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A new class of two-layer Green-Naghdi systems with improved frequency dispersion

Vincent Duchêne* Samer Israwi† Raafat Talhouk†

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Abstract

We introduce a new class of Green-Naghdi models for the propagation of internal waves between two $(1+1)$ -dimensional layers of homogeneous, immiscible, ideal, incompressible, irrotational fluids, vertically delimited by a flat bottom and a rigid lid. These models are tailored to improve the frequency dispersion of the original Green-Naghdi model, and in particular to manage high-frequency Kelvin-Helmholtz instabilities. Our models preserve the Hamiltonian structure, symmetry groups and conserved quantities of the original model. We provide a rigorous justification of a class of our models thanks to consistency, well-posedness and stability results. These results apply in particular to the original Green-Naghdi model as well as to the Saint-Venant (hydrostatic shallow water) system with surface tension.

1 Introduction

1.1 Motivation

This work is dedicated to the study of a bi-fluidic system which consists in two layers of homogeneous, immiscible, ideal and incompressible fluids under only the external influence of gravity. Such a configuration is commonly used in oceanography, where variations of temperature and salinity induce a density stratification; see [22] and references therein.

A striking property of the above setting, in contrast with the water-wave case (namely only one layer of homogeneous fluid with a free surface) is that the Cauchy problem for the governing equations is ill-posed outside of the analytic framework when surface tension is neglected [19, 23, 24]. This ill-posedness is caused by the formation of high-frequency (*i.e.* small wavelength) Kelvin-Helmholtz instabilities which are triggered by any non-trivial velocity shear. Recently, Lannes [25] showed that a small amount of surface tension is sufficient to durably regularize the high-frequency component of the flow, while the main low-frequency component remains mostly unaffected.

This result explains why, occasionally, surface tension may be harmlessly neglected in asymptotic models, that is simplified models constructed from smallness assumptions on physical properties of the flow. This is typically expected to be the case for shallow-water models, since the shallow-water regime implies that the main component of the flow is located at low frequencies; and in particular for the two-layer extension of the classical Green-Naghdi model introduced by Miyata [31, 32],

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Mal'tseva [29] and Choi and Camassa [10].¹ Somewhat surprisingly, it turns out that the original Green-Naghdi system, as already noticed in [10], behaves similarly as the full-Euler system: a linear stability analysis shows that surface tension is needed in order to avoid the appearance of high-frequency instabilities. However, a more precise study [27] indicates that the model actually overestimates Kelvin-Helmholtz instabilities, in the sense that the threshold on the velocity shear above which instabilities are triggered is always smaller for the original Green-Naghdi system than for the full Euler system.

Many attempts have been made in order to “regularize” the Green-Naghdi model, that is proposing new models with formally the same precision as the original model, but which are not subject to high-frequency Kelvin-Helmholtz instabilities, even without surface tension [33, 9, 12, 6, 16, 27]. The strategies adopted in these works rely on change of unknowns and/or Benjamin-Bona-Mahony type tricks; see [26, Section 5.2] for a thorough presentation of such methods in the water-wave setting. In this work, we present a new class of modified Green-Naghdi systems obtained through a different, somewhat simpler mean. We find numerous advantages in our method:

- The original Green-Naghdi model is only lightly modified, and the physical interpretations of variables and identities of the original model are preserved.
- The method is quite flexible. It allows in particular to construct models which completely suppress large-frequency Kelvin-Helmholtz instabilities; or a model which conforms perfectly with the linear stability analysis of the full Euler system.
- The rich structure of the original Green-Naghdi system (Hamiltonian formulation, groups of symmetry, conserved quantities) is maintained. This is generally not the case when change of unknowns or BBM tricks are involved; see discussion in [11, 18].

Our models may be viewed as Green-Naghdi systems with improved frequency dispersion. In particular, one of our models shares with full dispersion models such as in [5] the property that their dispersion relation is the same as the one of the full Euler system. A noteworthy feature of our models is that, by construction, they involve non-local operators (Fourier multipliers). Such operators are common in deep-water models, such as the Benjamin-Ono or Whitham equations for instance, but appear to be original in the shallow-water setting.

In the present work, we motivate our models through the study of Kelvin-Helmholtz instabilities, by linearizing the systems around solutions with constant shear. This formal study is supported by numerical simulations, which demonstrate how the predictions of the modified Green-Naghdi models may vary dramatically depending on their large-frequency dispersion properties, and the significant influence of small surface tension. We also provide a rigorous analysis for a class of our models by proving the well-posedness of the Cauchy problem as well as consistency and stability results, which together offer the full justification of our asymptotic models, in the sense described in [26]. This includes the original Green-Naghdi model as well as the Saint-Venant (hydrostatic shallow-water) system with surface tension; such results are new as far as we know.

For the sake of simplicity, our study is restricted to the setting of a flat bottom, rigid lid and one-dimensional horizontal variable. The construction of our models, however, is straightforwardly extended to the two-dimensional case. We also expect that our strategy can be favorably applied to more general configurations (non-trivial topography, free surface, multi-layer, *etc.*)

1.2 The full Euler system

For the sake of completeness and in order to fix the notations, we briefly recall the governing equations of a two-layer flow in our configuration, that we call *full Euler system*. We let the interested reader refer to [5, 1, 15] for more details.

¹In the following, we will refer to this system, precisely recalled thereafter, as *the original Green-Naghdi system*, in contrast with our *modified Green-Naghdi systems*.

The setting consists in two layers (infinite in the horizontal variable, vertically delimited by a flat rigid lid and a flat bottom) of immiscible, homogeneous, ideal, incompressible and irrotational fluid under only the external influence of gravity.

We assume that the interface between the two layers is given as the graph of a function, $\zeta(t, x)$, so that the domain of the two fluids at time t is given as

$$\Omega_1^t \stackrel{\text{def}}{=} \{(x, y), \zeta(t, x) \leq z \leq d_1\}, \quad \Omega_2^t \stackrel{\text{def}}{=} \{(x, z), -d_2 \leq z \leq \zeta(t, x)\}.$$

Here and thereafter, the subscript $i = 1$ (resp. $i = 2$) always refer to the upper (resp. lower) layer. The fluids being irrotational, we consider the velocity potentials in each layer, that we denote ϕ_i . Finally, P_i denotes the pressure inside each layer.

Let a be the maximum amplitude of the deformation of the interface. We denote by λ a characteristic horizontal length, say the wavelength of the interface. Then the typical velocity of small propagating internal waves (or wave celerity) is given by

$$c_0 = \sqrt{g \frac{(\rho_2 - \rho_1)d_1d_2}{\rho_2d_1 + \rho_1d_2}},$$

where d_1 (resp. d_2) is the depth of the upper (resp. lower) layer and ρ_1 (resp. ρ_2) its mass density. g denotes the acceleration of gravity. Consequently, we introduce the dimensionless variables

$$\tilde{z} \stackrel{\text{def}}{=} \frac{z}{d_1}, \quad \tilde{x} \stackrel{\text{def}}{=} \frac{x}{\lambda}, \quad \tilde{t} \stackrel{\text{def}}{=} \frac{c_0}{\lambda} t,$$

the dimensionless unknowns

$$\tilde{\zeta}(\tilde{t}, \tilde{x}) \stackrel{\text{def}}{=} \frac{\zeta(t, x)}{a}, \quad \tilde{\phi}_i(\tilde{t}, \tilde{x}, \tilde{z}) \stackrel{\text{def}}{=} \frac{d_1}{a\lambda c_0} \phi_i(t, x, z), \quad \tilde{P}_i(\tilde{t}, \tilde{x}, \tilde{z}) \stackrel{\text{def}}{=} \frac{d_1}{a\rho_2 c_0^2} P_i(t, x, z) \quad (i = 1, 2),$$

as well as the following dimensionless parameters

$$\gamma \stackrel{\text{def}}{=} \frac{\rho_1}{\rho_2}, \quad \epsilon \stackrel{\text{def}}{=} \frac{a}{d_1}, \quad \mu \stackrel{\text{def}}{=} \frac{d_1^2}{\lambda^2}, \quad \delta \stackrel{\text{def}}{=} \frac{d_1}{d_2}, \quad \text{Bo} \stackrel{\text{def}}{=} \frac{g(\rho_2 - \rho_1)\lambda^2}{\sigma}.$$

The last parameter is the Bond number, and measures the ratio of gravity forces over capillary forces (σ is the surface tension coefficient). After applying the above scaling, but withdrawing the tildes for the sake of readability, the system may be written as

$$(1.1) \quad \begin{cases} \mu \partial_x^2 \phi_i + \partial_z^2 \phi_i = 0 & \text{in } \Omega_i^t \quad (i = 1, 2), \\ \partial_z \phi_1 = 0 & \text{on } \{(x, z), z = 1\}, \\ \partial_z \phi_2 = 0 & \text{on } \{(x, z), z = -\delta^{-1}\}, \\ \partial_t \zeta = \frac{1}{\mu} \sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \partial_n \phi_1 = \frac{1}{\mu} \sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \partial_n \phi_2 & \text{on } \{(x, z), z = \epsilon \zeta(t, x)\}, \\ \partial_t \phi_i + \frac{\epsilon}{2} |\nabla_{x,z}^\mu \phi_i|^2 = -\frac{P}{\gamma_i} - \frac{\gamma + \delta}{1 - \gamma} z & \text{in } \Omega_i^t \quad (i = 1, 2), \\ \llbracket P(t, x) \rrbracket = -\frac{\gamma + \delta}{\text{Bo}} \frac{k(\epsilon \sqrt{\mu} \partial_x \zeta)}{\epsilon \sqrt{\mu}} & \text{on } \{(x, z), z = \epsilon \zeta(t, x)\}, \end{cases}$$

where $k(\partial_x \zeta) \stackrel{\text{def}}{=} -\partial_x \left(\frac{1}{\sqrt{1 + |\partial_x \zeta|^2}} \partial_x \zeta \right)$, $\llbracket P(t, x) \rrbracket \stackrel{\text{def}}{=} \lim_{\varkappa \rightarrow 0} \left(P(t, x, \epsilon \zeta(t, x) + \varkappa) - P(t, x, \epsilon \zeta(t, x) - \varkappa) \right)$, $\gamma_i = \frac{\rho_i}{\rho_2}$, $\nabla_{x,z}^\mu \stackrel{\text{def}}{=} (\sqrt{\mu} \partial_x, \partial_z)^\top$ and $\left(\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} \partial_n \phi_i \right) \Big|_{z=\epsilon \zeta} = -\mu \epsilon (\partial_x \zeta) (\partial_x \phi_i) \Big|_{z=\epsilon \zeta} + (\partial_z \phi_i) \Big|_{z=\epsilon \zeta}$.

We may conveniently rewrite the above system as two evolution equations, thanks to the use of Dirichlet-Neumann operators. In order to do so, we define

$$\psi(t, x) \stackrel{\text{def}}{=} \phi_1(t, x, \epsilon \zeta(t, x)),$$

and

$$\begin{aligned} G^\mu \psi &= G^\mu[\epsilon\zeta]\psi \stackrel{\text{def}}{=} \sqrt{1 + \mu|\epsilon\partial_x\zeta|^2}(\partial_n\phi_1) \Big|_{z=\epsilon\zeta} = -\mu\epsilon(\partial_x\zeta)(\partial_x\phi_1) \Big|_{z=\epsilon\zeta} + (\partial_z\phi_1) \Big|_{z=\epsilon\zeta}, \\ H^{\mu,\delta}\psi &= H^{\mu,\delta}[\epsilon\zeta]\psi \stackrel{\text{def}}{=} (\phi_2) \Big|_{z=\epsilon\zeta} = \phi_2(t, z, \epsilon\zeta(t, x)), \end{aligned}$$

where ϕ_1 and ϕ_2 are uniquely defined (up to an additive constant for ϕ_2) as the solutions of the Laplace problems implied by (1.1). The full system (1.1) then becomes

$$(1.2) \quad \begin{cases} \partial_t\zeta - \frac{1}{\mu}G^\mu\psi = 0, \\ \partial_t\left(\partial_x H^{\mu,\delta}\psi - \gamma\partial_x\psi\right) + (\gamma + \delta)\partial_x\zeta + \frac{\epsilon}{2}\partial_x\left(|\partial_x H^{\mu,\delta}\psi|^2 - \gamma|\partial_x\psi|^2\right) \\ \qquad \qquad \qquad = \mu\epsilon\partial_x\mathcal{N}^{\mu,\delta} - \frac{\gamma + \delta}{\text{Bo}}\frac{\partial_x(k(\epsilon\sqrt{\mu}\partial_x\zeta))}{\epsilon\sqrt{\mu}}, \end{cases}$$

$$\text{where } \mathcal{N}^{\mu,\delta} = \mathcal{N}^{\mu,\delta}[\epsilon\zeta, \psi] \stackrel{\text{def}}{=} \frac{\left(\frac{1}{\mu}G^\mu\psi + \epsilon(\partial_x\zeta)(\partial_x H^{\mu,\delta}\psi)\right)^2 - \gamma\left(\frac{1}{\mu}G^\mu\psi + \epsilon(\partial_x\zeta)(\partial_x\psi)\right)^2}{2(1 + \mu|\epsilon\partial_x\zeta|^2)}.$$

1.3 Our new class of modified Green-Naghdi models

The original Green-Naghdi system may be obtained from (1.2) by replacing the Dirichlet-Neumann operators with their truncated asymptotic expansions with respect to μ , the shallowness parameter. We let the reader refer to [17] and references therein for full details.

The Green-Naghdi system is usually written in terms of layer-averaged horizontal velocities, that is defining

$$\bar{u}_1(t, x) = \frac{1}{h_1(t, x)} \int_{\epsilon\zeta}^1 \partial_x\phi_1(t, x, z)dz, \quad \bar{u}_2(t, x) = \frac{1}{h_2(t, x)} \int_{-\delta^{-1}}^{\epsilon\zeta} \partial_x\phi_2(t, x, z)dz.$$

Here and thereafter, $h_1 = h_1(\epsilon\zeta) = 1 - \epsilon\zeta$ (resp. $h_2 = h_2(\epsilon\zeta) = \delta^{-1} + \epsilon\zeta$) always denotes the depth of the upper (resp. lower) layer.

One benefit of such a choice of unknowns is the *exact* identities (in contrast with $\mathcal{O}(\mu^2)$ approximations) due to mass conservation (see [17, Proposition 3 and (23)]):

$$(1.3) \quad \partial_t\zeta = \partial_x(h_1\bar{u}_1) = -\partial_x(h_2\bar{u}_2).$$

These identities are then supplemented with the following $\mathcal{O}(\mu^2)$ approximations:

$$(1.4) \quad \partial_t\left(\bar{u}_i + \mu\mathcal{Q}_i[\epsilon\zeta]\bar{u}_i\right) + \frac{\gamma + \delta}{1 - \gamma}\partial_x\zeta + \frac{\epsilon}{2}\partial_x(|\bar{u}_i|^2) = \mu\epsilon\partial_x(\mathcal{R}_i[\epsilon\zeta, \bar{u}_i]) - \frac{1}{\gamma_i}\partial_x P_i,$$

where $P_2 - P_1 = -\frac{\gamma+\delta}{\text{Bo}}\partial_x\left(\frac{1}{\sqrt{1+\mu\epsilon^2|\partial_x\zeta|^2}}\partial_x\zeta\right)$ and

$$(1.5) \quad \mathcal{Q}_i[\epsilon\zeta]\bar{u}_i \stackrel{\text{def}}{=} \frac{-1}{3}h_i^{-1}\partial_x(h_i^3\partial_x\bar{u}_i), \quad \mathcal{R}_i[\epsilon\zeta, \bar{u}_i] \stackrel{\text{def}}{=} \frac{1}{2}(h_i\partial_x\bar{u}_i)^2 + \frac{1}{3}h_i^{-1}\bar{u}_i\partial_x(h_i^3\partial_x\bar{u}_i).$$

Equations (1.3)-(1.4) form a closed system which corresponds to the classical two-layer Green-Naghdi model introduced in [32, 10],² with our choice of scaling in the non-dimensionalization procedure.

² These equations may be found with slightly different but equivalent formulations in the literature. Notice in particular that

$$\begin{aligned} \mu\partial_t(\mathcal{Q}_i[\epsilon\zeta]\bar{u}_i) - \mu\epsilon\partial_x(\mathcal{R}_i[\epsilon\zeta, \bar{u}_i]) &= \mu\mathcal{Q}_i[\epsilon\zeta]\partial_t\bar{u}_i - \frac{\mu\epsilon}{3}h_i^{-1}\partial_x\left(h_i^3(\bar{u}_i\partial_x^2\bar{u}_i - \epsilon(\partial_x\bar{u}_i)^2)\right) \\ &= \frac{-\mu}{3}h_i^{-1}\partial_x\left(h_i^3(\partial_x\partial_t\bar{u}_i + \epsilon\bar{u}_i\partial_x^2\bar{u}_i - \epsilon(\partial_x\bar{u}_i)^2)\right). \end{aligned}$$

We find it convenient to use the following unknown:

$$w \stackrel{\text{def}}{=} \frac{h_1 h_2}{h_1 + \gamma h_2} (\bar{u}_2 - \gamma \bar{u}_1) = -h_1 \bar{u}_1 = h_2 \bar{u}_2,$$

where the last identities are a consequence (in the one-dimensional setting) of (1.3) and the assumption $\bar{u}_i \rightarrow 0$ ($|x| \rightarrow \infty$). One can then rewrite system (1.3)-(1.4) with only two evolution equations:

$$(1.6) \quad \begin{cases} \partial_t \zeta + \partial_x w = 0, \\ \partial_t \left(\frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu \mathcal{Q}[\epsilon \zeta] w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left(\frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) \\ = \mu \epsilon \partial_x (\mathcal{R}[\epsilon \zeta, w]) + \frac{\gamma + \delta}{\text{Bo}} \partial_x^2 \left(\frac{1}{\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2}} \partial_x \zeta \right) \end{cases}$$

with $\mathcal{Q}[\epsilon \zeta] w \stackrel{\text{def}}{=} \mathcal{Q}_2[\epsilon \zeta](h_2^{-1} w) - \gamma \mathcal{Q}_1[\epsilon \zeta](-h_1^{-1} w)$ and $\mathcal{R}[\epsilon \zeta, w] \stackrel{\text{def}}{=} \mathcal{R}_2[\epsilon \zeta, h_2^{-1} w] - \gamma \mathcal{R}_1[\epsilon \zeta, -h_1^{-1} w]$.

Our new class of modified Green-Naghdi models are now obtained by slightly modifying the dispersion components: we replace the operators \mathcal{Q} and \mathcal{R} in (1.6) with the following.

$$(1.7) \quad \begin{cases} \mathcal{Q}^F[\epsilon \zeta] w \stackrel{\text{def}}{=} -\frac{1}{3} h_2^{-1} \partial_x F_2^\mu \{ h_2^3 \partial_x F_2^\mu \{ h_2^{-1} w \} \} - \frac{\gamma}{3} h_1^{-1} \partial_x F_1^\mu \{ h_1^3 \partial_x F_1^\mu \{ h_1^{-1} w \} \}, \\ \mathcal{R}^F[\epsilon \zeta, w] \stackrel{\text{def}}{=} \frac{1}{3} w h_2^{-2} \partial_x F_2^\mu \{ h_2^3 \partial_x F_2^\mu \{ h_2^{-1} w \} \} - \frac{\gamma}{3} w h_1^{-2} \partial_x F_1^\mu \{ h_1^3 \partial_x F_1^\mu \{ h_1^{-1} w \} \} \\ + \frac{1}{2} (h_2 \partial_x F_2^\mu \{ h_2^{-1} w \})^2 - \frac{\gamma}{2} (h_1 \partial_x F_1^\mu \{ h_1^{-1} w \})^2. \end{cases}$$

where $F_i^\mu \stackrel{\text{def}}{=} F_i(\sqrt{\mu} D)$ ($i = 1, 2$) is a Fourier multiplier:

$$\widehat{F_i^\mu \varphi} = F_i(\sqrt{\mu} \xi) \widehat{\varphi}(\xi).$$

The choice of the Fourier multipliers does not need to be precised yet. Natural properties for our purpose, however, include $F_i(0) = 1$ and $F_i'(0) = 0$ (so that $F_i^\mu - \text{Id}$ is formally of size $\mathcal{O}(\mu)$), $F_i(k) = F_i(|k|)$ and $0 \leq F_i \leq 1$. A class of Fourier multipliers for which our rigorous results hold is precised in Definition 1.1, thereafter, and we present three relevant examples below.

- $F_i^{\text{id}}(\sqrt{\mu} D) \equiv 1$ yields, of course, to the original model of [32, 10], namely (1.3)-(1.4). The classical Green-Naghdi model can therefore be treated as a particular case in all our results.
- $F_i^{\text{reg}}(\sqrt{\mu} D) = (1 + \mu \theta_i |D|^2)^{-1/2}$, with $\theta_i > 0$ is an operator of order -1 , and $\sqrt{\mu} \partial_x F_i^\mu$ is a bounded operator in L^2 , uniformly with respect to $\mu \geq 0$. As a consequence, this choice yields a well-posed system for sufficiently small and regular data, even in absence of surface tension.
- $F_i^{\text{imp}}(\sqrt{\mu} D) = \sqrt{\frac{3}{\delta_i^{-1} \sqrt{\mu} |D| |\tanh(\delta_i^{-1} \sqrt{\mu} |D|)} - \frac{3}{\delta_i^{-2} \mu |D|^2}}$, with convention $\delta_1 = 1, \delta_2 = \delta$. The modified Green-Naghdi system with this choice conforms perfectly with the full Euler system, as far as the linear stability analysis of Section 3 is concerned. In particular, its dispersion relation is the same as the one of the full Euler system. One may thus expect (at least qualitatively) an improved precision when only weak nonlinearities ($\epsilon \ll 1$) are involved.

1.4 Outline of the paper

Some elementary properties of our models are studied in Section 2. More precisely, we show that all of our models enjoy a Hamiltonian structure, symmetry groups and conserved quantities, consistently with the already known properties of the original Green-Naghdi model (which themselves are inherited from the full Euler system).

In Section 3, we recall the linear analysis of Kelvin-Helmholtz instabilities for the full Euler system, and extend the study to our models. In particular, we recover that the classical Green-Naghdi model overestimates Kelvin-Helmholtz instabilities, whereas our modified model with the choice $F_i = F_i^{\text{imp}}$ recovers perfectly the behavior of the full Euler system.

Section 4 is dedicated to numerical illustrations of this phenomenon. We give two examples (with and without surface tension) where the original, improved and regularized Green-Naghdi models predict very different behavior. Roughly speaking, the flows are very similar as long as no instabilities are present, but the threshold above which Kelvin-Helmholtz instabilities are triggered varies dramatically from one model to the other.

The main and rigorous results concerning our models are restricted to the following class of Fourier multipliers in (1.7).

Definition 1.1. *The operator $F_i^\mu = F_i(\sqrt{\mu}D)$ ($i = 1, 2$) is admissible if it satisfies:*

- i. $F_i : \mathbb{R} \rightarrow \mathbb{R}^+$ is even and positive;*
- ii. F_i is of twice differentiable, $F_i(0) = 1$, $F_i'(0) = 0$ and $\sup_{k \in \mathbb{R}} |F_i''(k)| \leq C_F < \infty$;*
- iii. $k \mapsto |k|F_i(k)$ is sub-additive, namely for any $k, l \in \mathbb{R}$, $|k + l|F_i(k + l) \leq |k|F_i(k) + |l|F_i(l)$.*

In that case, one can define appropriate pairs $K_{F_i} \in \mathbb{R}^+$ and $\sigma \in [0, 1]$ such that

$$(1.8) \quad \forall k \in \mathbb{R}, \quad F_i(k) \leq K_{F_i} |k|^{-\sigma}.$$

Proposition 1.2. *The three aforementioned examples, namely $F_i^{\text{id}}, F_i^{\text{reg}}, F_i^{\text{imp}}$ are admissible, and satisfy (1.8) with (respectively) $\sigma = 0, 1, 1/2$.*

Section 5 is dedicated to the proof that for any admissible choice F_i , the Cauchy problem for system (1.6)-(1.7) with sufficiently regular initial data is well-posed under some hyperbolicity conditions. Roughly speaking, we show that provided

$$\Upsilon_F = \epsilon^2(1 + (\gamma K_1 + K_{F_2})(\mu \text{Bo})^{1-\sigma}) \text{ is sufficiently small,}$$

then our system is well-posed (in the sense of Hadamard) for sufficiently regular and bounded initial data, on a time interval uniform with respect to compact sets of parameters; see Theorem 5.1 for details.

In section 6, we supplement the above result with consistency (Proposition 6.1) and stability (Proposition 6.2) results, which together offer the full justification of our models (Proposition 6.3).

Finally, we present in Section A some improved results in the limiting case $\mu = 0$, that is on the so-called Saint-Venant, or shallow-water system (with surface tension). Section B is dedicated to the detailed presentation of our functional setting and some notations; and Section C provides preliminary results concerning our functional spaces.

We conclude this section with the proof of Proposition 1.2.

Lemma 1.3. *A sufficient condition for F_i^μ to be admissible is, in addition to i. and ii.,*

- iii'. $k \mapsto kF_i(k)$ is non-decreasing on \mathbb{R}^+ and $k \mapsto F_i(k)$ is non-increasing on \mathbb{R}^+ .*

Proof. Let $k + l \geq k \geq l \geq 0$. Since $F_i(k)$ is non-increasing on \mathbb{R}^+ , one has $0 \leq F_i(k + l) \leq F_i(k) \leq F_i(l)$, and therefore $(k + l)F_i(k + l) \leq kF_i(k) + lF_i(l) \leq kF_i(k) + lF_i(l)$. This shows $k \in \mathbb{R}^+ \mapsto kF_i(k) = |k|F_i(k)$ is sub-additive.

Since F_i is even, $k \mapsto F_i(k)$ is non-decreasing for $k \in \mathbb{R}^-$, and one shows in the same way that $k \in \mathbb{R}^- \mapsto |k|F_i(k)$ is sub-additive.

There remains the case $k \leq 0 \leq l$. In that situation, since F_i is even and $k \in \mathbb{R}^+ \mapsto kF_i$ is non-decreasing, one has $|k + l|F_i(k + l) = |k + l|F_i(|k + l|) \leq (|k| + |l|)F_i(|k| + |l|)$ and the sub-additivity in \mathbb{R}^+ yields the desired result. \square

Proposition 1.2 is now obtained as a direct consequence of Lemma 1.3.

Only the last example requires clarifications. That $F_i^{\text{imp}}(0) = 1$ and $F_i^{\text{imp}'}(0) = 0$ is given by standard Taylor expansion at the origin: $F_i^{\text{imp}}(k) = 1 - \frac{1}{15}\delta_i^{-2}k^2 + \mathcal{O}(k^4)$.

We first remark that $f : k \mapsto \frac{k}{\tanh(k)}$ is non-decreasing for $k \in \mathbb{R}^+$, since $f'(k) = \frac{\cosh(k)\sinh(k) - k}{\sinh(k)^2}$, and $\cosh(k) \times \sinh(k) \geq 1 \times k$ for $k \in \mathbb{R}^+$. It follows that $k \in \mathbb{R}^+ \mapsto kF_i^{\text{imp}}(k) = \sqrt{3}\delta_i \sqrt{f(\delta_i^{-1}k) - 1}$ is non-decreasing.

Now, we have to prove that $k \mapsto F_i^{\text{imp}}(k)$ is non-increasing for $k \in \mathbb{R}^+$. Equivalently, we show that $g : k \mapsto \frac{1}{k \tanh(k)} - \frac{1}{k^2}$ is non-increasing. Notice

$$g'(k) = \frac{-k^2 - k \sinh(k) \cosh(k) + 2 \sinh(k)^2}{k^3 \sinh(k)^2} = \frac{-k^2 - \frac{1}{2}k \sinh(2k) + \cosh(2k) - 1}{k^3 \sinh(k)^2}.$$

One can show that $h(k) = -k^2 - \frac{1}{2}k \sinh(2k) + \cosh(2k) - 1$ is negative for $k > 0$ by differentiating several times with respect to k . One easily checks that $h(0) = h'(0) = h''(0) = 0$ and one has

$$h'''(k) = 2 \sinh(2k) - 4k \cosh(2k) = 2 \cosh(2k)(\tanh(2k) - 2k) < 0 \quad (k > 0).$$

It follows, iteratively, that $h''(k) > 0$, $h'(k) > 0$, and finally $h(k) > 0$ for any $k > 0$. We conclude that g and therefore $k \mapsto F_i^{\text{imp}}(k)$ are non-increasing for $k \in \mathbb{R}^+$.

By Lemma 1.3, $k \mapsto |k|F_i^{\text{imp}}(k)$ is sub-additive, and the proof of Proposition 1.2 is complete.

2 Hamiltonian structure, group of symmetries and conserved quantities

It is known from the seminal work of Zakharov [43] that the full Euler system (with one layer) admits a Hamiltonian structure. This Hamiltonian structure has been extended to the two-layer case in [3, 28, 13]. The classical Green-Naghdi system also admits a Hamiltonian structure, which is directly inherited from the one of the full Euler system [14, 2]. We show in Section 2.1 that this structure is preserved in our models. Such a Hamiltonian structure has not been exhibited for regularized Green-Naghdi system in the literature [33, 9, 6, 27], with the noteworthy exception of [12].

We then enumerate the group of symmetries of the system (Section 2.2) that originates from the full Euler system (see [4]), and deduce the related conserved quantities (Section 2.3).

2.1 Hamiltonian formulation

Let us recall the well-known Hamiltonian structure of the full Euler system. The discussion below is loosely based on [14]. We introduce

$$H(\zeta, v) = \frac{1}{2} \int_{\mathbb{R}} (\gamma + \delta)\zeta^2 + \frac{2(\gamma + \delta)}{\mu\epsilon^2 \text{Bo}} (\sqrt{1 + \mu\epsilon^2 |\partial_x \zeta|^2} - 1) - v\mathcal{G}_1(\gamma\mathcal{G}_2 + \mathcal{G}_1)^{-1}\mathcal{G}_2 v,$$

where $\mathcal{G}_i = \mathcal{G}_i[\epsilon\zeta]$ are such that $\partial_x(\mathcal{G}_i \partial_x \varphi) = G_i \varphi$, and G_i are Dirichlet-to-Neumann operators:

- $G_1 : \varphi \mapsto \frac{1}{\mu}(\partial_n \phi) \Big|_{z=\epsilon\zeta}$, where ϕ is the unique solution to

$$\mu \partial_x^2 \phi + \partial_z^2 \phi = 0 \text{ in } \Omega_1^t, \quad \phi(x, \epsilon\zeta) = \varphi \quad \text{and} \quad \partial_z \phi(x, 1) = 0.$$

- $G_2 : \varphi \mapsto \frac{1}{\mu}(\partial_n \phi) \Big|_{z=\epsilon\zeta}$, where ϕ is the unique solution to

$$\mu \partial_x^2 \phi + \partial_z^2 \phi = 0 \text{ in } \Omega_2^t, \quad \phi(x, \epsilon\zeta) = \varphi \quad \text{and} \quad \partial_z \phi(x, -\delta^{-1}) = 0.$$

The operators G_i are well-defined if $h_1(\epsilon\zeta), h_2(\epsilon\zeta) \geq h_0 > 0$ (see [26, Chapter 3]), and consequently $\mathcal{G}_i \partial_x \varphi$ as well, thanks to the identity [26, Prop. 3.35]

$$G_i[\epsilon\zeta]\psi = -\partial_x(h_i \bar{u}_i) \quad (i = 1, 2).$$

In that way, recalling the construction of the full Euler in Section 1.2, we define

$$\begin{aligned} \psi(t, x) &\stackrel{\text{def}}{=} \phi_1(t, x, \epsilon\zeta(t, x)) \\ v(t, x) &\stackrel{\text{def}}{=} \partial_x(\phi_2(t, x, \epsilon\zeta(t, x)) - \gamma\phi_1(t, x, \epsilon\zeta(t, x))), \end{aligned}$$

and deduce $v = \mathcal{G}_2^{-1} \mathcal{G}_1 \partial_x \psi - \gamma \partial_x \psi = \mathcal{G}_2^{-1} (\mathcal{G}_1 + \gamma \mathcal{G}_2) \partial_x \psi$.

With these definitions, one can check that the full Euler system (1.2) can be written as ³

$$\partial_t \zeta = -\partial_x \left(\frac{\delta H}{\delta v} \right), \quad \partial_t v = -\partial_x \left(\frac{\delta H}{\delta \zeta} \right),$$

with the usual notation for functional derivatives of Fréchet differentiable functionals, *i.e.* for any $\Phi \in \mathcal{S}(\mathbb{R})$,

$$\left(\frac{\delta H}{\delta \zeta}, \Phi \right)_{L^2} = \lim_{\varkappa \rightarrow 0} \frac{H(\zeta + \varkappa \Phi, v) - H(\zeta, v)}{\varkappa} \quad ; \quad \left(\frac{\delta H}{\delta v}, \Phi \right)_{L^2} = \lim_{\varkappa \rightarrow 0} \frac{H(\zeta, v + \varkappa \Phi) - H(\zeta, v)}{\varkappa}.$$

Let us now deduce the Hamiltonian structure of the Green-Naghdi systems. We use the following $\mathcal{O}(\mu^2)$ approximations (see *e.g.* [26, Prop. 3.37] when $F_i^\mu \equiv 1$, the general case adds only a $\mathcal{O}(\mu^2)$ perturbation):

$$\mathcal{G}_i^{-1} = -\mathcal{A}_i + \mathcal{O}(\mu^2), \quad \mathcal{A}_i[\epsilon\zeta] : w \mapsto \frac{1}{h_i} w - \mu \frac{1}{3} h_i^{-1} \partial_x F_i^\mu \{ h_i^3 \partial_x F_i^\mu \{ h_i^{-1} w \} \}.$$

Notice that $\mathcal{A}[\epsilon\zeta] \stackrel{\text{def}}{=} \gamma \mathcal{A}_1[\epsilon\zeta] + \mathcal{A}_2[\epsilon\zeta]$ is a symmetric, coercive thus positive definite operator, if $h_1(\epsilon\zeta), h_2(\epsilon\zeta) \geq h_0 > 0$ (see Section 5.2 below). The full Euler's Hamiltonian becomes, plugging the above (truncated) approximation,

$$(2.1) \quad H_{\text{GN}}(\zeta, v) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}} (\gamma + \delta) \zeta^2 + \frac{2(\gamma + \delta)}{\mu \epsilon^2 \text{Bo}} (\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} - 1) + v \mathcal{A}[\epsilon\zeta]^{-1} v.$$

Consistently with Section 1.3 and the above approximation $\mathcal{G}_i^{-1} = -\mathcal{A}_i$, one may introduce the notation

$$w \stackrel{\text{def}}{=} \mathcal{A}[\epsilon\zeta]^{-1} v \quad \text{and} \quad \bar{u}_i \stackrel{\text{def}}{=} (-1)^i w / h_i \quad (i = 1, 2).$$

The new Hamiltonian may then be written as

$$\begin{aligned} H_{\text{GN}}(\zeta, v) &= \frac{1}{2} \int_{\mathbb{R}} (\gamma + \delta) \zeta^2 + \frac{2(\gamma + \delta)}{\mu \epsilon^2 \text{Bo}} (\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} - 1) + w \mathcal{A}[\epsilon\zeta] w \\ &= \frac{1}{2} \int_{\mathbb{R}} (\gamma + \delta) \zeta^2 + \frac{2(\gamma + \delta)}{\mu \epsilon^2 \text{Bo}} (\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} - 1) + \gamma h_1 |\bar{u}_1|^2 + h_2 |\bar{u}_2|^2 \\ &\quad + \mu \frac{\gamma}{3} h_1 (h_1 \partial_x F_1^\mu \bar{u}_1)^2 + \mu \frac{1}{3} h_2 (h_2 \partial_x F_2^\mu \bar{u}_2)^2. \end{aligned}$$

³The first identity is easily seen from the above, since

$$\frac{\delta H}{\delta v} = \mathcal{G}_1 (\gamma \mathcal{G}_2 + \mathcal{G}_1)^{-1} \mathcal{G}_2 v = \mathcal{G}_1 \partial_x \psi = \frac{1}{\mu} (\partial_n \phi) \Big|_{z=\epsilon\zeta}.$$

The second is less straightforward, we let the reader refer to [3, 13, 14] for more details.

Let us now check the Hamiltonian structure of our modified systems (1.6)-(1.7) by computing the fractional derivatives, $\frac{\delta H_{\text{GN}}}{\delta \zeta}$ and $\frac{\delta H_{\text{GN}}}{\delta v}$. One has immediately

$$\frac{\delta H_{\text{GN}}}{\delta v} = \mathcal{A}[\epsilon \zeta]^{-1} v = w.$$

Calculations are longer but straightforward for the second term. One has

$$\begin{aligned} \frac{\delta H_{\text{GN}}}{\delta \zeta} &= (\gamma + \delta) \zeta - \frac{\gamma + \delta}{\text{Bo}} \partial_x \left(\frac{(\partial_x \zeta)}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{1/2}} \right) - \frac{1}{2} (\mathcal{A}[\epsilon \zeta]^{-1} v) \frac{\delta \mathcal{A}}{\delta \zeta} (\mathcal{A}[\epsilon \zeta]^{-1} v) \\ &= (\gamma + \delta) \zeta - \frac{\gamma + \delta}{\text{Bo}} \partial_x \left(\frac{(\partial_x \zeta)}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{1/2}} \right) + \frac{\epsilon}{2} \frac{h_1^2 - \gamma h_2^2}{h_1^2 h_2^2} |w|^2 - \mu \epsilon \mathcal{R}^{\text{F}}[\epsilon \zeta, w]. \end{aligned}$$

One now recognizes the Hamiltonian structures of system (1.6)-(1.7):

$$\partial_t \zeta = -\partial_x \left(\frac{\delta H_{\text{GN}}}{\delta v} \right) \quad ; \quad \partial_t v = -\partial_x \left(\frac{\delta H_{\text{GN}}}{\delta \zeta} \right).$$

2.2 Symmetry groups

Based on the work of [4], one may list symmetry groups of our systems. Most of the symmetry groups of the full Euler system have no equivalent for the Green-Naghdi model, because they involve variations on the vertical variable, which is not accessible anymore. The most physical symmetries, however, remain. We list them below.

Assume (ζ, \bar{u}_i, P_i) (with $i = 1, 2$) satisfies (1.3)-(1.4). Then for any $\varkappa \in \mathbb{R}$, $(\zeta^\varkappa, \bar{u}_i^\varkappa, P_i^\varkappa)$ also satisfies (1.3)-(1.4), where

i. Horizontal translation

$$(\zeta^\varkappa, \bar{u}_i^\varkappa, P_i^\varkappa)(t, x) \stackrel{\text{def}}{=} \left(\zeta(t, x - \varkappa), \bar{u}_i(t, x - \varkappa), P_i(t, x - \varkappa) \right)$$

ii. Time translation

$$(\zeta^\varkappa, \bar{u}_i^\varkappa, P_i^\varkappa)(t, x) \stackrel{\text{def}}{=} \left(\zeta(t - \varkappa, x), \bar{u}_i(t - \varkappa, x), P_i(t - \varkappa, x) \right)$$

iii. Variation of base-level for potential pressure

$$(\zeta^\varkappa, \bar{u}_i^\varkappa, P_i^\varkappa)(t, x) \stackrel{\text{def}}{=} \left(\zeta(t, x), \bar{u}_i(t, x), P_i(t, x) + \varkappa \right)$$

iv. Horizontal Galilean boost

$$(\zeta^\varkappa, \bar{u}_i^\varkappa, P_i^\varkappa)(t, x) \stackrel{\text{def}}{=} \left(\zeta(t, x - \varkappa t), \bar{u}_i(t, x - \varkappa t) + \varkappa, P_i(t, x - \varkappa t) \right)$$

It is interesting to notice that when working with formulation (1.6). i.,ii.,iii. induce symmetry groups as well (although iii. is trivial), but not iv.. Indeed, because the Galilean boost breaks the conditions $\bar{u}_i \rightarrow 0$ at infinity, the identity $w = -h_1 \bar{u}_1 = h_2 \bar{u}_2$ is invalid for any non-trivial \varkappa . Such a discrepancy is attributable to the rigid-lid assumption, and in particular vanishes in the limit $\gamma \rightarrow 0$ (one layer of fluid); see also the discussion of the following section.

2.3 Conserved quantities

Again, we find it more convenient at first to work with formulation (1.3)-(1.4)-(1.7), and deduce the conserved quantities of (1.6) afterwards.

The first obviously conserved quantity, given by (1.3), is the (excess of) mass:

$$\frac{d}{dt} \mathcal{Z} = 0, \quad \mathcal{Z}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \zeta(t, x) dx.$$

Equations (1.4) yield other conserved quantities: the ‘‘horizontal velocity mass’’

$$\frac{d}{dt} \mathcal{V}_i = 0, \quad \mathcal{V}_i \stackrel{\text{def}}{=} \int_{\mathbb{R}} \bar{u}_i + \mu \mathcal{Q}_i^F[\epsilon \zeta] \bar{u}_i dx \quad (i = 1, 2).$$

Choi and Camassa [10] observed a similar conservation law of the original model, and related this result to the irrotationality assumption of the full Euler system. Indeed, by the discussion of Section 2.1, one has

$$\mathcal{V}_i \approx \int_{\mathbb{R}} \partial_x(\phi_i(t, x, \epsilon \zeta(t, x))) dx,$$

where ϕ_1 (resp. ϕ_2) is the velocity potential of the upper (resp. lower) layer, and the approximation is of size $\mathcal{O}(\mu^2)$. Thus one has by construction $\frac{d}{dt} \mathcal{V}_i = \mathcal{O}(\mu^2)$, and it turns out that this approximately conserved quantity is actually exactly conserved by the Green-Naghdi flow (see also [20]). Of course, the linear combination

$$\mathcal{V}_2 - \gamma \mathcal{V}_1 = \int_{\mathbb{R}} \frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu \mathcal{Q}^F[\epsilon \zeta] w \, dx$$

is a conserved quantity of system (1.6)-(1.7).

After long but straightforward manipulations, one may check that the total horizontal momentum satisfies

$$\frac{d}{dt} \mathcal{M} = - \int_{-\infty}^{\infty} h_1 \partial_x P_1 + h_2 \partial_x P_2 = [h_1 P_1 + h_2 P_2]_{-\infty}^{+\infty}, \quad \mathcal{M} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \gamma h_1 \bar{u}_1 + h_2 \bar{u}_2 dx.$$

The horizontal momentum is in general *not* conserved. This somewhat unintuitive result is a consequence of the rigid-lid assumption (the momentum is conserved in the one-layer case with free surface), and has been thoroughly studied in [7, 8].

One has the conservation of total energy:

$$\begin{aligned} \frac{d}{dt} \mathcal{E} = 0, \quad \mathcal{E} \stackrel{\text{def}}{=} \int_{\mathbb{R}} H_{\text{GN}} dx &= \int_{\mathbb{R}} (\gamma + \delta) \zeta^2 + \frac{2(\gamma + \delta)}{\mu \epsilon^2 \text{Bo}} (\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2} - 1) \\ &+ \gamma h_1 |\bar{u}_1|^2 + h_2 |\bar{u}_2|^2 + \mu \frac{\gamma}{3} h_1 (h_1 \partial_x F_1^\mu \bar{u}_1)^2 + \mu \frac{1}{3} h_2 (h_2 \partial_x F_2^\mu \bar{u}_2)^2 dx. \end{aligned}$$

The conservation of energy may be deduced from the Hamiltonian structure of the system; see *e.g.* [39]. Let us denote for simplicity $U = (\zeta, v)^\top$, $\delta H_{\text{GN}} = (\frac{\delta H_{\text{GN}}}{\delta \zeta}, \frac{\delta H_{\text{GN}}}{\delta v})^\top$ and $J = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$, so that system (1.6)-(1.7) reads simply (by the discussion of Section 2.1)

$$(2.2) \quad \partial_t U = J \delta H_{\text{GN}}.$$

One deduces

$$0 = \{H_{\text{GN}}, H_{\text{GN}}\} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \delta H_{\text{GN}} \cdot J \delta H_{\text{GN}} \, dx = \int_{\mathbb{R}} \delta H_{\text{GN}} \cdot \partial_t U \, dx = \int_{\mathbb{R}} \partial_t (H_{\text{GN}}(\zeta, v)) \, dx,$$

where the first identity owes to the skew-symmetry of J , and the last identity follows from the invariance of the Hamiltonian with respect to time translation. This yields immediately

$$\frac{d}{dt} \int_{\mathbb{R}} H_{\text{GN}}(\zeta, v) \, dx = \frac{d}{dt} \mathcal{E} = 0.$$

Similarly, the invariance of the Hamiltonian with respect to space translation yields

$$\{H_{\text{GN}}, I\} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \delta H_{\text{GN}} \cdot J \delta I \, dx = \int_{\mathbb{R}} \delta H_{\text{GN}} \cdot (-\partial_x U) \, dx = \int_{\mathbb{R}} -\partial_x (H_{\text{GN}}(\zeta, v)) \, dx = 0,$$

where $I(\zeta, v) \stackrel{\text{def}}{=} \zeta v$, so that $\delta I = (\frac{\delta I}{\delta \zeta}, \frac{\delta I}{\delta v})^\top = (v, \zeta)^\top$. By (2.2) and the skew-symmetry of J , the horizontal impulse is thus conserved:

$$\frac{d}{dt} \mathcal{I} \stackrel{\text{def}}{=} \frac{d}{dt} \int_{\mathbb{R}} I(\zeta, v) \, dx = \int_{\mathbb{R}} \delta \mathcal{I} \cdot \partial_t U \, dx = \int_{\mathbb{R}} \delta \mathcal{I} \cdot J \delta H_{\text{GN}} \, dx = -\{H_{\text{GN}}, \mathcal{I}\} = 0.$$

This conserved quantity of the bi-fluidic Green-Naghdi model seems to have been unnoticed until now. In the one-layer case, that is when $\gamma = 0$, it is related to the momentum (which is conserved in this situation) through the horizontal velocity mass:

$$\delta^{-1} \mathcal{V} + \epsilon \mathcal{I} = \int_{\mathbb{R}} h_2 v \, dx = \int_{\mathbb{R}} h_2 \bar{u}_2 + \mu h_2 \mathcal{Q}_2^{\mathbb{F}}[\epsilon \zeta] \bar{u}_2 \, dx = \int_{\mathbb{R}} h_2 \bar{u}_2 \, dx = \mathcal{M}.$$

When considering the case of one-layer with free surface, the symmetry with respect to Galilean boost yields an additional conserved quantity, which is the counterpart of the ‘‘horizontal coordinate of mass centroid times mass’’ for the full Euler system as defined in [4], namely

$$\mathcal{C}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} C(t, x) \, dx, \quad \text{with } C(t, x) = \zeta x - t h_2 v \quad \text{or, equivalently, } C(t, x) = \zeta x - t w.$$

The conservation of \mathcal{C} can be deduced as above, or simply from the conservation of momentum in the one-layer case. Indeed, one deduces from (1.3)

$$\frac{d}{dt} \mathcal{C} = \int_{\mathbb{R}} x \partial_t \zeta - w - t \partial_t w \, dx = \int_{\mathbb{R}} -x \partial_x w - w \, dx - t \frac{d}{dt} \int_{\mathbb{R}} w \, dx = -t \frac{d}{dt} \mathcal{M}.$$

3 Kelvin-Helmholtz instabilities

In this section, we formally investigate the conditions for the appearance of Kelvin-Helmholtz instabilities for the full Euler system as well as for our Green-Naghdi models. In order to do so, we linearize the system at stake around the solution of constant shear (flat interface, $\zeta = 0$; and constant, horizontal velocity in each layer). This yields a linear system which may be then explicitly solved through Fourier analysis. In particular, one obtains sufficient and necessary conditions for the existence of *unstable modes*, that is planewave solutions growing exponentially in time. We will say that the system suffers from Kelvin-Helmholtz instabilities when such modes exist.

Such a study has been thoroughly worked out by Lannes and Ming [27] for the full Euler system and the original Green-Naghdi model (as well as some regularized models derived therein), and the fact that the original Green-Naghdi model overestimates Kelvin-Helmholtz instabilities was highlighted. Thus we only briefly recall the result concerning the full Euler system in Section 3.1, and extend the results of the original Green-Naghdi system to our class of modified systems in Section 3.2. The fact that our models can be tailored to improve their behavior with respect to Kelvin-Helmholtz instabilities is brought to light, and serves as the main motivation for this work.

3.1 The full Euler system

We linearize (1.1) around the constant shear solution: $(\zeta = 0 + \varkappa \tilde{\zeta}, \phi_1 = \underline{u}_1 x + \varkappa \tilde{\phi}_1, \phi_2 = \underline{u}_2 x + \varkappa \tilde{\phi}_2)$ where \underline{u}_1 and \underline{u}_2 are constants, and $\varkappa \ll 1$. Notice that by (1.3), one has necessarily $\underline{u}_1 + \delta^{-1} \underline{u}_2 = 0$, and therefore $\underline{u}_1 = \frac{-v}{\gamma + \delta}$ and $\underline{u}_2 = \frac{\delta v}{\gamma + \delta}$, where $v \stackrel{\text{def}}{=} \underline{u}_2 - \gamma \underline{u}_1$. When withdrawing $\mathcal{O}(\varkappa^2)$ terms, one obtains the following linear system (see [27]):

$$(3.1) \quad \begin{cases} \partial_t \tilde{\zeta} + c(D) \partial_x \tilde{\zeta} + b(D) \partial_x \tilde{v} = 0, \\ \partial_t \tilde{v} + a(D) \partial_x \tilde{\zeta} + c(D) \partial_x \tilde{v} = 0 \end{cases}$$

where $\tilde{v} \stackrel{\text{def}}{=} \partial_x((\tilde{\phi}_2 - \gamma \tilde{\phi}_1)|_{z=\epsilon_\zeta})$, and

$$c(k) \stackrel{\text{def}}{=} \frac{\delta \tanh(\sqrt{\mu}|k|) - \gamma \tanh(\sqrt{\mu}\delta^{-1}|k|)}{\tanh(\sqrt{\mu}|k|) + \gamma \tanh(\sqrt{\mu}\delta^{-1}|k|)} \frac{\epsilon v}{\gamma + \delta}, \quad b(k) \stackrel{\text{def}}{=} \frac{1}{\sqrt{\mu}|k|} \frac{\tanh(\sqrt{\mu}|k|) \tanh(\sqrt{\mu}\delta^{-1}|k|)}{\tanh(\sqrt{\mu}|k|) + \gamma \tanh(\sqrt{\mu}\delta^{-1}|k|)}$$

and

$$a(k) \stackrel{\text{def}}{=} (\gamma + \delta) \left(1 + \frac{|k|^2}{\text{Bo}}\right) - \frac{\sqrt{\mu}|k|\gamma}{\tanh(\sqrt{\mu}|k|) + \gamma \tanh(\sqrt{\mu}\delta^{-1}|k|)} \frac{(\delta + 1)^2}{(\delta + \gamma)^2} |\epsilon v|^2.$$

Since $b(k) > 0$, the mode with wavenumber k is stable (namely the planewave solutions $e^{i(kx - \omega_\pm(k)t)}$ satisfy $\omega_\pm(k) \in \mathbb{R}$) if and only if $a(k) > 0$.

For small values of k , this yields the necessary condition

$$\gamma \epsilon^2 |v|^2 \frac{\delta(\delta + 1)^2}{(\delta + \gamma)^3} < \gamma + \delta.$$

For large values of k , one approximates $\tanh(\sqrt{\mu}|k|) + \gamma \tanh(\sqrt{\mu}\delta^{-1}|k|) \approx 1 + \gamma$, and deduce

$$\min_{|k|} \{a(k)\} \approx (\gamma + \delta) - \frac{\gamma^2 \mu \text{Bo} (\delta + 1)^4}{4(1 + \gamma)^2 (\gamma + \delta)^5} \epsilon^4 |v|^4.$$

The full Euler system is therefore stable for each wavenumber provided

$$\Upsilon |v|^2 \stackrel{\text{def}}{=} \gamma(1 + \sqrt{\mu \text{Bo}}) \epsilon^2 |v|^2 \quad \text{is sufficiently small.}$$

Again, we refer to [27] for much a more involved analysis.

3.2 Our class of Green-Naghdi systems

When linearizing (1.6)-(1.7) around the constant shear solution, $U \stackrel{\text{def}}{=} (\zeta, w)^\top \stackrel{\text{def}}{=} (0 + \varkappa \tilde{\zeta}, \underline{w} + \varkappa \tilde{w})^\top$, where \underline{w} is constant, one obtains the following system:

$$(3.2) \quad \begin{cases} \partial_t \tilde{\zeta} + \partial_x \tilde{w} = 0, \\ \bar{b}^F(D) \partial_t \tilde{w} - \bar{c}^F(D) \partial_t \tilde{\zeta} + \bar{a}^F(D) \partial_x \tilde{\zeta} + \bar{c}^F(D) \partial_x \tilde{w} = 0, \end{cases}$$

with

$$\bar{b}^F(k) \stackrel{\text{def}}{=} \gamma + \delta + \mu \frac{|F_2^\mu|^2 + \gamma \delta |F_1^\mu|^2}{3\delta} |k|^2, \quad \bar{c}^F(k) = \epsilon \underline{w} (\delta^2 - \gamma) + \mu \epsilon \underline{w} \frac{|F_2^\mu|^2 - \gamma |F_1^\mu|^2}{3} |k|^2,$$

and

$$\bar{a}^F(k) \stackrel{\text{def}}{=} (\gamma + \delta) - \epsilon^2 |\underline{w}|^2 (\delta^3 + \gamma) - \mu \epsilon^2 |\underline{w}|^2 \frac{\delta |F_2^\mu|^2 + \gamma |F_1^\mu|^2}{3} |k|^2 + \frac{\gamma + \delta}{\text{Bo}} |k|^2$$

where (with a slight abuse of notations) $F_i^\mu = F_i(\sqrt{\mu}k)$.

The stability criterion is more easily seen when rewriting system (3.2) with unknown ⁴

$$\tilde{v} = (\gamma + \delta)\tilde{w} - \epsilon\underline{w}(\delta^2 - \gamma)\tilde{\zeta} - \mu\left\{\frac{|F_2^\mu|^2 + \gamma\delta|F_1^\mu|^2}{3\delta}\partial_x^2\tilde{w} - \epsilon\underline{w}\frac{|F_2^\mu|^2 - \gamma|F_1^\mu|^2}{3}\partial_x^2\tilde{\zeta}\right\}.$$

Indeed, one obtains in that case

$$(3.3) \quad \begin{cases} \partial_t\tilde{\zeta} + c^F(D)\partial_x\tilde{\zeta} + b^F(D)\partial_x\tilde{v} = 0, \\ \partial_t\tilde{v} + a^F(D)\partial_x\tilde{\zeta} + c^F(D)\partial_x\tilde{v} = 0 \end{cases}$$

with

$$c^F(k) = \epsilon\underline{w}\frac{\frac{\delta^2 - \gamma}{\gamma + \delta} + \mu\frac{|F_2^\mu|^2 - \gamma|F_1^\mu|^2}{3(\gamma + \delta)}|k|^2}{1 + \mu\frac{|F_2^\mu|^2 + \gamma\delta|F_1^\mu|^2}{3\delta(\gamma + \delta)}|k|^2}, \quad b^F(k) = \frac{\frac{1}{\gamma + \delta}}{1 + \mu\frac{|F_2^\mu|^2 + \gamma\delta|F_1^\mu|^2}{3\delta(\gamma + \delta)}|k|^2}$$

and

$$a^F(k) = (\gamma + \delta) + \frac{\gamma + \delta}{\text{Bo}}|k|^2 - |\epsilon\underline{w}|^2\frac{\gamma(\delta + 1)^2}{\delta(\gamma + \delta)}\frac{(\delta^2 + \frac{1}{3}\mu|k|^2|F_2^\mu|^2)(1 + \frac{1}{3}\mu|k|^2|F_1^\mu|^2)}{1 + \mu\frac{|F_2^\mu|^2 + \gamma\delta|F_1^\mu|^2}{3\delta(\gamma + \delta)}|k|^2}.$$

As for the full Euler system, since $b^F(k) > 0$, the mode with wavenumber k is stable (namely the planewave solutions $e^{i(kx - \omega_\pm(k)t)}$ satisfy $\omega_\pm(k) \in \mathbb{R}$) if and only if $a^F(k) > 0$.

Let us quickly discuss the three examples introduced in Section 1.3.

- In the case of the original Green-Naghdi system, $F_i^{\text{id}}(\sqrt{\mu}D) \equiv 1$, the condition to ensure that all modes are stable is (see [27] for a more detailed discussion)

$$\Upsilon_{\text{GN}}|\underline{w}|^2 \stackrel{\text{def}}{=} \gamma(1 + \mu\text{Bo})\epsilon^2|\underline{w}|^2 \quad \text{is sufficiently small.}$$

This is more stringent than the similar condition of the full Euler system in the oceanographic context, where one expects $\mu\text{Bo} \gg 1$.

- If $F_i^{\text{imp}}(\sqrt{\mu}D) = \sqrt{\frac{3}{\delta_i^{-1}\sqrt{\mu}|D|\tanh(\delta_i^{-1}\sqrt{\mu}|D|)} - \frac{3}{\delta_i^{-2}\mu|D|^2}}$ (with convention $\delta_1 = 1, \delta_2 = \delta$), then the linearized system (3.3) is *exactly* (3.1) (recall that by (1.3), $\underline{w} = \frac{1}{\gamma + \delta}\underline{v}$):

$$a^F(k) = a(k) \quad ; \quad b^F(k) = b(k) \quad ; \quad c^F(k) = c(k)$$

In particular, the stability criterion of this Green-Naghdi model corresponds to the one of the full Euler system. As previously mentioned, this also shows that the model has the same dispersion relation as the full Euler system, as this corresponds to setting $\underline{w} = 0$. Models with such a property were already obtained and discussed; see [42, 5, 37, 34] and references therein, but never to our knowledge in the shallow-water regime.

- In the case $F_i^{\text{reg}}(\sqrt{\mu}D) = \frac{1}{\sqrt{1 + \mu\theta_i|D|^2}}$, one remarks that

$$(\gamma + \delta) > |\epsilon\underline{w}|^2\frac{\gamma(\delta + 1)^2}{\delta(\gamma + \delta)}\left(\delta^2 + \frac{1}{3\theta_2}\right)\left(1 + \frac{1}{3\theta_1}\right),$$

⁴This change of unknown is not without signification. It consists in writing the system with the ‘‘original’’ variables of the full Euler system: $\zeta, v = \partial_x((\phi_2 - \gamma\phi_1)|_{z=c_\zeta})$, or more precisely $\mathcal{O}(\mu^2)$ approximations of these variables, instead of using the flux w . It is interesting to compare our nonlinear results and in particular Theorem 5.1 with the naive sufficient condition for stability, $\bar{a}^F(k) > 0$ which comes from (3.2). We see that our hyperbolicity conditions are natural generalizations of this instability criterion, and explains the discrepancy with respect the sharp condition $a^F(k) > 0$ (in particular when $\gamma \rightarrow 0$); see also Remark 5.3.

is a sufficient condition to ensure that all modes are stable, and *does not require the presence of surface tension*. A natural choice is $\theta_i = \frac{1}{15\delta_i^2}$ with convention $\delta_1 = 1, \delta_2 = \delta$, motivated by the fact that the Taylor expansion of the dispersion relation around $\mu = 0$ fits the one of the improved model, and therefore the one of the full Euler system, at augmented order $\mathcal{O}(\mu^3)$, instead of the $\mathcal{O}(\mu^2)$ precision of the original Green-Naghdi system.

In Figure 1, we plot the instability curves corresponding to $a^F(k)$ for the three above examples. More precisely, for fixed $k \in \mathbb{R}$, we plot the value of $\epsilon^2|w|^2$ above which $a^F(k) > 0$, and thus instabilities are triggered. One clearly sees a great discrepancy for large wavenumbers. In particular the minimum of the curve, which corresponds to the domain where all wavenumbers are stable, not only varies for each model but also is obtained at different values of k .

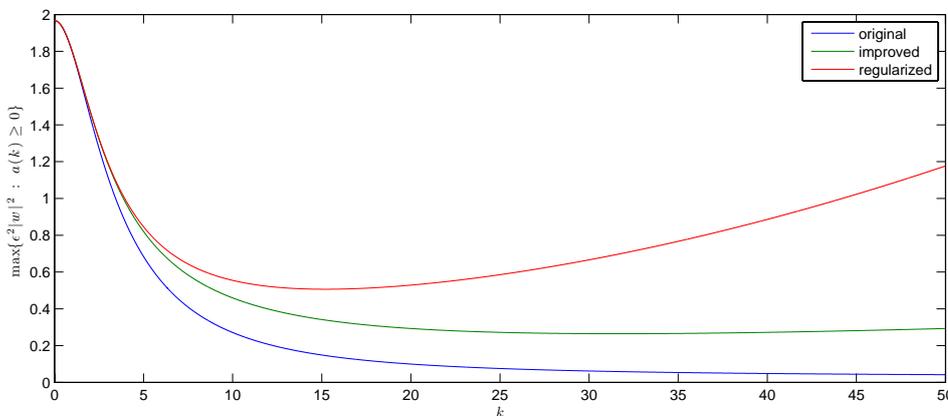


Figure 1: Instability curves of the modified Green-Naghdi models with $F_i = 1$ (original), $F_i = F_i^{\text{reg}}$ (regularized) and $F_i = F_i^{\text{imp}}$ (improved). The last one coincides with the full Euler system counterpart. The dimensionless parameters are $\gamma = 0.95$, $\delta = 0.5$, $\epsilon = 0.5$, $\mu = 0.1$, $\text{Bo} = 10^3$.

4 Numerical illustrations

We numerically compute several of our Green-Naghdi systems, with and without surface tension, in order to observe how the different frequency dispersion may affect the appearance of Kelvin-Helmholtz instabilities.

As in Figure 1, we focus on the three aforementioned examples: $F_i = 1$ (original), $F_i = F_i^{\text{reg}}$ with $\theta_i = \frac{1}{15\delta_i^2}$ (regularized) and $F_i = F_i^{\text{imp}}$ (improved). Values for the dimensionless parameters are $\gamma = 0.95$, $\delta = 0.5$, $\epsilon = 0.5$, $\mu = 0.1$; and $\text{Bo}^{-1} = 10^{-3}$ (with surface tension) or $\text{Bo}^{-1} = 0$ (without surface tension).

The initial data is $\zeta(0, x) = -e^{-4|x+1/3|^2}$ and $w(0, x) = e^{-4|x-1/3|^2}$, so as to produce a smooth and localized flow, but to avoid any cancellation due to symmetries.

Figures 2 and 3 represent the predicted flow at time $t = 1$ and $t = 1.5$, in the situation with surface tension. Figure 4 represents the predicted flow at time $t = 1$ in the situation without surface tension. Each time, the left panel plots the flux, $w(t, x)$ (or rather $1 + w$ for the sake of readability) as well as the interface deformation, $\zeta(t, x)$; while the right panel plots the spatial Fourier transform of the interface deformation, $\widehat{\zeta}(t, k)$. The dashed line represents the initial data, and the three colored lines the predictions of each model.

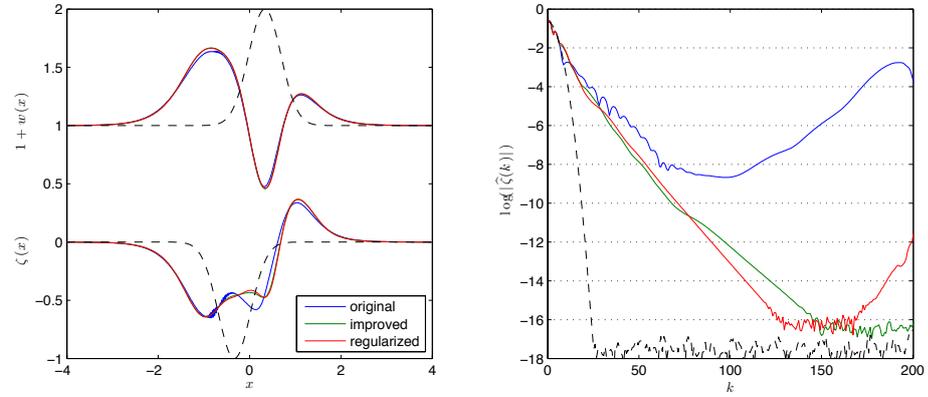


Figure 2: Prediction of the Green-Naghdi models, with surface tension, at time $t = 1$.

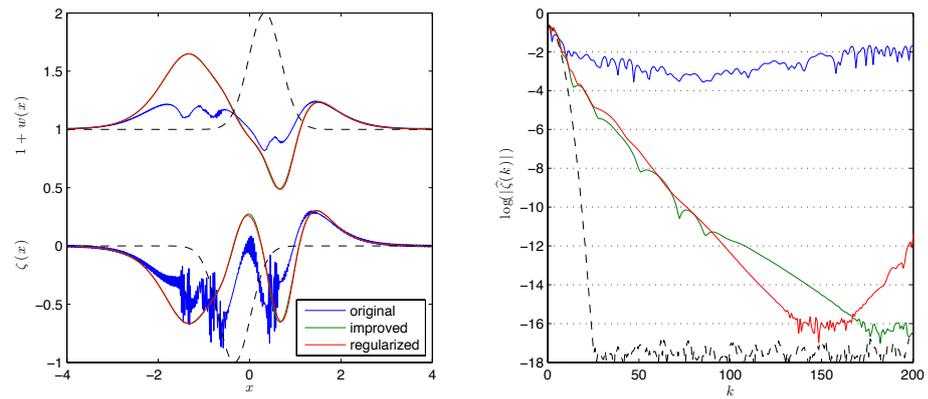


Figure 3: Prediction of the Green-Naghdi models, with surface tension, at time $t = 1.5$.

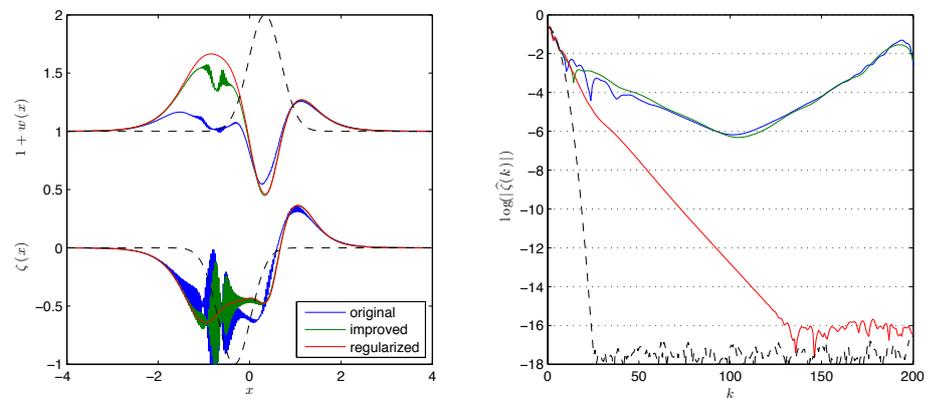


Figure 4: Prediction of the Green-Naghdi models, without surface tension, at time $t = 1$.

Discussion In the situation with surface tension, we see that at time $t = 1$ (Figure 2), the predictions of the three models are similar. Only the original model shows small but clear discrepancy, and in particular early signs of instabilities on the deformation of the interface (located, remarkably, where the flux w has highest amplitude). This situation is clearer when looking at the Fourier transform, right panel. We see the existence of a strong large frequency component which has grown from machine precision noise. As expected, modes with higher wavenumbers grow faster.

The regularized model also exhibits a non-trivial (although very small) high-frequency component. This component is however stable in time and of the size of the precision of the time evolution (Runge-Kutta) scheme. It is not produced by Kelvin-Helmholtz instabilities, but rather by numerical errors. It does not appear when surface tension is absent (Figure 4).

At later time $t = 1.5$ (Figure 3), the Kelvin-Helmholtz instabilities have completely destroyed the flow of the original model. The flows predicted by the regularized and improved models remain smooth and very similar. When running the numerical simulation for much larger time, our computations indicate that the flow of the regularized and improved models remains smooth for any positive time.

When surface tension is neglected from the models, we see (Figure 4) that at time $t = 1$, Kelvin-Helmholtz instabilities have appeared for both the original and the improved model (again, located mostly where the flux has maximal amplitude). The flow predicted by the improved model, however, remains smooth and is very similar to the flow with surface tension.

Numerical scheme Let us now briefly present our numerical scheme. It is very natural in our context to use spectral methods [41] as for the space discretization, since Fourier multipliers thus do not require any particular attention. Such methods yield an exponential accuracy with respect to the spatial mesh size for smooth data. In our simulations, we used $2^9 = 512$ equally distributed points (with periodic boundary conditions) on $x \in [-4; 4]$. As for the time evolution, we use the Matlab solver `ode45`, which is based on the fourth and fifth order Runge-Kutta-Merson method [38], with a relative tolerance of 10^{-10} and absolute tolerance of 10^{-12} . It is convenient to solve the system written in terms of ζ and $v \stackrel{\text{def}}{=} \frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu \mathcal{Q}^F[\epsilon \zeta] w$, although this requires to solve at each time step w as a function of ζ and v .

In Table 1 we display the numerical variations, between time $t = 1$ and initial time $t = 0$, of the conserved quantities (discussed in Section 2.3) as a very rough mean to appreciate the precision of the numerical scheme. One sees that the agreement is excellent, except when the horizontal impulse is concerned. In that case, one sees a great sensibility to the presence of large frequency components. In other words, when Kelvin-Helmholtz instabilities have induced a strong high-frequency component (*i.e.* for the classical Green-Naghdi model, or the improved Green-Naghdi model without surface tension), then the size of this component, which we can see in Figures 2 and 4, is reflected in the precision of the numerical scheme. It is remarkable that the other conserved quantities do not suffer from such a loss of precision.

	With surface tension			Without surface tension		
	original	regularized	improved	original	regularized	improved
Mass \mathcal{Z}	$2.3315 \cdot 10^{-15}$	$1.7764 \cdot 10^{-15}$	$2.2204 \cdot 10^{-15}$	$8.8818 \cdot 10^{-16}$	$2.5535 \cdot 10^{-15}$	$6.6613 \cdot 10^{-16}$
Velocity \mathcal{V}	$-1.199 \cdot 10^{-14}$	$-1.1102 \cdot 10^{-14}$	$-8.6597 \cdot 10^{-15}$	$-1.7319 \cdot 10^{-14}$	$-1.1768 \cdot 10^{-14}$	$-9.3259 \cdot 10^{-15}$
Impulse \mathcal{I}	$-3.3755 \cdot 10^{-4}$	$-4.5686 \cdot 10^{-14}$	$-1.2024 \cdot 10^{-13}$	$5.5151 \cdot 10^{-2}$	$-1.068 \cdot 10^{-13}$	$1.8664 \cdot 10^{-2}$
Energy \mathcal{E}	$-2.5935 \cdot 10^{-13}$	$-1.7319 \cdot 10^{-13}$	$-3.6282 \cdot 10^{-13}$	$-4.5741 \cdot 10^{-14}$	$-3.2574 \cdot 10^{-13}$	$-3.0531 \cdot 10^{-13}$

Table 1: Difference between conserved quantities at time $t = 1$ and time $t = 0$.

5 Well-posedness analysis

This section is dedicated to the proof of the main result of this work, namely the well-posedness theory for the class of modified Green-Naghdi systems introduced in Section 1.3 and that we recall:

$$(5.1) \quad \begin{cases} \partial_t \zeta + \partial_x w = 0, \\ \partial_t \left(\frac{h_1 + \gamma h_2}{h_1 h_2} w + \mu \mathcal{Q}^F[\epsilon \zeta] w \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left(\frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) \\ = \mu \epsilon \partial_x (\mathcal{R}^F[\epsilon \zeta] w) + \frac{\gamma + \delta}{\text{Bo}} \partial_x^2 \left(\frac{1}{\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2}} \partial_x \zeta \right), \end{cases}$$

where

$$\begin{aligned} \mathcal{Q}^F[\epsilon \zeta] w &= -\frac{1}{3} h_2^{-1} \partial_x F_2^\mu \{ h_2^3 \partial_x F_2^\mu \{ h_2^{-1} w \} \} - \frac{\gamma}{3} h_1^{-1} \partial_x F_1^\mu \{ h_1^3 \partial_x F_1^\mu \{ h_1^{-1} w \} \}, \\ \mathcal{R}^F[\epsilon \zeta, w] &= \frac{1}{3} w h_2^{-2} \partial_x F_2^\mu \{ h_2^3 \partial_x F_2^\mu \{ h_2^{-1} w \} \} - \frac{\gamma}{3} w h_1^{-2} \partial_x F_1^\mu \{ h_1^3 \partial_x F_1^\mu \{ h_1^{-1} w \} \} \\ &\quad + \frac{1}{2} (h_2 \partial_x F_2^\mu \{ h_2^{-1} w \})^2 - \frac{\gamma}{2} (h_1 \partial_x F_1^\mu \{ h_1^{-1} w \})^2. \end{aligned}$$

Here and thereafter, we always denote $h_1 = h_1(\epsilon \zeta) = 1 - \epsilon \zeta$ and $h_2 = h_2(\epsilon \zeta) = \delta^{-1} + \epsilon \zeta$. Let us also recall that F_i^μ ($i = 1, 2$) denotes a Fourier multiplier:

$$F_i^\mu = F_i(\sqrt{\mu} D) \quad \text{i.e.} \quad \widehat{F_i^\mu f}(\xi) = F_i(\sqrt{\mu} \xi) \widehat{f}(\xi).$$

In this section (and subsequently in Section 6), we restrict ourselves to admissible Fourier multipliers F_i^μ , as defined in Definition 1.1. This permits the functional analysis detailed in Section C.

For reasons explained below, our energy space involves both space and time derivatives of the unknowns. For $U = (\zeta, w)^\top$, we define

$$(5.2) \quad E^0(U) \stackrel{\text{def}}{=} |\zeta|_{X_{\text{Bo}^{-1}}^0}^2 + |w|_{Y_{F^\mu}^0}^2, \quad E^N(U) \stackrel{\text{def}}{=} \sum_{|\alpha|=0}^N E^0(\partial^\alpha U) \stackrel{\text{def}}{=} |\zeta|_{X_{\text{Bo}^{-1}}^N}^2 + |w|_{Y_{F^\mu}^N}^2,$$

where N always denotes an integer and α a multi-index. The functional setting and in particular the definitions of functional spaces $X_{\text{Bo}^{-1}}^N$ and $Y_{F^\mu}^N$ are given in Section B, in annex of this paper.

Finally, in addition to $\gamma, \mu, \epsilon, \delta, \text{Bo}^{-1} \geq 0$, it is convenient to introduce the following dimensionless parameters:

$$(5.3) \quad \Upsilon_F \stackrel{\text{def}}{=} \epsilon^2 (1 + (\gamma K_{F_1} + K_{F_2})(\mu \text{Bo})^{1-\sigma}) < \infty$$

where σ, K_{F_1}, K_{F_2} are specified in Definition 1.1, (1.8); and

$$(5.4) \quad \mathbf{m} \stackrel{\text{def}}{=} \max\{\epsilon, \gamma, \delta, \delta^{-1}, \mu, \text{Bo}^{-1}\} < \infty.$$

Theorem 5.1 (Well-posedness). *Let $U^0 \stackrel{\text{def}}{=}} (\zeta^0, w^0)^\top \in X_{\text{Bo}^{-1}}^N \times Y_{F^\mu}^N$ with $N \geq 4$, satisfying*

$$\begin{aligned} h_1^0 &\stackrel{\text{def}}{=} 1 - \epsilon \zeta^0 \geq h_0 > 0, & h_2^0 &\stackrel{\text{def}}{=} \delta^{-1} + \epsilon \zeta^0 \geq h_0 > 0; \\ (\gamma + \delta) - \epsilon^2 \max_{x \in \mathbb{R}} \{ ((h_2^0)^{-3} + \gamma (h_1^0)^{-3}) |w^0|^2 \} &\geq k_0 > 0. \end{aligned}$$

One can define $K = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, \epsilon |\zeta|_{H_x^3})$ such that if $\Upsilon_F |w^0|_{Z_{F^\mu}^1}^2 \leq K^{-1}$, there exists $T > 0$ and a unique $U \stackrel{\text{def}}{=}} (\zeta, w)^\top \in C_w^0([0, T]; X_{\text{Bo}^{-1}}^N \times Y_{F^\mu}^N)$ solution to (5.1) and $U|_{t=0} = U^0$. Moreover, there exists $\mathbf{C}_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, E^N(U^0))$ such that

$$T^{-1} \leq \mathbf{C}_0 \times \left(\epsilon + \Upsilon_F^{1/2} |w^0|_{Z_{F^\mu}^1} + \Upsilon_F |w^0|_{Z_{F^\mu}^2}^2 \right) \quad \text{and} \quad \sup_{t \in [0, T]} E^N(U) \leq \mathbf{C}_0 \times E^N(U^0).$$

Before the proving this result, let us discuss a few of its implications.

Remark 5.2 (Initial data). *Since our functional spaces involve time derivatives, it is not a priori clear how to define $|\zeta^0|_{Y_{F^\mu}^0}$ and $|w^0|_{Y_{F^\mu}^N}$. As it is manifest from the proof, the definition of $(\partial^\alpha U^0)|_{t=0}$ for sufficiently regular $\zeta^0(x), w^0(x)$ is given by system (5.1) itself. More precisely, for $\alpha = (0, \alpha_2)$, then the definition is clear. We then define $(\partial^\alpha U^0)|_{t=0}$ for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 > 0$ by finite induction on α_1 , through the identities obtained from (5.1) differentiated $|\alpha_1| - 1$ times with respect to time. These identities are exactly the ones given in Lemma 5.5, and are uniquely solved by Lemma 5.7 (below).*

Remark 5.3 (Domain of hyperbolicity and time of existence). *Hypotheses on the initial data ensure that the flow lies in the “domain of hyperbolicity” of the system; see Lemma 5.6. They may be seen as the nonlinear version of the stability criterion presented in Section 3.2, as they provide sufficient conditions for Kelvin-Helmholtz instabilities not to appear. However, remark that our “Kelvin-Helmholtz instability parameter”, Υ_F , is not multiplied by γ , in contrast with Υ and Υ_{GN} in Section 3, as well as the nonlinear result for the full Euler system obtained by Lannes [25, (5.1)]. The latter results imply that the large-frequency Kelvin-Helmholtz instabilities disappear in the limit $\gamma \rightarrow 0$, so that surface tension is not necessary for the well-posedness of the system when $\gamma = 0$. We do not recover such property with our rigorous analysis, although numerical simulations indicate that our models are well-posed when $\gamma = 0$ and $Bo = \infty$, as long as the non-vanishing depth condition is satisfied.*

A second setback is that the time of existence involves $\Upsilon_F^{1/2}|w^0|_{Z_{F^\mu}^1}$, and not only $\Upsilon_F|w^0|_{Z_{F^\mu}^2}$. In practice, this means that when $\Upsilon_F \ll 1$, and in particular when $\Upsilon_F \leq \epsilon \ll 1$, then the time of existence of our result is significantly smaller than the one in [26, Theorem 6].

However, let us note that our conclusions, in particular with the choice $F_i = F_i^{\text{imp}}$ where $\sigma = 1/2$, are in complete agreement with aforementioned results in the oceanographic setting of internal waves, where one expects large values of ϵ and $\gamma \approx 1$.

We believe that the above limitations originate from the choice of unknowns used when quasilinearizing the equation. This was quickly discussed in footnote 4 in Section 3.2 as for the occurrence of γ . The restriction on the time of existence originates from estimates (5.6) and (5.7) in Lemma 5.5, and more precisely the lack of an analogue of [25, Lemma 7] thanks to which “good unknowns” could be constructed. We show in Section A how the techniques used in this work, applied to the Saint-Venant system (that is setting $\mu = 0$) written with different unknowns, yields sharp results. It would be interesting to obtain similar results on Green-Naghdi systems, but would possibly require to construct new models with different variables.

Remark 5.4 (Regularized systems). *In the case $\sigma = 1$, one sees that Theorem 5.1 does not depend on Bo (through Υ_F). In particular, the results hold true even when surface tension is neglected, that is $Bo^{-1} = 0$. Notice that we recover in that case the “quasilinear timescale” $T^{-1} \lesssim \epsilon$. However, without the surface tension component, our strategy relying on the use of space-time energy is certainly not needed, and we expect that classical methods can be applied to prove the well-posedness for initial data in Sobolev spaces: $(\zeta^0, w^0)^\top \in H^s \times H^s$, $s > 3/2$.*

Strategy and outline Our strategy is similar to the one used for the full Euler system with surface tension by Lannes [26, 25], and originates from an idea of Rousset and Tzvetkov [35, 36]. The main difference with respect to the traditional methods for quasilinear systems is that we treat time derivatives in the same way as space derivatives. In particular, the main tool of the analysis is the control of a *space-time* energy. The reason for such a strategy is that

- the two unknowns, ζ and w , are controlled in different functional spaces, one being continuously embedded in the other but to the price of a non-uniform constant (see Lemma C.1), and the inclusion being strict;

- the most singular term of the system, namely the one which involves the operator of highest order, comes from the surface tension component, and couples the two unknowns (it appears as an off-diagonal component of the quasilinearized system).

This is why one cannot use standard energy methods in Sobolev-based functional spaces, as commutator estimates fail to control all coupling terms.

More precisely, our strategy is as follows. In Lemma 5.5 below, we “quasilinearize” the system. We differentiate several times the equations with respect to space and time, and extract the leading order components. The quasilinear system we consider is the complete system of all the equations satisfied by the original unknowns and their space-time derivatives up to sufficiently high order. Thus only L^2 -type estimates on the aforementioned linear “block” systems will be required. In Section 5.2, we study the operators involved in the block systems. In particular, we derive sufficient conditions for hyperbolicity in Lemma 5.6, that is assumptions on the data which allow to exhibit a coercive symmetrizer of the system. Thanks to such results, one obtains as usual some a priori energy estimates in Section 5.3. Finally, in Section 5.4, we explain how to deduce from these energy estimates the well-posedness of the linear block systems (Lemma 5.12), and in turn the well-posedness of the nonlinear system (Theorem 5.1).

5.1 Quasilinearization of the Green-Naghdi systems

The following Lemma introduces the quasilinear systems which are central in our analysis.

Lemma 5.5. *Let $U = (\zeta, w)^\top \in X_{\text{Bo}-1}^N \times Y_{\text{F}\mu}^N$ with $N \geq 4$, solution to (5.1) and satisfying*

$$(5.5) \quad h_1(\epsilon\zeta) = 1 - \epsilon\zeta \geq h_0 > 0, \quad h_2(\epsilon\zeta) = \delta^{-1} + \epsilon\zeta \geq h_0 > 0.$$

For any $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| \leq N$, denote $U^{(\alpha)} \stackrel{\text{def}}{=} (\partial^\alpha \zeta, \partial^\alpha w)^\top$ and $\zeta^{(\tilde{\alpha})} \stackrel{\text{def}}{=} (\partial^{\alpha-e_1} \zeta, \partial^{\alpha-e_2} \zeta)^\top$ (if $\alpha_j = 0$, then $\partial^{\alpha-e_j} \zeta = 0$ by convention). Then one can define $r^{(\alpha)} = r^{(\alpha)}[\epsilon\zeta, \epsilon w] \in (Y_{\text{F}\mu}^0)^\star$ such that $U^{(\alpha)}$ satisfies:

- If $|\alpha| = N$,

$$\begin{cases} \partial_t \zeta^{(\alpha)} + \partial_x w^{(\alpha)} = 0, \\ \mathfrak{b}[\epsilon\zeta] \partial_t w^{(\alpha)} + \partial_x \mathfrak{a}[\epsilon\zeta, \epsilon w] \zeta^{(\alpha)} + \partial_x \tilde{\mathfrak{a}}_\alpha[\epsilon\zeta] \zeta^{(\tilde{\alpha})} + \mathfrak{c}[\epsilon\zeta, \epsilon w] \partial_x w^{(\alpha)} = r^{(\alpha)}, \end{cases}$$

- If $|\alpha| \leq N - 1$,

$$\begin{cases} \partial_t \zeta^{(\alpha)} + \partial_x w^{(\alpha)} = 0, \\ \mathfrak{b}[\epsilon\zeta] \partial_t w^{(\alpha)} + \partial_x \mathfrak{a}[\epsilon\zeta, \epsilon w] \zeta^{(\alpha)} + \mathfrak{c}[\epsilon\zeta, \epsilon w] \partial_x w^{(\alpha)} = r^{(\alpha)}, \end{cases}$$

where

$$\begin{aligned} \mathfrak{a}[\epsilon\zeta, \epsilon w] \bullet &\stackrel{\text{def}}{=} \left((\gamma + \delta) - \epsilon^2 \frac{h_1^3 + \gamma h_2^3}{(h_1 h_2)^3} |w|^2 \right) \times \bullet - \mu \epsilon^2 (d_1 \mathcal{R}_2^{\text{F}}[\epsilon\zeta, w] - \gamma d_1 \mathcal{R}_1^{\text{F}}[\epsilon\zeta, w]) \bullet \\ &\quad - \frac{\gamma + \delta}{\text{Bo}} \partial_x \left(\frac{\partial_x \bullet}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{3/2}} \right) \\ \mathfrak{b}[\epsilon\zeta] \bullet &\stackrel{\text{def}}{=} \frac{h_1 + \gamma h_2}{h_1 h_2} \bullet + \mu (\mathcal{Q}_2^{\text{F}}[\epsilon\zeta] + \gamma \mathcal{Q}_1^{\text{F}}[\epsilon\zeta]) \bullet, \\ \mathfrak{c}[\epsilon\zeta, \epsilon w] \bullet &\stackrel{\text{def}}{=} \epsilon \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} w \times \bullet - \mu \epsilon (d \mathcal{Q}_2^{\text{F}}[\epsilon\zeta](w) + \gamma d \mathcal{Q}_1^{\text{F}}[\epsilon\zeta](w)) \bullet \\ &\quad - \mu \epsilon (d_2 \mathcal{R}_2^{\text{F}}[\epsilon\zeta, w] - \gamma d_2 \mathcal{R}_1^{\text{F}}[\epsilon\zeta, w]) \bullet, \end{aligned}$$

with $\mathcal{Q}_i^F, d\mathcal{Q}_i^F, d_1\mathcal{R}_i^F, d_2\mathcal{R}_i^F$ defined in (5.12),(5.14),(5.18),(5.19) below; and

$$\check{\mathbf{a}}_\alpha[\epsilon\zeta]\zeta^{(\check{\alpha})} \stackrel{\text{def}}{=} \frac{\gamma + \delta}{\text{Bo}} \partial_x \left(\sum_{j \in \{1,2\}} 3\alpha_j \mu \epsilon^2 \frac{(\partial_x \partial^{e_j} \zeta)(\partial_x \zeta)(\partial_x \zeta_j^{(\check{\alpha})})}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{5/2}} \right).$$

Moreover, $r^{(\alpha)} = r^{(\alpha)}[\epsilon\zeta, \epsilon w]$ satisfies

$$(5.6) \quad |r^{(\alpha)}|_{(Y_{F\mu}^0)^*} \leq C(\mathbf{m}, h_0^{-1}, E^N(U)) \times E^{|\alpha|}(U)^{1/2} \times (\epsilon + \Upsilon_F^{1/2} |w|_{Z_{F\mu}^1} + \Upsilon_F |w|_{Z_{F\mu}^2}),$$

and

$$(5.7) \quad |r^{(\alpha)}[\epsilon\zeta_1, \epsilon w_1] - r^{(\alpha)}[\epsilon\zeta_2, \epsilon w_2]|_{(Y_{F\mu}^0)^*} \leq C(\mathbf{m}, h_0^{-1}, E^N(U_1), E^N(U_2)) \times E^{|\alpha|}(U_1 - U_2)^{1/2} \\ \times (\epsilon + \Upsilon_F^{1/2} |w_1|_{Z_{F\mu}^1} + \Upsilon_F |w_1|_{Z_{F\mu}^2}).$$

Proof. The proof simply consists in differentiating α times the Green-Naghdi system (5.1). The higher order terms contribute to \mathbf{a} , \mathbf{b} , \mathbf{c} and $\check{\mathbf{a}}$, while lower order terms contribute to $r^{(\alpha)}$. In the following, we explain how the estimates concerning $r^{(\alpha)}$ are obtained, by treating separately the first order terms, dispersive terms and the surface tension term.

Contribution from the first order terms, $\partial_t \left(\frac{h_1 + \gamma h_2}{h_1 h_2} w \right) + \frac{\epsilon}{2} \partial_x \left(\frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right)$.

Consider the identity

$$\frac{\epsilon}{2} \partial_x \left(\frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) = \epsilon \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} w \partial_x w - \epsilon^2 \frac{h_1^3 + \gamma h_2^3}{(h_1 h_2)^3} |w|^2 \partial_x \zeta.$$

It follows, by Leibniz's rule,

$$(5.8) \quad \partial^\alpha \left(\frac{\epsilon}{2} \partial_x \left(\frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |w|^2 \right) \right) = \partial_x \left(-\epsilon^2 \frac{h_1^3 + \gamma h_2^3}{(h_1 h_2)^3} |w|^2 \partial^\alpha \zeta \right) + \epsilon \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} w \partial_x \partial^\alpha w + \epsilon r_1^{(\alpha)},$$

with

$$r_1^{(\alpha)} = \sum_{n=0}^{|\alpha|+1} \sum_{\beta_i, \beta'_j} \epsilon^n C^{(\beta_i, \beta'_j)} G^{(n)}(\epsilon\zeta) \left(\prod_{i=1}^n \partial^{\beta_i} \zeta \right) \left(\prod_{j=1}^2 \partial^{\beta'_j} w \right) \stackrel{\text{def}}{=} \sum_{n=0}^{|\alpha|+1} \sum_{\beta_i, \beta'_j} \epsilon^n C^{(\beta_i, \beta'_j)} r_1^{(\beta_i, \beta'_j)},$$

where (β_i, β'_j) is any $n + 2$ -tuple of multi-index satisfying

$$1 \leq |\beta_1| \leq \dots \leq |\beta_n| \leq |\alpha|, \quad 0 \leq |\beta'_1| \leq |\beta'_2| \leq |\alpha| \quad \text{and} \quad \sum_{i=1}^n \beta_i + \sum_{j=1}^2 \beta'_j = \alpha + (0, 1),$$

$C^{(\beta_i, \beta'_j)}$ is a constant and $G^{(n)}$ the n -th derivative of $G(X) = \frac{h_1^2(X) - \gamma h_2^2(X)}{(h_1 h_2)^2(X)} = \frac{(1-X)^2 - \gamma(\delta^{-1} + X)^2}{(1-X)^2(\delta^{-1} + X)^2}$.

We estimate each of these terms as follows :

- if $|\beta_n| = |\alpha|$, then $0 \leq |\beta_1|, \dots, |\beta_{n-1}|, |\beta'_1|, |\beta'_2| \leq 1$, and

$$|r_1^{(\beta_i, \beta'_j)}|_{L^2} \leq |G^{(n)}(\epsilon\zeta)|_{L^\infty} |\partial^{\beta_n} \zeta|_{L^2} \left(\prod_{i=1}^{n-1} |\partial^{\beta_i} \zeta|_{L^\infty} \right) \left(\prod_{j=1}^2 |\partial^{\beta'_j} w|_{L^\infty} \right);$$

- otherwise $0 \leq |\beta_1|, \dots, |\beta_n|, |\beta'_1| \leq |\alpha| - 1$, and

$$|r_1^{(\beta_i, \beta'_j)}|_{L^2} \leq |G^{(n)}(\epsilon\zeta)|_{L^\infty} \left(\prod_{i=1}^n |\partial^{\beta_i} \zeta|_{L^\infty} \right) \left(|\partial^{\beta'_1} w|_{L^\infty} |\partial^{\beta'_2} \zeta|_{L^2} \right).$$

One has $|G^{(n)}(\epsilon\zeta)|_{L^\infty} \leq C(\mathbf{m}, h_0^{-1})$ since $\epsilon\zeta$ satisfies (5.5); and by Sobolev embedding,

$$|\partial^\beta u|_{L^\infty} \leq |\partial^\beta u|_{H_x^1} \leq |\partial^\beta u|_{H^1} \leq \min\{|u|_{X_{\text{Bo}^{-1}}^{1+|\beta|}}, |u|_{Y_{F^\mu}^{1+|\beta|}}\}$$

(recall that by definition, $|\partial^\beta u|_{H^1}^2 \stackrel{\text{def}}{=} |\partial^\beta u|_{L^2}^2 + |\partial_x \partial^\beta u|_{L^2}^2 + |\partial_t \partial^\beta u|_{L^2}^2$). We deduce immediately, since $N \geq 4$ and $|\alpha| \leq N$,

$$(5.9) \quad |r_1^{(\alpha)}|_{L^2} \leq C(\mathbf{m}, h_0^{-1}, E^N(U)) \times E^{|\alpha|}(U)^{1/2}.$$

Similarly, we write

$$(5.10) \quad \begin{aligned} \partial^\alpha \partial_t \left(\frac{h_1 + \gamma h_2}{h_1 h_2} w \right) &= \frac{h_1 + \gamma h_2}{h_1 h_2} \partial_t \partial^\alpha w - \epsilon w \frac{h_1^2 - \gamma h_2^2}{h_1 h_2} \partial_t \partial^\alpha \zeta + \epsilon r_2^{(\alpha)} \\ &= \frac{h_1 + \gamma h_2}{h_1 h_2} \partial_t \partial^\alpha w + \epsilon w \frac{h_1^2 - \gamma h_2^2}{h_1 h_2} \partial_x \partial^\alpha w + \epsilon r_2^{(\alpha)}, \end{aligned}$$

where we used $\partial_t \zeta = -\partial_x w$, from the first equation of (5.1). One obtains as above the following estimate:

$$(5.11) \quad |r_2^{(\alpha)}|_{L^2} \leq C(\mathbf{m}, h_0^{-1}, E^N(U)) \times E^{|\alpha|}(U)^{1/2}.$$

Contribution from the dispersive terms, $\mu \partial_t (\mathcal{Q}^F[\epsilon\zeta]w) - \mu \epsilon \partial_x (\mathcal{R}^F[\epsilon\zeta]w)$.

Define (with a slight abuse of notation with respect to (1.5))

$$(5.12) \quad \mathcal{Q}_i^F[\epsilon\zeta]w \stackrel{\text{def}}{=} -\frac{1}{3} h_i^{-1} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-1} w\}\},$$

so that $\mathcal{Q}^F[\epsilon\zeta]w = \mathcal{Q}_2^F[\epsilon\zeta]w + \gamma \mathcal{Q}_1^F[\epsilon\zeta]w$. Differentiating $\alpha + \mathbf{e}_1$ times and using $\partial_t \zeta = -\partial_x w$ yields

$$(5.13) \quad \begin{aligned} \partial^\alpha \partial_t \mathcal{Q}_i^F[\epsilon\zeta]w &= \mathcal{Q}_i^F[\epsilon\zeta] \partial^\alpha \partial_t w + d \mathcal{Q}_i^F[\epsilon\zeta](w) (\epsilon \partial^\alpha \partial_t \zeta) + r_{3,i}^{(\alpha)} \\ &= \mathcal{Q}_i^F[\epsilon\zeta] \partial^\alpha \partial_t w - d \mathcal{Q}_i^F[\epsilon\zeta](w) (\epsilon \partial^\alpha \partial_x w) + r_{3,i}^{(\alpha)} \end{aligned}$$

where we defined

$$(5.14) \quad \begin{aligned} (-1)^i d \mathcal{Q}_i^F[\epsilon\zeta](w) \bullet &= \frac{1}{3} h_i^{-2} \left(\partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-1} w\}\} \right) \times \bullet \\ &\quad - h_i^{-1} \partial_x F_i^\mu \{h_i^2 \partial_x F_i^\mu \{h_i^{-1} w\} \times \bullet\} + \frac{1}{3} h_i^{-1} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-2} w \times \bullet\}\}; \end{aligned}$$

and

$$r_{3,i}^{(\alpha)} = \sum_{\beta_j} C^{(\beta_j)} (\partial^{\beta_1} h_i^{-1}) \partial_x F_i^\mu \{(\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{(\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w)\}\} \stackrel{\text{def}}{=} \sum_{\beta_j} C^{(\beta_j)} r_{3,i}^{(\beta_j)},$$

where $C^{(\beta_j)}$ is a constant and (β_j) is any 4-tuple of multi-index satisfying

$$0 \leq |\beta_1|, |\beta_2|, |\beta_3|, |\beta_4| \leq |\alpha| \quad \text{and} \quad \sum_{j=1}^4 \beta_j = \alpha + \mathbf{e}_1,$$

We estimate each of these terms by assuming that $U = (\zeta, w)^\top \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$, so that for any $f \in \mathcal{S}(\mathbb{R})$, the following identities are immediately valid:

$$\begin{aligned} \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F\mu}^0)^*} &= \left((\partial^{\beta_1} h_i^{-1}) \partial_x F_i^\mu \{ (\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \}, f \right)_{L^2} \\ &= - \left((\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \}, \partial_x F_i^\mu \{ (\partial^{\beta_1} h_i^{-1}) f \} \right)_{L^2}. \end{aligned}$$

The estimates hold as well for $U = (\zeta, w)^\top \in X_{\text{Bo}^{-1}}^N \times Y_{F\mu}^N$ ($N \geq 4$) and $f \in Y_{F\mu}^0$ by density of $\mathcal{S}(\mathbb{R})$ in $Y_{F\mu}^0$ and $X_{\text{Bo}^{-1}}^0$ using standard continuity arguments.

- if $|\beta_1| = |\alpha|$, then $0 \leq |\beta_2|, |\beta_3|, |\beta_4| \leq 1$ and

$$\gamma^{2-i} \left| \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F\mu}^0)^*} \right| \leq \gamma^{2-i} \mu \left| \partial^{\beta_1} h_i^{-1} \right|_{L^2} \left| \partial_x F_i^\mu \{ (\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \} \right|_{L^\infty} |f|_{L^2}.$$

Notice first, since $|\beta_1| = |\alpha| \geq 1$ (otherwise this term does not appear), there exists $j \in \{1, 2\}$ such that $\mathbf{e}_j \leq \beta_1$ and

$$\left| \partial^{\beta_1} h_i^{-1} \right|_{L^2} = \left| \partial^{\beta_1 - \mathbf{e}_j} (h_i^{-2} \epsilon \partial^{\mathbf{e}_j} \zeta) \right|_{L^2} \leq C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{W^{|\alpha|-1, \infty}}) \times \epsilon |\zeta|_{H^{|\alpha|}}.$$

Now, using several times Lemma C.3, and since $\partial^{\beta_2} h_i^3 = 3h_i^2 \epsilon \partial^{\beta_2} \zeta$ if $|\beta_2| = 1$ or $\partial^{\beta_2} h_i^3 = h_i^3$ if $|\beta_2| = 0$ (and similarly for $\partial^{\beta_3} h_i^{-1}$), one has

$$\begin{aligned} &\gamma^{2-i} \mu \left| \partial_x F_i^\mu \{ (\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \} \right|_{L^\infty} \\ &\leq \sqrt{\gamma^{2-i} \mu} \left| (\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \right|_{Z_{F\mu}^0} \\ &\leq \sqrt{\gamma^{2-i} \mu} C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{H^3}) \left| \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \right|_{Z_{F\mu}^0} \\ &\leq \sqrt{\gamma^{2-i} \mu} C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{H^3}) \left| (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \right|_{Z_{F\mu}^1} \\ &\leq \sqrt{\gamma^{2-i} \mu} C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{H^4}) |w|_{Z_{F\mu}^2}. \end{aligned}$$

Therefore, since $\max\{4, |\alpha|\} \leq N$,

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F\mu}^0)^*} \right| \leq C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{H^N}) \times \epsilon \sqrt{\gamma^{2-i} \mu} |w|_{Z_{F\mu}^2} |\zeta|_{X_{\text{Bo}^{-1}}^{|\alpha|}} |f|_{L^2}.$$

- if $|\beta_2| = |\alpha|$, then $0 \leq |\beta_1|, |\beta_3|, |\beta_4| \leq 1$ and

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F\mu}^0)^*} \right| \leq \gamma^{2-i} \mu \left| \partial^{\beta_2} h_i^3 \right|_{L^2} \left| \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \right|_{L^\infty} \left| \partial_x F_i^\mu \{ (\partial^{\beta_1} h_i^{-1}) f \} \right|_{L^2}.$$

One has as above $\left| \partial^{\beta_2} h_i^3 \right|_{L^2} \leq \epsilon |\zeta|_{H^{|\alpha|}} C(\mathbf{m}, |\epsilon \zeta|_{W^{|\alpha|-1}})$. As for the other terms, using Lemma C.3, and $\partial^{\beta_1} h_i^{-1} = -\epsilon h_i^{-2} \partial^{\beta_1} \zeta$ if $|\beta_1| = 1$ or $\partial^{\beta_1} h_i^{-1} = h_i^{-1}$ if $|\beta_1| = 0$, one has

$$\sqrt{\gamma^{2-i} \mu} \left| \partial_x F_i^\mu \{ (\partial^{\beta_1} h_i^{-1}) f \} \right|_{L^2} \leq \left| (\partial^{\beta_1} h_i^{-1}) f \right|_{Y_{F\mu}^0} \leq C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{H^3}) |\epsilon \zeta|_{Z_{F\mu}^1}^{|\beta_1|} |f|_{Y_{F\mu}^0}.$$

The last term is treated identically and one obtains eventually

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F\mu}^0)^*} \right| \leq C(\mathbf{m}, h_0^{-1}, |\epsilon \zeta|_{H^N}) \times \epsilon |w|_{Z_{F\mu}^1} |\zeta|_{X_{\text{Bo}^{-1}}^{|\alpha|}} |f|_{Y_{F\mu}^0}.$$

- if $|\beta_4| = |\alpha|$, then $0 \leq |\beta_1|, |\beta_2|, |\beta_3| \leq 1$ and

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F\mu}^0)^*} \right| \leq \left| \partial^{\beta_2} h_i^3 \right|_{L^\infty} \sqrt{\gamma^{2-i} \mu} \left| \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^{-1}) (\partial^{\beta_4} w) \} \right|_{L^2} \sqrt{\gamma^{2-i} \mu} \left| \partial_x F_i^\mu \{ (\partial^{\beta_1} h_i^{-1}) f \} \right|_{L^2}.$$

One has $|\partial^{\beta_2} h_i^3|_{L^\infty} \leq C(\mathbf{m}, |\epsilon\zeta|_{L^\infty}) |\epsilon\zeta|_{W^{1,\infty}}^{|\beta_2|}$. We have already seen that

$$\sqrt{\gamma^{2-i}\mu} |\partial_x F_i^\mu \{(\partial^{\beta_1} h_i^{-1})f\}|_{L^2} \leq C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{H^3}) |\epsilon\zeta|_{Z_{F^\mu}^1}^{|\beta_1|} |f|_{Y_{F^\mu}^0}$$

and similarly

$$\sqrt{\gamma^{2-i}\mu} |\partial_x F_i^\mu \{(\partial^{\beta_3} h_i^{-1})(\partial^{\beta_4} w)\}|_{L^2} \leq C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{H^3}) |\epsilon\zeta|_{Z_{F^\mu}^1}^{|\beta_3|} |w|_{Y_{F^\mu}^{|\alpha|}}.$$

Since $|\beta_1| + |\beta_2| + |\beta_3| = 1$, it follows

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F^\mu}^0)^*} \right| \leq C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{H^3}) \times \epsilon |\zeta|_{Z_{F^\mu}^1} |w|_{Y_{F^\mu}^{|\alpha|}} |f|_{Y_{F^\mu}^0}.$$

- if $|\beta_3| = |\alpha|$, one obtains as above

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F^\mu}^0)^*} \right| \leq C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{H^3}, |\epsilon\zeta|_{W^{|\alpha|-1,\infty}}) \times \epsilon |w|_{Z_{F^\mu}^1} |\zeta|_{X_{F^\mu}^{|\alpha|}} |f|_{Y_{F^\mu}^0}.$$

By Lemma C.1 and (5.3), it follows

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F^\mu}^0)^*} \right| \leq C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{H^N}) \times \Upsilon_F^{1/2} |w|_{Z_{F^\mu}^1} |\zeta|_{X_{B_0^{-1}}^{|\alpha|}} |f|_{Y_{F^\mu}^0}.$$

- otherwise $0 \leq |\beta_1|, |\beta_2|, |\beta_3|, |\beta_4| \leq |\alpha| - 1$.

If $|\beta_1| \leq |\alpha| - 2$, we use

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F^\mu}^0)^*} \right| \leq |\partial^{\beta_2} h_i^3|_{L^\infty} \sqrt{\gamma^{2-i}\mu} |\partial_x F_i^\mu \{(\partial^{\beta_3} h_i^{-1})(\partial^{\beta_4} w)\}|_{L^2} \sqrt{\gamma^{2-i}\mu} |\partial_x F_i^\mu \{(\partial^{\beta_1} h_i^{-1})f\}|_{L^2}.$$

If $|\beta_1| = |\alpha| - 2$, then we use

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{L^2} \right| \leq \mu |\partial^{\beta_1} h_i^{-1}|_{L^2} |\partial_x F_i^\mu \{(\partial^{\beta_2} h_i^3) \partial_x F_i^\mu \{(\partial^{\beta_3} h_i^{-1})(\partial^{\beta_4} w)\}\}|_{L^\infty} |f|_{L^2}.$$

Proceeding as above, this yields

$$\left| \gamma^{2-i} \mu \langle r_{3,i}^{(\beta_j)}, f \rangle_{(Y_{F^\mu}^0)^*} \right| \leq C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{H^{|\alpha|}}) \times \epsilon |w|_{Y_{F^\mu}^{|\alpha|}} |\zeta|_{X_{B_0^{-1}}^{|\alpha|}} |f|_{Y_{F^\mu}^0}.$$

Plugging these estimates into (5.13), we proved

$$(5.15) \quad \mu \partial^\alpha \partial_t (\mathcal{Q}^F[\epsilon\zeta]w) = \mu (\mathcal{Q}_2[\epsilon\zeta] + \gamma \mathcal{Q}_1[\epsilon\zeta]) \partial^\alpha \partial_t w - \mu (\mathrm{d}\mathcal{Q}_2[\epsilon\zeta](w) + \gamma \mathrm{d}\mathcal{Q}_1[\epsilon\zeta](w)) (\epsilon \partial^\alpha \partial_x w) + r_3^{(\alpha)},$$

with (recalling $\max\{|\alpha|, 4\} \leq N$)

$$(5.16) \quad |r_3^{(\alpha)}|_{(Y_{F^\mu}^0)^*} \leq C(\mathbf{m}, h_0^{-1}, E^N(U)) \times E^{|\alpha|}(U)^{1/2} \times (\epsilon + \Upsilon_F^{1/2} |w|_{Z_{F^\mu}^1}).$$

The other contribution is treated similarly. We define

$$(5.17) \quad \mathcal{R}_i^F[\epsilon\zeta, \epsilon w] \stackrel{\text{def}}{=} \frac{1}{3} w h_i^{-2} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-1} w\}\} + \frac{1}{2} (h_i \partial_x F_i^\mu \{h_i^{-1} w\})^2,$$

and

$$(5.18) \quad \begin{aligned} (-1)^i \mathrm{d}_1 \mathcal{R}_i^F[\epsilon\zeta, w] \bullet &\stackrel{\text{def}}{=} -\frac{2}{3} w h_i^{-3} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu (h_i^{-1} w)\} \times \bullet \\ &+ w h_i^{-2} \partial_x F_i^\mu \{h_i^2 \partial_x F_i^\mu \{h_i^{-1} w\} \times \bullet\} - \frac{1}{3} w h_i^{-2} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-2} w \bullet\}\} \\ &+ (h_i \partial_x F_i^\mu \{h_i^{-1} w\}) \times \left((\partial_x F_i^\mu \{h_i^{-1} w\}) \times \bullet - (h_i \partial_x F_i^\mu \{h_i^{-2} w \times \bullet\}) \right), \end{aligned}$$

and

$$(5.19) \quad d_2 \mathcal{R}_i^F[\epsilon \zeta, w] \bullet \stackrel{\text{def}}{=} \frac{1}{3} \left(h_i^{-2} \partial_x F_i^\mu \{ h_i^3 \partial_x F_i^\mu \{ h_i^{-1} w \} \} \right) \times \bullet + \frac{1}{3} w h_i^{-2} \partial_x F_i^\mu \{ h_i^3 \partial_x F_i^\mu \{ h_i^{-1} \times \bullet \} \} \\ + (h_i \partial_x F_i^\mu \{ h_i^{-1} w \}) \times (h_i \partial_x F_i^\mu \{ h_i^{-1} \times \bullet \}).$$

It follows

$$\partial_x \partial^\alpha \mathcal{R}_i^F[\epsilon \zeta, w] = \partial_x (d_1 \mathcal{R}_i[\epsilon \zeta, w] \partial^\alpha \zeta) + \partial_x (d_2 \mathcal{R}_i^F[\epsilon \zeta, w] \partial^\alpha w) + r_{4,i}^{(\alpha)},$$

with

$$r_{4,i}^{(\alpha)} = \sum_{(\beta_j)} C^{(\beta_j)} (\partial^{\beta_1} w) (\partial^{\beta_2} h_i^{-2}) \partial_x F_i^\mu \{ (\partial^{\beta_3} h_i^3) \partial_x F_i^\mu \{ (\partial^{\beta_4} h_i^{-1}) (\partial^{\beta_5} w) \} \} \\ + \sum_{(\beta'_j)} C^{(\beta'_j)} ((\partial^{\beta'_1} h_i) \partial_x F_i^\mu \{ (\partial^{\beta'_2} h_i^{-1}) (\partial^{\beta'_3} w) \}) ((\partial^{\beta'_4} h_i) \partial_x F_i^\mu \{ (\partial^{\beta'_5} h_i^{-1}) (\partial^{\beta'_6} w) \})$$

where $C^{(\beta_j)}, C^{(\beta'_j)}$ are constants and $(\beta_j), (\beta'_j)$ are tuples of multi-index satisfying

$$0 \leq |\beta_j|, |\beta'_j| \leq |\alpha| \quad \text{and} \quad \sum_{j=1}^5 \beta_j = \sum_{j=1}^6 \beta'_j = \alpha + \mathbf{e}_2,$$

All these terms may be estimated as previously, and one obtains without any additional difficulty

(5.20)

$$\mu \epsilon \partial^\alpha \partial_x (\mathcal{R}^F[\epsilon \zeta, w]) = \mu \epsilon^2 \partial_x \left((d_1 \mathcal{R}_2[\epsilon \zeta, w] - \gamma d_1 \mathcal{R}_1[\epsilon \zeta, w]) \partial^\alpha \zeta \right) + \mu \epsilon \partial_x \left((d_2 \mathcal{R}_2[\epsilon \zeta, w] - \gamma d_2 \mathcal{R}_1[\epsilon \zeta, w]) \partial^\alpha w \right) + r_4^{(\alpha)},$$

with

$$(5.21) \quad |r_4^{(\alpha)}|_{(Y_{F^\mu}^0)^*} \leq C(\mathbf{m}, h_0^{-1}, E^N(U)) \times E^{|\alpha|}(U)^{1/2} \times (\epsilon + \Upsilon_F |w|_{Z_{F^\mu}^1}^2).$$

Contribution from the surface tension term, $\frac{\gamma + \delta}{\text{Bo}} \partial_x^2 \left(\frac{1}{\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2}} \partial_x \zeta \right)$.

Let us denote $s(\partial_x \zeta) = \frac{1}{\sqrt{1 + \mu \epsilon^2 |\partial_x \zeta|^2}} \partial_x \zeta$ and notice $\partial s = \frac{1}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{3/2}} \partial \partial_x \zeta$. It follows

$$(5.22) \quad \partial^\alpha \partial_x^2 s = \partial_x^2 \left(\frac{1}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{3/2}} \partial^\alpha \partial_x \zeta \right) - \sum_{j=1}^2 3\alpha_j \partial_x^2 \left(\frac{\mu \epsilon^2 (\partial_x \zeta) (\partial^{\mathbf{e}_j} \partial_x \zeta)}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{5/2}} \partial^{\alpha - \mathbf{e}_j} \partial_x \zeta \right) + r_5^{(\alpha)},$$

with

$$r_5^{(\alpha)} = \sum_{k=1}^{N+1} \frac{(\mu \epsilon^2)^k}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{k+3/2}} \sum_{(\beta_j)} C^{(\beta_j)} r_k^{(\beta_j)}, \quad r_k^{(\beta_j)} \stackrel{\text{def}}{=} \prod_{j=1}^{2k+1} \partial^{\beta_j} \partial_x \zeta,$$

where for any $k \in \{1, \dots, N+1\}$, (β_j) is a $2k+1$ -uple such that for all $j \in \{1, \dots, 2k+1\}$, one has

$$0 \leq |\beta_1| \leq \dots \leq |\beta_{2k+1}| \leq |\alpha| \leq N \quad \text{and} \quad \sum_{j=1}^{2k+1} \beta_j = \alpha + 2\mathbf{e}_2,$$

and $C^{(\beta_j)}$ is a constant.

Assume first that $|\beta_{2k+1}| = N$. Then for any $j \in \{1, \dots, k\}$, $|\beta_j| \leq 2$. It follows

$$\frac{1}{\text{Bo}} |r_k^{(\beta_j)}|_{L^2} \leq \frac{1}{\text{Bo}^{1/2}} C(|\partial_x \zeta|_{W^{2,\infty}}) |\partial^{\beta_{2k+1}} \zeta|_{X_{\text{Bo}^{-1}}^0} \leq \frac{1}{\text{Bo}^{1/2}} C(|\zeta|_{H^4}) |\zeta|_{X_{\text{Bo}^{-1}}^N}.$$

Now, if $|\beta_{2k+1}| = N - 1$, then either $|\beta_{2k}| = 3$ and $|\beta_j| = 0$ for any $j \leq 2k - 1$, or $|\beta_j| \leq 2$ for any $j \leq 2k$. The latter case is estimated as above, while in the former case, one has

$$\frac{1}{\text{Bo}} |r_k^{(\beta_j)}|_{L^2} \leq \frac{1}{\text{Bo}^{1/2}} C(|\partial_x \zeta|_{W^{2,\infty}}) |\partial^{\beta_{2k}} \zeta|_{W_{\text{Bo}^{-1}}^0} |\partial^{\beta_{2k+1}} \partial_x \zeta|_{L^2} \leq \frac{1}{\text{Bo}^{1/2}} C(|\zeta|_{X_{\text{Bo}^{-1}}^4}) |\zeta|_{X_{\text{Bo}^{-1}}^N}.$$

Otherwise, one has $|\beta_j| \leq N - 2$ for any $j \in \{1, \dots, 2k + 1\}$, and in that case,

$$\frac{1}{\text{Bo}} |r_5^{(\beta_j)}|_{L^2} \leq \frac{1}{\text{Bo}} C(|\zeta|_{X_{\text{Bo}^{-1}}^N}) |\zeta|_{X_{\text{Bo}^{-1}}^N}.$$

Altogether, this yields for $N \geq 4$

$$(5.23) \quad \frac{1}{\text{Bo}} |r_5^{(\alpha)}|_{L^2} \leq \frac{\mu \epsilon^2}{\text{Bo}^{1/2}} C(\text{Bo}^{-1}, \mu \epsilon^2, |\zeta|_{X_{\text{Bo}^{-1}}^N}) |\zeta|_{X_{\text{Bo}^{-1}}^N}.$$

Finally, there remains to estimate for $1 \leq |\alpha| \leq N - 1$,

$$\begin{aligned} \partial_x \check{\mathbf{a}}_\alpha \zeta^{(\check{\alpha})} &\stackrel{\text{def}}{=} -\frac{\gamma + \delta}{\text{Bo}} \sum_{j=1}^2 3\alpha_j \partial_x^2 \left(\frac{\mu \epsilon^2 (\partial_x \zeta) (\partial^{\mathbf{e}_j} \partial_x \zeta)}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{5/2}} \partial^{\alpha - \mathbf{e}_j} \partial_x \zeta \right) \\ &= -\frac{\gamma + \delta}{\text{Bo}} \sum_{j=1}^2 3\alpha_j \left(\frac{\mu \epsilon^2 (\partial_x \zeta) (\partial^{\mathbf{e}_j} \partial_x \zeta)}{(1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{5/2}} \partial^{\alpha - \mathbf{e}_j} \partial_x^3 \zeta \right) + r_6^{(\alpha)}. \end{aligned}$$

The remainder $r_6^{(\alpha)}$ is easily estimated as above, and we use

$$\frac{1}{\text{Bo}} |(\partial_x \zeta) (\partial^{\mathbf{e}_j} \partial_x \zeta) \partial^{\alpha - \mathbf{e}_j} \partial_x^3 \zeta|_{L^2} \leq |\partial_x \zeta|_{L^\infty} \frac{1}{\text{Bo}^{1/2}} |\partial^{\mathbf{e}_j} \partial_x \zeta|_{L^2} \frac{1}{\text{Bo}^{1/2}} |\partial^{\alpha - \mathbf{e}_j} \partial_x^3 \zeta|_{L^2},$$

for the most singular term. Since $N \geq 4$, one has for $1 \leq |\alpha| \leq N - 1$,

$$(5.24) \quad |\partial_x \check{\mathbf{a}}_\alpha \zeta^{(\check{\alpha})}|_{L^2} \leq \mu \epsilon^2 C(\text{Bo}^{-1}, \mu \epsilon^2, |\zeta|_{X_{\text{Bo}^{-1}}^N}) |\zeta|_{X_{\text{Bo}^{-1}}^{|\alpha|}}.$$

The definition of the operators $\mathbf{a}, \mathbf{b}, \mathbf{c}, \check{\mathbf{a}}_\alpha^{(\alpha)}, r^{(\alpha)}$ and estimate (5.6) now follows from (5.8)-(5.9), (5.10)-(5.11), (5.15)-(5.16), (5.20)-(5.21), (5.22)-(5.23) as well as (5.24) when $1 \leq |\alpha| \leq N - 1$.

Estimate (5.7) is obtained identically, using in particular the trivial estimates

$$|H(\epsilon \zeta_1) - H(\epsilon \zeta_2)|_{L^\infty} \leq H_{1,h_0} \epsilon |\zeta_1 - \zeta_2|_{L^\infty}, \quad |H(\epsilon \zeta_1) - H(\epsilon \zeta_2)|_{L^2} \leq H_{1,h_0} \epsilon |\zeta_1 - \zeta_2|_{L^2},$$

and

$$|\widehat{H}(\epsilon \zeta_1) - \widehat{H}(\epsilon \zeta_2)|_{L^1} \leq |H(\epsilon \zeta_1) - H(\epsilon \zeta_2)|_{H^1} \leq C(H_{2,h_0}, |\epsilon \zeta_1|_{W^{1,\infty}}, |\epsilon \zeta_2|_{W^{1,\infty}}) \epsilon |\zeta_1 - \zeta_2|_{H^1},$$

where H and H_{1,h_0} are as in Lemma C.3.

This concludes our proof of Lemma 5.5. □

5.2 Preliminary results

In this section, we prove that the operator $\mathbf{a}[\epsilon \zeta, \epsilon w]$ (resp. $\mathbf{b}[\epsilon \zeta]$), introduced in Lemma 5.5, is symmetric, continuous and coercive with respect to the space $X_{\text{Bo}^{-1}}^0$ (resp. $Y_{F^\mu}^0$), provided that some conditions are satisfied by $(\epsilon \zeta, \epsilon w)$. These requirements can be seen as sufficient conditions for the hyperbolicity of the system, and permit to control the energy solutions to the quasilinear system for positive times (Section 5.3), and eventually prove the well-posedness of our system (Section 5.4).

Lemma 5.6. *Let $(\zeta, w)^\top \in H_x^3 \times Z_{F^\mu}^1$ be such that $\epsilon\zeta$ satisfies (5.5) with $h_0 > 0$. Then $\mathbf{a}[\epsilon\zeta, \epsilon w] \in \mathcal{L}(X_{\text{Bo}^{-1}}^0; (X_{\text{Bo}^{-1}}^0)^\star)$, $\mathbf{b}[\epsilon\zeta] \in \mathcal{L}(Y_{F^\mu}^0; (Y_{F^\mu}^0)^\star)$ and $\mathbf{c}[\epsilon\zeta, \epsilon w] \in \mathcal{L}(Y_{F^\mu}^0; (Y_{F^\mu}^0)^\star)$. Moreover, there exists $K_0, K_1 = C(\mathbf{m}, h_0^{-1}, \epsilon|\zeta|_{H_x^3})$ such that*

$$\begin{aligned} \forall f, g \in X_{\text{Bo}^{-1}}^0, & \quad |\langle \mathbf{a}[\epsilon\zeta, \epsilon w]f, g \rangle_{(X_{\text{Bo}^{-1}}^0)^\star}| \leq K_1(1 + \Upsilon_F |w|_{Z_{F^\mu}^1}^2) |f|_{X_{\text{Bo}^{-1}}^0} |g|_{X_{\text{Bo}^{-1}}^0} \\ \forall f, g \in Y_{F^\mu}^0, & \quad |\langle \mathbf{b}[\epsilon\zeta]f, g \rangle_{(Y_{F^\mu}^0)^\star}| \leq K_1 |f|_{Y_{F^\mu}^0} |g|_{Y_{F^\mu}^0} \\ \forall f, g \in Y_{F^\mu}^0, & \quad |\langle \mathbf{c}[\epsilon\zeta, \epsilon w]f, g \rangle_{(Y_{F^\mu}^0)^\star}| \leq \epsilon K_1 |w|_{Z_{F^\mu}^1} |f|_{Y_{F^\mu}^0} |g|_{Y_{F^\mu}^0}, \\ \forall f \in Y_{F^\mu}^0, & \quad \langle \mathbf{b}[\epsilon\zeta]f, f \rangle_{(Y_{F^\mu}^0)^\star} \geq \frac{1}{K_0} |f|_{Y_{F^\mu}^0}^2. \end{aligned}$$

Assume additionally that there exists $k_0 > 0$ such that

$$(5.25) \quad (\gamma + \delta) - \epsilon^2 \max_{x \in \mathbb{R}} \{(h_2^{-3} + \gamma h_1^{-3})|w|^2\} \geq k_0 > 0.$$

Then there exists $K, K'_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, \epsilon|\zeta|_{H_x^3})$ such that if

$$(5.26) \quad \Upsilon_F |w|_{Z_{F^\mu}^1}^2 \leq K^{-1},$$

then

$$\forall f \in X_{\text{Bo}^{-1}}^0, \quad \langle \mathbf{a}[\epsilon\zeta, \epsilon w]f, f \rangle_{(X_{\text{Bo}^{-1}}^0)^\star} \geq \frac{1}{K'_0} |f|_{X_{\text{Bo}^{-1}}^0}^2.$$

Proof. We establish each result for $f, g \in \mathcal{S}(\mathbb{R})$ so that all the terms are obviously well-defined and in particular the $(X^\star - X)$ duality product (with $X = X_{\text{Bo}^{-1}}^0$ or $Y_{F^\mu}^0$) coincides with the L^2 scalar product; the result for $f, g \in X_{\text{Bo}^{-1}}^0$ or $Y_{F^\mu}^0$ is then obtained by density of $\mathcal{S}(\mathbb{R})$ in $X_{\text{Bo}^{-1}}^0$ and $Y_{F^\mu}^0$, and continuous linear extension.

One has, after integration by parts,

$$(5.27) \quad \begin{aligned} \langle \mathbf{b}[\epsilon\zeta]f, g \rangle_{(Y_{F^\mu}^0)^\star} &= (\mathbf{b}[\epsilon\zeta]f, g)_{L^2} \\ &= \int_{\mathbb{R}} \frac{h_1 + \gamma h_2}{h_1 h_2} f g + \frac{\mu}{3} h_2^3 (\partial_x F_2 \{h_2^{-1} f\}) (\partial_x F_2 \{h_2^{-1} g\}) + \frac{\mu\gamma}{3} h_1^3 (\partial_x F_1 \{h_1^{-1} f\}) (\partial_x F_1 \{h_1^{-1} g\}) \, dx. \end{aligned}$$

It follows easily

$$|\langle \mathbf{b}[\epsilon\zeta]f, g \rangle_{(Y_{F^\mu}^0)^\star}| \leq K_1 |f|_{Y_{F^\mu}^0} |g|_{Y_{F^\mu}^0}.$$

We write again for the coercivity inequality,

$$\langle \mathbf{b}[\epsilon\zeta]f, f \rangle_{(Y_{F^\mu}^0)^\star} = \int_{\mathbb{R}} \frac{h_1 + \gamma h_2}{h_1 h_2} |f|^2 + \frac{\mu}{3} h_2^3 |\partial_x F_2 \{h_2^{-1} f\}|^2 + \frac{\mu\gamma}{3} h_1^3 |\partial_x F_1 \{h_1^{-1} f\}|^2 \, dx.$$

It follows immediately, since $\epsilon\zeta$ satisfies (5.5),

$$\langle \mathbf{b}[\epsilon\zeta]f, f \rangle_{(Y_{F^\mu}^0)^\star} \geq \frac{1 + \gamma}{1 + \delta^{-1}} |f|_{L^2}^2 + \frac{\mu h_0^3}{3} |\partial_x F_2 \{h_2^{-1} f\}|_{L^2}^2 + \frac{\mu\gamma h_0^3}{3} |\partial_x F_1 \{h_1^{-1} f\}|_{L^2}^2.$$

Now, by Lemma C.2, one has

$$\begin{aligned} |\partial_x F_1 f|_{L^2} &= |\partial_x F_1 \{(1 + \epsilon\zeta)h_1^{-1} f\}|_{L^2} \leq (|\partial_x F_1 \{h_1^{-1} f\}|_{L^2} + |\partial_x F_1 \{\epsilon\zeta h_1^{-1} f\}|_{L^2}) \\ &\leq (1 + \epsilon|\zeta|_{Z_{F^\mu}^0}) |\partial_x F_1 \{h_1^{-1} f\}|_{L^2}, \end{aligned}$$

and similarly for $|\partial_x F_2 \{h_2^{-1} f\}|_{L^2}^2$. We conclude

$$|f|_{Y_{F^\mu}^0}^2 \leq C(\mathbf{m}, h_0^{-1}, \epsilon|\zeta|_{Z_{F^\mu}^0}) \times \langle \mathbf{b}[\epsilon\zeta]f, f \rangle_{(Y_{F^\mu}^0)^\star}.$$

By similar argumentation, one easily shows that the operator $\mathfrak{c}[\epsilon\zeta, \epsilon w]$ is well-defined and continuous from $Y_{\mathbb{F}\mu}^0$ to $(Y_{\mathbb{F}\mu}^0)^*$, and satisfies the third estimate of the statement.

We show now the coercivity of $\mathfrak{a}[\epsilon\zeta, \epsilon w]$ under additional assumption (5.25). We write

$$\begin{aligned} \langle \mathfrak{a}[\epsilon\zeta, \epsilon w]f, f \rangle_{(X_{\text{Bo}^{-1}}^0)^*} &= (\mathfrak{a}[\epsilon\zeta, \epsilon w]f, f)_{L^2} \\ &= \int_{\mathbb{R}} \left((\gamma + \delta) - \epsilon^2 \frac{h_1^3 + \gamma h_2^3}{(h_1 h_2)^3} |w|^2 \right) |f|^2 + \frac{\gamma + \delta}{\text{Bo}} \frac{|\partial_x f|^2}{(1 + \mu\epsilon |\partial_x \zeta|^2)^{3/2}} dx + \mu\epsilon^2 (R_2 - \gamma R_1), \end{aligned}$$

with

$$\begin{aligned} R_i &= \left(\left(h_i (\partial_x \mathbb{F}_i^\mu \{h_i^{-1} w\})^2 - \frac{2}{3} w h_i^{-3} \partial_x \mathbb{F}_i^\mu \{h_i^3 \partial_x \mathbb{F}_i^\mu \{h_i^{-1} w\}\} \right) \times f, f \right)_{L^2} \\ &\quad + \frac{1}{3} \left(h_i^3 \partial_x \mathbb{F}_i^\mu \{h_i^{-2} w f\}, \partial_x \mathbb{F}_i^\mu \{h_i^{-2} w f\} \right)_{L^2} - 2 \left(\partial_x \mathbb{F}_i^\mu \{h_i^{-2} w\} \times f, (h_i^2 \partial_x \mathbb{F}_i^\mu \{h_i^{-1} w\}) \times f \right)_{L^2}. \end{aligned}$$

Using Cauchy-Schwarz inequality and Lemmata C.2, and C.3, one has the following estimate

$$\begin{aligned} \mu\epsilon^2 |R_2 - \gamma R_1| &\leq \epsilon^2 |w|_{Z_{\mathbb{F}\mu}^1}^2 C(\mathfrak{m}, h_0^{-1}, \epsilon |\zeta|_{H_x^3}) |f|_{Y_{\mathbb{F}\mu}^0}^2 \\ &\leq \Upsilon_{\mathbb{F}} |w|_{Z_{\mathbb{F}\mu}^1}^2 C(\mathfrak{m}, h_0^{-1}, \epsilon |\zeta|_{H_x^3}) |f|_{X_{\text{Bo}^{-1}}^0}^2, \end{aligned}$$

where the last identity follows from Lemma C.1.

From (5.25), one has immediately

$$\langle \mathfrak{a}[\epsilon\zeta, \epsilon w]f, f \rangle_{(X_{\text{Bo}^{-1}}^0)^*} - \mu\epsilon^2 (R_2 - \gamma R_1) \geq \min \left\{ k_0, \frac{\gamma + \delta}{(1 + \mu\epsilon |\partial_x \zeta|_{L^\infty}^2)^{3/2}} \right\} \times |f|_{X_{\text{Bo}^{-1}}^0}^2.$$

The existence of K'_0, K such that (5.26) implies

$$|f|_{X_{\text{Bo}^{-1}}^0}^2 \leq K'_0 \langle \mathfrak{a}[\epsilon\zeta, \epsilon w]f, f \rangle_{(X_{\text{Bo}^{-1}}^0)^*}$$

is now straightforward. Again, the previous inequality is still true for $f \in X_{\text{Bo}^{-1}}^0$ by continuity and density arguments.

One shows similarly that $\mathfrak{a}[\epsilon\zeta, \epsilon w] : X_{\text{Bo}^{-1}}^0 \rightarrow (X_{\text{Bo}^{-1}}^0)^*$ is well-defined and continuous, and satisfies the first estimate of the statement. This concludes the proof of Lemma 5.6. \square

The following Lemma is a direct consequence of Lemma 5.6.

Lemma 5.7. *Let $(\zeta, w)^\top \in H_x^3 \times Z_{\mathbb{F}\mu}^1$ be such that $\epsilon\zeta$ satisfies (5.5). Then $\mathfrak{b}[\epsilon\zeta] : Y_{\mathbb{F}\mu}^0 \rightarrow (Y_{\mathbb{F}\mu}^0)^*$ is a topological isomorphism with:*

$$\forall f \in Y_{\mathbb{F}\mu}^0, \quad |(\mathfrak{b}[\epsilon\zeta])^{-1} f|_{Y_{\mathbb{F}\mu}^0} \leq K_0 |f|_{(Y_{\mathbb{F}\mu}^0)^*},$$

with K_0 as in Lemma 5.6.

If, additionally, $(\epsilon\zeta, \epsilon w)$ satisfies (5.25)-(5.26), then $\mathfrak{a}[\epsilon\zeta, \epsilon w] : X_{\text{Bo}^{-1}}^0 \rightarrow (X_{\text{Bo}^{-1}}^0)^*$ is a topological isomorphism with:

$$\forall f \in X_{\text{Bo}^{-1}}^0, \quad |(\mathfrak{a}[\epsilon\zeta, \epsilon w])^{-1} f|_{X_{\text{Bo}^{-1}}^0} \leq K'_0 |f|_{(X_{\text{Bo}^{-1}}^0)^*},$$

with K'_0 as in Lemma 5.6.

Proof. By Lemma 5.6, $\mathbf{b}[\epsilon\zeta] : Y_{F^\mu}^0 \rightarrow (Y_{F^\mu}^0)^*$ is well-defined, continuous and coercive. We then deduce by the operator version of Lax-Milgram theorem that $\mathbf{b}[\epsilon\zeta]$ is an isomorphism from $Y_{F^\mu}^0$ onto $(Y_{F^\mu}^0)^*$. The continuity of the inverse follows from the continuity and coercivity of $\mathbf{b}[\epsilon\zeta]$:

$$|\mathbf{b}[\epsilon\zeta]^{-1}f|_{Y_{F^\mu}^0}^2 \leq K_0 \langle \mathbf{b}[\epsilon\zeta] \mathbf{b}[\epsilon\zeta]^{-1}f, \mathbf{b}[\epsilon\zeta]^{-1}f \rangle_{(Y_{F^\mu}^0)^*} \leq K_0 |f|_{(Y_{F^\mu}^0)^*} |\mathbf{b}[\epsilon\zeta]^{-1}f|_{Y_{F^\mu}^0}.$$

The whole discussion is identical for $\mathbf{a}[\epsilon\zeta, \epsilon w]$, replacing $Y_{F^\mu}^0$ with $X_{\text{Bo}-1}^0$, and K_0 with K'_0 . \square

We conclude this section with the following result.

Lemma 5.8. *Let $(\zeta, w)^\top \in H_x^3 \times Z_{F^\mu}^1$ be such that $\epsilon\zeta$ satisfies (5.5). Then the operator $\mathbf{a}[\epsilon\zeta, \epsilon w] : X_{\text{Bo}-1}^0 \rightarrow (X_{\text{Bo}-1}^0)^*$ is symmetric:*

$$\forall f, g \in X_{\text{Bo}-1}^0, \quad \langle \mathbf{a}[\epsilon\zeta, \epsilon w]f, g \rangle_{(X_{\text{Bo}-1}^0)^*} = \langle \mathbf{a}[\epsilon\zeta, \epsilon w]g, f \rangle_{(X_{\text{Bo}-1}^0)^*}.$$

The same result holds true for $\mathbf{b}[\epsilon\zeta]$ and $\mathbf{c}[\epsilon\zeta, \epsilon w]$, replacing $X_{\text{Bo}-1}^0$ with $Y_{F^\mu}^0$.

Proof. The symmetry property for $\mathbf{b}[\epsilon\zeta]$ is straightforwardly seen from (5.27). The other operators require a slight rewriting. In particular, notice

$$\begin{aligned} (-1)^i d_1 \mathcal{R}_i[\epsilon\zeta, w] \bullet &= \left(h_i (\partial_x F_i^\mu \{h_i^{-1}w\})^2 - \frac{2}{3} w h_i^{-3} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-1}w\}\} \right) \times \bullet \\ &+ (h_i^{-2}w) \times \partial_x F_i^\mu \{h_i^2 \partial_x F_i^\mu \{h_i^{-1}w\}\} \times \bullet - (h_i^2 \partial_x F_i^\mu \{h_i^{-1}w\}) \times \partial_x F_i^\mu \{(h_i^{-2}w) \times \bullet\} \\ &\quad - \frac{1}{3} (h_i^{-2}w) \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{(h_i^{-2}w) \bullet\}\} \end{aligned}$$

and

$$\begin{aligned} ((-1)^i d_2 \mathcal{Q}_i[\epsilon\zeta](w) + d_2 \mathcal{R}_i[\epsilon\zeta, w]) \bullet &= \left(\frac{2}{3} h_i^{-2} \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-1}w\}\} \right) \times \bullet \\ &- h_i^{-1} \partial_x F_i^\mu \{(h_i^2 \partial_x F_i^\mu \{h_i^{-1}w\}) \times \bullet\} + (h_i^2 \partial_x F_i^\mu \{h_i^{-1}w\}) \times \partial_x F_i^\mu \{h_i^{-1} \times \bullet\} \\ &+ \frac{1}{3} h_i^{-1} \times \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{(h_i^{-2}w) \times \bullet\}\} + \frac{1}{3} (h_i^{-2}w) \times \partial_x F_i^\mu \{h_i^3 \partial_x F_i^\mu \{h_i^{-1} \times \bullet\}\} \end{aligned}$$

are obviously symmetric, since $\partial_x F_i^\mu$ is skew-symmetric. The result is now clear. \square

5.3 A priori estimates

We now consider the quasi-linearized system arising from Lemma 5.5:

$$(5.28) \quad \begin{cases} \partial_t \dot{\zeta} + \partial_x \dot{w} = r_1, \\ \mathbf{b} \partial_t \dot{w} + \partial_x \mathbf{a} \dot{\zeta} + \partial_x \check{\mathbf{a}}_\alpha \check{\zeta} + \mathbf{c} \partial_x \dot{w} = r_2, \end{cases}$$

where we denote for conciseness $\mathbf{a} = \mathbf{a}[\epsilon\zeta, \epsilon w]$ (and similarly for $\check{\mathbf{a}}_\alpha, \mathbf{b}, \mathbf{c}$), as defined in Lemma 5.5, and r_1, r_2 are remainder terms to be precised.

More precisely, we introduce a regularized version of (5.28). Denote $J_\nu = (1 - \nu \partial_x^2)^{-1/2}$ and consider

$$(5.29) \quad \begin{cases} \partial_t \dot{\zeta} + J_\nu^2 \partial_x \dot{w} = r_1, \\ \mathbf{b} \partial_t \dot{w} + J_\nu^2 \partial_x \mathbf{a} \dot{\zeta} + J_\nu^2 \partial_x \check{\mathbf{a}}_\alpha \check{\zeta} + J_\nu \mathbf{c} J_\nu \partial_x \dot{w} = r_2. \end{cases}$$

The following Lemma provides an a priori control of the energy of any solution $(\dot{\zeta}, \dot{w})$ over a uniformly bounded from below time interval. This in turn allows to obtain the existence and

uniqueness of a solution to the initial value problem of the linearized system (5.28), as stated in Lemma 5.12. Lemma 5.11, below, offers a similar estimate on the difference between two solutions, and will be used in the proof of the well-posedness and stability of the nonlinear system (5.1); Theorem 5.1 and Proposition 6.2.

Lemma 5.9. *Let $\dot{U} \stackrel{\text{def}}{=}} (\dot{\zeta}, \dot{w})^\top$, $U \stackrel{\text{def}}{=} (\zeta, w)^\top \in L^\infty([0, T]; X_{\text{Bo}^{-1}}^4 \times Y_{\text{F}\mu}^4)$, $\check{\zeta} \in L^\infty([0, T]; (X_{\text{Bo}^{-1}}^1)^2)$ and $r = (r_1, r_2)^\top \in L^1([0, T]; X_{\text{Bo}^{-1}}^0 \times (Y_{\text{F}\mu}^0)^*)$ satisfying (5.29) with $\nu \in [0, 1]$. Assume moreover that $U(t)$ satisfies (5.5), (5.25) and (5.26) with $h_0^{-1}, k_0^{-1}, K^{-1}$ uniformly for $t \in [0, T]$. Then one has*

$$E^0(\dot{U})^{1/2} \leq \mathbf{C}_0 (E^0(\dot{U}|_{t=0})^{1/2} + (\mu\epsilon^2) \|\check{\zeta}\|_{L^\infty([0, T]; (X_{\text{Bo}^{-1}}^0)^2)}) e^{\lambda t} + \mathbf{C}_0 \int_0^t f(t') e^{\lambda(t-t')} dt',$$

with

$$\lambda = \mathbf{C}_0 \times \left(\epsilon + \Upsilon_{\text{F}} \|w\|_{L^\infty([0, T]; Z_{\text{F}\mu}^2)}^2 \right), \quad f(t) = |r|_{X_{\text{Bo}^{-1}}^0 \times (Y_{\text{F}\mu}^0)^*} + \mu\epsilon^2 |\check{\zeta}|_{(X_{\text{Bo}^{-1}}^1)^2}$$

and $\mathbf{C}_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, \|U\|_{L^\infty([0, T]; X_{\text{Bo}^{-1}}^4 \times Y_{\text{F}\mu}^4)})$.

Remark 5.10. *The energy estimate is uniform with respect to $\nu \in [0, 1]$. It holds in particular for solutions to the non-regularized system (5.28).*

Proof. Since $U, \dot{U} \in L^\infty([0, T]; X_{\text{Bo}^{-1}}^4 \times Y_{\text{F}\mu}^4)$, all the components of equation (5.29) are obviously well-defined in L^2 . We compute the L^2 inner product of the first equation with $\mathbf{a}\dot{\zeta} + \check{\mathbf{a}}_\alpha\check{\zeta}$, and add the L^2 inner product of the second equation with \dot{w} . Recalling that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are symmetric (by Lemma 5.8), and since \mathbf{J}_ν is symmetric and ∂_x is skew-symmetric, we obtain after straightforward manipulations

$$(5.30) \quad \frac{d}{dt} \left(\frac{1}{2} (\mathbf{a}\dot{\zeta}, \dot{\zeta})_{L^2} + (\dot{\zeta}, \check{\mathbf{a}}_\alpha\check{\zeta})_{L^2} + \frac{1}{2} (\mathbf{b}\dot{w}, \dot{w})_{L^2} \right) = \\ \frac{1}{2} ([\partial_t, \mathbf{a}]\dot{\zeta}, \dot{\zeta})_{L^2} + (\dot{\zeta}, \partial_t(\check{\mathbf{a}}_\alpha\check{\zeta}))_{L^2} + \frac{1}{2} ([\partial_t, \mathbf{b}]\dot{w}, \dot{w})_{L^2} + \frac{1}{2} ([\partial_x, \mathbf{c}]\mathbf{J}_\nu\dot{w}, \mathbf{J}_\nu\dot{w})_{L^2} \\ + (r_1, \mathbf{a}\dot{\zeta} + \check{\mathbf{a}}_\alpha\check{\zeta})_{L^2} + (r_2, \dot{w})_{L^2}.$$

We estimate below each of the components of the right-hand-side. These estimates follow from the product estimates of Section C, as in the proof of Lemma 5.5. For the sake of conciseness, we do not detail all calculations but rather provide the precise estimates for each component.

(I) $\stackrel{\text{def}}{=} ([\partial_t, \mathbf{a}]\dot{\zeta}, \dot{\zeta})_{L^2}$. One has, by definition,

$$[\partial_t, \mathbf{a}]\dot{\zeta} = -\epsilon^2 \dot{\zeta} \partial_t (G(\epsilon\zeta)|w|^2) - \mu\epsilon^2 \left([\partial_t, \mathbf{d}_1 \mathcal{R}_2^{\text{F}}[\epsilon\zeta, w]]\dot{\zeta} - \gamma [\partial_t, \mathbf{d}_1 \mathcal{R}_1^{\text{F}}[\epsilon\zeta, w]]\dot{\zeta} \right) \\ - \frac{\gamma + \delta}{\text{Bo}} \partial_x \left(\partial_t \left(\frac{1}{(1 + \mu\epsilon^2 |\partial_x \zeta|^2)^{3/2}} \right) \partial_x \dot{\zeta} \right),$$

where $G(\epsilon\zeta) = \frac{h_1^3 + \gamma h_2^3}{(h_1 h_2)^3}$ and $\mathcal{R}_i^{\text{F}}[\epsilon\zeta, w]$ is defined in (5.17).

The first component of $([\partial_t, \mathbf{a}]\dot{\zeta}, \dot{\zeta})_{L^2}$ is easily estimated:

$$\left| (-\epsilon^2 \dot{\zeta} \partial_t (G(\epsilon\zeta)|w|^2), \dot{\zeta})_{L^2} \right| \leq C_0 \epsilon^2 |w|_{W^{1, \infty}}^2 |\dot{\zeta}|_{L^2}^2,$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, |\epsilon\zeta|_{W^{1, \infty}})$.

The third term follows from one integration by parts:

$$\left| \left(-\frac{\gamma + \delta}{\text{Bo}} \partial_x \left(\partial_t \left((1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{-3/2} \right) \partial_x \dot{\zeta} \right), \dot{\zeta} \right)_{L^2} \right| = \frac{\gamma + \delta}{\text{Bo}} \left| \left(\partial_t \left((1 + \mu \epsilon^2 |\partial_x \zeta|^2)^{-3/2} \right) \partial_x \dot{\zeta}, \partial_x \dot{\zeta} \right)_{L^2} \right| \leq \mu \epsilon^2 C_0 \frac{1}{\text{Bo}} |\partial_x \dot{\zeta}|_{L^2}^2,$$

with $C_0 = C(\mathbf{m}, |\zeta|_{W^{2,\infty}})$.

Finally, the second term of $([\partial_t, \mathbf{a}]\dot{\zeta}, \dot{\zeta})$ is more involved, as $[\partial_t, \mathbf{d}_1 \mathcal{R}_1^F[\epsilon \zeta, w]]$ is the sum of many terms. However, they may all be treated as in the proof of Lemma 5.5. Using integration by parts if necessary, one may ensure that the operator $\partial_x F_i^\mu$ applies only once to each $\dot{\zeta}$ and since much regularity is assumed on $\zeta \in X_{\text{Bo}^{-1}}^4$, Lemmata C.2 and C.3 yield

$$\left| \mu \epsilon^2 \gamma^{2-i} \left([\partial_t, \mathbf{d}_1 \mathcal{R}_i^F[\epsilon \zeta, w]] \dot{\zeta}, \dot{\zeta} \right)_{L^2} \right| \leq C_0 \epsilon^2 |w|_{Z_{F^\mu}^2}^2 |\dot{\zeta}|_{Y_{F^\mu}^0}^2,$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, |\zeta|_{X_{\text{Bo}^{-1}}^4})$. Using Lemma C.1, one deduces

$$\left| \mu \epsilon^2 \gamma^{2-i} \left([\partial_t, \mathbf{d}_1 \mathcal{R}_i^F[\epsilon \zeta, w]] \dot{\zeta}, \dot{\zeta} \right)_{L^2} \right| \leq C_0 \Upsilon_F |w|_{Z_{F^\mu}^2}^2 |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2.$$

Altogether, we proved

$$(5.31) \quad |(I)| \leq C_0 \left(\mu \epsilon^2 + \Upsilon_F |w|_{Z_{F^\mu}^2}^2 \right) |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2,$$

where $C_0 = C(\mathbf{m}, h_0^{-1}, |\zeta|_{X_{\text{Bo}^{-1}}^4})$.

(II) $\stackrel{\text{def}}{=} ([\partial_t, \mathbf{b}]\dot{w}, \dot{w})_{L^2}$. One has, by definition,

$$[\partial_t, \mathbf{b}]\dot{w} = \partial_t \left(\frac{h_1 + \gamma h_2}{h_1 h_2} \right) \dot{w} + \mu \epsilon (d\mathcal{Q}_2^F[\epsilon \zeta](\dot{w}) + \gamma d\mathcal{Q}_1^F[\epsilon \zeta](\dot{w})) \partial_t \zeta,$$

where $d\mathcal{Q}_i^F$ is defined in (5.14). The first term is estimated as

$$\left| \left(\partial_t \left(\frac{h_1 + \gamma h_2}{h_1 h_2} \right) \dot{w}, \dot{w} \right)_{L^2} \right| \leq \epsilon C_0 |\dot{w}|_{L^2}^2,$$

where $C_0 = C(\mathbf{m}, h_0^{-1}, |\partial_t \zeta|_{L^\infty})$. For the second term we have after integration by parts and by triangular inequality

$$\left| \left(d\mathcal{Q}_i^F[\epsilon \zeta](\dot{w}) \partial_t \zeta, \dot{w} \right)_{L^2} \right| \leq \frac{2}{3} \left| \left(h_i^3 \partial_x F_i^\mu \{h_i^{-1} \dot{w}\}, \partial_x F_i^\mu \{h_i^{-2} \partial_t \zeta \dot{w}\} \right)_{L^2} \right| + \left| \left(h_i^2 (\partial_t \zeta) \partial_x F_i^\mu \{h_i^{-1} \dot{w}\}, \partial_x F_i^\mu \{h_i^{-1} \dot{w}\} \right)_{L^2} \right|.$$

By Lemmata C.2 and C.3, one immediately deduces

$$(5.32) \quad |(II)| \leq \epsilon C_0 |\dot{w}|_{Y_{F^\mu}^0}^2,$$

where $C_0 = C(\mathbf{m}, h_0^{-1}, |\zeta|_{X_{\text{Bo}^{-1}}^4})$.

(III) $\stackrel{\text{def}}{=} ([\partial_x, \mathbf{c}]J_\nu \dot{w}, J_\nu \dot{w})_{L^2}$. One may proceed similarly as above, and one obtains without any additional difficulty

$$(5.33) \quad |(III)| \leq \epsilon C_0 |J_\nu \dot{w}|_{Y_{F^\mu}^0}^2 \leq \epsilon C_0 |\dot{w}|_{Y_{F^\mu}^0}^2,$$

where $C_0 = C(\mathbf{m}, h_0^{-1}, |\zeta|_{X_{\text{Bo}^{-1}}^4}, |w|_{Y_{F\mu}^4})$, and the last inequality follows from $\|J_\nu\|_{L^2 \rightarrow L^2} = 1$.

(IV) $\stackrel{\text{def}}{=} (\dot{\zeta}, \partial_t(\check{\mathbf{a}}_\alpha \check{\zeta}))_{L^2}$. After one integration by parts, one has

$$(IV) = -3\mu\epsilon^2 \frac{\gamma + \delta}{\text{Bo}} \sum_{j \in \{1,2\}} \alpha_j \left(\partial_x \dot{\zeta}, \partial_t \left(\frac{(\partial_x \partial^{e_j} \zeta)(\partial_x \zeta)(\partial_x \check{\zeta}_j)}{(1 + \mu\epsilon^2 |\partial_x \zeta|^2)^{5/2}} \right) \right)_{L^2}.$$

Recall $\check{\zeta} = (\check{\zeta}_0, \check{\zeta}_1)^\top \in X_{\text{Bo}^{-1}}^1 \times X_{\text{Bo}^{-1}}^1$, so we easily deduce by Cauchy-Schwarz inequality

$$(5.34) \quad |(IV)| \leq \mu\epsilon^2 C(\mathbf{m}, |\zeta|_{W^{3,\infty}}) |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0} |\check{\zeta}|_{(X_{\text{Bo}^{-1}}^1)^2}.$$

(V) $\stackrel{\text{def}}{=} (r_1, \mathbf{a}\dot{\zeta} + \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} + (r_2, \dot{w})_{L^2}$. The remainder terms are straightforward to estimate, using in particular the estimates of Lemma 5.6. One obtains

$$(5.35) \quad |(V)| \leq C_0 |r_1|_{X_{\text{Bo}^{-1}}^0} |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0} + \mu\epsilon^2 C(\mathbf{m}, |\zeta|_{W^{2,\infty}}) |r_1|_{X_{\text{Bo}^{-1}}^0} |\dot{\zeta}|_{(X_{\text{Bo}^{-1}}^0)^2} + |r_2|_{(Y_{F\mu}^0)^*} |\dot{w}|_{Y_{F\mu}^0},$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, \epsilon |\zeta|_{H^3})$.

Altogether, plugging (5.31), (5.32), (5.33), (5.34), (5.35) into (5.30) yields

$$(5.36) \quad \frac{d}{dt} \left(\frac{1}{2} (\mathbf{a}\dot{\zeta}, \dot{\zeta})_{L^2} + (\dot{\zeta}, \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} + \frac{1}{2} (\mathbf{b}\dot{w}, \dot{w})_{L^2} \right) \leq C_0 \left(\epsilon + \Upsilon_F |w|_{Z_{F\mu}^2} \right) E^0(\dot{U}) + C_0 C_1 E^0(\dot{U})^{1/2},$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, E^4(U))$, and $C_1 = |r|_{X_{\text{Bo}^{-1}}^0 \times (Y_{F\mu}^0)^*} + \mu\epsilon^2 |\check{\zeta}|_{(X_{\text{Bo}^{-1}}^1)^2}$.

The proof is concluded as follows. By Lemma 5.6, there exists $K_0, K_1 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, E^4(U))$ such that

$$(5.37) \quad \frac{1}{K_0} E^0(\dot{U}) \leq \frac{1}{2} (\mathbf{a}\dot{\zeta}, \dot{\zeta})_{L^2} + \frac{1}{2} (\mathbf{b}\dot{w}, \dot{w})_{L^2} \leq K_1 E^0(\dot{U}).$$

Notice then that one has

$$\left| (\dot{\zeta}, \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} \right| \leq \mu\epsilon^2 C_2 |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0} |\check{\zeta}|_{X_{\text{Bo}^{-1}}^0},$$

with $C_2 = C(|\partial_x \zeta|_{\mathbf{m}, W^{2,\infty}})$. It follows, for any $M > 0$,

$$\left| (\dot{\zeta}, \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} \right| \leq \frac{1}{2} \mu\epsilon^2 C_2 \times (M^{-1} |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2 + M |\check{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2).$$

Using the above with $M = \mu\epsilon^2 C_2 K_0$ and recalling $E^0(\dot{U}) \geq |\dot{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2$ by definition (5.2), (5.37) yields

$$\frac{1}{2K_0} E^0(\dot{U}) - \frac{1}{2} K_0 (\mu\epsilon^2 C_2)^2 |\check{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2 \leq \frac{1}{2} (\mathbf{a}\dot{\zeta}, \dot{\zeta})_{L^2} + (\dot{\zeta}, \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} + \frac{1}{2} (\mathbf{b}\dot{w}, \dot{w})_{L^2}$$

and

$$\frac{1}{2} (\mathbf{a}\dot{\zeta}, \dot{\zeta})_{L^2} + (\dot{\zeta}, \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} + \frac{1}{2} (\mathbf{b}\dot{w}, \dot{w})_{L^2} \leq (K_1 + \frac{1}{2K_0}) E^0(\dot{U}) + \frac{1}{2} K_0 (\mu\epsilon^2 C_2)^2 |\check{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2.$$

Thus defining $\widetilde{\mathbf{M}} \stackrel{\text{def}}{=} \max_{t \in [0, T]} \left\{ \frac{1}{2} K_0 (\mu\epsilon^2 C_2)^2 |\check{\zeta}|_{X_{\text{Bo}^{-1}}^0}^2 \right\}$ and

$$\widetilde{E}^0(\dot{U}) \stackrel{\text{def}}{=} \frac{1}{2} (\mathbf{a}\dot{\zeta}, \dot{\zeta})_{L^2} + (\dot{\zeta}, \check{\mathbf{a}}_\alpha \check{\zeta})_{L^2} + \frac{1}{2} (\mathbf{b}\dot{w}, \dot{w})_{L^2},$$

one has

$$(5.38) \quad \frac{1}{2K_0} E^0(\dot{U}) - \widetilde{\mathbf{M}} \leq \widetilde{E}^0(\dot{U}) \leq (K_1 + \frac{1}{2K_0}) E^0(\dot{U}) + \widetilde{\mathbf{M}}.$$

The estimate (5.36) may therefore be reformulated as

$$\frac{d}{dt} (\widetilde{E}^0(\dot{U}) + \widetilde{\mathbf{M}}) \leq 2K_0 C_0 \left(\epsilon + \Upsilon_{\mathbb{F}} |w|_{Z_{\mathbb{F}\mu}^2} \right) \left(\widetilde{E}^0(\dot{U}) + \widetilde{\mathbf{M}} \right) + \sqrt{2K_0} C_0 C_1 \left(\widetilde{E}^0(\dot{U}) + \widetilde{\mathbf{M}} \right)^{1/2}.$$

We deduce as usual

$$\left(\widetilde{E}^0(\dot{U}) + \widetilde{\mathbf{M}} \right)^{1/2} \leq \left(\widetilde{E}^0(\dot{U}|_{t=0}) + \widetilde{\mathbf{M}} \right)^{1/2} e^{\lambda t} + \mathbf{C}_0 \int_0^t C_1(t') e^{\lambda(t-t')} dt',$$

where λ, \mathbf{C}_0 are as in the statement of the Lemma. Using (5.38) and augmenting \mathbf{C}_0 if necessary, the energy estimate is now straightforward. \square

Lemma 5.11. *Define two tuple of solutions to (5.28), (\dot{U}_1, U_1, r_1) and (\dot{U}_2, U_2, r_2) , satisfying the same properties as in Lemma 5.9 (with $\zeta_1 = \zeta_2 = 0$). Then one has*

$$E^0(\dot{U}_1 - \dot{U}_2)^{1/2} \leq \mathbf{C}_0 E^0(\dot{U}_1|_{t=0} - \dot{U}_2|_{t=0})^{1/2} e^{\lambda t} + \mathbf{C}_0 \int_0^t f(t') e^{\lambda(t-t')} dt',$$

with

$$\lambda = \mathbf{C}_0 \times \left(\epsilon + \Upsilon_{\mathbb{F}} \|w_1\|_{L^\infty([0, T]; Z_{\mathbb{F}\mu}^2)}^2 \right), \quad f(t) = |r_1 - r_2|_{X_{\mathbb{B}o-1}^0 \times (Y_{\mathbb{F}\mu}^0)^*} + \epsilon |\dot{U}_2|_{(W_x^{3, \infty})^2} |U_1 - U_2|_{X_{\mathbb{B}o-1}^2 \times Y_{\mathbb{F}\mu}^2}$$

$$\text{and } \mathbf{C}_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, \|U_1\|_{L^\infty([0, T]; X_{\mathbb{B}o-1}^4 \times Y_{\mathbb{F}\mu}^4)}, \|U_2\|_{L^\infty([0, T]; X_{\mathbb{B}o-1}^4 \times Y_{\mathbb{F}\mu}^4)}).$$

Proof. The difference between the two solutions satisfies the system

$$\begin{cases} \partial_t(\dot{\zeta}_1 - \dot{\zeta}_2) + \partial_x(\dot{w}_1 - \dot{w}_2) = r_1^1 - r_2^1, \\ \mathbf{b}_1 \partial_t(\dot{w}_1 - \dot{w}_2) + \partial_x \mathbf{a}_1(\dot{\zeta}_1 - \dot{\zeta}_2) + \mathbf{c}_1 \partial_x(\dot{w}_1 - \dot{w}_2) = r_1^2 - r_2^2 + r_{\text{diff}}, \end{cases}$$

where we denote $\mathbf{a}_i = \mathbf{a}[\epsilon \zeta_i, \epsilon w_i]$ (and similarly for $\mathbf{b}_i, \mathbf{c}_i$), and

$$r_{\text{diff}} \stackrel{\text{def}}{=} (\mathbf{b}_2 - \mathbf{b}_1) \partial_t \dot{w}_2 + (\partial_x \mathbf{a}_2 - \partial_x \mathbf{a}_1) \dot{\zeta}_2 + (\mathbf{c}_2 - \mathbf{c}_1) \partial_x \dot{w}_2 \stackrel{\text{def}}{=} \sum_{i=1}^3 r_{\text{diff}}^{(i)}.$$

The Lemma is a straightforward consequence of Lemma 5.9 (with $\nu = 0$), once r_{diff} is estimated. We focus on the most difficult term, namely $r_{\text{diff}}^{(2)} = (\partial_x \mathbf{a}_2 - \partial_x \mathbf{a}_1) \dot{\zeta}_2$; the other terms are obtained similarly.

Let $f \in Y_{\mathbb{F}\mu}^0$. One has

$$\begin{aligned} (r_{\text{diff}}^{(2)}, f)_{L^2} &= \int_{\mathbb{R}} -\epsilon^2 f \partial_x \left(\left(G(\epsilon \zeta_2) |w_2|^2 - G(\epsilon \zeta_1) |w_1|^2 \right) \dot{\zeta}_2 \right) \\ &\quad - \mu \epsilon^2 f \partial_x \left((\mathbf{d}_1 \mathcal{R}_2[\epsilon \zeta_2, w_2] - \mathbf{d}_1 \mathcal{R}_2[\epsilon \zeta_1, w_1] - \gamma \mathbf{d}_1 \mathcal{R}_1[\epsilon \zeta_2, w_2] + \gamma \mathbf{d}_1 \mathcal{R}_1[\epsilon \zeta_1, w_1]) \dot{\zeta}_2 \right) \\ &\quad + \frac{\gamma + \delta}{\mathbf{B}o} f \partial_x^2 \left(\left(\frac{1}{(1 + \mu \epsilon^2 |\partial_x \zeta_2|^2)^{3/2}} - \frac{1}{(1 + \mu \epsilon^2 |\partial_x \zeta_1|^2)^{3/2}} \right) \partial_x \dot{\zeta}_2 \right) \end{aligned}$$

where $G(\epsilon \zeta) \stackrel{\text{def}}{=} \frac{h_1^3 + \gamma h_2^3}{(h_1 h_2)^3}$.

Since $|G(\epsilon\zeta_2 - \epsilon\zeta_1)|_{L^2} \leq \epsilon|\zeta_2 - \zeta_1|_{L^2} \times \sup_{y \in [\epsilon\zeta_2, \epsilon\zeta_1]} G'(y)$, it is straightforward that

$$|\partial_x \left((G(\epsilon\zeta_2)|w_2|^2 - G(\epsilon\zeta_1)|w_1|^2)\dot{\zeta}_2 \right)|_{L^2} \leq C_0(\epsilon|\zeta_1 - \zeta_2|_{H^1} + |w_1 - w_2|_{H^1})|\dot{\zeta}_2|_{W_x^{1,\infty}},$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, |\zeta_1|_{L^\infty}, |\zeta_2|_{L^\infty}, |w_1|_{L^\infty}, |w_2|_{L^\infty})$.

Similarly,

$$\begin{aligned} \frac{1}{\text{Bo}^{-1}} |\partial_x^2 \left((1 + \mu\epsilon^2|\partial_x\zeta_2|^2)^{-3/2} - (1 + \mu\epsilon^2|\partial_x\zeta_1|^2)^{-3/2} \right) \partial_x \dot{\zeta}_2 \Big|_{L^2} \\ \leq \mu\epsilon^2 C_0 |\partial_x \zeta_1 - \partial_x \zeta_2|_{X_{\text{Bo}^{-1}}^1} |\partial_x \dot{\zeta}_2|_{W_x^{2,\infty}}, \end{aligned}$$

with $C_0 = C(\mathbf{m}, |\partial_x \zeta_1|_{W_x^{1,\infty}}, |\partial_x \zeta_2|_{W_x^{1,\infty}})$.

As for the last component, recall $d_1 \mathcal{R}_i$ is defined in (5.17). Proceeding as in the proof of Lemma 5.5, we obtain

$$\begin{aligned} \gamma^{2-i} \mu\epsilon^2 \left| (d_1 \mathcal{R}_i[\epsilon\zeta_2, w_2]\dot{\zeta}_2 - d_1 \mathcal{R}_i[\epsilon\zeta_1, w_1]\dot{\zeta}_2, f) \right|_{L^2} \\ \leq \epsilon^2 C_0 \times (|w_1 - w_2|_{Y_{F\mu}^1} + \epsilon|\zeta_1 - \zeta_2|_{Y_{F\mu}^1}) |\dot{\zeta}_2|_{Z_{F\mu}^1} |f|_{Y_{F\mu}^0} \end{aligned}$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, |\zeta_1|_{H_x^3}, |\zeta_2|_{H_x^3}, |w_1|_{Z_{F\mu}^1}, |w_2|_{Z_{F\mu}^1})$.

Altogether, we find

$$|r_{\text{diff}}^{(2)}|_{(Y_{F\mu}^0)^*} \leq \epsilon C_0 |\dot{\zeta}_2|_{W_x^{3,\infty}} |U_2 - U_1|_{X_{\text{Bo}^{-1}}^2 \times Y_{F\mu}^2},$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, |U_1|_{X_{\text{Bo}^{-1}}^4 \times Y_{F\mu}^4}, |U_2|_{X_{\text{Bo}^{-1}}^4 \times Y_{F\mu}^4})$.

All the other terms in r_{diff} are estimated in the same way, and Lemma 5.11 now directly follows from Lemma 5.9. \square

5.4 Well-posedness results; proof of Theorem 5.1

In this section we conclude the proof of the main result of the paper, Theorem 5.1, namely the well-posedness of the Cauchy problem for our class of system (5.1). We first prove in Lemma 5.12 the existence and uniqueness of solutions of the linearized system (5.28) for smooth data, and provide a uniform energy estimate. A solution of the nonlinear system (5.1) is then constructed using a Picard iteration scheme. Uniqueness, and continuous dependence with respect to the initial data follow from Lemma 5.11.

Lemma 5.12. *Let $\zeta, w, \check{\zeta}, r^1, r^2 \in H^\infty([0, T] \times \mathbb{R})$ be such that (5.5), (5.25), (5.26) hold. Then for any $\dot{U}^0 \stackrel{\text{def}}{=}} (\dot{\zeta}^0, \dot{w}^0)^\top \in H_x^\infty(\mathbb{R})^2$, there exists a unique solution $\dot{U} \stackrel{\text{def}}{=}} (\dot{\zeta}, \dot{w})^\top \in H^\infty([0, T] \times \mathbb{R})^2$ satisfying (5.28) and $\dot{U}|_{t=0} = \dot{U}^0$.*

Remark 5.13. *One could assume only continuity in time and finite (but large enough) regularity in space on $\zeta, w, \check{\zeta}, r$, but this is unnecessary since Lemma 5.12 is always used with smooth data.*

Proof. We first consider the regularized system introduced in (5.29) and that we recall. For any $\nu > 0$, define $J_\nu \stackrel{\text{def}}{=}} (1 - \nu\partial_x^2)^{-1/2}$ and consider

$$(5.39) \quad \begin{cases} \partial_t \dot{\zeta}_\nu + J_\nu^2 \partial_x \dot{w}_\nu = r^1, \\ \mathfrak{b} \partial_t \dot{w}_\nu + J_\nu^2 \partial_x \mathfrak{a} \dot{\zeta} + J_\nu^2 \partial_x \check{\mathfrak{a}}_\alpha \check{\zeta} + J_\nu \mathfrak{c} J_\nu \partial_x \dot{w}_\nu = r^2; \end{cases}$$

or, equivalently (recall that, by Lemma 5.7, $\mathfrak{b}^{-1} : (Y_{\mathbb{F}\mu}^0)^* \rightarrow Y_{\mathbb{F}\mu}^0$ is well-defined and continuous)

$$(5.40) \quad \begin{cases} \partial_t \dot{\zeta}_\nu + J_\nu^2 \partial_x \dot{w}_\nu = r^1, \\ \partial_t \dot{w}_\nu + \mathfrak{b}^{-1} J_\nu^2 \partial_x \dot{\alpha} \dot{\zeta} + \mathfrak{b}^{-1} J_\nu^2 \partial_x \check{\alpha}_\alpha \check{\zeta} + \mathfrak{b}^{-1} J_\nu \mathfrak{c} J_\nu \partial_x \dot{w}_\nu = \mathfrak{b}^{-1} r^2. \end{cases}$$

One straightforwardly checks that system (5.40) is a system of ordinary differential equations on $X_{\mathbb{B}o^{-1}}^0 \times Y_{\mathbb{F}\mu}^0$, which is solved uniquely by Cauchy-Lipschitz theorem. More precisely, for any $\nu > 0$ and $r = (r^1, r^2) \in C^0([0, T]; X_{\mathbb{B}o^{-1}}^0 \times (Y_{\mathbb{F}\mu}^0)^*)$, $\check{\zeta} \in C^0([0, T]; (X_{\mathbb{B}o^{-1}}^1)^2)$ and $\dot{U}^0 \in X_{\mathbb{B}o^{-1}}^0 \times Y_{\mathbb{F}\mu}^0$, there exists a unique $\dot{U}_\nu \stackrel{\text{def}}{=} (\dot{\zeta}_\nu, \dot{w}_\nu)^\top \in C^1([0, T]; X_{\mathbb{B}o^{-1}}^0 \times Y_{\mathbb{F}\mu}^0)$, solution to (5.40) with $\dot{U}|_{t=0} = (\dot{\zeta}^0, \dot{w}^0)^\top$.

Differentiating N times (5.39) and proceeding as in the proof of Lemma 5.5, one can check that $\partial_x^N \dot{U}_\nu$ satisfies (5.39) with obvious modifications to r^1, r^2 and $\check{\zeta}$. Thus, by the above argument, $\partial_x^N \dot{U}_\nu \in C^1([0, T]; X_{\mathbb{B}o^{-1}}^0 \times Y_{\mathbb{F}\mu}^0)$, and it follows (since N may be chosen arbitrarily large) that $\dot{U}_\nu \in C^1([0, T]; H_x^\infty(\mathbb{R}))$. In particular, $\partial_t U_\nu|_{t=0} \in H_x^\infty$.

Applying the above argument to $\partial_t U_\nu$ after differentiating (5.39) with respect to time, one deduces $\partial_t \dot{U}_\nu \in C^1([0, T]; H_x^\infty(\mathbb{R}))$, and by induction $U_\nu \in H^\infty([0, T] \times \mathbb{R})$.

Applying the estimate of Lemma 5.9 to $\partial_x^N U_\nu$ with $N \in \mathbb{N}$ given, one has

$$E^0(\partial_x^N \dot{U}_\nu) \leq M,$$

with $M = C(\mathfrak{m}, h_0^{-1}, k_0^{-1}, K, T, E^0(\partial_x^N \dot{U}^0), \|(\zeta, w, \check{\zeta}, r)\|_{H^\infty([0, T] \times \mathbb{R}^6)})$ uniform with respect to $\nu > 0$.

Let us now consider $V_{\nu, \nu'} = \dot{U}_\nu - \dot{U}_{\nu'}$. $V_{\nu, \nu'}$ satisfies (5.39) with $\check{\zeta} = 0, V_{\nu, \nu'}|_{t=0} = 0$ and

$$r_{\nu, \nu'}^1 = (J_\nu^2 - J_{\nu'}^2) \partial_x \dot{w}_{\nu'}, \quad r_{\nu, \nu'}^2 = (J_\nu^2 - J_{\nu'}^2) \partial_x \dot{\alpha} \dot{\zeta}_{\nu'} + (J_\nu^2 - J_{\nu'}^2) \partial_x \check{\alpha}_\alpha \check{\zeta} + (J_\nu \mathfrak{c} J_\nu - J_{\nu'} \mathfrak{c} J_{\nu'}) \partial_x \dot{w}_{\nu'}.$$

Since for any $s \in \mathbb{R}$, $\|J_\nu\|_{H_x^s \rightarrow H_x^s} = 1$ and $\|J_\nu - J_{\nu'}\|_{H_x^s \rightarrow H_x^s} \rightarrow 0$ ($\nu \rightarrow \nu'$) and thanks to the above energy estimates, one has $|r_{\nu, \nu'}|_{X_{\mathbb{B}o^{-1}}^0 \times (Y_{\mathbb{F}\mu}^0)^*} \leq |r_{\nu, \nu'}|_{L^2} \rightarrow 0$ ($\nu \rightarrow \nu'$). Thus applying Lemma 5.9 to $V_{\nu, \nu'}$, one deduces that \dot{U}_ν is a Cauchy sequence of $C^0([0, T]; X_{\mathbb{B}o^{-1}}^0 \times Y_{\mathbb{F}\mu}^0)$. Therefore there exists a limit, $\dot{U} \in C^0([0, T]; X_{\mathbb{B}o^{-1}}^0 \times Y_{\mathbb{F}\mu}^0)$, which is a solution of the non-regularized (*i.e.* $\nu = 0$) system, namely (5.28).

The above energy estimates on $\partial_x^N \dot{U}_\nu$ being uniform with respect to ν , one has $\dot{U} \in L^\infty([0, T]; H_x^\infty)$. By (5.28), we deduce $\partial_t \dot{U} \in L^\infty([0, T]; H_x^\infty)$, and by induction $\dot{U} \in H^\infty([0, T] \times \mathbb{R})$.

Uniqueness of the solution follows when applying the energy estimate of Lemma 5.9 to the difference between two solutions. \square

We can now conclude this section with the proof of our main result, Theorem 5.1.

Proof of Theorem 5.1. We construct the solution of our problem as the limit of a Picard iteration scheme. We first define Friedrichs mollifiers, $j_\varkappa = \mathbf{1}(|D| \leq \varkappa)$, in order to regularize the initial data:

$$\mathbf{U}_n|_{t=0} = \mathbf{U}_n^0 \stackrel{\text{def}}{=} \{(\partial^\alpha j_{2^n} \zeta^0, \partial^\alpha j_{2^n} w^0)\}_{|\alpha| \leq N}.$$

For each $n \geq 1$, we define, thanks to Lemma 5.12, $\mathbf{U}_n \stackrel{\text{def}}{=} \{(\zeta_n^{(\alpha)}, w_n^{(\alpha)})\}_{|\alpha| \leq N}$ as the unique solution to $\mathbf{U}_n|_{t=0} = \mathbf{U}_n^0$ as well as (5.28), where (using the notations and definitions of Lemma 5.5) $\mathfrak{a} = \mathfrak{a}[\epsilon \zeta_{n-1}, \epsilon w_{n-1}]$ and similarly for $\mathfrak{b}, \mathfrak{c}, r = r^{(\alpha)}$; $\check{\alpha} \check{\zeta} = 0$ if $|\alpha| \leq N - 1$ and $\check{\alpha} \check{\zeta} = \check{\alpha}_\alpha \check{\zeta}_{n-1}^{(\check{\alpha})}$ otherwise. This iteration scheme is initialized with smooth and time-constant $\mathbf{U}_0 = \mathbf{U}_0^0$.

Since we mollified the initial data, Lemma 5.12 defines at each step $\mathbf{U}_n \in C([0, T_n]; H_x^\infty)$, where

$$T_n(h'_0, k'_0, K', M') \stackrel{\text{def}}{=} \max \left\{ T \geq 0, \text{ such that } E^N(\zeta_n, w_n)^{1/2} \leq M' E^N(U^0)^{1/2} \right.$$

$$\left. \text{and } (\zeta_n, w_n) \text{ satisfies (5.5), (5.25), (5.26) with } h'_0, k'_0, K' \right\}.$$

One has $T_n > 0$ as soon as $h'_0 < h_0$, $k'_0 < k_0$, $K' > K$ and $M' > 1$, by standard continuity arguments. We will now prove that T_n can be bounded from below, uniformly with respect to $n \in \mathbb{N}$.

By Lemma 5.9, we have the energy estimate for $U_n^{(\alpha)} \stackrel{\text{def}}{=} (\zeta_n^{(\alpha)}, w_n^{(\alpha)})^\top$ with any $|\alpha| \leq N$:

$$E^0(U_n^{(\alpha)})^{1/2} \leq \mathbf{C}_0 (E^0(U_n^{(\alpha)}|_{t=0})^{1/2} + \mu \epsilon^2 M' E^N(U^0)^{1/2}) e^{\lambda t} + \mathbf{C}_0 \int_0^t f(t') e^{\lambda(t-t')} dt',$$

for any $t \in [0, T_{n-1}(h'_0, k'_0, K', M')]$ and with

$$\lambda = \mathbf{C}_0 \times \left(\epsilon + \Upsilon_{\mathbb{F}} \|w_{n-1}\|_{L^\infty([0, T]; Z_{\mathbb{F}\mu}^2)}^2 \right), \quad f(t) = |r^{(\alpha)}|_{(Y_{\mathbb{F}\mu}^0)^*} + \mu \epsilon^2 M' E^N(U^0)^{1/2},$$

and where $\mathbf{C}_0 = C(\mathbf{m}, (h'_0)^{-1}, (k'_0)^{-1}, K', M')$.

Notice that $\partial^\alpha U_n \neq U_n^{(\alpha)}$ but one can check (differentiating the equations satisfied by U_n) that $\partial^\alpha U_n$ satisfies (5.28) with a remainder term $\tilde{r}^{(\alpha)}[\epsilon \zeta_n, \epsilon w_n, \epsilon \zeta_{n-1}, \epsilon w_{n-1}]$ which is estimated identically as in Lemma 5.5. This yields, for any $t \in [0, \min\{T_{n-1}, T_n\}]$,

$$E^N(U_n)^{1/2} \leq \mathbf{C}_0 e^{\lambda t} E^N(U^0)^{1/2} \left(1 + \mathbf{C}'_0 M' t \times (\epsilon + \Upsilon_{\mathbb{F}}^{1/2} |w_{n-1}|_{Z_{\mathbb{F}\mu}^1} + \Upsilon_{\mathbb{F}} |w_{n-1}|_{Z_{\mathbb{F}\mu}^2}^2) \right),$$

with λ, \mathbf{C}_0 as above, and $\mathbf{C}'_0 = C(\mathbf{m}, (h'_0)^{-1}, M', E^N(U^0))$.

This allows to define $M^*, \frac{1}{T^*} = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, E^N(U^0))$, independent of n , such that

$$(\epsilon + \Upsilon_{\mathbb{F}}^{1/2} |w_{n-1}|_{Z_{\mathbb{F}\mu}^1} + \Upsilon_{\mathbb{F}} |w_{n-1}|_{Z_{\mathbb{F}\mu}^2}^2) t \leq \min\{T^*, T_{n-1}, T_n\} \implies E^N(U_n)^{1/2} \leq M^* E^N(U^0)^{1/2}.$$

In particular, one has on the above defined time interval, $E^{N-1}(\partial_t U_n)^{1/2} \leq M^* E^N(U^0)^{1/2}$, and therefore

$$|\epsilon \zeta_n - \epsilon \zeta_n|_{t=0}|_{L^\infty}(t) \leq \epsilon \int_0^t |\partial_t \zeta|_{L^\infty} \leq M^* E^N(U^0)^{1/2} \times (\epsilon t).$$

Thus, restricting T^* if necessary, (5.5) is uniformly satisfied with $h'_0 = h_0/2 > 0$. Similarly, one guarantees that (5.25), (5.26) hold with $k'_0 = k_0/2$ and $K' = 2K$, and $|w_{n-1}|_{Z_{\mathbb{F}\mu}^2} \leq 2|w^0|_{Z_{\mathbb{F}\mu}^2}$.

Altogether, this proves that there exists $M^*, \frac{1}{T^*}, \mathbf{C}_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, E^N(U^0))$ such that

$$T_n(h_0/2, k_0/2, 2K, M^*) \geq T^*/\lambda', \quad \lambda' \stackrel{\text{def}}{=} \epsilon + \Upsilon_{\mathbb{F}}^{1/2} |w^0|_{Z_{\mathbb{F}\mu}^1} + \Upsilon_{\mathbb{F}} |w^0|_{Z_{\mathbb{F}\mu}^2}^2,$$

uniformly with respect to $n \in \mathbb{N}$; and that for any $t \in [0, T^*/\lambda']$, one has

$$(5.41) \quad E^N(U_n)^{1/2} \leq M^* E^N(U^0)^{1/2}.$$

Let us now consider $V_n = U_n - U_{n-1}$. Notice first that

$$E^j(V_n|_{t=0}) = E^j((U_n - U_{n-1})|_{t=0}) \lesssim 2^{-2n(N-j)} E^N(U^0).$$

One can control $E^0(V_n)$ from Lemma 5.11, using the above, the estimate on $r_n^{(\alpha)} - r_{n-1}^{(\alpha)}$ given by Lemma 5.5 as well as the energy estimate (5.41). Similar estimates on $\partial^\alpha V_n$ for $0 \leq |\alpha| \leq 2$ yield

$$E^2(V_n)^{1/2} \leq \mathbf{C}_0 2^{-n(N-2)} e^{\lambda' t} + \mathbf{C}_0 \lambda' \int_0^t E^2(V_{n-1})^{1/2} e^{\lambda'(t-t')} dt',$$

with \mathbf{C}_0, λ' as above. Therefore, restricting $T^\sharp \leq T^*$ if necessary, the sequence $U_n = U^0 + \sum_{j=1}^n V_j$ converges in $C^0([0, T^\sharp/\lambda']; X_{\mathbb{B}o^{-1}}^2 \times Y_{\mathbb{F}\mu}^2)$.

Using that U_n is uniformly bounded in $C^0([0, T^\sharp/\lambda']; X_{\mathbb{B}o^{-1}}^N \times Y_{\mathbb{F}\mu}^N)$, standard interpolation arguments yields that U_n converges strongly in $C^0([0, T^\sharp/\lambda']; X_{\mathbb{B}o^{-1}}^{N-1} \times Y_{\mathbb{F}\mu}^{N-1})$. The limit $U = \lim_{n \rightarrow \infty} U_n$

belongs to $L^\infty([0, T^\sharp/\lambda']; X_{\text{Bo}^{-1}}^N \times Y_{\text{F}^\mu}^N) \cap C^0([0, T^\sharp/\lambda']; X_{\text{Bo}^{-1}}^{N-1} \times Y_{\text{F}^\mu}^{N-1})$ and then by classical argument belong to $C_w^0([0, T^\sharp/\lambda']; X_{\text{Bo}^{-1}}^N \times Y_{\text{F}^\mu}^N)$. It is now straightforward to check that U satisfies system (5.28), and therefore (by Lemma 5.5) (5.1).

By passing to the limit the energy estimate (5.41), one deduces the energy estimate of the statement. The uniqueness of the solution is a consequence of Lemma 5.11, applied to the difference between two solutions (see also Proposition 6.2, below). Theorem 5.1 is proved. \square

6 Full justification of our models

We show in this section how the above well-posedness analysis can be supplemented with consistency and stability results, which together provide the full justification of our models, (5.1). We recall that $0 \leq \gamma, \mu, \epsilon, \delta, \delta^{-1}, \text{Bo}^{-1} < \infty$ and that F_i is admissible, in the sense of Definition 1.1.

Proposition 6.1 (Consistency). *Let $U \stackrel{\text{def}}{=} (\zeta, \psi)^\top$ be a solution of the full Euler system (1.2) such that such that there exists $C_0, T > 0$ with*

$$\|\zeta\|_{L^\infty([0, T]; H_x^{s+5})} + \|\partial_t \zeta\|_{L^\infty([0, T]; H_x^{s+4})} + \|\partial_x \psi\|_{L^\infty([0, T]; H_x^{s+\frac{11}{2}})} + \|\partial_t \partial_x \psi\|_{L^\infty([0, T]; H_x^{s+\frac{9}{2}})} \leq C_0,$$

for given $s \geq t_0 + 1/2$, $t_0 > 1/2$. Moreover, assume that there exists $h_0 > 0$ such that (5.5) holds.

Define w by $\partial_x w = -\frac{1}{\mu} G^\mu[\epsilon\zeta]\psi = -\partial_t \zeta$. Then $(\zeta, w)^\top$ satisfies (by definition) the first equation of (5.1), and the second up to a remainder, r , bounded as

$$\|r\|_{L^\infty([0, T]; H_x^s)} \leq C_1 \mu^2,$$

with $C_1 = C(\mathbf{m}, h_0^{-1}, C_0, C_F)$.

Proof. The Proposition has been stated and proved, in the case of the original Green-Naghdi system, $\text{F}_i^{\text{id}} \equiv 1$, in [15, Proposition 2.4]. By triangular inequality, there only remains to estimate

$$r_{\mathcal{Q}} \stackrel{\text{def}}{=} \mu |\partial_t (\mathcal{Q}^{\text{Fid}}[\epsilon\zeta]w - \mathcal{Q}^{\text{F}}[\epsilon\zeta]w)|_{H_x^s}$$

and

$$r_{\mathcal{R}} \stackrel{\text{def}}{=} \mu \epsilon |\partial_x (\mathcal{R}^{\text{Fid}}[\epsilon\zeta, w] - \mathcal{R}^{\text{F}}[\epsilon\zeta, w])|_{H_x^s}.$$

We first show that $w \in L^\infty([0, T]; H_x^{s+5})$. By definition, $\partial_x w = -\partial_t \zeta \in L^\infty([0, T]; H_x^{s+4})$, and $w \in L^\infty([0, T]; L^2)$ is a consequence of the identity (1.3) and the uniform control of \bar{u}_i ; see *e.g.* [17, Proposition 4]. Similarly, one has $\partial_t w \in L^\infty([0, T]; H_x^{s+4})$.

Since by Definition 1.1, F_i'' is uniformly bounded, $\text{F}_i(0) = 1$ and $\text{F}_i'(0) = 0$; thus one has

$$|\text{F}_i(\sqrt{\mu}k) - 1| \leq \frac{1}{2} C_F \mu |k|^2.$$

The remainders $r_{\mathcal{Q}}$ and $r_{\mathcal{R}}$ are now treated as follows. One has

$$r_{\mathcal{Q}} \leq \sum_{i=1}^2 \frac{\mu}{3} |\partial_t (h_i^{-1} \partial_x (\text{F}_i^\mu - \text{Id}) \{h_i^3 \partial_x \text{F}_i^\mu \{h_i^{-1} w\}\})|_{H_x^s} + \frac{\mu}{3} |\partial_t (h_i^{-1} \partial_x \{h_i^3 \partial_x (\text{F}_i^\mu - \text{Id}) \{h_i^{-1} w\}\})|_{H_x^s}.$$

Since H_x^s is an algebra for $s > 1/2$ and by Lemma C.3, one has immediately

$$r_{\mathcal{Q}} \leq \mu^2 C_F C(\mathbf{m}, h_0^{-1}, |\partial_t \zeta|_{L^\infty([0, T]; H_x^{s+4})}, |\partial_t w|_{L^\infty([0, T]; H_x^{s+4})}).$$

Similarly, one can check

$$r_{\mathcal{R}} \leq \mu^2 C_F C(\mathbf{m}, h_0^{-1}, |\zeta|_{L^\infty([0, T]; H_x^{s+5})}, |w|_{L^\infty([0, T]; H_x^{s+5})}),$$

and the Proposition is proved. \square

Proposition 6.2 (Stability). *Let $N \geq 4$ and $U_i = (\zeta_i, w_i)^\top \in L^\infty([0, T]; X_{\text{Bo}^{-1}}^N \times Y_{\text{F}^\mu}^N)$ solution to (5.1) with remainder terms $(0, r_i)^\top$. Assume ζ_i satisfies (5.5), (5.25), (5.26) with $h_0, k_0, K > 0$. Set $2 \leq n \leq N - 1$ and assume that $\partial^\alpha r_i \in L^1([0, T]; (Y_{\text{F}^\mu}^0)^*)$ for any $|\alpha| \leq n$.*

Then there exists $0 < T^ \leq T$ such that for all $t \in [0, T^*]$*

$$E^n(U_1 - U_2)^{1/2} \leq \mathbf{C}_0 E^n(U_1|_{t=0} - U_2|_{t=0})^{1/2} e^{\lambda t} + \mathbf{C}_0 \int_0^t e^{\epsilon \lambda(t-t')} f_n(t') dt'.$$

with

$$\lambda = \mathbf{C}_0 \times \left(\epsilon + \Upsilon_{\text{F}} \|w_1\|_{L^\infty([0, T]; Z_{\text{F}^\mu}^2)}^2 \right), \quad f_n(t) = \sum_{|\alpha| \leq n} |\partial^\alpha r_1 - \partial^\alpha r_2|_{(Y_{\text{F}^\mu}^0)^*},$$

and $\mathbf{C}_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, K, \|U_1\|_{L^\infty([0, T]; X_{\text{Bo}^{-1}}^4 \times Y_{\text{F}^\mu}^4)}, \|U_2\|_{L^\infty([0, T]; X_{\text{Bo}^{-1}}^4 \times Y_{\text{F}^\mu}^4)})$. Moreover, one has

$$(T^*)^{-1} \leq \mathbf{C}_0 (\epsilon + \Upsilon_{\text{F}}^{1/2} \|w_1\|_{L^\infty([0, T]; Z_{\text{F}^\mu}^1)}^2 + \Upsilon_{\text{F}} \|w_1\|_{L^\infty([0, T]; Z_{\text{F}^\mu}^2)}^2).$$

Proof. By Lemma 5.5, for any $|\alpha| \leq n \leq N - 1$, $U_i^{(\alpha)} \stackrel{\text{def}}{=} (\partial^\alpha \zeta_i, \partial^\alpha w_i)^\top$ satisfies (5.28) with remainder terms $\tilde{r}_i^{(\alpha)} \stackrel{\text{def}}{=} r_i^{(\alpha)} + \partial^\alpha r_i \in L^1([0, T]; (Y_{\text{F}^\mu}^0)^*)$, and

$$|\tilde{r}_1^{(\alpha)} - \tilde{r}_2^{(\alpha)}|_{(Y_{\text{F}^\mu}^0)^*} \leq |\partial^\alpha r_1 - \partial^\alpha r_2|_{(Y_{\text{F}^\mu}^0)^*} + C_0 \times (\epsilon + \Upsilon_{\text{F}}^{1/2} |w_1|_{Z_{\text{F}^\mu}^1} + \Upsilon_{\text{F}} |w_1|_{Z_{\text{F}^\mu}^1}^2) \times E^{|\alpha|} (U_1 - U_2)^{1/2},$$

with $C_0 = C(\mathbf{m}, h_0^{-1}, E^N(U_1), E^N(U_2))$. By Lemma 5.11, one has

$$E^0(U_1^{(\alpha)} - U_2^{(\alpha)})^{1/2} \leq \mathbf{C}_0 E^0(U_1^{(\alpha)}|_{t=0} - U_2^{(\alpha)}|_{t=0})^{1/2} e^{\lambda t} + \mathbf{C}_0 \int_0^t f^{(\alpha)}(t') e^{\lambda(t-t')} dt',$$

with λ and \mathbf{C}_0, λ as in the statement and

$$f^{(\alpha)}(t) = |\tilde{r}_1^{(\alpha)} - \tilde{r}_2^{(\alpha)}|_{(Y_{\text{F}^\mu}^0)^*} + \epsilon |U_2|_{(W_x^{3,\infty})_2} |U_1 - U_2|_{X_{\text{Bo}^{-1}}^2 \times Y_{\text{F}^\mu}^2}.$$

Since $n \geq 2$ and $N \geq 4$, one can restrict T^* as in the statement and augment \mathbf{C}_0 if necessary so that the estimate holds. \square

The following Proposition is now a straightforward consequence of Theorem 5.1 and Propositions 6.1 and 6.2.

Proposition 6.3 (Full justification). *Let $U^0 \equiv (\zeta^0, w^0)^\top \in X_{\text{Bo}^{-1}}^N \times Y_{\text{F}^\mu}^N$ with N sufficiently large, and satisfying (5.5), (5.25), (5.26). Define ψ^0 with $\partial_x w^0 = -\frac{1}{\mu} G^\mu[\epsilon \zeta^0] \psi^0$ and assume that $(\zeta^0, \psi^0)^\top$ satisfies the hypotheses of Theorem 5 in [25]. Then there exists $C, T > 0$ such that*

- *There exists a unique solution $U \equiv (\zeta, \psi)^\top$ to the full Euler system (1.2), defined on $[0, T]$ and with initial data $(\zeta^0, \psi^0)^\top$ (provided by Theorem 5 in [25]);*
- *There exists a unique solution $U_{\text{F}} \equiv (\zeta_{\text{F}}, w_{\text{F}})^\top$ to our modified Green-Naghdi model (5.1), defined on $[0, T]$ and with initial data $(\zeta^0, w^0)^\top$ (provided by Theorem 5.1);*
- *Defining $\partial_x w = -\frac{1}{\mu} G^\mu[\epsilon \zeta] \psi = -\partial_t \zeta$, one has for any $t \in [0, T]$,*

$$\|(\zeta, w) - (\zeta_{\text{F}}, w_{\text{F}})\|_{L^\infty([0, t]; X_{\text{Bo}^{-1}}^0 \times Y_{\text{F}^\mu}^0)} \leq C \mu^2 t.$$

A The Saint-Venant system

The Saint-Venant system (with surface tension) is obtained from our Green-Naghdi models (5.1) by setting $\mu = 0$. The results of Sections 5 thus apply as a particular case. However, it is possible to obtain sharper results by considering the system obtained after the following change of variable $\bar{v} \stackrel{\text{def}}{=} \frac{h_1 + \gamma h_2}{h_1 h_2} w = \bar{u}_2 - \gamma \bar{u}_1$:

$$(A.1) \quad \begin{cases} \partial_t \zeta + \partial_x (H(\epsilon \zeta) \bar{v}) = 0, \\ \partial_t \bar{v} + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x (H'(\epsilon \zeta) |\bar{v}|^2) - \frac{\gamma + \delta}{\text{Bo}} \partial_x^3 \zeta = 0, \end{cases}$$

where we denote $H(\epsilon \zeta) = \frac{h_1 h_2}{h_1 + \gamma h_2}$.

In the following, we quickly review the steps of the method developed in Section 5, pointing out the differences, and providing results without proof.

The analogue of Lemma 5.5 is the following:

Lemma A.1. *Let $U = (\zeta, \bar{v})^\top \in X_{\text{Bo}^{-1}}^N \times H^N$ with $N \geq 2$, solution to (A.1) and satisfying*

$$(A.2) \quad h_1(\epsilon \zeta) = 1 - \epsilon \zeta \geq h_0 > 0, \quad h_2(\epsilon \zeta) = \delta^{-1} + \epsilon \zeta \geq h_0 > 0.$$

For any $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| \leq N$, denote $U^{(\alpha)} \stackrel{\text{def}}{=} (\partial^\alpha \zeta, \partial^\alpha \bar{v})^\top$ and $\bar{v}^{(\check{\alpha})} \stackrel{\text{def}}{=} (\partial^{\alpha - e_1} \bar{v}, \partial^{\alpha - e_2} \bar{v})^\top$ (if $\alpha_j = 0$, then $\partial^{\alpha - e_j} \bar{v} = 0$ by convention). Then $U^{(\alpha)}$ satisfies:

$$\begin{cases} \partial_t \zeta^{(\alpha)} + \partial_x (H(\epsilon \zeta) \bar{v}^{(\alpha)}) + \epsilon H'(\epsilon \zeta) \bar{v} \partial_x \zeta^{(\alpha)} + \partial_x \check{\mathfrak{d}}_\alpha [\epsilon \zeta] \bar{v}^{(\check{\alpha})} = r_1^{(\alpha)}, \\ \partial_t \bar{v}^{(\alpha)} + \partial_x \mathfrak{a}_{\text{SV}}[\epsilon \zeta, \bar{v}] \zeta^{(\alpha)} + \epsilon H'(\epsilon \zeta) \bar{v} \partial_x \bar{v}^{(\alpha)} = r_2^{(\alpha)}, \end{cases}$$

with $\check{\mathfrak{d}}_\alpha [\epsilon \zeta] \bar{v}^{(\check{\alpha})} \stackrel{\text{def}}{=} \sum_{j \in \{1, 2\}} \alpha_j H'(\epsilon \zeta) (\epsilon \partial^{e_j} \zeta) (\partial^{\alpha - e_j} \bar{v})$ and

$$\mathfrak{a}_{\text{SV}}[\epsilon \zeta, \bar{v}] \bullet \stackrel{\text{def}}{=} \left((\gamma + \delta) + \frac{\epsilon^2}{2} H''(\epsilon \zeta) |\bar{v}|^2 \right) \bullet - \frac{\gamma + \delta}{\text{Bo}} \partial_x^2 \bullet,$$

and $r^{(\alpha)}[\epsilon \zeta, \bar{v}] = (r_1^{(\alpha)}[\epsilon \zeta, \bar{v}], r_2^{(\alpha)}[\epsilon \zeta, \bar{v}])^\top \in X_{\text{Bo}^{-1}}^0 \times L^2$ satisfies

$$\|r^{(\alpha)}[\epsilon \zeta, \bar{v}]\|_{X_{\text{Bo}^{-1}}^0 \times L^2} \leq \epsilon C(\mathfrak{m}, h_0^{-1}, |\zeta|_{X_{\text{Bo}^{-1}}^N}, |\bar{v}|_{H^N}) \times (|\zeta|_{X_{\text{Bo}^{-1}}^N} + |\bar{v}|_{H^N}).$$

Notice that we have to keep a second-order term on the first equation, namely $\partial_x \check{\mathfrak{d}}_\alpha [\epsilon \zeta] \bar{v}^{(\check{\alpha})}$. This is due to the fact that we have to control the remainder term $r_1^{(\alpha)}[\epsilon \zeta, \bar{v}] \in X_{\text{Bo}^{-1}}^0$, so as to be able to bootstrap the energy estimates in the appropriate space: $E^N(\zeta, \bar{v}) = |\zeta|_{X_{\text{Bo}^{-1}}^N}^2 + |\bar{v}|_{H^N}^2$.

One has immediately the following analogue of Lemma 5.6.

Lemma A.2. *Let $(\zeta, \bar{v})^\top \in L^\infty \times L^\infty$ be such that $\epsilon \zeta$ satisfies (A.2) with $h_0 > 0$, and*

$$(A.3) \quad (\gamma + \delta) + \frac{\epsilon^2}{2} H''(\epsilon \zeta) |\bar{v}|^2 = (\gamma + \delta) - \gamma \epsilon^2 \frac{(h_1 + h_2)^2}{(h_1 + \gamma h_2)^3} |\bar{v}|^2 \geq k_0 > 0.$$

Then there exists $K_0, K_1 = C(\mathfrak{m}, h_0^{-1}, k_0^{-1}, \epsilon |\zeta|_{L^\infty})$ such that

$$\begin{aligned} \forall f, g \in X_{\text{Bo}^{-1}}^0, & \quad |\langle \mathfrak{a}_{\text{SV}}[\epsilon \zeta, \bar{v}] f, g \rangle_{(X_{\text{Bo}^{-1}}^0)^*}| \leq K_1 |f|_{X_{\text{Bo}^{-1}}^0} |g|_{X_{\text{Bo}^{-1}}^0}, \\ \forall f, g \in L^2, & \quad |(H(\epsilon \zeta) f, g)_{L^2}| \leq K_1 |f|_{L^2} |g|_{L^2}, \\ \forall f \in X_{\text{Bo}^{-1}}^0, & \quad \langle \mathfrak{a}_{\text{SV}}[\epsilon \zeta, \bar{v}] f, f \rangle_{(X_{\text{Bo}^{-1}}^0)^*} \geq \frac{1}{K_0} |f|_{X_{\text{Bo}^{-1}}^0}^2, \\ \forall f \in L^2, & \quad (H(\epsilon \zeta) f, f)_{L^2} \geq \frac{1}{K_0} |f|_{L^2}^2. \end{aligned}$$

A priori energy estimates are obtained by adding the L^2 inner product of the first equation with $\mathbf{a}_{\text{SV}}\zeta^{(\alpha)}$, and the one of second one with $H(\epsilon\zeta)\bar{v}^{(\alpha)} + \check{\delta}_\alpha[\epsilon\zeta]\bar{v}^{(\check{\alpha})}$, and following the proof of Lemmata 5.9 and 5.11.

Applying the strategy of Section 5.4, one then obtains the following analogue of Theorem 5.1.

Theorem A.3. *Let $N \geq 2$ and $U^0 \stackrel{\text{def}}{=} }(\zeta^0, \bar{v}^0)^\top \in X_{\text{Bo}^{-1}}^N \times H^N$, satisfying (A.2),(A.3) with $h_0, k_0 > 0$. Then there exists $T > 0$ and a unique solution $U \stackrel{\text{def}}{=} }(\zeta, \bar{v})^\top \in C_w^0([0, T]; X_{\text{Bo}^{-1}}^N \times H^N)$ satisfying (A.1). Moreover, there exists $\mathbf{C}_0 = C(\mathbf{m}, h_0^{-1}, k_0^{-1}, |U^0|_{X_{\text{Bo}^{-1}}^N \times H^N})$ such that*

$$T^{-1} \leq \mathbf{C}_0 \times \epsilon \quad \text{and} \quad \sup_{t \in [0, T]} \left(|\zeta|_{X_{\text{Bo}^{-1}}^N} + |\bar{v}|_{H^N} \right) \leq \mathbf{C}_0 \left(|\zeta^0|_{X_{\text{Bo}^{-1}}^N} + |\bar{v}^0|_{H^N} \right).$$

The following remarks indicate that the above result is sharp in many ways, in contrast with the discussion of Remark 5.3.

Remark A.4. *Theorem A.3 is valid uniformly with respect to the parameter Bo^{-1} , and the result holds in particular in the case without surface tension: $\text{Bo}^{-1} = 0$. This case is however straightforward as the Saint-Venant system is then a quasilinear system, and the result was stated in particular in [21]. (A.3) corresponds exactly to the hyperbolicity condition provided therein.*

Remark A.5. *Setting $\zeta = 0$ in (A.3) yields*

$$(\gamma + \delta) - \gamma \epsilon^2 \frac{\delta(\delta + 1)^2}{(\delta + \gamma)^3} |\bar{v}|^2 > 0.$$

One thus recovers the necessary and sufficient condition for stability of all models provided in Section 3 when setting $\mu = 0$ (recall $w = \frac{h_1 h_2}{h_1 + \gamma h_2} \bar{v}$).

Remark A.6. *In the limit $\gamma \rightarrow 0$, notice that (A.3) is automatically satisfied: Kelvin-Helmholtz instabilities disappear in the water-wave case, and (A.2) suffices to ensure the stability of the flow.*

B Notations and functional setting

The notation $a \lesssim b$ means that $a \leq C_0 b$, where C_0 is a nonnegative constant whose exact expression is of no importance. We denote by $C(\lambda_1, \lambda_2, \dots)$ a nonnegative constant depending on the parameters $\lambda_1, \lambda_2, \dots$ and whose dependence on the λ_j is always assumed to be nondecreasing.

In this paper, we sometimes work with norms involving derivatives in both space and time variables. We find it convenient to use the following sometimes non-standard notations.

- For $1 \leq p < \infty$, we denote $L_x^p = L^p = L^p(\mathbb{R})$ the standard Lebesgue spaces associated with the norm

$$|f|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

The real inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1, f_2)_{L^2} = \int_{\mathbb{R}} f_1(x) f_2(x) dx.$$

The space $L^\infty = L_x^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm

$$|f|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}} |f(x)| < \infty.$$

- We acknowledge the fact that only space derivatives are involved by the use of a subscript. For $k \in \mathbb{N}$, we denote by $W_x^{k,\infty}(\mathbb{R}) = \{f, \text{ s.t. } \forall 0 \leq j \leq k, \partial_x^j f \in L^\infty(\mathbb{R})\}$ endowed with its canonical norm.

For any real constant $s \in \mathbb{R}$, $H_x^s = H_x^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H_x^s} = |\Lambda^s f|_{L^2} < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1 - \partial_x^2)^{1/2}$. We denote $H_x^\infty = \bigcap_{N \in \mathbb{N}} H_x^N$.

- In absence of subscript, the derivatives are with respect to space and time, and thus apply to functions defined on $(t, x) \in [0, T) \times \mathbb{R}$. Thus for $N \in \mathbb{N}$, $W^{N,\infty}$ is the space of functions endowed with the following norm:

$$|f|_{W^{N,\infty}} = \sum_{|\alpha| \leq N} |\partial^\alpha f|_{L_x^\infty},$$

where we use the standard multi-index notation: $\alpha \in \mathbb{N}^2$, $\partial^{(\alpha_1, \alpha_2)} = \partial_t^{\alpha_1} \partial_x^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. In particular, $\partial^{\mathbf{e}_1} = \partial^{(1,0)} = \partial_t$ and $\partial^{\mathbf{e}_2} = \partial^{(0,1)} = \partial_x$.

Similarly, H^N is the space of functions endowed with

$$|f|_{H^N}^2 = \sum_{|\alpha| \leq N} |\partial^\alpha f|_{L_x^2}^2.$$

We denote $H^\infty = \bigcap_{N \in \mathbb{N}} H^N$.

- Given $\mu, \gamma, \text{Bo}^{-1} \geq 0$ and \mathbf{F}_i ($i = 1, 2$) admissible functions (in the sense of Definition 1.1), we define $X_{\text{Bo}^{-1}}^0$, $Y_{\mathbf{F}^\mu}^0$, $W_{\text{Bo}^{-1}}^0$, $Z_{\mathbf{F}^\mu}^0$ as the completion of the Schwartz space, $\mathcal{S}(\mathbb{R})$, for the following norms:

$$\begin{aligned} |f|_{X_{\text{Bo}^{-1}}^0}^2 &\stackrel{\text{def}}{=} |f|_{L^2}^2 + \frac{1}{\text{Bo}} |\partial_x f|_{L^2}^2, \\ |f|_{Y_{\mathbf{F}^\mu}^0}^2 &\stackrel{\text{def}}{=} |f|_{L^2}^2 + \mu\gamma |\partial_x \mathbf{F}_1^\mu f|_{L^2}^2 + \mu |\partial_x \mathbf{F}_2^\mu f|_{L^2}^2, \\ |f|_{W_{\text{Bo}^{-1}}^0} &\stackrel{\text{def}}{=} |\widehat{f}|_{L^1} + \frac{1}{\text{Bo}} |\widehat{\partial_x f}|_{L^1}, \\ |f|_{Z_{\mathbf{F}^\mu}^0} &\stackrel{\text{def}}{=} |\widehat{f}|_{L^1} + \sqrt{\mu\gamma} |\widehat{\partial_x \mathbf{F}_1^\mu f}|_{L^1} + \sqrt{\mu} |\widehat{\partial_x \mathbf{F}_2^\mu f}|_{L^1}. \end{aligned}$$

For $N \in \mathbb{N}$ we define consistently with above the norms controlling space and time derivatives:

$$\begin{aligned} |f|_{X_{\text{Bo}^{-1}}^N}^2 &\stackrel{\text{def}}{=} \sum_{|\alpha| \leq N} |\partial^\alpha f|_{X_{\text{Bo}^{-1}}^0}^2, & |f|_{Y_{\mathbf{F}^\mu}^N}^2 &\stackrel{\text{def}}{=} \sum_{|\alpha| \leq N} |\partial^\alpha f|_{Y_{\mathbf{F}^\mu}^0}^2, \\ |f|_{W_{\text{Bo}^{-1}}^N} &\stackrel{\text{def}}{=} \sum_{|\alpha| \leq N} |\partial^\alpha f|_{W_{\text{Bo}^{-1}}^0}, & |f|_{Z_{\mathbf{F}^\mu}^N} &\stackrel{\text{def}}{=} \sum_{|\alpha| \leq N} |\partial^\alpha f|_{Z_{\mathbf{F}^\mu}^0}. \end{aligned}$$

- Denoting X any of the previously defined functional spaces, we denote by X^* its topological dual, endowed with the norm $|\varphi|_{X^*} = \sup_{|f|_X \leq 1} |\varphi(f)|$; and by $\langle \cdot, \cdot \rangle_{(X)^*}$ the $(X^* - X)$ duality brackets.
- For any function $u = u(t, x)$ defined on $[0, T) \times \mathbb{R}$ with $T > 0$, and any of the previously defined functional spaces, X , we denote $L^\infty([0, T); X)$ the space of functions such that $u(t, \cdot)$ is controlled in X , uniformly for $t \in [0, T)$, and use double bar symbol for the associated norm:

$$\|u\|_{L^\infty([0, T); X)} = \text{ess sup}_{t \in [0, T)} |u(t, \cdot)|_X < \infty.$$

For $k \in \mathbb{N}$, $C^k([0, T); X)$ denotes the space of X -valued continuous functions on $[0, T)$ with continuous derivatives up to the order k . Finally, $C_w^0([0, T); X)$ is the space of continuous functions with values in X , given the weak topology.

C Functional analysis

In this section, we provide tools (injections and product estimates) similar to the classical ones concerning Sobolev spaces, for the functional spaces $X_{\text{Bo}^{-1}}^N, Y_{F_i}^N, Z_{F_i}^N$, as defined in Section B.

Let us fix $\mu, \gamma, \text{Bo}^{-1} \geq 0$ and F_i ($i = 1, 2$) admissible functions (in the sense of Definition 1.1). In particular there exists $K_{F_i} > 0$ and $\sigma \in [0, 1]$ such that

$$(C.1) \quad |F_i(\xi)|^2 \leq \min \{1, K_{F_i} |\xi|^{-2\sigma}\}.$$

The following standard injections, which will be frequently used, sometimes without notice:

$$(C.2) \quad |f|_{L^\infty} \leq |\widehat{f}|_{L^1} \lesssim |f|_{H_x^{t_0}} \quad (t_0 > 1/2); \quad \text{thus} \quad |f|_{Z_{F_i}^N} \lesssim |f|_{Y_{F_i}^{N+1}} \quad \text{and} \quad |f|_{Z_{F_i}^N} \lesssim |f|_{H^{N+t_0+1}}.$$

One immediately sees that the space $X_{\text{Bo}^{-1}}^0$ is continuously embedded in $Y_{F_i}^0$; the following Lemma precises the norm of the inclusion map.

Lemma C.1. *If F_i satisfies (C.1), then*

$$(C.3) \quad \forall f \in X_{\text{Bo}^{-1}}^0(\mathbb{R}), \quad |f|_{Y_{F_i}^0}^2 \leq (1 + (\gamma K_{F_1} + K_{F_2})(\mu \text{Bo})^{1-\sigma}) |f|_{X_{\text{Bo}^{-1}}^0}^2.$$

Proof. The inequality is a simple consequence of Parseval's identity and Young's inequality:

$$\mu |\partial_x F_i(\sqrt{\mu}D)f|_{L^2}^2 \leq K_{F_i} \mu^{1-\sigma} \int |\xi|^{2-2\sigma} |\widehat{f}(\xi)|^2 d\xi \leq K_{F_i} (\mu \text{Bo})^{1-\sigma} \int (1 + \frac{1}{\text{Bo}} |\xi|^2) |\widehat{f}(\xi)|^2 d\xi,$$

where we used $\text{Bo}^\sigma |\xi|^{2-2\sigma} \leq \sigma \text{Bo} + (1 - \sigma) |\xi|^2$. \square

Sobolev spaces H_x^N and $W_x^{N,\infty}$ enjoy straightforward product estimates, which are immediately extended to $X_{\text{Bo}^{-1}}^N$ and $W_{\text{Bo}^{-1}}^N$:

$$(C.4) \quad |fg|_{X_{\text{Bo}^{-1}}^0} \lesssim |g|_{W_{\text{Bo}^{-1}}^0} |f|_{X_{\text{Bo}^{-1}}^0} \lesssim |g|_{X_{\text{Bo}^{-1}}^1} |f|_{X_{\text{Bo}^{-1}}^0}, \quad |fg|_{W_{\text{Bo}^{-1}}^0} \lesssim |f|_{W_{\text{Bo}^{-1}}^0} |g|_{W_{\text{Bo}^{-1}}^0};$$

and therefore

$$\forall N \geq 1, \quad |fg|_{X_{\text{Bo}^{-1}}^N} \lesssim |f|_{X_{\text{Bo}^{-1}}^N} |g|_{X_{\text{Bo}^{-1}}^N}.$$

Spaces $Y_{F_i}^N$ and $Z_{F_i}^N$ enjoys similar estimates, thanks to the sub-additivity property of admissible functions (recall Definition 1.1).

Lemma C.2. *Let $F_i : \mathbb{R} \rightarrow \mathbb{R}^+$ ($i = 1, 2$) be admissible functions. Then for any $1 \leq p, q, \tilde{p}, \tilde{q}, r \leq \infty$ satisfying $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}}$, one has*

$$(C.5) \quad |\partial_x \widehat{F_i^\mu \{fg\}}|_{L^r} \leq |\widehat{f}|_{L^p} |\widehat{\partial_x F_i^\mu g}|_{L^q} + |\widehat{g}|_{L^{\tilde{p}}} |\widehat{\partial_x F_i^\mu f}|_{L^{\tilde{q}}}.$$

It follows in particular:

$$(C.6) \quad |fg|_{Y_{F_i}^0} \lesssim |g|_{Z_{F_i}^0} |f|_{Y_{F_i}^0} \lesssim |g|_{Y_{F_i}^1} |f|_{Y_{F_i}^0},$$

$$(C.7) \quad \forall N \geq 1, \quad |fg|_{Y_{F_i}^N} \lesssim |f|_{Y_{F_i}^N} |g|_{Y_{F_i}^N},$$

$$(C.8) \quad |fg|_{Z_{F_i}^0} \lesssim |f|_{Z_{F_i}^0} |g|_{Z_{F_i}^0}.$$

Proof. From the sub-additivity, one has $\sqrt{\mu}|\xi|F_i(\sqrt{\mu}\xi) \leq \sqrt{\mu}|\eta|F_i(\sqrt{\mu}\eta) + \sqrt{\mu}|\xi - \eta|F_i(\sqrt{\mu}(\xi - \eta))$. Thus

$$\begin{aligned} |\partial_x \widehat{F_i^\mu \{fg\}}|_{L^r}^r &= \int_{\mathbb{R}} (|\xi|F_i(\sqrt{\mu}\xi))^r |\widehat{f} \star \widehat{g}|^r(\xi) d\xi = \int_{\mathbb{R}} d\xi \left| \int_{\mathbb{R}} d\eta |\xi|F_i(\sqrt{\mu}\xi) \widehat{f}(\eta) \overline{\widehat{g}(\xi - \eta)} \right|^r \\ &\leq \int d\xi \left| \int d\eta |\eta|F_i(\sqrt{\mu}\eta) |\widehat{f}|(\eta) |\widehat{g}|(\xi - \eta) + |\xi - \eta|F_i(\sqrt{\mu}(\xi - \eta)) |\widehat{f}|(\eta) |\widehat{g}|(\xi - \eta) \right|^r \\ &\leq \int d\xi \left| (|\partial_x \widehat{F_i^\mu f}| \star |\widehat{g}|)(\xi) + (|\widehat{f}| \star |\partial_x \widehat{F_i^\mu g}|)(\xi) \right|^r, \end{aligned}$$

where we used that $|\partial_x \widehat{F_i^\mu f}|(\xi) = |i\xi F_i(\sqrt{\mu}\xi) \widehat{f}|(\xi) = |\xi|F_i(\sqrt{\mu}\xi) |\widehat{f}|(\xi)$ since $F_i(\sqrt{\mu}\xi) \geq 0$. Estimate (C.5) follows from Young's inequality for convolutions.

Estimate (C.6) is deduced with $r = p = \tilde{q} = 2$ and $\tilde{p} = q = 1$, and using (C.2).

Estimate (C.7) follows from the above result and triangular inequality,

$$|fg|_{Y_{F^\mu}^N} \leq \sum_{|\alpha+\beta| \leq N} C_{\alpha,\beta,N} |(\partial^\alpha f)(\partial^\beta g)|_{Y_{F^\mu}^0} \lesssim \sum_{|\alpha| \leq N-1, |\beta| \leq N} |\partial^\alpha f|_{Y_{F^\mu}^1} |\partial^\beta f|_{Y_{F^\mu}^0}.$$

Estimate (C.8) follows from (C.5) with $p = \tilde{p} = q = \tilde{q} = r = 1$. \square

The following Lemma allows to estimate products which are not covered by the above Lemma because one of the element is regular but has a non-zero limit at infinity (typically a rational fraction of $h_1 = 1 - \epsilon\zeta$ and $h_2 = \delta^{-1} + \epsilon\zeta$).

Lemma C.3. *Let $H \in C^\infty(-\delta^{-1}, 1)$ and $\epsilon\zeta \in L^\infty$ such that*

$$h_1(\epsilon\zeta) = 1 - \epsilon\zeta \geq h_0 > 0, \quad h_2(\epsilon\zeta) = \delta^{-1} + \epsilon\zeta \geq h_0 > 0.$$

Then, denoting $H_{n,h_0} \stackrel{\text{def}}{=} |H|_{C^n([- \delta^{-1} + h_0, 1 - h_0])}$ and fixing $t_0 > 1/2$, one has

- *For any $s \geq 0$, if $\zeta \in H^s$ and $f \in H_x^s$, then one has with $n \in \mathbb{N}, n \geq \max\{s, t_0\}$:*

$$|H(\epsilon\zeta)f|_{H_x^s} \leq C(h_0^{-1}, H_{n,h_0}, |\epsilon\zeta|_{H_x^{\max\{s, t_0\}}}) |f|_{H_x^s}.$$

- *For any $\widehat{f} \in L^1$, one has*

$$|\widehat{H(\epsilon\zeta)f}|_{L^1} \leq C(h_0^{-1}, H_{1,h_0}, |\epsilon\zeta|_{H_x^{t_0}}) |\widehat{f}|_{L^1}.$$

- *For any $N \in \mathbb{N}$, if $\zeta \in H_x^{t_0+1+N}$ and $f \in Z_{F^\mu}^N$, then one has*

$$|H(\epsilon\zeta)f|_{Z_{F^\mu}^N} \leq C(h_0^{-1}, H_{2+N,h_0}, |\epsilon\zeta|_{H_x^{t_0+1+N}}) |f|_{Z_{F^\mu}^N}.$$

- *For any $N \in \mathbb{N}$, if $\zeta \in H_x^{t_0+1+N}$ and $f \in Y_{F^\mu}^N$, then one has*

$$|H(\epsilon\zeta)f|_{Y_{F^\mu}^N} \leq C(h_0^{-1}, H_{2+N,h_0}, |\epsilon\zeta|_{H_x^{t_0+1+N}}) |f|_{Y_{F^\mu}^N}.$$

Proof. In each case, we decompose $H(\epsilon\zeta)f = H(0)f + (H(\epsilon\zeta) - H(0))f = H(0)f + G_{h_0}(\epsilon\zeta)f$ where G_{h_0} is such that $G_{h_0} \in C^\infty(\mathbb{R})$, $G_{h_0}(x) = H(x) - H(0)$ for $x \in [-\delta^{-1} + h_0, 1 - h_0]$ and $G_{h_0}(x) = 0$ for $x \in \mathbb{R} \setminus [-\delta^{-1}, 1]$. It is clear that, since $\min\{h_1(\epsilon\zeta), h_2(\epsilon\zeta)\} \geq h_0 > 0$, one can construct such a G_{h_0} satisfying additionally: for any $n \in \mathbb{N}$, $|G_{h_0}|_{C^n} = C(h_0^{-1}, H_{n,h_0})$.

The first estimate is a direct consequence of a classical Schauder-type estimates in Sobolev spaces; see *e.g.* [40].

The other estimates are then deduced. Indeed, one has

$$|\widehat{H(\epsilon\zeta)}f|_{L^1} \leq |\widehat{H(0)}f|_{L^1} + |\widehat{G_{h_0}(\epsilon\zeta)} \star \widehat{f}|_{L^1} \leq H(0)|\widehat{f}|_{L^1} + |\widehat{G_{h_0}(\epsilon\zeta)}|_{L^1}|\widehat{f}|_{L^1}.$$

The second estimate now follows from (C.2) and applying the above result:

$$|\widehat{G_{h_0}(\epsilon\zeta)}|_{L^1} \lesssim |G_{h_0}(\epsilon\zeta)|_{H_x^{t_0}} \leq C(h_0^{-1}, |H|_{C^1}, |\epsilon\zeta|_{H_x^{t_0}}).$$

The third estimate is estimated similarly. First, since we have seen that $Z_{F\mu}^0$ is an algebra,

$$|H(\epsilon\zeta)f|_{Z_{F\mu}^0} \leq |H(0)f|_{Z_{F\mu}^0} + |G_{h_0}(\epsilon\zeta)f|_{Z_{F\mu}^0} \leq H(0)|f|_{Z_{F\mu}^0} + |G_{h_0}(\epsilon\zeta)|_{Z_{F\mu}^0}|f|_{Z_{F\mu}^0}.$$

Since $|u|_{Z_{F\mu}^0} \leq |u|_{H_x^{t_0+1}}$ for any $u \in H_x^{t_0+1}$, one deduces the desired result for $N = 0$ as above. The case $N \geq 1$ is obtained by induction, deriving N times $H(\epsilon\zeta)f$ and applying Leibniz's rule.

The last estimate is obtained identically since by Lemma C.2

$$|H(\epsilon\zeta)f|_{Y_{F\mu}^0} \leq |H(0)f|_{Y_{F\mu}^0} + |G_{h_0}(\epsilon\zeta)f|_{Y_{F\mu}^0} \leq H(0)|f|_{Y_{F\mu}^0} + |G_{h_0}(\epsilon\zeta)|_{Y_{F\mu}^0}|f|_{Y_{F\mu}^0}.$$

The proof is now complete. □

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