_r(x)\Lambda_r, \tag{1.4}$$

and the convergence is uniform on $[0,1]$. The coefficients $\lambda_r(x)$ are given by

$$\lambda_r(x) = x(r) - \frac{x(r^+)+x(r^-)}{2}, \quad r \in D_j, j \geqslant 1, \tag{1.5}$$

and $\lambda_0(x) = x(0)$, $\lambda_1(x) = x(1)$.

Ciesielski proved (cf. [Cie60]) that $\{\Lambda_r; r \in D\}$ is a Schauder basis of $\mathcal{H}^o_\alpha[0,1]$ and the norms $\|\cdot\|_\alpha$ and the sequential norm defined by

$$\|x\|^{\mathrm{seq}}_\alpha := \sup_{j\geqslant 0} 2^{j\alpha} \max_{r\in D_j} |\lambda_r(x)|, \tag{1.6}$$

are equivalent.

Considering the sequential norm, we can show (see Theorem 3 in [Suq99]) that a sequence $(\xi_n)_{n\geqslant 1}$ of random elements of $\mathcal{H}^o_\alpha$ vanishing at 0 is tight if and only if for each positive ε,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \mu\left\{\sup_{j\geqslant J} 2^{j\alpha} \max_{r\in D_j} |\lambda_r(\xi_n)| > \varepsilon\right\} = 0. \tag{1.7}$$

Notation 1.1. In the sequel, we will denote $r_{k,j} := k2^{-j}$ and $u_{k,j} := [nr_{k,j}]$ (or r_k and u_k for short). Notice that $u_{k+1,j} - u_{k,j} = [nr_{k,j} + n2^{-j}] - u_{k,j} \leqslant 2n2^{-j}$ if $j \leqslant \log n$, where $\log n$ denotes the binary logarithm of n and for a real number x, $[x]$ is the unique integer for which $[x] \leqslant x < [x]+1$.

Remark 1.2. Since for each $x \in \mathcal{H}_{1/2-1/p}[0,1]$, each $j \geqslant 1$ and each $r \in D_j$,

$$|\lambda_r(x)| \leqslant \frac{|x(r^+)-x(r)|}{2} + \frac{|x(r)-x(r^-)|}{2} \leqslant \\ \leqslant \max\left\{\left|x(r^+)-x(r)\right|, \left|x(r)-x(r^-)\right|\right\}, \tag{1.8}$$

for a function f, the sequential norm of $n^{-1/2}S^{\mathrm{pl}}_n(f)$ does not exceed

$$\sup_{j\geqslant 1} 2^{\alpha j} n^{-1/2} \max_{0\leqslant k<2^j} \left|S^{\mathrm{pl}}_n(f, r_{k+1,j}) - S^{\mathrm{pl}}_n(f, r_{k,j})\right|. \tag{1.9}$$

Now, we state the result obtained by Račkauskas and Suquet in [RS03].

Theorem 1.3. *Let $p > 2$ and let $(f \circ T^j)_{j \geqslant 0}$ be an i.i.d. centered sequence with unit variance. Then the condition*

$$\lim_{t \to \infty} t^p \mu \{|f| > t\} = 0 \tag{1.10}$$

is equivalent to the weak convergence of the sequence $(n^{-1/2} S_n^{\mathrm{pl}}(f))_{n \geqslant 1}$ to a standard Brownian motion in the space $\mathcal{H}^o_{1/2-1/p}[0,1]$.

1.2. **Some facts about the $\mathbb{L}^{p,\infty}$ spaces.** In the rest of the paper, χ denotes the indicator function. Let $p > 2$. We define the $\mathbb{L}^{p,\infty}$ space as the collection of functions $f \colon \Omega \to \mathbb{R}$ such that the quantity

$$\|f\|_{p,\infty}^p := \sup_{t>0} t^p \mu \{|f| > t\} < \infty. \tag{1.11}$$

This quantity is denoted like a norm, while it is not a norm (the triangle inequality may fail, for example if $X = [0,1]$ endowed with the Lebesgue measure, $f(x) := x^{-1/p}$ and $g(x) := f(1-x)$; in this case $\|f+g\|_{p,\infty} \geqslant 2^{1+1/p}$ but $\|f\|_{p,\infty} + \|g\|_{p,\infty} = 2$). However, there exists a constant κ_p such that for each f,

$$\|f\|_{p,\infty} \leqslant \sup_{A : \mu(A) > 0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \chi_A] \leqslant \kappa_p \|f\|_{p,\infty} \tag{1.12}$$

and $N_p(f) := \sup_{A:\mu(A)>0} \mu(A)^{-1+1/p} \mathbb{E}[|f| \chi_A]$ defines a norm. The first inequality in (1.12) can be seen from the estimate $t\mu \{|f| > t\} \leqslant \mathbb{E}[|f| \chi \{|f| > t\}]$; for the second one, we write

$$\mathbb{E}[|f| \chi_A] = \int_0^{+\infty} \mu(\{|f| > t\} \cap A) \, \mathrm{d}t \leqslant \int_0^{+\infty} \min\{\mu\{|f| > t\}, \mu(A)\} \, \mathrm{d}t, \tag{1.13}$$

and we bound the integrand by $\min\left\{t^{-p} \|f\|_{p,\infty}^p, \mu(A)\right\}$.

A function f satisfies (1.10) if and only if it belongs to the closure of bounded functions with respect to N_p. Indeed, if f satisfies (1.10), then the sequence $(f\chi|f < n|)_{n \geqslant 1}$ converges to f in $\mathbb{L}^{p,\infty}$. If $N_p(f-g) < \varepsilon$ with g bounded, then

$$\limsup_{t \to \infty} t^p \mu \{|f| > t\} \leqslant \limsup_{t \to \infty} t^p \mu \{|f - g| > t/2\} \leqslant 2^p \varepsilon. \tag{1.14}$$

We now provide two technical lemmas about $\mathbb{L}^{p,\infty}$ spaces. The first one will be used in the proof of the weak invariance principle for martingales, since we will have to control the tail function of the random variables involved in the construction of the truncated martingale (cf.(3.111)). The second one will provides an estimation of the $\mathbb{L}^{p,\infty}$ norm of a simple function, which will be used in the proof of Theorem 2.1, since the function m is contructed as a series of simple functions.

Lemma 1.4. *If $\lim_{t\to\infty} t^p \mu \{|f| > t\} = 0$, then for each sub-$\sigma$-algebra $\mathcal{A}$, we have $\lim_{t\to\infty} t^p \mu \{\mathbb{E}[|f| \mid \mathcal{A}] > t\} = 0$.*

Proof. For simplicity, we assume that f is non-negative. For a fixed t, the set $\{\mathbb{E}[f \mid \mathcal{A}] > t\}$ belongs to the σ-algebra $\mathcal{A}$, hence

$$t\mu \{\mathbb{E}[f \mid \mathcal{A}] > t\} \leqslant \mathbb{E}[\mathbb{E}[f \mid \mathcal{A}] \chi \{\mathbb{E}[f \mid \mathcal{A}] > t\}] = \mathbb{E}[f\chi \{\mathbb{E}[f \mid \mathcal{A}] > t\}]. \tag{1.15}$$

By definition of N_p,

$$\mathbb{E}[f\chi \{\mathbb{E}[f \mid \mathcal{A}] > t\}] \leqslant N_p(f\chi \{\mathbb{E}[f \mid \mathcal{A}] > t\}) \, \mu \{\mathbb{E}[f \mid \mathcal{A}] > t\}^{1-1/p}, \tag{1.16}$$

hence

$$t^p \mu \{\mathbb{E}[f \mid \mathcal{A}] > t\} \leqslant N_p(f\chi \{\mathbb{E}[f \mid \mathcal{A}] > t\})^p. \tag{1.17}$$

Notice that

$$\forall s > 0, \quad N_p\left(f\chi\left\{\mathbb{E}[f \mid \mathcal{A}] > t\right\}\right) \leqslant s\mu\left\{\mathbb{E}[f \mid \mathcal{A}] > t\right\}^{1/p} + N_p\left(f\chi\left\{f > s\right\}\right), \tag{1.18}$$

hence

$$\limsup_{t\to\infty} \mathbb{E}[f\chi\left\{\mathbb{E}[|f| \mid \mathcal{A}] > t\right\}] \leqslant N_p(f\chi\left\{f > s\right\} \leqslant \kappa_p \sup_{x\geqslant s} x^p \mu\left\{f > x\right\}. \tag{1.19}$$

By the assumption on the function f, the right hand side goes to 0 as s goes to infinity, which concludes the proof of the lemma. □

Lemma 1.5. *Let $f := \sum_{i=0}^{N} a_i\chi(A_i)$, where the family $(A_i)_{i=0}^{N}$ is pairwise disjoint and $0 \leqslant a_N < \cdots < a_0$. Then*

$$\|f\|_{p,\infty}^p \leqslant \max_{0\leqslant j\leqslant N} a_j^p \sum_{i=0}^{j} \mu(A_i). \tag{1.20}$$

Proof. We have the equality

$$\mu\left\{f > t\right\} = \sum_{j=0}^{N} \chi_{(a_{j+1},a_j]}(t) \sum_{i=0}^{j} \mu(A_i), \tag{1.21}$$

where $a_{N+1} := 0$, therefore

$$t^p\mu\left\{f > t\right\} \leqslant \max_{0\leqslant j\leqslant N} a_j^p \sum_{i=0}^{j} \mu(A_i). \tag{1.22}$$

□

2. Main results

The goal of the paper is to give a sharp sufficient condition on the moments of a strictly stationary martingale difference sequence which guarantees the weak invariance principle in $\mathcal{H}_\alpha^o[0,1]$ for a fixed α.

We first show that Theorem 1.3 does not extend to strictly stationary ergodic martingale difference sequences, that is, sequences of the form $(m \circ T^i)_{i\geqslant 0}$ such that m is $\mathcal{M}$ measurable and $\mathbb{E}[m \mid T\mathcal{M}] = 0$ for some sub-σ-algebra $\mathcal{M}$ satisfying $T\mathcal{M} \subset \mathcal{M}$.

An application of Kolmogorov's continuity criterion shows that if $(m \circ T^i)_{i\geqslant 0}$ is a martingale difference sequence such that $m \in \mathbb{L}^{p+\delta}$ for some positive δ and $p > 2$, then the partial sum process $(n^{-1/2}S_n^{\mathrm{pl}}(m))_{n\geqslant 1}$ is tight in $\mathcal{H}_{1/2-1/p}^o[0,1]$ (see [KR91]).

We provide a condition on the quadratic variance which improves the previous approach (since the previous condition can be replaced by $m \in \mathbb{L}^p$). Then using martingale approximation we can provide a Hannan type condition which guarantees the weak invariance principle in $\mathcal{H}_\alpha^o[0,1]$.

Theorem 2.1. *Let $p > 2$ and $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system with positive entropy. There exists a function $m\colon \Omega \to \mathbb{R}$ and a σ-algebra $\mathcal{M}$ for which $T\mathcal{M} \subset \mathcal{M}$ such that:*

- *the sequence $(m \circ T^i)_{i\geqslant 0}$ is a martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i\geqslant 0}$;*
- *the convergence $\lim_{t\to+\infty} t^p\mu\left\{|m| > t\right\} = 0$ takes place;*
- *the sequence $(n^{-1/2}S_n^{\mathrm{pl}}(m))_{n\geqslant 1}$ is not tight in $\mathcal{H}_{1/2-1/p}^o[0,1]$.*

Theorem 2.2. *Let $(\Omega, \mathcal{F}, \mu, T)$ be a dynamical system, $\mathcal{M}$ a sub-σ-algebra of $\mathcal{F}$ such that $T\mathcal{M} \subset \mathcal{M}$ and $\mathcal{I}$ the collection of sets $A \in \mathcal{F}$ such that $T^{-1} = A$.*

Let $p > 2$ and let $(m \circ T^j, T^{-i}\mathcal{M})$ be a strictly stationary martingale difference sequence. Assume that $t^p \mu\{|m| > t\} \to 0$ and $\mathbb{E}[m^2 \mid T\mathcal{M}] \in \mathbb{L}^{p/2}$. Then

$$n^{-1/2} S_n^{\mathrm{pl}}(m) \to \eta \cdot W \text{ in distribution in } \mathcal{H}^o_{1/2-1/p}[0,1], \tag{2.1}$$

where the random variable η is given by

$$\eta = \lim_{n\to\infty} \mathbb{E}[S_n^2 \mid \mathcal{I}]/n \text{ in } \mathbb{L}^1 \tag{2.2}$$

and η is independent of the process $(W_t)_{t\in[0,1]}$.

In particular, (2.1) takes place if m belongs to $\mathbb{L}^p$.

The key point of the proof of Theorem 2.2 is an inequality in the spirit of Doob's one, which gives $n^{-1}\mathbb{E}\left[\max_{1\leqslant j\leqslant n} S_j(m)^2\right] \leqslant 2\mathbb{E}[m^2]$. It is used in order to establish tightness of the sequence $(n^{-1/2} S_n^{\mathrm{pl}}(m))_{n\geqslant 1}$ in the space $C[0,1]$.

Proposition 2.3. *Let $p > 2$. There exists a constant C_p depending only on p such that if $(m \circ T^i)_{i\geqslant 1}$ is a martingale difference sequence, then the following inequality holds:*

$$\sup_{n\geqslant 1} \left\| \left\| n^{-1/2} S_n^{\mathrm{pl}}(m) \right\|_{\mathcal{H}^o_{1/2-1/p}} \right\|_{p,\infty}^p \leqslant C_p \left(\|m\|_{p,\infty}^p + \mathbb{E}\left(\mathbb{E}[m^2 \mid T\mathcal{M}]\right)^{p/2} \right). \tag{2.3}$$

Remark 2.4. As Theorem 2.1 shows, the condition $\lim_{t\to\infty} t^p \mu\{|m| > t\} = 0$ alone for martingale difference sequences is not sufficient to obtain the weak convergence of $n^{-1/2} S_n^{\mathrm{pl}}(m)$ in $\mathcal{H}^o_\alpha[0,1]$ for $\alpha = 1/2 - 1/p$. For the constructed m in Theorem 2.1, the quadratic variance is κm^2 for some constant κ and m does not belong to the $\mathbb{L}^p$ space.

By Lemma A.2 in [MRS12], the Hölder norm of a polygonal line is reached at two vertices, hence, for a function g,

$$\left\| n^{-1/2} S_n^{\mathrm{pl}}(g - g\circ T) \right\|_{\mathcal{H}^o_{1/2-1/p}} = n^{-1/p} \max_{1\leqslant i<j\leqslant n} \frac{\left| g\circ T^j - g \circ T^i \right|}{(j-i)^{1/2-1/p}} \tag{2.4}$$

$$\leqslant 2n^{-1/p} \max_{1\leqslant j\leqslant n} \left| g \circ T^j \right|. \tag{2.5}$$

As a consequence, if g belongs to $\mathbb{L}^p$, then the sequence $\left(\left\| n^{-1/2} S_n^{\mathrm{pl}}(g - g\circ T) \right\|_{\mathcal{H}^o_{1/2-1/p}} \right)_{n\geqslant 1}$ converges to 0 in probability . Therefore, we can exploit a martingale-coboundary decomposition in $\mathbb{L}^p$.

Corollary 2.5. *Let $p > 2$ and let f be an $\mathcal{M}$-measurable function which can be written as*

$$f = m + g - g \circ T, \tag{2.6}$$

where $m, g \in \mathbb{L}^p$ and $(m\circ T^i)_{i\geqslant 0}$ is a martingale difference sequence for the filtration $(T^{-i}\mathcal{M})_{i\geqslant 0}$. Then $n^{-1/2} S_n^{\mathrm{pl}}(f) \to \eta W$ in distribution in $\mathcal{H}^o_{1/2-1/p}[0,1]$, where η is given by (2.2) and independent of W.

We define for a function h the operators $\mathbb{E}_k(h) := \mathbb{E}[h \mid T^k\mathcal{M}]$ and $P_i(h) := \mathbb{E}_i(h) - \mathbb{E}_{i+1}(h)$. The condition $\sum_{i=0}^\infty \|P_i(f)\|_2$ was introduced by Hannan in [Han73] in order to deduce a central limit theorem. It actually implies the weak invariance principle (see Corollary 2 in [DMV07]).

Theorem 2.6. *Let $p > 2$ and let f be an $\mathcal{M}$-measurable function such that*

$$\mathbb{E}\left[f \mid \bigcap_{i\in\mathbb{Z}} T^i\mathcal{M}\right] = 0 \text{ and} \tag{2.7}$$

$$\sum_{i\geqslant 0} \|P_i(f)\|_p < \infty. \tag{2.8}$$

Then $n^{-1/2}S_n^{\mathrm{pl}}(m) \to \eta W$ in distribution in $\mathcal{H}^o_{1/2-1/p}[0,1]$, where η is given by (2.2) and independent of W.

3. Proofs

3.1. **Proof of Theorem 2.1.** We need a result about dynamical systems of positive entropy for the construction of a counter-example.

Lemma 3.1. *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic probability measure preserving system of positive entropy. There exists two T-invariant sub-σ-algebras $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$ and a function $g\colon \Omega \to \mathbb{R}$ such that:*

- *the σ-algebras $\mathcal{B}$ and $\mathcal{C}$ are independent;*
- *the function g is $\mathcal{B}$-measurable, takes the values -1, 0 and 1, has zero mean and the process $(g \circ T^n)_{n\in\mathbb{Z}}$ is independent;*
- *the dynamical system $(\Omega, \mathcal{C}, \mu, T)$ is aperiodic.*

This is Lemma 3.8 from [LV01].

We consider the following four increasing sequences of positive integers $(I_l)_{l\geqslant 1}$, $(J_l)_{l\geqslant 1}$, $(n_l)_{l\geqslant 1}$ and $(L_l)_{l\geqslant 1}$. We define $k_l := 2^{I_l+J_l}$ and impose the conditions:

$$\sum_{l=1}^{\infty} \frac{1}{L_l} < \infty; \tag{3.1}$$

$$\lim_{l\to\infty} J_l \cdot \mu\left\{|\mathcal{N}| \geqslant 4^{1/p}\frac{L_l}{\|g\|_2}\right\} = 1; \tag{3.2}$$

$$\lim_{l\to\infty} J_l 2^{-I_l/2} = 0; \tag{3.3}$$

$$\lim_{l\to\infty} n_l \sum_{i>l} \frac{k_i}{n_i} = 0; \tag{3.4}$$

$$\text{for each } l, \quad \sum_{i=1}^{l-1} \frac{k_l}{L_i}\left(\frac{n_i}{2^{I_i}}\right)^{1/p} < \frac{n_l^{1/p}}{2}. \tag{3.5}$$

Here $\mathcal{N}$ denotes a random variable whose distribution is standard normal. Such sequences can be constructed as follows: first pick a sequence $(L_l)_{l\geqslant 1}$ satisfying (3.1), for example $L_l = l^2$. Then construct $(J_l)_{l\geqslant 1}$ such that (3.2) holds. Once the sequence $(J_l)_{l\geqslant 1}$ is constructed, define $(I_l)_{l\geqslant 1}$ satisfying (3.3). Now the sequence $(k_l)_{l\geqslant 1}$ is completely determined. Noticing that (3.4) is satisfied if the series $\sum_l k_l n_{l-1}/n_l$ converges, we construct the sequence $(n_l)_{l\geqslant 1}$ by induction; once $n_i, i \leqslant l-1$ are defined, we choose n_l such that $n_l \geqslant l^2 k_l n_{l-1}$ and (3.5) holds.

Using Rokhlin's lemma, we can find for any integer $l \geqslant 1$ a measurable set $C_l \in \mathcal{C}$ such that the sets $T^{-i}C_l$, $i = 0, \ldots, n_l - 1$ are pairwise disjoint and $\mu\left(\bigcup_{i=0}^{n_l-1} T^{-i}C_l\right) > 1/2$.

For a fixed l, we define

$$k_{l,j} := 2^{I_l+J_l-j}, \quad 0 \leqslant j \leqslant J_l, \tag{3.6}$$

$$k_{l,j} := 2^{I_l+J_l-j}, \quad 0 \leqslant j \leqslant J_l \text{ and} \tag{3.7}$$

$$f_l := \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left(\frac{n_l}{k_{l,J_l-j}}\right)^{1/p} \chi\left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i}C_l\right) + \\ + \frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l}}\right)^{1/p} \chi\left(\bigcup_{i=0}^{k_{l,J_l}-1} T^{-i}C_l\right),$$

$$f := \sum_{l=1}^{+\infty} f_l, \quad m := g \cdot f, \tag{3.8}$$

where g is the function obtained by Lemma 3.1.

Proposition 3.2. *We have the estimate* $\|f_l\|_{p,\infty} \leqslant \kappa'_p L_l^{-1}$ *for some constant* κ'_p *depending only on* p. *As a consequence,* $\lim_{t\to\infty} t^p \mu\{|m| > t\} = 0$.

Proof. Notice that

$$\left\| \frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l}}\right)^{1/p} \chi\left(\bigcup_{i=0}^{k_{l,J_l}-1} T^{-i}C_l\right)\right\|_{p,\infty}^p = \frac{1}{L_l^p}\frac{n_l}{k_{l,J_l}} k_{l,J_l} \cdot \mu(C_l) \leqslant \frac{1}{L_l^p}. \tag{3.9}$$

Next, using Lemma 1.5 with $N := J_l - 1$, $a_j := \frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l-j}}\right)^{1/p}$ and $A_j := \bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}-1} T^{-i}C_l$, we obtain

$$\left\| \frac{1}{L_l} \sum_{j=0}^{J_l-1} \left(\frac{n_l}{k_{l,J_l-j}}\right)^{1/p} \chi\left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}} T^{-i}C_l\right)\right\|_{p,\infty}^p \leqslant \max_{0\leqslant j\leqslant J_l-1} \left(\frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l-j}}\right)^{1/p}\right)^p \sum_{i=0}^{j} \mu(A_j) \tag{3.10}$$

$$\leqslant \frac{1}{L_l^p} \max_{0\leqslant j\leqslant J_l-1} \frac{n_l}{k_{l,J_l-j}} \sum_{i=0}^{j} \frac{k_{l,J_l-i}}{n_l} \tag{3.11}$$

$$= \frac{1}{L_l^p} \max_{0\leqslant j\leqslant J_l-1} \sum_{i=0}^{j} \frac{2^{I_l+i}}{2^{I_l+j}} \tag{3.12}$$

$$\leqslant \frac{2}{L_l^p}, \tag{3.13}$$

hence by (1.12), (3.9) and (3.13),

$$\|f_l\|_{p,\infty} \leqslant N_p\left(\frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l}}\right)^{1/p}\chi\left(\bigcup_{i=0}^{k_{l,J_l}-1}T^{-i}C_l\right)\right)+ \tag{3.14}$$

$$+N_p\left(\frac{1}{L_l}\sum_{j=0}^{J_l-1}\left(\frac{n_l}{k_{l,J_l-j}}\right)^{1/p}\chi\left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}}T^{-i}C_l\right)\right) \tag{3.15}$$

$$\leqslant \kappa_p\left\|\frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l}}\right)^{1/p}\chi\left(\bigcup_{i=0}^{k_{l,J_l}-1}T^{-i}C_l\right)\right\|_{p,\infty}+ \tag{3.16}$$

$$+\kappa_p\left\|\frac{1}{L_l}\sum_{j=0}^{J_l-1}\left(\frac{n_l}{k_{l,J_l-j}}\right)^{1/p}\chi\left(\bigcup_{i=k_{l,J_l-j}}^{k_{l,J_l-j-1}}T^{-i}C_l\right)\right\|_{p,\infty} \tag{3.17}$$

$$\leqslant \frac{1}{L_l}\kappa_p\left(1+2^{1/p}\right) \tag{3.18}$$

We thus define $\kappa'_p := \kappa_p\left(1+2^{1/p}\right)$.

We fix $\varepsilon > 0$; using (3.1), we can find an integer l_0 such that $\sum_{l>l_0} 1/L_l < \varepsilon$. Since the function $\sum_{l=1}^{l_0} gf_l$ is bounded, we have,

$$\limsup_{t\to\infty} t^p\mu\left\{|m|>t\right\} \leqslant \limsup_{t\to\infty} t^p\mu\left\{\left|\sum_{l=1}^{l_0} gf_l\right|>\frac{t}{2}\right\}+2^p\left\|\sum_{l>l_0} gf_l\right\|_{p,\infty}^p \tag{3.19}$$

$$=2^p\left\|\sum_{l>l_0} gf_l\right\|_{p,\infty}^p \tag{3.20}$$

$$\leqslant\left(2\sum_{l>l_0} N_p(f_l)\right)^p \tag{3.21}$$

$$\leqslant \kappa'_p\left(\sum_{l>l_0}\frac{1}{L_l}\right)^p \tag{3.22}$$

$$\leqslant \kappa'_p\varepsilon^p, \tag{3.23}$$

where the second inequality comes from inequalities (1.12). Since ε is arbitrary, the proof of Lemma 3.2 is complete. □

We denote by $\mathcal{M}$ the σ-algebra generated by $\mathcal{C}$ and the random variables $g\circ T^k$, $k\leqslant 0$. It satisfies $\mathcal{M}\subset T^{-1}\mathcal{M}$.

Proposition 3.3. *The sequence $(m\circ T^i)_{i\geqslant 0}$ is a (stationary) martingale difference sequence with respect to the filtration $(T^{-i}\mathcal{M})_{i\geqslant 0}$.*

Proof. We have to show that $\mathbb{E}[m\mid T\mathcal{M}]=0$. Since the σ-algebra $\mathcal{C}$ is T-invariant, we have $T\mathcal{M}=\sigma(\mathcal{C}\cup\sigma(g\circ T^k, k\leqslant -1))$. This implies

$$\mathbb{E}[m\mid T\mathcal{M}]=\mathbb{E}[gf\mid T\mathcal{M}]=f\cdot\mathbb{E}[g\mid T\mathcal{M}]. \tag{3.24}$$

Since g is centered and independent of $T\mathcal{M}$, Proposition 3.3 is proved. □

It remains to prove that the process $(n^{-1/2}S_n^{\mathrm{pl}}(m))_{n\geqslant 1}$ is not tight in $\mathcal{H}^o_{1/2-1/p}[0,1]$.

Proposition 3.4. *Under conditions* (3.2), (3.3) *and* (3.4), *there exists an integer* l_0 *such that for* $l \geqslant l_0$

$$P_l := \mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(gf_l)-S_u(gf_l)|}{v^{1/2-1/p}}\geqslant 1\right\}\geqslant\frac{1}{16}. \tag{3.25}$$

Proof. Let us fix an integer $l\geqslant 1$. Assume that $\omega\in T^{-s}C_l$, where $k_l\leqslant s\leqslant n_l-1$. Since $T^u\omega$ belongs to $T^{-(s-u)}C_l$ we have for $s-n_l\leqslant u\leqslant s$

$$(f_l\circ T^u)(\omega)=\begin{cases}\frac{1}{L_l}\left(\frac{n_l}{k_{l,J_l}}\right)^{1/p}, & \text{if } s-k_{l,J_l}<u\leqslant s;\\ \frac{1}{L_l}\left(\frac{n_l}{k_{l,j}}\right)^{1/p}, & \text{if } s-k_{l,j-1}<u\leqslant s-k_{l,j}, \text{ and } 1\leqslant j\leqslant J_l;\\ 0, & \text{if } s-n_l\leqslant u<s-k_l.\end{cases} \tag{3.26}$$

As a consequence,

$$\begin{aligned}T^{-s}C_l\cap\left\{\frac{1}{n_l^{1/p}}\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{s-k_{l,j-1}+1}(gf_l)-S_{s-k_{l,j}}(gf_l)\right|}{(k_{l,j-1}-1-k_{l,j})^{1/2-1/p}}\geqslant 1\right\}\\ =T^{-s}C_l\cap\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{s-k_{l,j-1}+1}(g)-S_{s-k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\}.\end{aligned} \tag{3.27}$$

Since for $k_l+1\leqslant s\leqslant n_l-k_l$ and $1\leqslant j\leqslant J_l$, we have $1\leqslant s-k_{l,j}\leqslant n_l-k_l$ and $1\leqslant k_{l,j-1}-1-k_{l,j}\leqslant k_l$, the inequality

$$\begin{aligned}\chi(T^{-s}C_l)\cdot\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{s-k_{l,j-1}+1}(gf_l)-S_{s-k_{l,j}}(gf_l)\right|}{(k_{l,j-1}-1-k_{l,j})^{1/2-1/p}}\\ \leqslant\chi(T^{-s}(C_l))\max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(gf_l)-S_u(gf_l)|}{v^{1/2-1/p}}\end{aligned} \tag{3.28}$$

takes place and since the sets $(T^{-s}C_l)_{s=0}^{n_k-1}$ are pairwise disjoint, we obtain the lower bound

$$P_l\geqslant\sum_{s=1}^{n_l-2k_l}\mu\left(T^{-(s+k_l)}(C_l)\cap\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{s+k_l-k_{l,j-1}+1}(gf_l)-S_{s+k_l-k_{l,j}}(gf_l)\right|}{(k_{l,j-1}-1-k_{l,j})^{1/2-1/p}}\geqslant 1\right\}\right). \tag{3.29}$$

Using the fact that T is measure-preserving, this becomes

$$P_l\geqslant(n_l-2k_l)\cdot\mu\left(T^{-k_l}(C_l)\cap\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_l-k_{l,j-1}-1}(gf_l)-S_{k_l-k_{l,j}}(gf_l)\right|}{(k_{l,j-1}-1-k_{l,j})^{1/2-1/p}}\geqslant 1\right\}\right), \tag{3.30}$$

and plugging (3.27) in the previous estimate, we get

$$P_l\geqslant(n_l-2k_l)\mu\left(T^{-k_l}(C_l)\cap\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_l-k_{l,j-1}-1}(g)-S_{k_l-k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\}\right). \tag{3.31}$$

The sets $\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_l-k_{l,j-1}+1}(g)-S_{k_l-k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\}$ and $T^{-k_l}C_l$ belong to the independent sub-σ-algebras $\mathcal{B}$ and $\mathcal{C}$ respectively, hence using the fact that the sequences $(g\circ T^i)_{i\geqslant 0}$ and $(g\circ T^{-i})_{i\geqslant 0}$ are identically distributed, we obtain

$$P_l\geqslant (n_l-2k_l)\mu\left(C_l\right)\mu\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_{l,j-1}-1}(g)-S_{k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\}. \tag{3.32}$$

By construction, we have $n_l\cdot\mu(C_l)=\mu\left(\bigcup_{i=0}^{n_l-1}T^{-i}C_l\right)>1/2$, hence

$$P_l\geqslant\frac{1}{2}\left(1-2\frac{k_l}{n_l}\right)\mu\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_{l,j-1}-1}(g)-S_{k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\}. \tag{3.33}$$

It remains to find a lower bound for

$$P_l':=\mu\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_{l,j-1}-1}(g)-S_{k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\}. \tag{3.34}$$

Let us define the set

$$E_j:=\left\{\frac{\left|S_{k_{l,j-1}-1}(g)-S_{k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}}\geqslant L_l\right\} \tag{3.35}$$

Since the sequence $(g\circ T^i)_{i\in\mathbb{Z}}$ is independent, the family $(E_j)_{1\leqslant j\leqslant J_l}$ is independent, hence

$$P_l'\geqslant 1-\prod_{j=1}^{J_l}(1-\mu(E_j)). \tag{3.36}$$

We define the quantity

$$c_j:=\mu\left\{|\mathcal{N}|\geqslant\frac{L_l}{\|g\|_2}\left(\frac{k_{l,j-1}}{k_{l,j}-1}\right)^{1/p}\right\} \tag{3.37}$$

(we recall that $\mathcal{N}$ denotes a standard normally distributed random variable). By the Berry-Esseen theorem, we have for each $j\in\{1,\dots,J_l\}$,

$$|\mu(E_j)-c_j|\leqslant\frac{1}{\|g\|_2^3}\frac{1}{(k_{l,j-1}-1)^{1/2}}\leqslant\frac{\sqrt{2}}{\|g\|_2^3}2^{-I_l/2}. \tag{3.38}$$

Plugging the estimate (3.38) into (3.36) and noticing that for an integer N and $(a_n)_{n=1}^N$, $(b_n)_{n=1}^N$ two families of numbers in the unit interval,

$$\left|\prod_{n=1}^N a_n-\prod_{n=1}^N b_n\right|\leqslant\sum_{n=1}^N|a_n-b_n|, \tag{3.39}$$

we obtain

$$P_l' \geqslant 1 - \prod_{j=1}^{J_l}(1-\mu(E_j)) + \prod_{j=1}^{J_l}(1-c_j) - \prod_{j=1}^{J_l}(1-c_j) \tag{3.40}$$

$$\geqslant 1 - \prod_{j=1}^{J_l}(1-c_j) - \sum_{j=1}^{J_l}|\mu(E_j)-c_j| \tag{3.41}$$

$$\geqslant 1 - \prod_{j=1}^{J_l}(1-c_j) - J_l\frac{\sqrt{2}}{\|g\|_2^3}2^{-I_l/2}. \tag{3.42}$$

Notice that

$$1 - \prod_{j=1}^{J_l}(1-c_j) \geqslant 1 - \max_{1\leqslant j\leqslant J_l}(1-c_j)^{J_l} \tag{3.43}$$

and since $(I_l)_{\geqslant}$ is increasing and $I_1 \geqslant 1$, we have

$$\frac{k_{l,j-1}}{k_{l,j}-1} = \frac{2}{1-k_{l,j}^{-1}} \leqslant \frac{2}{1-2^{-I_l}} \leqslant 4 \tag{3.44}$$

it follows by (3.37) that $c_j \geqslant \mu\left\{|\mathcal{N}| \geqslant 4^{1/p}\frac{L_l}{\|g\|_2}\right\}$ for $1 \leqslant j \leqslant J_l$. We thus have

$$P_l' \geqslant 1 - \left(1 - \mu\left\{|\mathcal{N}| \geqslant 4^{1/p}\frac{L_l}{\|g\|_2}\right\}\right)^{J_l} - J_l\frac{\sqrt{2}}{\|g\|_2^3}2^{-I_l/2}. \tag{3.45}$$

Using the elementary inequality

$$1-(1-t)^n \geqslant nt - \frac{n(n-1)}{2}t^2 \tag{3.46}$$

valid for a positive integer n and $t \in [0,1]$, we obtain

$$P_l' \geqslant J_l\mu\left\{|\mathcal{N}| \geqslant 4^{1/p}\frac{L_l}{\|g\|_2}\right\} - \frac{J_l^2}{2}\left(\mu\left\{|\mathcal{N}| \geqslant 4^{1/p}\frac{L_l}{\|g\|_2}\right\}\right)^2 - J_l\frac{\sqrt{2}}{\|g\|_2^3}2^{-I_l/2}. \tag{3.47}$$

By conditions (3.3) and (3.2), there exists an integer l_0' such that if $l \geqslant l_0'$, then

$$\mu\left\{\max_{1\leqslant j\leqslant J_l}\frac{\left|S_{k_{l,j-1}-1}(g) - S_{k_{l,j}}(g)\right|}{(k_{l,j}-1)^{1/2-1/p}k_{l,j-1}^{1/p}} \geqslant L_l\right\} \geqslant \frac{1}{4}. \tag{3.48}$$

Combining (3.33) with (3.48), we obtain for $l \geqslant l_0'$

$$P_l \geqslant \frac{1}{8}\left(1 - 2\frac{k_l}{n_l}\right). \tag{3.49}$$

By condition (3.4), we thus get that $P_l \geqslant 1/16$ for $l \geqslant l_0$, where $l_0 \geqslant l_0'$ and $k_l/n_l \leqslant 1/4$ if $l \geqslant l_0$.

This concludes the proof of Proposition 3.4. □

Proposition 3.5. *Under conditions* (3.1), (3.2), (3.3), (3.4) *and* (3.5), *we have for l large enough*

$$\mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(m) - S_u(m)|}{v^{1/2-1/p}} \geqslant \frac{1}{2}\right\} \geqslant \frac{1}{32}. \tag{3.50}$$

Since the Hölder modulus of continuity of a piecewise linear function is reached at vertices, we derive the following corollary.

Corollary 3.6. *If $l \geqslant l_0$, then*

$$\mu\left\{\omega_{1/2-1/p}\left(\frac{1}{\sqrt{n_l}}S^{\mathrm{pl}}_{n_l}(m), \frac{k_l}{n_l}\right) \geqslant \frac{1}{2}\right\} \geqslant \frac{1}{32}. \tag{3.51}$$

Therefore, for each positive δ, we have

$$\limsup_{n\to\infty} \mu\left\{\omega_{1/2-1/p}\left(\frac{1}{\sqrt{n}}S^{\mathrm{pl}}_{n}(m), \delta\right) \geqslant \frac{1}{2}\right\} \geqslant \frac{1}{32}, \tag{3.52}$$

and the process $(n^{-1/2}S^{\mathrm{pl}}_n(m))_{n\geqslant 1}$ is not tight in $\mathcal{H}^o_{1/2-1/p}[0,1]$.

Proof of Proposition 3.5. Let l_0 be the integer given by Proposition 3.4 and let $l \geqslant l_0$. We define $m'_l := \sum_{i=1}^{l-1} gf_i$ and $m''_l := \sum_{i=l+1}^{+\infty} gf_i$.

We define for $i \geqslant 1$,

$$M_{l,i} := \frac{1}{n_l^{1/p}} \max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}} \frac{|S_{u+v}(gf_i) - S_u(gf_i)|}{v^{1/2-1/p}}. \tag{3.53}$$

Let i be an integer such that $i < l$. Notice that for $1 \leqslant u \leqslant n_l - k_l$ and $v \leqslant k_l$, we have

$$|S_{u+v}(gf_i) - S_u(gf_i)| = U^u(|S_v(gf_i)|), \tag{3.54}$$

where $U(h)(\omega) = h\left(T\left(\omega\right)\right)$ and since

$$|S_v(gf_i)| \leqslant v\left\|gf_i\right\|_\infty \leqslant \frac{k_l}{L_i}\left(\frac{n_i}{2^{I_i}}\right)^{1/p}, \tag{3.55}$$

the estimate

$$M_{l,i} \leqslant \frac{k_l}{L_i n_l^{1/p}}\left(\frac{n_i}{2^{I_i}}\right)^{1/p} \tag{3.56}$$

holds. Since

$$\frac{1}{n_l^{1/p}} \max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leqslant \sum_{i=1}^{l-1} M_{l,i}, \tag{3.57}$$

we have by (3.56),

$$\frac{1}{n_l^{1/p}} \max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leqslant \sum_{i=1}^{l-1} \frac{k_l}{L_i n_l^{1/p}}\left(\frac{n_i}{2^{I_i}}\right)^{1/p}. \tag{3.58}$$

By (3.5), the following bound takes place:

$$\frac{1}{n_l^{1/p}} \max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}} \frac{|S_{u+v}(m'_l) - S_u(m'_l)|}{v^{1/2-1/p}} \leqslant \frac{1}{2}. \tag{3.59}$$

The following set inclusions hold

$$\left\{\frac{1}{n_l^{1/p}} \max_{\substack{1\leqslant u\leqslant n_l-k_l\\ 1\leqslant v\leqslant k_l}} \frac{|S_{u+v}(m''_l) - S_u(m''_l)|}{v^{1/2-1/p}} \neq 0\right\} \subset \bigcup_{i>l}\{M_{l,i} \neq 0\} \tag{3.60}$$

$$\subset \bigcup_{i>l}\bigcup_{u=1}^{n_l}\{U^u(gf_i) \neq 0\}. \tag{3.61}$$

We thus have

$$\mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(m_l'')-S_u(m_l'')|}{v^{1/2-1/p}}\neq 0\right\}\leqslant\sum_{i>l}n_l\cdot\mu\left\{gf_i\neq 0\right\} \tag{3.62}$$

$$\leqslant n_l\sum_{i>l}\mu\left\{f_i\neq 0\right\} \tag{3.63}$$

$$=n_l\sum_{i>l}(k_i+1)\mu(C_i) \tag{3.64}$$

$$\leqslant 2n_l\sum_{i>l}\frac{k_i}{n_i}. \tag{3.65}$$

and by (3.4), it follows that

$$\mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(m_l'')-S_u(m_l'')|}{v^{1/2-1/p}}\neq 0\right\}\leqslant\frac{1}{32} \tag{3.66}$$

Accounting (3.59), we thus have

$$\begin{aligned}\mu&\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(m)-S_u(m)|}{v^{1/2-1/p}}\geqslant\frac{1}{2}\right\}\\ &\geqslant\mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(gf_l+m_l'')-S_u(gf_l+m_l'')|}{v^{1/2-1/p}}\geqslant 1\right\}\\ &\geqslant\mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(gf_l)-S_u(gf_l)|}{v^{1/2-1/p}}\geqslant 1\right\}\\ &\qquad-\mu\left\{\frac{1}{n_l^{1/p}}\max_{\substack{1\leqslant u\leqslant n_l-k_l\\1\leqslant v\leqslant k_l}}\frac{|S_{u+v}(m_l'')-S_u(m_l'')|}{v^{1/2-1/p}}\neq 0\right\},\end{aligned} \tag{3.67}$$

hence combining Proposition 3.4 with (3.66), we obtain the conclusion of Proposition 3.5. □

Theorem 2.1 follows from Corollary 3.6 and Propositions 3.2 and 3.3.

3.2. **Proof of Theorem 2.2 and Proposition 2.3.**

Proof of Proposition 2.3. Let us fix a positive t. Recall the equivalence between $\|x\|_\alpha$ and $\|x\|_\alpha^{\mathrm{seq}}$ and Notation 1.1. By Remark 1.2, we have to show that for some constant C depending only on p and each integer $n\geqslant 1$,

$$\begin{aligned}P(n,t):=t^p\mu&\left\{\sup_{j\geqslant 1}2^{\alpha j}n^{-1/2}\max_{0\leqslant k<2^j}\left|S_n^{\mathrm{pl}}(m,r_{k+1,j})-S_n^{\mathrm{pl}}(m,r_{k,j})\right|>t\right\}\leqslant\\ &\leqslant C\left(\|m\|_{p,\infty}^p+\mathbb{E}\left(\mathbb{E}[m^2\mid T\mathcal{M}]\right)^{p/2}\right)\end{aligned} \tag{3.68}$$

In the proof, we shall denote by C_p a constant depending only on p which may change from line to line.

We define

(3.69)

$$P_1(n,t) := \mu\left\{\sup_{1\leqslant j\leqslant \log n} 2^{\alpha j} n^{-1/2} \max_{0\leqslant k<2^j} \left|S_n^{\mathrm{pl}}(m, r_{k+1,j}) - S_n^{\mathrm{pl}}(m, r_{k,j})\right| > t\right\}, \text{ and}$$

$$P_2(n,t) := \mu\left\{\sup_{j>\log n} 2^{\alpha j} n^{-1/2} \max_{0\leqslant k<2^j} \left|S_n^{\mathrm{pl}}(m, r_{k+1,j}) - S_n^{\mathrm{pl}}(m, r_{k,j})\right| > t\right\}, \tag{3.70}$$

hence

$$P(n,t) \leqslant t^p P_1(n,t/2) + t^p P_2(n,t/2). \tag{3.71}$$

We estimate $P_2(n,t)$. For $j > \log n$, we have the inequality

$$r_{k+1,j} - r_{k,j} = (k+1)2^{-j} - k2^{-j} = 2^{-j} < 1/n, \tag{3.72}$$

hence if $r_{k,j}$ belongs to the interval $[l/n, (l+1)/n)$ for some $l \in \{0, \dots, n-1\}$, then

- either $r_{k+1,j} \in [l/n, (l+1)/n)$, and in this case,

$$\left|S_n^{\mathrm{pl}}(m, r_{k+1,j}) - S_n^{\mathrm{pl}}(m, r_{k,j})\right| = \left|m \circ T^{l+1}\right| 2^{-j} n \leqslant 2^{-j} n \max_{1\leqslant l\leqslant n} \left|U^l(m)\right|; \tag{3.73}$$

- or $r_{k+1,j}$ belongs to the interval $[(l+1)/n, (l+2)/n)$. The estimates

$$\begin{aligned}\left|S_n^{\mathrm{pl}}(m, r_{k+1,j}) - S_n^{\mathrm{pl}}(m, r_{k,j})\right| &\leqslant \left|S_n^{\mathrm{pl}}(m, r_{k+1,j}) - S_n^{\mathrm{pl}}(m, (l+1)/n)\right| + \\ &+ \left|S_n^{\mathrm{pl}}(m, (l+1)/n) - S_n^{\mathrm{pl}}(m, r_{k,j})\right| \leqslant 2^{1-j} n \max_{1\leqslant l\leqslant n} \left|U^l(m)\right|\end{aligned} \tag{3.74}$$

  hold.

Considering these two cases, we obtain

$$P_2(n,t) \leqslant \mu\left\{\sup_{j>\log n} 2^{\alpha j} n 2^{1-j} n^{-1/2} \max_{1\leqslant l\leqslant n} \left|U^l(m)\right| > t\right\} \tag{3.75}$$

$$\leqslant \mu\left\{2n^{\alpha-1/2} \max_{1\leqslant l\leqslant n} \left|U^l(m)\right| > t\right\} \tag{3.76}$$

$$\leqslant n\mu\left\{2n^{-1/p} |m| > t\right\} \tag{3.77}$$

$$\leqslant \frac{2^p}{t^p} \sup_{x>0} x^p \mu\left\{|m| > x\right\}. \tag{3.78}$$

Therefore, establishing inequality (3.68) reduces to find a constant C depending only on p such that

$$\sup_n \sup_t t^p P_1(n,t) \leqslant C\left(\|m\|_{p,\infty}^p + \mathbb{E}\left(\mathbb{E}[m^2 \mid T\mathcal{M}]\right)^{p/2}\right) \tag{3.79}$$

We define $u_{k,j} := [nr_{k,j}]$ for $k < 2^j$ and $j \geqslant 1$ (see Notation 1.1).

Notice that the inequalities

$$\left|S_{u_{k,j}}(m) - S_n^{\mathrm{pl}}(m, r_{k,j})\right| \leqslant \left|U^{u_{k,j}+1}(m)\right| \quad \text{and} \tag{3.80}$$

$$\left|S_n^{\mathrm{pl}}(m, r_{k+1,j}) - S_{u_{k+1,j}}(m)\right| \leqslant \left|U^{u_{k+1,j}+1}(m)\right| \tag{3.81}$$

take place because if $j \leqslant \log n$, then

$$u_{k,j} \leqslant nr_{k,j} \leqslant u_{k,j} + 1 \leqslant u_{k+1,j} \leqslant nr_{k+1,j} \leqslant u_{k+1,j} + 1. \tag{3.82}$$

Therefore, $P_1(n,t) \leqslant P_{1,1}(n,t) + P_{1,2}(n,t)$, where

$$P_{1,1}(n,t) := \mu\left\{\max_{1\leqslant j\leqslant \log n} 2^{\alpha j} n^{-1/2} \max_{0\leqslant k<2^j} \left|S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)\right| > t/2\right\}, \tag{3.83}$$

$$P_{1,2}(n,t) := \mu\left\{\max_{1\leqslant j\leqslant \log n} 2^{\alpha j} n^{-1/2} \max_{1\leqslant l\leqslant n} \left|U^l(m)\right| > t/4\right\}. \tag{3.84}$$

Notice that

$$P_{1,2}(n,t) \leqslant \mu\left\{n^{\alpha-1/2} \max_{1\leqslant l\leqslant n} \left|U^l(m)\right| > t/4\right\} \tag{3.85}$$

$$\leqslant n\mu\left\{|m| > n^{1/p} t/4\right\} \tag{3.86}$$

$$\leqslant 4^p t^{-p} \sup_{x\geqslant 0} x^p \mu\left\{|m| > x\right\}, \tag{3.87}$$

hence (3.79) will follow from the existence of a constant C depending only on p such that

$$\sup_n \sup_t t^p P_{1,1}(n,t) \leqslant C\left(\|m\|_{p,\infty}^p + \mathbb{E}\left(\mathbb{E}[m^2 \mid T\mathcal{M}]\right)^{p/2}\right). \tag{3.88}$$

We estimate $P_{1,1}(n,t)$ in the following way:

$$P_{1,1}(n,t) \leqslant \sum_{j=1}^{\log n} 2^j \max_{0\leqslant k<2^j} \mu\left\{\left|S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)\right| > tn^{1/2} 2^{-1-\alpha j}\right\} \tag{3.89}$$

We define for $1 \leqslant j \leqslant \log n$ and $0 \leqslant k < 2^j$ the quantity

$$P(n,j,k,t) := \mu\left\{\left|S_{u_{k+1,j}}(m) - S_{u_{k,j}}(m)\right| > tn^{1/2} 2^{-1-\alpha j}\right\}. \tag{3.90}$$

If $(f \circ T^j)_{j\geqslant 0}$ is a strictly stationary sequence, we define

$$Q_{f,n}(u) := \mu\left\{\max_{1\leqslant j\leqslant n} \left|f \circ T^j\right| > u\right\} + \mu\left\{\left(\sum_{i=1}^n U^i \mathbb{E}[f^2 \mid T\mathcal{M}]\right)^{1/2} > u\right\}. \tag{3.91}$$

The following inequality is Theorem 1 of [Nag03]. It allows us to express the tail function of a martingale by that of the increments and the quadratic variance.

Theorem 3.7. *Let m be an $\mathcal{M}$-measurable function such that $\mathbb{E}[m \mid T\mathcal{M}] = 0$. Then for each positive y and each integer n,*

$$\mu\left\{|S_n(m)| > y\right\} \leqslant c(q,\eta) \int_0^1 Q_{m,n}(\varepsilon_q u \cdot y) u^{q-1} \mathrm{d}u, \tag{3.92}$$

where $q > 0$, $\eta > 0$, $\varepsilon_q := \eta/q$ and $c(q,\eta) := q\exp(3\eta e^{\eta+1} - \eta - 1)/\eta$.

We shall use (3.92) with $q := p+1$, $\eta = 1$ and $y := n^{1/2} 2^{-1-\alpha j} t$ in order to estimate $P(n,j,k,t)$:

$$\begin{aligned} P(n,j,k,t) \leqslant{}& C_p \int_0^1 \mu\left\{\max_{1\leqslant i\leqslant u_{k+1,j}-u_{k,j}} \left|U^i(m)\right| > n^{1/2} 2^{-1-\alpha j} t u \varepsilon_{p+1}\right\} u^p \mathrm{d}u \\ &+ C_p \int_0^1 \mu\left\{\left(\sum_{i=u_{k,j}+1}^{u_{k+1,j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right)^{1/2} > n^{1/2} 2^{-1-\alpha j} u t \varepsilon_{p+1}\right\} u^p \mathrm{d}u. \end{aligned} \tag{3.93}$$

Exploiting the inequality $u_{k+1,j} - u_{k,j} \leqslant 2n2^{-j}$, we get from the previous bound

$$P(n,j,k,t) \leqslant C_p \int_0^1 \mu\left\{\max_{1\leqslant i\leqslant 2n2^{-j}} \left|U^i(m)\right| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1}\right\} u^p \mathrm{d}u + C_p \int_0^1 \mu\left\{\left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right)^{1/2} > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1}\right\} u^p \mathrm{d}u. \tag{3.94}$$

We define for $j \leqslant \log n$, $t \geqslant 0$ and $u \in (0,1)$,

$$P'(n,j,t,u) := \mu\left\{\max_{1\leqslant i\leqslant 2n2^{-j}} \left|U^i(m)\right| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1}\right\}, \quad \text{and} \tag{3.95}$$

$$P''(n,j,t,u) := \mu\left\{\left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right)^{1/2} > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1}\right\}. \tag{3.96}$$

Using the fact that the random variables $U^i(m), 1 \leqslant i \leqslant 2n2^{-j}$ are identically distributed, we derive the bound

$$P'(n,j,t,u) \leqslant 2n2^{-j}\mu\left\{|m| > n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1}\right\}, \tag{3.97}$$

hence

$$P'(n,j,t,u) \leqslant 2n2^{-j}(n^{1/2}2^{-1-\alpha j}tu\varepsilon_{p+1})^{-p} \|m\|_{p,\infty}^p = 2^{p+1}\varepsilon_{p+1}^{-p}n^{1-p/2}2^{j(-1+p\alpha)}t^{-p}u^{-p} \|m\|_{p,\infty}^p. \tag{3.98}$$

Since α and p are linked by the relationship $1/2 - 1/p = \alpha$, we have $p\alpha = p/2 - 1$ hence

$$\int_0^1 P'(n,j,t,u)u^p \mathrm{d}u \leqslant C_p t^{-p} n^{1-p/2} 2^{j(p/2-2)} \|m\|_{p,\infty}^p. \tag{3.99}$$

Notice the following set equalities:

$$\left\{\left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right)^{1/2} > \varepsilon_{p+1}un^{1/2}2^{-1-\alpha j}t\right\} = \left\{\frac{1}{2n2^{-j}}\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}]) > 2^{-3}\varepsilon_{p+1}^2u^2 2^{2j/p}t^2\right\} \tag{3.100}$$

and that $n2^{-j} \geqslant 1$ (because $j \leqslant \log n$), hence

$$\left\{\left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right)^{1/2} > \varepsilon_{p+1}un^{1/2}2^{-1-\alpha j}t\right\} \subseteq \subseteq \bigcup_{N\geqslant 2}\left\{\frac{1}{N}\sum_{i=1}^{N} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}]) > 2^{-3}\varepsilon_{p+1}^2u^2 2^{2j/p}t^2\right\}, \tag{3.101}$$

from which it follows

$$\mu\left\{\left(\sum_{i=1}^{2n2^{-j}} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right)^{1/2} > \varepsilon_{p+1} u n^{1/2} 2^{-1-\alpha j} t\right\} \leqslant \tag{3.102}$$

$$\leqslant \mu\left\{\sup_{N\geqslant 2}\frac{1}{N}\sum_{i=1}^{N} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{2j/p} t^2\right\}.$$

Combining (3.99) and (3.102), we obtain

$$\max_{0\leqslant k<2^j} P(n,j,k,t) \leqslant C_p t^{-p} n^{1-p/2} 2^{j(p/2-2)} \|m\|_{p,\infty}^p \tag{3.103}$$

$$+ C_p \int_0^1 \mu\left\{\sup_{N\geqslant 2}\frac{1}{N}\sum_{i=1}^{N} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{2j/p} t^2\right\} u^p \mathrm{d}u,$$

hence by (3.89) and (3.90),

$$P_{1,1}(n,t) \leqslant C_p t^{-p} \|m\|_{p,\infty}^p \sum_{j=1}^{\log n} 2^j 2^{j(p/2-2)} n^{1-p/2} + \tag{3.104}$$

$$+ C_p \int_0^1 \sum_{j=1}^{\log n} 2^j \mu\left\{\sup_{N\geqslant 2}\frac{1}{N}\sum_{i=1}^{N} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}]) > 2^{-3}\varepsilon_{p+1}^2 u^2 2^{2j/p} t^2\right\} u^p \mathrm{d}u.$$

From the elementary bounds

$$\sum_{j=1}^{\log n} 2^{j(p/2-1)} n^{1-p/2} \leqslant (1-2^{1-p/2})^{-1} \tag{3.105}$$

$$\sum_{j\geqslant 1} 2^j \mu\left\{|g| > 2^{2j/p}\right\} \leqslant 2\mathbb{E}\,|g|^{p/2}, \quad \text{for any non-negative function } g, \tag{3.106}$$

with

$$g := 2^3 \varepsilon_{p+1}^{-2} u^{-2} \sup_{N\geqslant 2}\frac{1}{N}\sum_{i=1}^{N} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}]), u \in (0,1) \tag{3.107}$$

we obtain

$$P_{1,1}(n,t) \leqslant C_p t^{-p} \|m\|_{p,\infty}^p + C_p t^{-p} \left\|\sup_{N\geqslant 2}\frac{1}{N}\sum_{i=1}^{N} U^i(\mathbb{E}[m^2 \mid T\mathcal{M}])\right\|_{p/2}^{p/2}. \tag{3.108}$$

As the Koopman operator U is an $\mathbb{L}^1$-$\mathbb{L}^\infty$ contraction, Theorem 1 of [Ste61] gives the existence of a constant A_p such that for each $h \in \mathbb{L}^{p/2}$,

$$\left\|\sup_{N\geqslant 1}\frac{1}{N}\sum_{j=1}^{N} U^j(h)\right\|_{p/2} \leqslant A_p \|h\|_{p/2}. \tag{3.109}$$

Applying (3.109) with $h := \mathbb{E}[m^2 \mid T\mathcal{M}]$, we get by (3.108)

$$P_{1,1}(n,t) \leqslant C_p t^{-p} \|m\|_{p,\infty}^p + C_p t^{-p} \mathbb{E}\left(\mathbb{E}[m^2 \mid T\mathcal{M}]\right)^{p/2}, \tag{3.110}$$

which establishes (3.79). This concludes the proof of Proposition 2.3. □

Proof of Theorem 2.2. The convergence of finite dimensional distributions can be proved using Theorem of [Bil68]. Its proof works for filtrations of the form $(T^{-i}\mathcal{M})_{i\geqslant 0}$ where $T\mathcal{M}\subset\mathcal{M}$ and also in the non-ergodic setting by considering the ergodic components.

We deduce tightness in Theorem 2.2 from Proposition 2.3 by a truncation argument. For a fixed R, we define

$$m_R := m\chi\{|m|\leqslant R\} - \mathbb{E}[m\chi\{|m|\leqslant R\}\mid T\mathcal{M}] \quad \text{and} \tag{3.111}$$

$$m'_R := m\chi\{|m|> R\} - \mathbb{E}[m\chi\{|m|> R\}\mid T\mathcal{M}]. \tag{3.112}$$

In this way, the sequences $(m_R\circ T^i)_{i\geqslant 0}$ and $(m'_R\circ T^i)_{i\geqslant 0}$ are martingale differences sequences and $m = m_R + m'_R$.

Since $|m_R|\leqslant 2R$ and $(m_R\circ T^i)_{i\geqslant 0}$ is a martingale difference sequence, the sequence $(n^{-1/2}S_n^{\mathrm{pl}}(m_R))_{n\geqslant 1}$ is tight in $\mathcal{H}^o_{1/2-1/p}[0,1]$. Consequently, for each positive ε, the following convergence takes place:

$$\lim_{J\to\infty}\limsup_{n\to\infty}\mu\left\{\sup_{j\geqslant J}2^{\alpha j}\max_{r\in D_j}\left|\lambda_r\left(S_n^{\mathrm{pl}}(m_R)\right)\right|>\varepsilon n^{1/2}\right\}=0. \tag{3.113}$$

Using Proposition 2.3, we derive the following bound, valid for each ε and each R,

$$\lim_{J\to\infty}\limsup_{n\to\infty}\mu\left\{\sup_{j\geqslant J}2^{\alpha j}\max_{r\in D_j}\left|\lambda_r\left(S_n^{\mathrm{pl}}(m)\right)\right|>\varepsilon n^{1/2}\right\}\leqslant \tag{3.114}$$

$$\leqslant C_p\varepsilon^{-p}\left(\sup_{t\geqslant 0}t^p\mu\{|m|\,\chi\{|m|>R\}>t\}+\sup_{t\geqslant 0}t^p\mu\{\mathbb{E}[|m|\,\chi\{|m|>R\}\mid T\mathcal{M}]>t\}\right)+$$
$$+\varepsilon^{-p}C_p\mathbb{E}\left(\left(\mathbb{E}[m^2\chi\{|m|>R\}\mid T\mathcal{M}]\right)^{p/2}\right).$$

The first term is $\sup_{t\geqslant R}t^p\mu\{|m|>t\}$, which goes to 0 as R goes to infinity.

The second term can be bounded by $\sup_{t\geqslant R}t^p\mu\{\mathbb{E}[|m|\mid T\mathcal{M}]>t\}$. Indeed, if $t\geqslant R$, we use the inclusion

$$\{\mathbb{E}[|m|\,\chi\{|m|>R\}\mid T\mathcal{M}]>t\}\subset\{\mathbb{E}[|m|\mid T\mathcal{M}]>t\}, \tag{3.115}$$

and if $t<R$, then accounting the fact that the random variable $\mathbb{E}[|m|\,\chi\{|m|>R\}\mid T\mathcal{M}]$ is greater than R, we get

$$\begin{aligned}\mathbb{E}[|m|\,\chi\{|m|>R\}\mid T\mathcal{M}] &= \mathbb{E}[|m|\,\chi\{|m|>R\}\mid T\mathcal{M}]\chi\{\mathbb{E}[|m|\mid T\mathcal{M}]>R\}\\ &\leqslant \mathbb{E}[|m|\mid T\mathcal{M}]\chi\{\mathbb{E}[|m|\mid T\mathcal{M}]>R\},\end{aligned} \tag{3.116}$$

from which it follows that

$$t^p\mu\{\mathbb{E}[|m|\,\chi\{|m|>R\}\mid T\mathcal{M}]>t\}\leqslant R^p\mu\{\mathbb{E}[|m|\mid T\mathcal{M}]>R\}. \tag{3.117}$$

By Lemma 1.4, the convergence

$$\lim_{R\to\infty}\sup_{t\geqslant R}t^p\mu\{\mathbb{E}[|m|\mid T\mathcal{M}]>t\}=0 \tag{3.118}$$

takes place.

The third term of (3.114) converges to 0 as R goes to infinity by monotone convergence.

This concludes the proof of tightness in Theorem 2.2.

□

3.3. **Proof of Theorem 2.6.** By (2.7), the equality $f = \sum_{i\geqslant 0} P_i(f)$ holds almost surely. For a fixed integer K, we define $f_K := \sum_{i=0}^{K} P_i(f)$. Then f_K satisfies the conditions of Corollary 2.5.

Indeed, we have the equalities

(3.119)
$$P_i(f) - P_0(U^i f) = \mathbb{E}[f \mid T^i\mathcal{M}] - \mathbb{E}[U^i f \mid \mathcal{M}] - \mathbb{E}[f \mid T^{i+1}\mathcal{M}] + \mathbb{E}[U^i f \mid T\mathcal{M}]$$

$$= (I - U^i)\mathbb{E}[f \mid T^i\mathcal{M}] - (I - U^i)\mathbb{E}[f \mid T^{i+1}\mathcal{M}] \tag{3.120}$$

and the later term can be expressed as a coboundary noticing that $(I - U^i) = (I - U)\sum_{k=0}^{i-1} U^k$. Since $P_i(f)$ belongs to the $\mathbb{L}^p$ space, we may write $f_K - \sum_{i=0}^{K} P_0(U^i f)$ as $(I - U)g_K$ where g_K belongs to the $\mathbb{L}^p$ space. Defining $m_K := \sum_{i=0}^{K} P_0(U^i(f))$, the sequence $(m_K \circ T^i)_{i\geqslant 0}$ is a martingale difference sequence hence for each positive ε,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \mu\left\{\sup_{j\geqslant J} 2^{\alpha j} \max_{r\in D_j} \left|\lambda_r\left(S_n^{\mathrm{pl}}(f_K)\right)\right| > \varepsilon n^{1/2}\right\} = 0. \tag{3.121}$$

Now, we have to show that the convergence in (3.121) holds if f_K is replaced by $f - f_K$. To this aim, we use the inclusion

$$\left\{\sup_{j\geqslant J} 2^{\alpha j} \max_{r\in D_j} \left|\lambda_r\left(S_n^{\mathrm{pl}}(f - f_K)\right)\right| > \varepsilon n^{1/2}\right\} \subseteq \tag{3.122}$$
$$\subseteq \left\{\sup_{j\geqslant 1} 2^{\alpha j} \max_{r\in D_j} \left|\lambda_r\left(S_n^{\mathrm{pl}}(f - f_K)\right)\right| > \varepsilon n^{1/2}\right\},$$

hence

(3.123)
$$\mu\left\{\sup_{j\geqslant J} 2^{\alpha j} \max_{r\in D_j} \left|\lambda_r\left(S_n^{\mathrm{pl}}(f - f_K)\right)\right| > \varepsilon n^{1/2}\right\} \leqslant \varepsilon^{-p} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\mathrm{pl}}(f - f_K) \right\|_{\mathcal{H}^o_{1/2-1/p}} \right\|_{p,\infty}^p$$

$$= \varepsilon^{-p} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\mathrm{pl}} \left(\sum_{i\geqslant K+1} P_i(f) \right) \right\|_{\mathcal{H}^o_{1/2-1/p}} \right\|_{p,\infty}^p, \tag{3.124}$$

from which it follows that

$$\mu\left\{\sup_{j\geqslant J} 2^{\alpha j} \max_{r\in D_j} \left|\lambda_r\left(S_n^{\mathrm{pl}}(f - f_K)\right)\right| > \varepsilon n^{1/2}\right\} \tag{3.125}$$
$$\leqslant \varepsilon^{-p} \left(\sum_{i\geqslant K+1} \left\| \left\| \frac{1}{\sqrt{n}} S_n^{\mathrm{pl}}\left(P_i(f)\right) \right\|_{\mathcal{H}^o_{1/2-1/p}} \right\|_{p,\infty} \right)^p.$$

Notice that for a fixed i, the sequence $(U^l(P_i(f)))_{l\geqslant 1}$ is a martingale difference sequence (with respect to the filtration $(T^{-i-l}\mathcal{M})_{l\geqslant 0}$). Therefore, by Proposition 2.3, we obtain

$$\left\| \left\| \frac{1}{\sqrt{n}} S_n^{\mathrm{pl}}\left(P_i(f)\right) \right\|_{\mathcal{H}^o_{1/2-1/p}} \right\|_{p,\infty} \leqslant C_p \left\| P_i(f) \right\|_p. \tag{3.126}$$

Plugging this estimate into (3.125), we obtain that for some constant C depending only on p,

(3.127)
$$\mu\left\{\sup_{j\geqslant J}2^{\alpha j}\max_{r\in D_j}\left|\lambda_r\left(S_n^{\mathrm{pl}}(f-f_K)\right)\right|>\varepsilon n^{1/2}\right\}\leqslant C\varepsilon^{-p}\left(\sum_{i\geqslant K+1}\|P_i(f)\|_p\right)^p.$$

Combining (3.121) and (3.127), we obtain for each K:

$$\lim_{J\to\infty}\limsup_{n\to\infty}\mu\left\{\sup_{j\geqslant J}2^{\alpha j}\max_{r\in D_j}\left|\lambda_r\left(S_n^{\mathrm{pl}}(f)\right)\right|>n^{1/2}\varepsilon\right\}\leqslant \tag{3.128}$$
$$\leqslant C\varepsilon^{-p}\left(\sum_{i\geqslant K+1}\|P_i(f)\|_p\right)^p.$$

Since K is arbitrary, we conclude the proof of Theorem 2.6 thanks to assumption (2.8).

Acknowledgements. The author is grateful to the referee for many comments which improved the readability of the paper.

The author would like to thank Dalibor Volný for many useful discussions which lead to the counter-example in Theorem 2.1, and also Alfredas Račkauskas and Charles Suquet for their support.

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