Protected quasi-locality in quantum systems with long-range interactions

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Lorenzo Cevolani, Giuseppe Carleo, and Laurent Sanchez-Palencia
Laboratoire Charles Fabry, Institut d'Optique, CNRS, Univ. Paris Sud 11, 2 avenue Augustin Fresnel, F-91127 Palaiseau cedex, France

We study the out-of-equilibrium dynamics of quantum systems with long-range interactions. Two different models describing, respectively, lattice bosons, and spins are considered. Our study is based on a combined approach based, on one hand, on accurate many-body numerical calculations and, on the other hand, on a quasi-particle microscopic theory. For sufficiently fast decaying long-range potentials, we find that the quantum speed limit set by the long-range Lieb-Robinson bounds is never attained and a purely ballistic behavior is found. For slowly decaying potentials, a radically different scenario is observed. In the bosonic case, a remarkable local spreading of correlations is still observed, despite the existence of infinitely fast traveling excitations in the system. This is in marked contrast with the spin case, where locality is broken. We finally provide a microscopic justification of the different regimes observed and of the origin of the protected locality in bosonic models.

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It is common wisdom that the propagation of a signal through a classical medium presents a distinct notion of causality, characterized by the progressive time growth of the spatial region explored by the signal. In spite of the intrinsically non-local nature of the quantum theory, this familiar notion of locality is preserved in a wide class of quantum systems with short-range interactions. A milestone example is provided by the Lieb-Robinson (LR) bounds, which set a ballistic limit to the propagation of information, with exponentially small leaks outside the locality cone [1, 2]. The existence of LR bounds has many fundamental implications for thermalization, entanglement scaling laws, and information transfer in quantum systems [3]. A renewed interest in these topics is currently sparked by the impressive progress in the time-dependent control of ultracold-atom systems. Direct observation of cone spreading of correlations was reported in Refs. [4, 5].

The extension of the notion of locality to quantum systems with long-range interactions constitutes a fundamental challenge. The paradigmatic model of long-range interactions considers an algebraic decay of some coupling term of the form $V(R) \sim 1/R^{\alpha}$ [6–10]. It applies either to the exchange coupling term in spin systems, as realized in cold ion crystals [11, 12], or to the two-body interactions in particle systems, as realized in ultracold gases of polar molecules [13], magnetic atoms [14], and Rydberg atoms [15]. A remarkable feature of long-range systems is that instantaneous propagation of information, in violation of locality, can take place when the exponent α is smaller than some threshold. This possibility is supported by the known extensions of the LR bounds to long-range interactions [16]. The latter yield "quasi-local" super-ballistic bounds with algebraic leaks for $\alpha > d$, where d is the dimension of the system, whereas for $\alpha < d$ no known generalized bounds exist, hence suggesting the breaking of quasi-locality. Evidence of the breaking of quasi-locality in 1D Ising spin systems has been reported theoretically [6, 7] and in cold ion crystals [12, 17]. However, many questions remain open. For instance, although the observations are compatible with the long-range bounds, the propagation was found to be much slower than expected [6]. Hence, the bounds are usually not saturated and it is not clear that they provide a universal criterion for the breaking of quasi-locality. So far only spin systems have been studied. Moreover the threshold value for the breaking of locality in these systems is debated, and contrasting results have been put forward [6, 7]. To progress on these questions, it is of crucial importance to provide a unified understanding of a wider class of systems and, at the same time, to understand the microscopic origin of the breaking of quasi-locality.

In this Letter, we study the out-of-equilibrium dynamics of one-dimensional (1D) interacting quantum systems with long-range algebraic interactions in two different models, namely the long-range transverse Ising (LRTI) and Bose-Hubbard (LRBH) models. On one hand, we perform *ab-initio* quantum many-body calculations based the time-dependent variational Monte Carlo (t-VMC) approach [18]. On the other hand, we provide a unified analytical framework based on quasi-particles analysis. Both the numerical and analytical results consistently show that the two systems behave dramatically different. For the LRTI model, while local spreading of correlations is found for $\alpha > 1$, we find that quasi-locality is broken for $\alpha < 1$. This effect is traced back to the divergences of both the quasi-particle energy and velocity, which induce infinitely fast oscillations and instantaneous propagation of correlations. Conversely, for the LRBH model, we find ballistic spreading of correlations for any value of α, in marked contrast with expectations based on the lack of LR bound. This effect is traced back to the fact that the quasi-particle energy remains finite, which cancels super-ballistic contributions. All the observed regimes are explained by the unified quasi-particle analysis. These results shed new light on the dynamics of long-range quantum systems. In particular, they show the prominent role of relevant microscopic features of the model in the spreading of quantum correlations.

Long-range transverse Ising model.– We start with the long-range transverse Ising model (LRTI), the Hamiltonian of

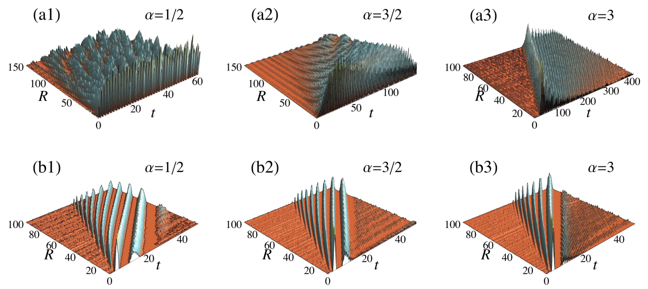

FIG. 1. Correlation spreading in long-range spin and boson models for various values of α. (a) Connected spin-spin correlation function in the LRTI model for the quench $V_i = 1/2 \to V_f = 1/10$. (b) Connected density-density correlation function in the LRBH model for the quench $U_i = V_i = 1 \to U_f = V_f = 1/4$. Results obtained using the t-VMC approach for systems of $L = 400$ sites (for visibility, only a part is shown). The length unit is the lattice spacing and the time units are for (a) the magnetic field h and for (b) the hopping J.

which reads

$$\mathscr{H} = -h\sum_i \sigma_i^x + \frac{V}{2}\sum_{i\neq j}\frac{\sigma_i^z\sigma_j^z}{|i-j|^\alpha}, \tag{1}$$

where σ_i^x, σ_i^z are the Pauli matrices, h is the transverse field, and V is the strength of the long-range spin exchange term. Hamiltonian (1) is the prototype for long-range interacting quantum systems [6, 7, 19]. Moreover, it is experimentally implemented in cold ion crystals with long-range lattice-mediated interactions [20]. Evidence of the breaking of quasi-locality in information spreading has been reported for the 1D LRTI model for sufficiently small exponents α [6, 7, 12, 17], consistently with the absence of a long-range LR bound (LLR) for $\alpha < 1$. It was pointed out, however, that a model-dependent form of a quasi-locality may occur for specific initial states [6].

Asymptotically reliable results to reveal quasi-locality require sufficiently long propagation times and sufficiently large systems. This is particularly crucial to determine precisely the nature of the dynamical regimes. To achieve this goal, we compute the unitary evolution of the correlation functions by means of the t-VMC approach [18, 21]. The latter permits to simulate the dynamics of correlated quantum systems with an accuracy comparable to tensor-network methods and proved numerically stable for unprecedented long times and large sizes.

In the t-VMC calculations, we use a Jastrow wavefunction with long-range spin-spin correlations at arbitrary distance [22]. In order to avoid misleading finite size effects, which are usually strong in these issues [23], periodic boundary conditions (PBC) are used. For a lattice of size L with PBC, the interaction potential is taken as the sum of the contributions resulting from all the periodic images of the finite system. The Fourier components of the effective interaction potential are $P(k) = 2\sum_{n=1}^{\infty}\frac{\cos(kn)}{n^\alpha} = 2\mathrm{Cl}_\alpha(k)$, where we have used the Poisson summation formula over the periodic images, k is an integer multiple of $2\pi/L$, and $\mathrm{Cl}_\alpha(k)$ is the Clausen cosine function. In order to have a well-behaved potential in the thermodynamic limit, we set $P(k=0) = 0$. For charged particles, it would correspond to the standard regularization procedure ensuring charge neutrality [24].

We consider global quenches of the strength of the long-range interaction, $V_i \to V_f$. The results for the time-connected average $G_c^{\sigma\sigma}(R,t) = G^{\sigma\sigma}(R,t) - G^{\sigma\sigma}(R,0)$ of the spin-spin correlation function $G^{\sigma\sigma}(R) = \langle \sigma_i^z \sigma_{i+R}^z \rangle$ are shown in Fig. 1 (a). We find three qualitatively different regimes. For $\alpha < 1$, Fig. 1 (a1), the propagation of correlations takes place on extremely short time scales and no cone-like structure emerges. This is the signature of an efficient microscopic mechanism leading to the breaking of locality in the system. For $\alpha > 2$, Fig. 1 (a3), a correlation cone with a well-determined velocity v clearly emerges. It is marked by a strong suppression of leaks in the region defined by $R/t > v$ and space-time oscillations in the region $R/t < v$. In the intermediate regime where $1 < \alpha < 2$, Fig. 1 (a2), a correlation cone structure traveling with a velocity v is still visible but prominent leaks in the region $R/t > v$ are also appearing.

In order to quantify more precisely the nature of the leaks in the quasi-local regimes we study the time-integrated absolute value of the correlation function $\bar{G}_c(R,t) = \frac{1}{t}\int_0^t dt' \, |G_c(R,t')|$. While it retains all the features of the signal propagation, it is less sensitive to time oscillations. In Fig. 2 (a) we show the behavior of $\bar{G}_c(R,t)$ for $\alpha = 3$. It clearly shows the sharp boundary of a ballistic cone. This is further assessed introducing a small cutoff ε and computing the first propagation time $t^\star(R)$ such that $\bar{G}_c(R,t^\star) > \varepsilon$. The result is almost independent of ε and we find the scaling $vt^\star = R^\beta$, with finite v and $\beta \simeq 1$ to very good precision and up to large system sizes and long propagation times. The presence of a ballistic spreading, with exponentially suppressed leaks, is a stronger realization of locality than what expected from the looser LLR bound, which instead allows for polynomially suppressed leaks. For $1 < \alpha < 2$ the same analysis of the leaks, shown in Fig. 2 (b), reveals instead that polynomial leaks appear with an exponent $\beta \simeq \alpha$, and a velocity v that vanishes with ε. This is compatible with the LLR bound [16].

Remarkably, the regimes we find here for a global quench of the magnetic field are the same qualitative regimes who have been identified for a local quench in the LRTI model in Ref. [7]. An analysis of a related spin model instead showed different characteristic exponents for the breaking of quasi-locality [6], with a complete understanding of the possible scenarios being debated. We will come to the microscopic origin of characteristic limiting exponents for the breaking of quasi-locality in the second part of this Letter.

Long-range Bose-Hubbard model.– We now turn to the long-range Bose-Hubbard (LRBH), which describes interacting spinless bosons in a periodic potential with nearest-neighbor tunneling and long-range two-body interactions.

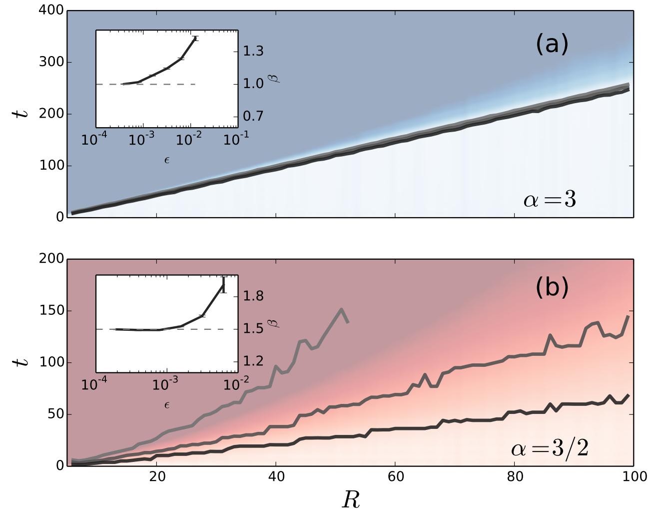

FIG. 2. Main panels: time-integrated spin-spin correlation functions $\bar{G}_c(R,t)$ for the same data as in Fig. 1(a) computed using the t-VMC approach for $L=400$ sites. Superimposed lines show $t^\star(\varepsilon)$ (see text), for ε ranging from 2×10^{-2} (lighter lines) down to 5×10^{-3} (darker lines). Insets: behavior of the fitted exponent β as a function of the cutoff parameter ε for system size $L=2^{14}$, computed within linear spin-wave.

The Hamiltonian reads

$$\mathscr{H} = -J\sum_{\langle i,j\rangle}\left(b_i^\dagger b_j + h.c.\right) + \frac{U}{2}\sum_i n_i\left(n_i-1\right) + \frac{V}{2}\sum_{i\neq j}\frac{n_i n_j}{|i-j|^\alpha}, \tag{2}$$

where $b_i(b_i^\dagger)$ destroys (creates) a boson on site i, $n_i = b_i^\dagger b_i$ is the number operator, J is the tunneling amplitude, U is the on-site interaction energy, and V is the strength of the interaction potential. The short-range ($V=0$) Bose-Hubbard Hamiltonian is now routinely realized in ultracold-atom experiments and has been used to observe ballistic cone spreading of correlations in the case of short-range interactions [25]. The Hamiltonian (2) we consider here is its long-range counterparts, which applies to polar molecules [13], magnetic atoms [14] and Rydberg atoms [15].

We perform t-VMC calculations similar to the previous case but now using a Jastrow wavefunction incorporating density-density correlations at arbitrary large distances [26]. For simplicity we choose $U=V$, we fix the density at half filling ($n=\frac{1}{2}$), and we consider the density-density correlation function $G(R)=\langle n_i n_{i+R}\rangle$. As for the LRTI model, we study global quenches in the interaction strength $V_i\rightarrow V_f$. The results for various values of the exponent α are shown in Fig. 1 (b). Surprisingly, we find here that the LRBH model exhibits the same qualitative behavior for all values of α, in marked contrast with the LRTI model. Within numerical precision, we always find a purely ballistic cone spreading of correlations at some velocity v. The LLR bound is therefore never saturated and, for every value of the exponent α, the spreading is qualitatively identical to the short-range case. Hence, in the LRBH model quasi-locality appears to be strongly protected even for very long-range interactions. This is further confirmed by a precise analysis of the leaks, along the same lines as for the LRTI model. It always yields a scaling of the form $vt^\star = R^\beta$ with $\beta = 1$ and the signal is exponentially suppressed out of the causality region $R > vt$.

The radically different behaviors of the LRTI and LRBH models are particularly striking because they share the same class of long-range interactions and are therefore subjected to the same universal LLR bounds [16]. Hence, a classification of the quasi-local regimes *needs to* be explicitly model-dependent and take into account the relevant microscopic degrees of freedom which are not captured by the generic LLR bounds.

Quasi-particle analysis.– In order to understand the different behaviors of the two models, let us use a general microscopic quasi-particle approach. The latter has a broad range of applications e.g. universal conformal theories [27], spin systems [7], superfluids [28], and Mott insulators [4]. A generic time-dependent two-body correlation function in a translationally invariant model with well-defined quasi-particle excitations can be written as

$$G_c(R,t) = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\mathscr{F}(k)\left\{\cos(kR) + -\frac{1}{2}\left[\cos\left(kR-2E_k^f t\right)+\cos\left(kR+2E_k^f t\right)\right]\right\}, \tag{3}$$

where E_k^f is the k-momentum quasi-particle energy of the post-quench Hamiltonian and $\mathscr{F}(k)$ is the weight associated to each quasi-particle. This general form states that the collective excitations of the system are coherent superpositions of pairs carrying excitations of momentum k and traveling in opposite directions. Whereas E_k^f depends only on the post-quench Hamiltonian, in general $\mathscr{F}(k)$ instead depends both on the pre- and post-quench Hamiltonians.

In the LRTI model and in the regime of large transverse field, we can apply linear spin wave theory [11]. The quasi-particle energy and weight read, respectively, $E_k^f = 2\sqrt{h[h+VP(k)]}$ and $\mathscr{F}^{\sigma\sigma}(k) = \frac{2P(k)(V_i-V_f)}{E_k^i\left[1+P(k)V_f/h\right]}$, where $P(k)$ are the Fourier components of the interaction potential. Let us then analyze the outcome of Eq. (3).

The ballistic behavior observed for $\alpha>2$ can be understood from stationary phase analysis. Along the line $R=vt$, it yields the dominant contribution

$$G_c(R,t) \simeq \frac{\mathscr{F}(k^*)}{4\sqrt{\pi\left|\partial_k^2 E_{k^*}^f\right|t}}\cos\left(k^*R-2E_{k^*}^f t+\phi\right), \tag{4}$$

where $k^\star$ is solution of $v=2v_g(k^\star)$, v_g is the group velocity, and ϕ is a time-independent phase. For $\alpha>2$, the group velocity is bounded and we find a ballistic cone spreading of correlations with a velocity that is given by twice the maximum group velocity. We have checked that it agrees well with the numerics where a cone propagation is also found. A quantitative analysis of the leaks within the spin-wave approach

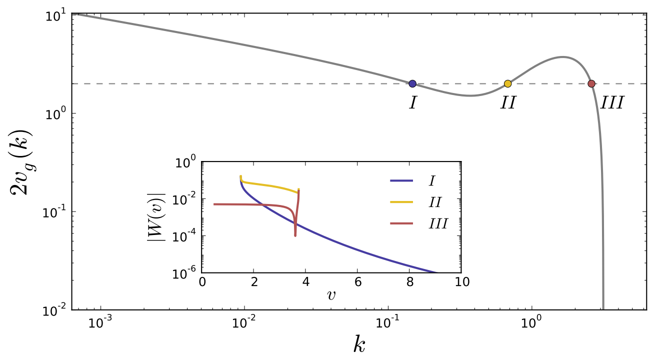

FIG. 3. Quasi-particle analysis of the group velocity for the LRBH model with exponent $\alpha = 1/2$, density $n = 1/2$ and $U = V = 1/4$. In the inset, the stationary phase weight corresponding to the three branches solution of the equation $2v_g(k) = v$ is shown.

confirms that they are exponentially suppressed. Moreover, the exponent β is found to smoothly approach unity upon reducing the cutoff parameter ε [see inset of Fig. 2 (a)], in agreement with the t-VMC results.

For $1 < \alpha < 2$ the situation is different because the group velocity is not bounded and exhibits the infrared divergence $v_g(k) \sim k^{-|2-\alpha|}$. The lack of a purely ballistic propagation is therefore traced back to the presence of infinitely fast traveling modes at low momenta. More precisely, the behavior of the leaks can be found from the asymptotic $R \to \infty$ expansion of Eq. (3), retaining only the contributions of the divergent velocities. For $\alpha = 3/2$, it can be computed exactly. It yields $G_c(R \to \infty, t) \simeq F(t) \times t/R^{\alpha}$, where $F(t)$ is a bounded function of time. This scaling explains the t-VMC results. It is further confirmed by the analysis of the leaks within the spin-wave approach. The exponent β is found to smoothly approach the interaction exponent α upon reducing the cutoff parameter ε [see inset in Fig. 2 (b)]. The same was found for other values of α between 1 and 2.

For $\alpha < 1$, both the quasi-particle energy and the group velocity diverge for $k \to 0$, respectively, as $E_k^f \sim k^{-\left|\frac{1-\alpha}{2}\right|}$ and $v_g(k) \sim k^{-\left|\frac{3-\alpha}{2}\right|}$. In particular, it is the energy divergence which sets the breaking of quasi-locality in this case. The latter gives rise to a rapidly oscillating factor of the form $\cos(t/k^{\frac{1-\alpha}{2}})$ in Eq. (3), which leads to a non-analytic $t = 0$ delta-kick in the thermodynamic limit. More precisely, an exact asymptotic expansion of the correlation function (3) can be derived in the limit of small propagation times t and large distances R. Keeping the relevant small quasi-momenta, it yields

$$G_c(R, t \to 0) \simeq \lim_{L \to \infty} A \frac{\sin\left(L^{\frac{1-\alpha}{2}} t\right)}{t} \frac{\cos(R/L)}{R^{2-\alpha}} + B \frac{t^2}{R^{\frac{1+\alpha}{2}}}, \quad (5)$$

where A and B are finite numerical constants, which depend on the microscopic parameters of the model. A remarkable consequence of this expression is that the first term yields an instantaneous contribution to the signal, on a time scale $\tau = 1/L^{\frac{(1-\alpha)}{2}}$, independent on the distance R and with an exponent set by the divergence of the quasi-particle energy. This implies that the system reacts on a time scale inversely proportional to the system size, yielding an efficient mechanism for the breaking of locality.

An analogous microscopic analysis can be carried out for the LRBH model. In the superfluid regime, the quantities E_k^f and $\mathscr{F}(k)$ are found by Bogoliubov analysis [28], which yields $E_k^f = \sqrt{\varepsilon_k \left[\varepsilon_k + 2n\left(U_f + V_f P(k)\right)\right]}$ and $\mathscr{F}^{nn}(k) = n^2 \frac{\left[(U_f - U_i) + (V_f - V_i)P(k)\right]\varepsilon_k}{\left[\varepsilon_k + 2n\left(U_f + V_f P(k)\right)\right] E_k^i}$, where $\varepsilon_k = 4\sin^2(k/2)$ is the free-particle lattice dispersion and n is the particle density.

For $\alpha > 1$, the origin of the observed ballistic behavior is traced back to the bounded quasi-particle velocities. The propagation of correlations is dominated by the stationary-phase points of Eq. (4) and the correlation cone velocity is given by twice the maximum group velocity.

For $\alpha < 1$, the group velocity diverges as $v_g(k \to 0) \sim k^{-\left|\frac{1-\alpha}{2}\right|}$, whereas the quasi-particle energy is always finite. Hence, at variance with the LRTI model, the correlation function does not exhibit any instantaneous kick at $t = 0$ such as that of Eq. (5), and quasi-locality is preserved. Moreover, although this case is formally analogous to the intermediate regime, with polynomial leaks, found for the LRTI model, a purely ballistic spreading is found within numerical precision in the LRBH model. To understand this, let us come back to the stationary-phase approach of Eq. (4). Due to the specific form of the group-velocity dispersion in the LRBH model, the equation $v = 2v_g(k^\star)$ has up to three separate solutions for given velocity v, see Fig. 3. The correlation function is thus dominated by three contributions (*I*, *II*, and *III*) of the form of Eq. (4). The behavior of the corresponding weights, $W(v) = \mathscr{F}^{\sigma\sigma}(k^\star)/\sqrt{\left|\partial_k^2 E^f(k^\star)\right|}$, along the three branches is shown in the inset of Fig. 3. The largest weights, corresponding to the velocities dominating the propagation, belong to the regular branches (*II* and *III*). The latter extend up to a certain maximum velocity $v_{\max}$, which effectively sets the correlation cone velocity v_c. The infinitely fast modes, corresponding instead to the branch *I*, have a weight which is polynomially suppressed as $W(v \to \infty) \propto v^{-\left|\frac{9-3\alpha}{2(1-\alpha)}\right|}$. However, for $v \simeq v_{\max}$, the weights of these modes are several order of magnitudes smaller than the quasi-local, ballistic modes. This separation of scales is therefore responsible of the effective suppression of the infinitely fast non-local modes in the LRBH model. Notice that in the LRTI model the irregular branch of the infinitely fast modes is never protected by a finite-velocity branch. This is a consequence of the monotonic behavior of the quasi-particle group velocity in the spin model, yielding a single branch of solutions for the stationary phase equation.

Conclusions.– We have studied the out-of-equilibrium dynamics of two basic models of interacting one-dimensional quantum systems with long range algebraic interactions. While they share the same class of long-range interactions, characterized by the exponent α, accurate many-body numerical calculations show that they exhibit radically different dynamical behaviors. The latter are consistently interpreted using quasi-particle analysis of the two models. For $\alpha > 2$ both LRTI and LRBH models show ballistic spreading of correlations with exponentially small leaks, hence leading to a strong form of quasi-locality. For $1 < \alpha < 2$ quasi-locality is still found in the two models. However, algebraic leaks appear in the LRTI model, which can be traced back to a divergent group velocity at low momenta. For $\alpha < 1$ quasi-locality is completely broken in the LRTI model. The dynamics in this regime is characterized by a response time scale that goes to zero with the system size and due to the divergence of the quasi-particle energy. Conversely, quasi-locality is preserved in the LRBH model. While the group velocity diverges, the quasi-particle energy stays always finite, and no instantaneous propagation of correlations can set in. Moreover, a ballistic spreading is observed, which can be ascribed to an efficient separation of scales, characterized by a vanishingly small weight for the super-ballistic branch.

Our results shed new light on the dynamics of long-range quantum systems. Yet many questions remain open. For instance, it would be instructive to understand the behavior of models belonging to other universality classes. Moreover, it is expected, on the basis of universal bounds, that the critical exponents for the breaking of locality depend on the system dimension [16]. It is thus of utmost interest to study the counterparts of the effects discussed here in dimension higher than one, which could be done by extending the present approach [21].

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