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► **To cite this version:**

Robert Eymard, Thierry Gallouët, Raphaèle Herbin. RTk mixed finite elements for some nonlinear problems. . Mathematics and Computers in Simulation, 2014, pp.1-20. 10.1016/j.matcom.2014.11.013 . hal-01127963

**HAL Id: hal-01127963**

**<https://hal.science/hal-01127963>**

Submitted on 9 Mar 2015

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# $\mathcal{RT}_k$ mixed finite elements for some nonlinear problems

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## Abstract

We show that the discrete operators and spaces of gradient discretizations can be designed so that the corresponding gradient scheme for a linear diffusion problem be identical to the Raviart–Thomas  $\mathcal{RT}_k$  mixed finite element method for both the primal mixed finite element formulation and the hybrid dual formulation. We then give the hybrid dual  $\mathcal{RT}_0$  scheme for the approximation of a nonlinear model for two-phase flow in porous media; its convergence is then known thanks to a recent proof of the convergence of gradient schemes for this problem.

*Keywords:* mixed finite element, two phase flow, gradient schemes

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## 1. Introduction

The numerical solution of environmental underground studies often involves models which require the approximation of linear and nonlinear heterogeneous and anisotropic diffusion operators for general piecewise regular coefficients and on general meshes [5, 2, 1, 4]. A wide number of numerical schemes based on several different approaches have been developed in the last fifteen years to this purpose. An illustration of the variety of these approaches may be found in the two benchmarks which were held in 2008 (two-dimensional case) and in 2011 (3D case) [15, 14]. The family of gradient schemes was introduced to synthesize

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some of these approaches and was proven to converge for a large number of nonlinear problems [12, 7]. This family contains several well-known schemes, such as conforming and lumped conforming schemes, the hybrid mixed mimetic family of schemes [7], and some discrete duality finite volume schemes [6]. The aim of this paper is to show that it also contains the  $\mathcal{RT}_k$  mixed finite element method [16, 17] in the case of a linear diffusion problem, and then apply this gradient discretization to two phase flow problems. Let us first recall the primal and dual formulations of the mixed finite element method [8] for the following linear anisotropic diffusion problem. Let  $\Omega$  be an open bounded connected polygonal subset of  $\mathbb{R}^d$ ,  $d \in \mathbb{N}^*$ , let  $\underline{\lambda}$  and  $\bar{\lambda} \in \mathbb{R}$ , such that  $0 < \underline{\lambda} \leq \bar{\lambda}$  and let  $\mathcal{M}_d(\underline{\lambda}, \bar{\lambda})$  denote the set of  $d \times d$  symmetric matrices with eigenvalues in  $(\underline{\lambda}, \bar{\lambda})$ . Assuming that  $\Lambda$  is a measurable function from  $\Omega$  to  $\mathcal{M}_d(\underline{\lambda}, \bar{\lambda})$ , and  $f \in L^2(\Omega)$ , let  $\bar{u} \in H_0^1(\Omega)$  be the solution to:

$$\int_{\Omega} \Lambda(\mathbf{x}) \nabla \bar{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \forall v \in H_0^1(\Omega). \quad (1.1)$$

We now consider a regular simplicial mesh  $\mathcal{T}$  of  $\Omega$ . We denote by  $\mathcal{E}$  the set of edges (in 2D) or faces (in 3D) of the mesh, and  $\mathcal{E}_{\text{int}}$  the subset of internal faces, i.e. such that there exists  $(K, L) \in \mathcal{T}^2$  such that  $\sigma$  is an interface to  $K$  and  $L$ , which we denote by  $\sigma = K|L$ . We introduce the following spaces

$$H_{\text{div}}(\Omega) = \{\boldsymbol{\varphi} \in L^2(\Omega)^d, \text{div} \boldsymbol{\varphi} \in L^2(\Omega)\},$$

$$\mathbf{V}_h = \{\mathbf{v} \in (L^2(\Omega))^d; \mathbf{v}|_K \in \mathcal{RT}_k(K), \forall K \in \mathcal{T}\}, \quad (1.2)$$

$$\mathbf{V}_h^{\text{div}} = \mathbf{V}_h \cap H_{\text{div}}(\Omega), \quad (1.3)$$

$$W_h = \{p \in L^2(\Omega); p|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}\}, \quad (1.4)$$

$$M_h^0 = \{\mu : \bigcup_{\sigma \in \mathcal{E}} \bar{\sigma} \rightarrow \mathbb{R}, \mu|_{\sigma} \in \mathcal{P}_k(\sigma), \mu|_{\partial\Omega} = 0\}, \quad (1.5)$$

where

- $\mathcal{RT}_k(K)$  is the Raviart-Thomas space of order  $k$  defined on  $K$  with dimension  $d \binom{k+d}{d} + \binom{k+d-1}{d-1}$ ,
- $\mathcal{P}_k(K)$  is the space of polynomials of  $d$  variables of degree  $k$ ; its dimension is  $\binom{k+d}{d}$ .

- $\mathcal{P}_k(\sigma)$  is the space of polynomials of  $d-1$  variables of degree  $k$ ; its dimension is  $\binom{k+d-1}{d-1}$ .

Let us now first recall the primal formulation of the mixed finite element method for Problem (1.1). The primal formulation reads

$$(\mathbf{v}, q) \in \mathbf{V}_h^{\text{div}} \times W_h, \quad (1.6a)$$

$$\int_{\Omega} \mathbf{w}(\mathbf{x}) \cdot \Lambda^{-1}(\mathbf{x}) \mathbf{v}(\mathbf{x}) d\mathbf{x} - \int_{\Omega} q(\mathbf{x}) \text{div} \mathbf{w}(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \mathbf{w} \in \mathbf{V}_h^{\text{div}}, \quad (1.6b)$$

$$\int_{\Omega} \psi(\mathbf{x}) \text{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \psi(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}, \quad \forall \psi \in W_h. \quad (1.6c)$$

We then introduce the Arnold-Brezzi formulation [3, 18].

$$(\mathbf{v}, q, \lambda) \in \mathbf{V}_h \times W_h \times M_h^0, \quad (1.7a)$$

$$\int_K \mathbf{w}(\mathbf{x}) \cdot \Lambda^{-1}(\mathbf{x}) \mathbf{v}(\mathbf{x}) d\mathbf{x} - \int_K q(\mathbf{x}) \text{div} \mathbf{w}(\mathbf{x}) d\mathbf{x} + \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \lambda(\mathbf{x}) \mathbf{w}|_K(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{x}) = 0, \quad \forall \mathbf{w} \in \mathbf{V}_h, \quad (1.7b)$$

$$\int_K \psi(\mathbf{x}) \text{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = \int_K \psi(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}, \quad \forall \psi \in W_h, \forall K \in \mathcal{T}, \quad (1.7c)$$

$$\int_{\sigma} \mu(\mathbf{x}) \mathbf{v}|_K(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{x}) + \int_{\sigma} \mu(\mathbf{x}) \mathbf{v}|_L(\mathbf{x}) \cdot \mathbf{n}_{L,\sigma} d\gamma(\mathbf{x}), \quad \forall \sigma = K|L \in \mathcal{E}_{\text{int}}, \forall \mu \in M_h^0, \quad (1.7d)$$

where  $\mathbf{n}_{K,\sigma}$  is the unit normal vector to  $\sigma$  outward  $K$ . It is shown in e.g. [3] that the problems (1.6) and (1.7) admit a unique solution, and that the solutions  $(\mathbf{v}, q)$  to (1.6) and (1.7) are identical. Moreover, let us recall the error estimate [8, Theorem 5.3 p. 39] (due to Brezzi): there exists  $\delta$ , only depending on  $\underline{\lambda}, \bar{\lambda}$ , on the regularity of the mesh and on  $\Omega$  such that

$$\|q - \bar{u}\|_{L^2(\Omega)} + \|\mathbf{v} - \Lambda \nabla \bar{u}\|_{H_{\text{div}}(\Omega)} \leq \delta \left( \inf_{\psi \in W_h} \|\psi - \bar{u}\|_{L^2(\Omega)} + \inf_{\mathbf{w} \in \mathbf{V}_h^{\text{div}}} \|\mathbf{w} - \Lambda \nabla \bar{u}\|_{H_{\text{div}}(\Omega)} \right). \quad (1.8)$$

In the sequel, we recall the definition of a gradient discretization, from which it is possible to build a gradient scheme for a large class of problems. We then

give two constructions of gradient discretization issued from the mixed finite element method, one from the primal formulation and one from the dual formulation, which lead to the same schemes in the case of a linear diffusion anisotropic heterogeneous problem. We then apply these results to the discretization of a nonlinear model of two-phase flow in porous media.

## 2. Gradient discretizations for diffusion problems

A *gradient discretization*  $\mathcal{D}$  for a second order elliptic problem posed on the domain  $\Omega$ , with homogeneous Dirichlet boundary conditions on the boundary  $\partial\Omega$ , is defined by  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ , where:

- the set of discrete unknowns  $X_{\mathcal{D},0}$  is a finite dimensional vector space on  $\mathbb{R}$ ,
- the linear mapping  $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^2(\Omega)$  is the reconstruction of an approximate function from the discrete unknowns (also often called “lifting operator”).
- the linear mapping  $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^2(\Omega)^d$  is the reconstruction of an approximate gradient from the discrete unknowns; we shall call it “discrete gradient operator”. It is chosen such that  $\|\cdot\|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot\|_{L^2(\Omega)^d}$  is a norm on  $X_{\mathcal{D},0}$ .

Let us now give the fundamental properties that we seek when designing a gradient discretization (or when recognizing a gradient discretization in an existing scheme) in order to be able to prove the convergence of the resulting gradient schemes for the approximation of linear and nonlinear steady or unsteady diffusion problems [7, 12, 9, 13, 11].

- **Coercivity.** Let  $C_{\mathcal{D}}$  be the norm of the linear mapping  $\Pi_{\mathcal{D}}$ , defined by

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}}v\|_{L^2(\Omega)}}{\|v\|_{\mathcal{D}}}. \quad (2.1)$$

A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretizations is said to be coercive if there exists  $C_P \in \mathbb{R}_+$  such that  $C_{\mathcal{D}_m} \leq C_P$  for all  $m \in \mathbb{N}$ .

- **Consistency.** Let  $S_{\mathcal{D}}$  be defined by:  $\varphi \in H^1(\Omega) \mapsto S_{\mathcal{D}}(\varphi) \in [0, +\infty)$  with

$$S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} (\|\Pi_{\mathcal{D}}v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}}v - \nabla\varphi\|_{L^2(\Omega)^d}). \quad (2.2)$$

A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretizations is said to be consistent if, for all  $\varphi \in H_0^1(\Omega)$ ,  $S_{\mathcal{D}_m}(\varphi)$  tends to 0 as  $m \rightarrow \infty$ .

• **Limit-conformity.** Let  $W_{\mathcal{D}}: H_{\text{div}}(\Omega) \times X_{\mathcal{D},0} \rightarrow [0, +\infty)$  be defined by

$$\begin{aligned} \forall (\varphi, u) \in H_{\text{div}}(\Omega) \times X_{\mathcal{D},0}, \\ W_{\mathcal{D}}(\varphi, u) = \int_{\Omega} (\nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} u(\mathbf{x}) \text{div} \varphi(\mathbf{x})) \, d\mathbf{x}. \end{aligned} \quad (2.3)$$

Note that for a conforming finite element method, we have  $W_{\mathcal{D}}(\varphi, u) = 0$ .

A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretizations is said to be limit-conforming if, for all sequence  $u_m \in X_{\mathcal{D}_m,0}$  such that  $\|u_m\|_{\mathcal{D}_m}$  is bounded, and for all  $\varphi \in H_{\text{div}}(\Omega)$ ,  $W_{\mathcal{D}_m}(\varphi, u_m)$  tends to 0 as  $m \rightarrow \infty$ .

• **Compactness.** A sequence  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  of gradient discretizations is said to be compact if, for all sequence  $u_m \in X_{\mathcal{D}_m,0}$  such that  $\|u_m\|_{\mathcal{D}_m}$  is bounded, the sequence  $(\Pi_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}$  is relatively compact in  $L^2(\Omega)$ .

Let us illustrate these concepts on Problem (1.1). Let  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  be a gradient discretization; then the related **gradient scheme** for the discretization of (1.1) is to look for  $u \in X_{\mathcal{D},0}$  such that

$$\int_{\Omega} \Lambda(\mathbf{x}) \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) \, d\mathbf{x}, \forall v \in X_{\mathcal{D},0}. \quad (2.4)$$

The coercivity, consistency and limit-conformity properties for a family of gradient discretizations are sufficient to ensure the convergence of  $\Pi_{\mathcal{D}} u$  to  $\bar{u}$  in  $L^2(\Omega)$  and that of  $\nabla_{\mathcal{D}} u$  to  $\nabla \bar{u}$  in  $L^2(\Omega)^d$  [11]. The compactness property is only needed for the convergence of gradient schemes in the case of nonlinear problems [7, 12, 9, 13].

### 3. Primal mixed finite element and gradient discretizations

Here we construct a gradient discretization (in the sense of Section 2) inspired from the primal mixed finite element formulation (1.6) of Problem (1.1). Let  $W_h$  be defined by (1.4) and let  $(\chi_i)_{i \in I}$  be a family of piecewise polynomial basis functions of degree  $k$  on each cell of the mesh, spanning  $W_h$ . Let us define

the gradient discretization  $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$  by:

$$X_{\mathcal{D},0} = \mathbb{R}^I, \quad (3.1a)$$

$$\Pi_{\mathcal{D}}u = \sum_{i \in I} u_i \chi_i, \forall u \in X_{\mathcal{D},0}, \quad (3.1b)$$

$$\forall u \in X_{\mathcal{D},0},$$

$$\Lambda \nabla_{\mathcal{D}}u \in \mathbf{V}_h^{\text{div}}, \quad (3.1c)$$

$$\int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathcal{D}}u(\mathbf{x}) d\mathbf{x} + \int_{\Omega} \Pi_{\mathcal{D}}u(\mathbf{x}) \text{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h^{\text{div}}. \quad (3.1d)$$

**Remark 3.1.** *As we mentioned in the preceding section, a gradient discretization only relies on the definition of discrete operators and therefore, should be problem independent. However we use here the tensor  $\Lambda$  to define the discrete gradient, which enables to ensure the  $H_{\text{div}}$  conformity in the sense that (3.1c) holds, so that the discrete gradient is problem dependent. Another natural way to obtain a gradient scheme is to construct a gradient discretization without  $\Lambda$  (which is thus problem independent); in this case, we can still prove the convergence, but the approximation is no longer  $H_{\text{div}}$  conforming; moreover, the resulting approximate gradient is  $H_{\text{div}}$  consistent, but not  $\Lambda \nabla_{\mathcal{D}}u$ , and one can expect lower convergence rate in the case of highly anisotropic and heterogeneous problems.*

In order for (3.1) to define a gradient discretization, the system (3.1d) should define one and only one  $\nabla_{\mathcal{D}}u$ , and  $\|\cdot\|_{\mathcal{D}} := \|\nabla_{\mathcal{D}} \cdot\|_{L^2(\Omega)^d}$  has to be a norm on  $X_{\mathcal{D},0}$ . The existence and uniqueness of  $\nabla_{\mathcal{D}}u$  results from the fact that (3.1d) is a square linear system, whose solution vanishes if the right-hand-side vanishes. The fact that it defines a norm results from the coercivity property shown in the next theorem 3.2.

Let us now recall some known results on the  $\mathcal{RT}_k$  mixed finite element schemes. The broken Sobolev space  $H^1(\mathcal{T})$  is the set of functions whose restriction to each simplex  $K$  of the mesh belongs to  $H^1$ . First recall that, for  $(\mathbf{V}_h^{\text{div}}, W_h)$  defined

by (1.3)-(1.4) , there exists [8, Lemma 3.5 page 17] an interpolation operator  $P_k$  defined by

$$\begin{aligned} P_k : \mathbf{H}_{\mathcal{T}} &= H_{\text{div}}(\Omega) \cap (H^1(\mathcal{T}))^d \rightarrow \mathbf{V}_h^{\text{div}}, \\ \forall p \in W_h, \forall \mathbf{v} \in \mathbf{H}_{\mathcal{T}}, \int_{\Omega} p(\mathbf{x}) \text{div}(\mathbf{v} - P_k \mathbf{v})(\mathbf{x}) d\mathbf{x} &= 0, \end{aligned} \quad (3.2)$$

and there exists  $\alpha > 0$ , only depending on the regularity of the mesh [8, Theorem 3.1], such that

$$\forall \mathbf{v} \in \mathbf{H}_{\mathcal{T}}, \|\mathbf{v} - P_k \mathbf{v}\|_{L^2(\Omega)^d} \leq \alpha h \left( \sum_{T \in \mathcal{T}} \|\mathbf{v}\|_{H^1(T)}^2 \right)^{1/2}, \quad (3.3)$$

where  $h$  denotes the size of the mesh  $\mathcal{T}$ . Let us recall how we deduce from the above properties the standard “inf-sup” condition [17]: let  $p \in W_h$ , let us prolong  $p$  by 0 on a ball  $B$  with radius  $R$  containing  $\Omega$ . Then there exists  $w \in H_0^1(B)$  such that

$$\forall q \in H_0^1(B), \int_B \nabla w(\mathbf{x}) \cdot \nabla q(\mathbf{x}) d\mathbf{x} = \int_B p(\mathbf{x}) q(\mathbf{x}) d\mathbf{x}. \quad (3.4)$$

Moreover  $w \in H^2(B)$  and there exists  $\beta$  only depending on  $d$  and  $R$  such that

$$\|w\|_{H^2(B)} \leq \beta \|p\|_{L^2(\Omega)}. \quad (3.5)$$

Therefore, since  $\nabla w \in \mathbf{H}_{\mathcal{T}}$ , we have from (3.3)

$$\|\nabla w - P_k \nabla w\|_{L^2(\Omega)^d} \leq \alpha h \beta \|p\|_{L^2(\Omega)},$$

which shows that

$$\|P_k \nabla w\|_{L^2(\Omega)^d} \leq (2R\alpha + 1)\beta \|p\|_{L^2(\Omega)}, \quad (3.6)$$

which leads to the inf-sup condition.

**Theorem 3.2.** *Let  $(\mathcal{T}_m)_{m \in \mathbb{N}}$  be a sequence of regular simplicial meshes in the sense of [8, Theorem 3.1 p.14] such that the size  $h_m$  of the mesh  $\mathcal{T}_m$ , tends to 0 as  $m \rightarrow \infty$ . Let  $\mathcal{D}_m = (X_{\mathcal{D}_m, 0}, \Pi_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$  be defined by (3.1) for each  $m \in \mathbb{N}$ , then the family  $(\mathcal{D}_m)_{m \in \mathbb{N}}$  is coercive, consistent, limit-conforming and compact in the sense of the definitions of Section 2.*



**Proof**

• **Coercivity.** Let  $u \in X_{\mathcal{D}_m,0}$  (and therefore that  $p = \Pi_{\mathcal{D}_m} u \in (W_h)_m$ ). Using (3.6), let  $w \in H_0^1(B)$  be defined by (3.4), and let  $\mathbf{v} = P_k \nabla w \in \mathbf{V}_h^{\text{div}}$ . Thanks to (3.2), we get that

$$\|p\|_{L^2(\Omega)}^2 = - \int_{\Omega} p(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x}.$$

From (3.1), we get

$$\|\Pi_{\mathcal{D}_m} u\|_{L^2(\Omega)}^2 = \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathcal{D}_m} u(\mathbf{x}) d\mathbf{x}.$$

Thanks to (3.6), we then get

$$\|\Pi_{\mathcal{D}_m} u\|_{L^2(\Omega)} \leq (\alpha D + 1) \beta \|\nabla_{\mathcal{D}_m} u\|_{L^2(\Omega)^d},$$

which proves the coercivity property.

• **Consistency.** Let us check the consistency property on the set  $\mathcal{R} = \{\varphi \in H_0^1(\Omega); \text{there exists } f \in C_c^\infty(\Omega) \text{ such that } \varphi \text{ is solution to (1.1)}\}$ . Let  $\varphi \in \mathcal{R}$ . Considering Problem (1.6) with  $f = -\operatorname{div}(\Lambda \nabla \varphi)$ , we define  $u \in X_{\mathcal{D}_m,0}$  by  $p = \Pi_{\mathcal{D}_m} u$  and  $\mathbf{v} = -\Lambda \nabla_{\mathcal{D}_m} u$ . Then, we get from (1.8)

$$\begin{aligned} & \|\Pi_{\mathcal{D}_m} u - \varphi\|_{L^2(\Omega)} + \|\Lambda \nabla_{\mathcal{D}_m} u - \Lambda \nabla \varphi\|_{H_{\operatorname{div}}(\Omega)} \\ & \leq \delta \left( \inf_{\psi \in (W_h)_m} \|\psi - \varphi\|_{L^2(\Omega)} + \inf_{\mathbf{w} \in (\mathbf{V}_h^{\operatorname{div}})_m} \|\mathbf{w} - \Lambda \nabla \varphi\|_{H_{\operatorname{div}}(\Omega)} \right). \end{aligned}$$

Since the right hand side of the above inequality tends to 0 as  $m \rightarrow \infty$ , we obtain that  $S_{\mathcal{D}_m}(\varphi)$  tends to 0 as  $m \rightarrow \infty$ . The proof of consistency is then concluded by density of  $\mathcal{R}$  in  $H_0^1(\Omega)$  (see Lemma 3.3 below).

• **Limit-conformity.** Let  $(u_m)_{m \in \mathbb{N}}$  such that  $u_m \in X_{\mathcal{D}_m,0}$  and  $\nabla_{\mathcal{D}_m} u_m$  remains bounded in  $L^2(\Omega)^d$  as  $m \rightarrow \infty$ . Let  $\boldsymbol{\varphi} \in H_{\operatorname{div}}(\Omega)$ , and  $\boldsymbol{\varphi}_m \in (\mathbf{V}_h^{\operatorname{div}})_m$  be an interpolation of  $\boldsymbol{\varphi}$  such that  $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_m\|_{H_{\operatorname{div}}(\Omega)}$  tends to 0 as  $m \rightarrow \infty$ . Then

$$\begin{aligned} W_{\mathcal{D}_m}(\boldsymbol{\varphi}, u_m) &= \int_{\Omega} (\nabla_{\mathcal{D}_m} u_m(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) + \Pi_{\mathcal{D}_m} u_m(\mathbf{x}) \operatorname{div} \boldsymbol{\varphi}(\mathbf{x})) d\mathbf{x} = \\ & \int_{\Omega} \left( \nabla_{\mathcal{D}_m} u_m(\mathbf{x}) \cdot (\boldsymbol{\varphi}(\mathbf{x}) - \boldsymbol{\varphi}_m(\mathbf{x})) + \Pi_{\mathcal{D}_m} u_m(\mathbf{x}) (\operatorname{div} \boldsymbol{\varphi}(\mathbf{x}) - \operatorname{div} \boldsymbol{\varphi}_m(\mathbf{x})) \right) d\mathbf{x}, \end{aligned}$$

thanks to (3.1). Applying the coercivity inequality, we get that the right term of the preceding inequality tends to 0 as  $m \rightarrow \infty$ , which shows the limit conformity of the sequence.

• **Compactness.** We consider a sequence  $(u_m)_{m \in \mathbb{N}}$  such that  $u_m \in X_{\mathcal{D}_m,0}$  and  $\nabla_{\mathcal{D}_m} u_m$  remains bounded in  $L^2(\Omega)^d$  as  $m \rightarrow \infty$ . Then, thanks to the coercivity property, we first extract a subsequence (specially denoted), such that  $\Pi_{\mathcal{D}_m} u_m$  weakly converges in  $L^2(\mathbb{R}^d)$  to some  $u \in L^2(\mathbb{R}^d)$  (prolonging by 0 outside  $\Omega$ ). Using the limit-conformity, we get that  $\nabla_{\mathcal{D}_m} u_m$  (prolonging by 0 outside  $\Omega$ ) weakly converges in  $L^2(\mathbb{R}^d)^d$  to  $\nabla u$ , which shows that  $u \in H_0^1(\Omega)$ . Let  $w_m \in H_0^1(B) \cap H^2(B)$  (resp.  $w \in H_0^1(B) \cap H^2(B)$ ) be defined by (3.4) for  $p = \Pi_{\mathcal{D}_m} u_m$  (resp.  $p = u$ ). A classical result is that  $w_m$  converges in  $H_0^1(B)$  to  $w$  (from the weak convergence of the gradient and the convergence of its norm). Applying (3.5), we get that

$$\|\nabla w_m\|_{H^1(\Omega)} \leq \beta \|\Pi_{\mathcal{D}_m} u_m\|_{L^2(\Omega)}. \quad (3.7)$$

Letting  $\mathbf{v} = P_k \nabla w_m$  in (3.1), we get

$$\int_{\Omega} P_k \nabla w_m(\mathbf{x}) \cdot \nabla_{\mathcal{D}_m} u_m(\mathbf{x}) d\mathbf{x} + \int_{\Omega} \Pi_{\mathcal{D}_m} u_m(\mathbf{x}) \operatorname{div} P_k \nabla w_m(\mathbf{x}) d\mathbf{x} = 0,$$

which provides, thanks to (3.2),

$$\int_{\Omega} P_k \nabla w_m(\mathbf{x}) \cdot \nabla_{\mathcal{D}_m} u_m(\mathbf{x}) d\mathbf{x} - \int_{\Omega} (\Pi_{\mathcal{D}_m} u_m(\mathbf{x}))^2 d\mathbf{x} = 0.$$

Thanks to (3.3) (using (3.7)) and to the convergence of  $\nabla w_m$  to  $\nabla w$  in  $L^2(\Omega)^d$ , we get that  $P_k \nabla w_m$  converges in  $L^2(\Omega)^d$  to  $\nabla w$ . By strong/weak convergence on the first term, we get

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{\Omega} (\Pi_{\mathcal{D}_m} u_m(\mathbf{x}))^2 d\mathbf{x} &= \int_{\Omega} \nabla w(\mathbf{x}) \cdot \nabla u(\mathbf{x}) d\mathbf{x} \\ &= - \int_{\Omega} \operatorname{div}(\nabla w(\mathbf{x})) u(\mathbf{x}) d\mathbf{x} = \int_{\Omega} u(\mathbf{x})^2 d\mathbf{x}. \end{aligned}$$

This shows the convergence of  $\Pi_{\mathcal{D}_m} u_m$  to  $u$  in  $L^2(\Omega)$ , hence concluding the proof of the compactness of the discretization.  $\square$

**Lemma 3.3 (A density result).** *Let  $\mathcal{R} = \{\varphi \in H_0^1(\Omega); \text{there exists } f \in C_c^\infty(\Omega) \text{ such that } \varphi \text{ solution of (1.1)}\}$ . Then  $\mathcal{R}$  is dense in  $H_0^1(\Omega)$ .*

**Proof** The mapping  $T : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  defined by  $u \mapsto T(u) = -\operatorname{div}\Lambda\nabla u$  is continuous and one-to-one thanks to the Lax-Milgram lemma. Therefore the inverse mapping  $T^{-1}$  is also continuous. Since  $C_c^\infty(\Omega)$  is dense in  $H^{-1}(\Omega)$  and  $\mathcal{R} = T^{-1}(C_c^\infty(\Omega))$ , the conclusion follows.  $\square$

We then have the following property, which establishes the link between the gradient discretization (3.1) and the primal form of the mixed finite element method.

**Theorem 3.4.** *Using the gradient discretization (3.1), the gradient scheme (2.4) for the approximation of Problem (1.1) is equivalent to the primal formulation (1.6) of the mixed finite element method.*

**Proof** Let  $u \in X_{\mathcal{D},0}$  be a solution to (2.4). Let us show that  $(\mathbf{v} = -\Lambda\nabla_{\mathcal{D}}u, q = \Pi_{\mathcal{D}}u)$  is the solution of (1.6). We first observe that (3.1d) ensures (1.6b). Let us now consider  $\psi \in W_h$ , which can therefore be written  $\psi = \Pi_{\mathcal{D}}v$ , with  $v \in X_{\mathcal{D},0}$ . We then write, thanks to (2.4),

$$\int_{\Omega} \Lambda(\mathbf{x})\nabla_{\mathcal{D}}u(\mathbf{x}) \cdot \nabla_{\mathcal{D}}v(\mathbf{x})d\mathbf{x} = \int_{\Omega} f(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}.$$

Setting  $\mathbf{v} = -\Lambda(\mathbf{x})\nabla_{\mathcal{D}}u$  in (3.1d) with  $u$  replaced by  $v$ , we have

$$\int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \nabla_{\mathcal{D}}v(\mathbf{x})d\mathbf{x} + \int_{\Omega} \Pi_{\mathcal{D}}v(\mathbf{x})\operatorname{div}\mathbf{v}(\mathbf{x})d\mathbf{x} = 0,$$

which implies

$$\int_{\Omega} \Lambda(\mathbf{x})\nabla_{\mathcal{D}}u(\mathbf{x}) \cdot \nabla_{\mathcal{D}}v(\mathbf{x})d\mathbf{x} = \int_{\Omega} \psi(\mathbf{x})\operatorname{div}\mathbf{v}(\mathbf{x})d\mathbf{x}.$$

This completes the proof of (1.6c). Reciprocally, considering the solution  $(\mathbf{v}, q)$  to (1.6), since  $q \in W_h$ , there exists a unique  $u \in X_{\mathcal{D},0}$  such that  $q = \Pi_{\mathcal{D}}u$ . From (1.6b), we get that  $\mathbf{v} = -\Lambda\nabla_{\mathcal{D}}u$ . Following the same computation lines as the beginning of this proof and letting  $\psi = \Pi_{\mathcal{D}}v$  for any  $v \in X_{\mathcal{D},0}$ , we get that (1.6c) implies (2.4), again using (3.1d) with  $u$  replaced by  $v$ .  $\square$

#### 4. Mixed hybrid formulation and gradient schemes

Here we construct a gradient discretization (in the sense of Section 2) inspired from the dual mixed finite element formulation (1.7) of Problem (1.1). Let  $W_h$

be defined by (1.4) and let again  $(\chi_i)_{i \in I}$  be a family of piecewise polynomial basis functions of degree  $k$  on each cell of the mesh, spanning  $W_h$ . Let  $M_h^0$  be defined by (1.5) and let  $(\xi_j)_{j \in J}$  be a family spanning  $M_h^0$ . Recall that  $\mathbf{V}_h$  is defined by (1.2). Let us define the gradient discretization  $\tilde{\mathcal{D}} = (X_{\tilde{\mathcal{D}},0}, \Pi_{\tilde{\mathcal{D}}}, \nabla_{\tilde{\mathcal{D}}})$  by:

$$X_{\tilde{\mathcal{D}},0} = \mathbb{R}^{I \cup J}, \quad (4.1a)$$

$$\Pi_{\tilde{\mathcal{D}}}u = \sum_{i \in I} u_i \chi_i \text{ and } \Gamma_{\tilde{\mathcal{D}}}u = \sum_{j \in J} u_j \xi_j, \quad \forall u \in X_{\tilde{\mathcal{D}},0}, \quad (4.1b)$$

$\forall u \in X_{\tilde{\mathcal{D}},0}, \Lambda \nabla_{\tilde{\mathcal{D}}}u \in \mathbf{V}_h$  and:

$$\begin{aligned} \int_K \mathbf{w}(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}}u(\mathbf{x}) d\mathbf{x} + \int_K \Pi_{\tilde{\mathcal{D}}}u(\mathbf{x}) \operatorname{div} \mathbf{w}(\mathbf{x}) d\mathbf{x} \\ - \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \Gamma_{\tilde{\mathcal{D}}}u(\mathbf{x}) \mathbf{w}|_K(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{x}) = 0, \quad \forall \mathbf{w} \in \mathbf{V}_h. \end{aligned} \quad (4.1c)$$

**Remark 4.1.** *In the case  $k = 0$ , the gradient discretization (4.1) has the same degrees of freedom as the SUSHI scheme [10], which is also a gradient discretization. Nevertheless, the definitions of the discrete gradients are different.*

As in the previous section, in order for (4.1) to define a gradient discretization, the system (4.1c) should define one and only one discrete gradient  $\nabla_{\tilde{\mathcal{D}}}u$ , and  $\|\cdot\|_{\tilde{\mathcal{D}}} := \|\nabla_{\tilde{\mathcal{D}}}\cdot\|_{L^2(\Omega)^d}$  has to be a norm on  $X_{\tilde{\mathcal{D}},0}$ . The existence and uniqueness of  $\nabla_{\tilde{\mathcal{D}}}u$  again results from the fact that (4.1c) provides a square linear system, whose solution vanishes if the right-hand-side vanishes. The fact that it defines a norm results, on one hand, from the coercivity property shown in Theorem 4.2 below, and on the other hand, on [18, Proposition 3.1 p. 15], whose consequence is that for given  $(u_i)_{i \in I}$  and  $\nabla_{\tilde{\mathcal{D}}}u$ , there exists one and only one  $(u_j)_{j \in J}$  such that (4.1c) hold.

**Theorem 4.2.** *Let  $(\mathcal{T}_m)_{m \in \mathbb{N}}$  be a sequence of regular simplicial meshes in the sense of [8, Theorem 3.1 p.14] such that the size  $h_m$  of the mesh  $\mathcal{T}_m$ , tends to 0 as  $m \rightarrow \infty$ . Let  $\tilde{\mathcal{D}}_m = (X_{\tilde{\mathcal{D}}_m,0}, \Pi_{\tilde{\mathcal{D}}_m}, \nabla_{\tilde{\mathcal{D}}_m})$  be defined by (4.1) for each  $m \in \mathbb{N}$ ,*

then the family  $(\tilde{\mathcal{D}}_m)_{m \in \mathbb{N}}$  is coercive, consistent, limit-conforming and compact in the sense of the definitions of Section 2.

**Proof**

Let us first denote by  $T : X_{\tilde{\mathcal{D}},0} \rightarrow X_{\mathcal{D},0}$ ,  $T(u) = (u_i)_{i \in I}$ , where we use the gradient discretization  $\mathcal{D}$  defined by (3.1). The first remark is that the definition of  $T$  implies  $\Pi_{\tilde{\mathcal{D}}}u = \Pi_{\mathcal{D}}T(u)$  a.e. in  $\Omega$ . We observe that, selecting  $\mathbf{w} = \Lambda \nabla_{\mathcal{D}}T(u) \in \mathbf{V}_h^{\text{div}} \subset \mathbf{V}_h$  in (4.1c) and summing on  $K \in \mathcal{T}$ , all the integrals on  $\sigma \in \mathcal{E}_{\text{int}}$  vanish, and we obtain

$$\int_{\Omega} \Lambda \nabla_{\mathcal{D}}T(u)(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}}u(\mathbf{x})d\mathbf{x} + \int_{\Omega} \Pi_{\tilde{\mathcal{D}}}u(\mathbf{x})\text{div}(\Lambda \nabla_{\mathcal{D}}T(u))(\mathbf{x})d\mathbf{x} = 0.$$

Now using (3.1d) with  $\mathbf{v} = \Lambda \nabla_{\mathcal{D}}T(u)$ , we get

$$\int_{\Omega} \Lambda \nabla_{\mathcal{D}}T(u)(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}}u(\mathbf{x})d\mathbf{x} = \int_{\Omega} \Lambda \nabla_{\mathcal{D}}T(u)(\mathbf{x}) \cdot \nabla_{\mathcal{D}}T(u)(\mathbf{x})d\mathbf{x}.$$

We thus obtain, thanks to the Cauchy-Schwarz inequality,

$$\int_{\Omega} \Lambda \nabla_{\mathcal{D}}T(u)(\mathbf{x}) \cdot \nabla_{\mathcal{D}}T(u)(\mathbf{x})d\mathbf{x} \leq \int_{\Omega} \Lambda \nabla_{\tilde{\mathcal{D}}}u(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}}u(\mathbf{x})d\mathbf{x},$$

which leads to

$$\|\nabla_{\mathcal{D}}T(u)\|_{L^2(\Omega)^d}^2 \leq \frac{\bar{\lambda}}{\underline{\lambda}} \|\nabla_{\tilde{\mathcal{D}}}u\|_{L^2(\Omega)^d}^2, \quad \forall u \in X_{\tilde{\mathcal{D}},0}. \quad (4.2)$$

- **Coercivity** follows from the coercivity of the gradient discretization  $\mathcal{D}$  and (4.2).
- **Consistency.** Thanks to [18, Proposition 3.1 p. 15], for any  $u \in X_{\mathcal{D},0}$ , there exists  $\tilde{u} \in X_{\tilde{\mathcal{D}},0}$  such that  $T(\tilde{u}) = u$  and  $\nabla_{\mathcal{D}}u = \nabla_{\tilde{\mathcal{D}}}\tilde{u}$ . The consistency of  $\tilde{\mathcal{D}}_m$  is then a consequence of that of  $\mathcal{D}_m$ .
- **Limit-conformity.** Similarly to the study of the limit-conformity of  $\mathcal{D}_m$ , we get

$$\begin{aligned} W_{\tilde{\mathcal{D}}_m}(\boldsymbol{\varphi}, u_m) &= \int_{\Omega} \left( \nabla_{\tilde{\mathcal{D}}_m} u_m(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) + \Pi_{\tilde{\mathcal{D}}_m} u_m(\mathbf{x}) \text{div} \boldsymbol{\varphi}(\mathbf{x}) \right) d\mathbf{x} = \\ &= \int_{\Omega} \left( \nabla_{\tilde{\mathcal{D}}_m} u_m(\mathbf{x}) \cdot (\boldsymbol{\varphi}(\mathbf{x}) - \boldsymbol{\varphi}_m(\mathbf{x})) + \Pi_{\tilde{\mathcal{D}}_m} u_m(\mathbf{x}) (\text{div} \boldsymbol{\varphi}(\mathbf{x}) - \text{div} \boldsymbol{\varphi}_m(\mathbf{x})) \right) d\mathbf{x}, \end{aligned}$$

thanks to (4.1c) and since the terms on  $\sigma \in \mathcal{E}_{\text{int}}$  vanish, using that  $\varphi_m \in (\mathbf{V}_h^{\text{div}})_m$ . So the conclusion is identical.

• **Compactness.** This property is again an immediate consequence of the compactness of the gradient discretization  $\mathcal{D}$  and (4.2).  $\square$

**Theorem 4.3.** *Using the gradient discretization (4.1), the gradient scheme (2.4) for the approximation of Problem (1.1) is equivalent to the Arnold-Brezzi formulation (1.7) of the mixed finite element method.*

**Proof** Let  $u \in X_{\tilde{\mathcal{D}},0}$  be a solution to (2.4). Let us show that  $(\mathbf{v} = -\Lambda \nabla_{\tilde{\mathcal{D}}} u, q = \Pi_{\tilde{\mathcal{D}}} u, \lambda = \Gamma_{\tilde{\mathcal{D}}} u)$  is the solution of (1.7). We first observe that (4.1c) ensures (1.7b). Let us now consider  $\psi \in W_h$  and  $\mu \in M_h^0$ . Consider a particular  $K \in \mathcal{T}$ , and take in (2.4) a function test  $v \in X_{\tilde{\mathcal{D}},0}$  such that  $\Pi_{\tilde{\mathcal{D}}} v|_K = \psi|_K$ ,  $\Pi_{\tilde{\mathcal{D}}} v|_L = 0$  for all  $L \in \mathcal{T} \setminus \{K\}$  and  $\Gamma_{\tilde{\mathcal{D}}} v = 0$ . We remark that, thanks to (4.1c), the support of  $\nabla_{\tilde{\mathcal{D}}} u$  is also reduced to  $K$ , and that we may write, thanks to (2.4),

$$\int_K \Lambda(\mathbf{x}) \nabla_{\tilde{\mathcal{D}}} u(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} v(\mathbf{x}) d\mathbf{x} = \int_K f(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x}.$$

Setting  $\mathbf{w} = \mathbf{v}$  in (4.1c) where  $u$  is replaced by  $v$ , we get, using  $\Gamma_{\tilde{\mathcal{D}}} v = 0$ ,

$$\int_K \mathbf{v}(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} v(\mathbf{x}) d\mathbf{x} + \int_K \Pi_{\tilde{\mathcal{D}}} v(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = 0,$$

which implies

$$\int_K \Lambda(\mathbf{x}) \nabla_{\tilde{\mathcal{D}}} u(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} v(\mathbf{x}) d\mathbf{x} = \int_K \Pi_{\tilde{\mathcal{D}}} v(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x} = \int_K \psi(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x}) d\mathbf{x}.$$

This completes the proof of (1.7c). Then, we let  $v \in X_{\tilde{\mathcal{D}},0}$  in (2.4) such that  $\Pi_{\tilde{\mathcal{D}}} v = 0$  and  $\Gamma_{\tilde{\mathcal{D}}} v|_{\sigma} = \lambda|_{\sigma}$  for a given  $\sigma = K|L \in \mathcal{E}_{\text{int}}$ , and  $\Gamma_{\tilde{\mathcal{D}}} v|_{\sigma'} = 0$  for all  $\sigma' \in \mathcal{E} \setminus \{\sigma\}$ . Again setting  $\mathbf{w} = \mathbf{v}$  in (4.1c) where  $u$  is replaced by  $v$ , we get, using  $\Pi_{\tilde{\mathcal{D}}} v = 0$ ,

$$\int_K \mathbf{v}(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} v(\mathbf{x}) d\mathbf{x} - \int_{\sigma} \Gamma_{\tilde{\mathcal{D}}} v(\mathbf{x}) \mathbf{v}|_K(\mathbf{x}) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{x}) = 0,$$

and

$$\int_L \mathbf{v}(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} v(\mathbf{x}) d\mathbf{x} - \int_{\sigma} \Gamma_{\tilde{\mathcal{D}}} v(\mathbf{x}) \mathbf{v}|_L(\mathbf{x}) \cdot \mathbf{n}_{L,\sigma} d\gamma(\mathbf{x}) = 0.$$

This implies (1.7d), using (2.4) and the fact that the support of  $\nabla_{\mathcal{D}}v$  is reduced to  $K \cup L$ .

Conversely, considering the solution  $(\mathbf{v}, q, \lambda)$  to (1.7), since  $q \in W_h$  and  $\lambda \in M_h^0$ , there exists a unique  $u \in X_{\mathcal{D},0}$  such that  $q = \Pi_{\mathcal{D}}u$  and  $\lambda = \Gamma_{\mathcal{D}}u$ . From (1.7b), we get that  $\mathbf{v} = -\Lambda \nabla_{\mathcal{D}}u$ . For any  $v \in X_{\mathcal{D},0}$ , letting  $\psi = \Pi_{\mathcal{D}}v$  and  $\mu = \Gamma_{\mathcal{D}}v$ , following the same computation lines as the beginning of this proof, we get that (1.7c) and (1.7d) imply (2.4) using (4.1c) where  $u$  is replaced by  $v$ .  $\square$

Here again, the fact that we wish to obtain a mixed finite element scheme has led us to a problem dependent discretization. For the linear problem (1.1), this is not a difficulty. However if the diffusion tensor depends on the unknown, it becomes very intricate to ensure the  $H_{\text{div}}$  conformity, and in fact quite useless since one can get the approximate continuity of the flux from the gradient discretization itself. This line of thought may lead to consider the scheme (4.1), replacing  $\Lambda \nabla_{\mathcal{D}}u \in \mathbf{V}_h$  by  $\nabla_{\mathcal{D}}u \in \mathbf{V}_h$  in (4.1c).

## 5. Application to two phase flow in porous media and Richards' equation

We are interested here in the approximation of  $(u, v)$ , solution to the incompressible two-phase flow problem in the space domain  $\Omega$  during the time period  $(0, T)$ , which was studied in [12] in the case of a general gradient discretization; here we propose to show that the specific gradient discretization (4.1) obtained from the dual mixed finite element formulation is well adapted to this problem. The continuous problem reads:

$$\Phi(\mathbf{x})\partial_t S(\mathbf{x}, p) - \text{div}(k_1(\mathbf{x}, S(\mathbf{x}, p))\Lambda(\mathbf{x})\nabla u) = f_1, \quad (5.1a)$$

$$\Phi(\mathbf{x})\partial_t(1 - S(\mathbf{x}, p)) - \text{div}(k_2(\mathbf{x}, S(\mathbf{x}, p))\Lambda(\mathbf{x})\nabla v) = f_2, \quad (5.1b)$$

$$p = u - v, \text{ for } \in \Omega \times (0, T), \quad (5.1c)$$

considered with the following initial condition:

$$S(\mathbf{x}, p(\mathbf{x}, 0)) = S(\mathbf{x}, p_{\text{mi}}(\mathbf{x})), \text{ for a.e. } x \in \Omega, \quad (5.1d)$$

together with the following non-homogeneous Dirichlet boundary conditions:

$$u(\mathbf{x}, t) = \bar{u}(\mathbf{x}) \text{ and } v(\mathbf{x}, t) = \bar{v}(\mathbf{x}) \text{ on } \partial\Omega \times (0, T). \quad (5.1e)$$

In Problem (5.1),  $u$  (resp.  $v$ ) denotes the pressure of the phase 1, called the wetting phase (resp. of the phase 2, which is the non-wetting phase),  $p$  is the difference between the two pressures, called the capillary pressure, the saturation of the phase 1 is denoted by  $S(\mathbf{x}, p)$  (it is called the “water content” in the framework of Richards’ equation), and where  $\Phi$ ,  $\Lambda$ ,  $k_i$ , and  $f_i$  ( $i = 1, 2$ ) respectively denote the porosity, the absolute permeability, the relative permeabilities, and the source terms, under the following assumptions:

- $\Omega$  is an open bounded connected polyhedral subset of  $\mathbb{R}^d$ ,

$$d \in \mathbb{N}^* \text{ and } T > 0, \quad (5.2a)$$

- $\Phi$  is a measurable function from  $\Omega$  to  $\mathbb{R}$  with

$$\Phi(\mathbf{x}) \in [\Phi_{\min}, \Phi_{\max}], \quad \Phi_{\max} \geq \Phi_{\min} > 0, \quad (5.2b)$$

- $\Lambda$  is a measurable function from  $\Omega$  to  $\mathcal{M}_d$ , where

$\mathcal{M}_d$  denotes the set of  $d \times d$  matrices, such that for a.e.  $\mathbf{x} \in \Omega$ ,

$\Lambda(\mathbf{x})$  is symmetric, and the set of its eigenvalues is included in  $[\underline{\lambda}, \bar{\lambda}]$ ,

$$\text{with } 0 < \underline{\lambda} \leq \bar{\lambda}, \quad (5.2c)$$

- $p_{\text{ini}} \in L^2(\Omega)$ , (5.2d)

- $S(\mathbf{x}, q) \in [0, 1]$  for all  $(\mathbf{x}, q) \in \Omega \times \mathbb{R}$  with  $S(\mathbf{x}, q) = S_j(q)$

for a.e.  $\mathbf{x} \in \Omega_j$  and all  $q \in \mathbb{R}$ , where  $S_j$  is a non decreasing Lipschitz

continuous function with constant  $L_S$ ,

$(\Omega_j)_{j \in J}$  is a family of disjoint connected polyhedral open sets such that

$$\bigcup_{j \in J} \overline{\Omega_j} = \bar{\Omega} \text{ where } J \text{ is a finite set,} \quad (5.2e)$$

- $f_i \in L^2(\Omega \times (0, T))$ ,  $i = 1, 2$ , (5.2f)

- $k_i(\mathbf{x}, s) \in [k_{\min}, k_{\max}]$  for  $(\mathbf{x}, s) \in \Omega \times [0, 1]$  with  $k_{\max} \geq k_{\min} > 0$  and

$$k_i(\cdot, s) \text{ measurable, } k_i(\mathbf{x}, \cdot) \text{ continuous, } i = 1, 2, \quad (5.2g)$$



$$\bullet \bar{u}, \bar{v} \in H^1(\Omega). \quad (5.2h)$$

**Remark 5.1 (Gravity terms).** *Gravity terms are taken into account in the mathematical analysis performed in [12]; we choose not to include them here for the sake of brevity and clarity.*

Assumptions (5.2) are quite general, except for  $k_{\min} > 0$  in Hypothesis (5.2g). This assumption is needed in the mathematical proof of convergence [12]. Assumption (5.2e) is compatible with the so-called Van Genuchten model  $S_j(p) = 1/((\max(-p, 0)/p_j)^{n_j} + 1)^{m_j}$ , with real parameters  $p_j, n_j, m_j > 0$ . The hypothesis that the function  $S(\mathbf{x}, p)$  is defined by given functions in a partition of the domain is classical. Problem (5.1) is considered under the following weak sense.

$$\begin{aligned} u - \bar{u}, v - \bar{v} \in L^2(0, T; H_0^1(\Omega)), \quad p = u - v; \quad \forall \varphi \in C_c^\infty(\Omega \times [0, T]), \\ \int_0^T \int_\Omega (-\Phi(\mathbf{x})S(\mathbf{x}, p(\mathbf{x}, t))\partial_t \varphi(\mathbf{x}, t) \\ + k_1(\mathbf{x}, S(\mathbf{x}, p(\mathbf{x}, t)))\Lambda(\mathbf{x})\nabla u(\mathbf{x}, t) \cdot \nabla \varphi(\mathbf{x}, t)) d\mathbf{x} dt \end{aligned} \quad (5.3a)$$

$$\begin{aligned} - \int_\Omega \Phi(\mathbf{x})S(\mathbf{x}, p_{\text{ini}}(\mathbf{x}))\varphi(\mathbf{x}, 0) d\mathbf{x} = \int_0^T \int_\Omega f_1(\mathbf{x}, t)\varphi(\mathbf{x}, t) d\mathbf{x} dt, \\ \int_0^T \int_\Omega (\Phi(\mathbf{x})S(\mathbf{x}, p(\mathbf{x}, t))\partial_t \varphi(\mathbf{x}, t) \\ + k_2(\mathbf{x}, S(\mathbf{x}, p(\mathbf{x}, t)))\Lambda(\mathbf{x})\nabla v(\mathbf{x}, t) \cdot \nabla \varphi(\mathbf{x}, t)) d\mathbf{x} dt \end{aligned} \quad (5.3b)$$

$$+ \int_\Omega \Phi(\mathbf{x})S(\mathbf{x}, p_{\text{ini}}(\mathbf{x}))\varphi(\mathbf{x}, 0) d\mathbf{x} = \int_0^T \int_\Omega f_2(\mathbf{x}, t)\varphi(\mathbf{x}, t) d\mathbf{x} dt,$$

where we denote by  $C_c^\infty(\Omega \times [0, T])$  the set of the restrictions of functions of  $C_c^\infty(\Omega \times (-\infty, T))$  to  $\Omega \times [0, T]$ . Alternately, we also consider the case of Richards' equation, which can be obtained from (5.1) in two ways. One can replace (5.1a) by

$$u(\mathbf{x}, t) = \bar{u}(\mathbf{x}) \text{ for } (\mathbf{x}, t) \in \Omega \times (0, T), \quad (5.4)$$

or one can replace (5.1b) by

$$v(\mathbf{x}, t) = \bar{v}(\mathbf{x}) \text{ for } (\mathbf{x}, t) \in \Omega \times (0, T). \quad (5.5)$$

Then the corresponding weak formulation of the problem is obtained by replacing (5.3a) by (5.4) or (5.3b) by (5.5). Note that this sense is an extension of the condition classically used in hydrogeological studies, which prescribes a constant condition with respect to time and space for the air pressure. As shown in the numerical examples of [12], this extension allows to use Richards' equation as a good approximation of the full two phase flow problem in other engineering frameworks.

We consider a time interval  $(0, T)$  and  $(t^{(n)})_{n=0, \dots, N}$  such that  $t^{(0)} = 0 < t^{(1)} \dots < t^{(N)} = T$ . We then set  $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$ , for  $n = 0, \dots, N - 1$ . Let us consider the following scheme for the approximation of Problem (5.1) (let us emphasize that this discretization is not based on the global pressure formulation of the problem). We consider the gradient discretization given by (4.1) (we emphasize that we cannot consider the gradient discretization given by (3.1), for an important reason explained below); recall that this discretization is partially problem dependent since it depends on the diffusion tensor  $\Lambda$ . The gradient scheme is then given by

$$\textit{Initialization: } s_{\tilde{\mathcal{D}}}^{(0)}(\mathbf{x}) = S(\mathbf{x}, \Pi_{\tilde{\mathcal{D}}} p^{(0)}(\mathbf{x})), \quad (5.6a)$$

For  $n = 0, \dots, N - 1$ :

$$u^{(n+1)} - \bar{u}_{\tilde{\mathcal{D}}} \in X_{\tilde{\mathcal{D}},0}, \quad v^{(n+1)} - \bar{v}_{\tilde{\mathcal{D}}} \in X_{\tilde{\mathcal{D}},0}, \quad (5.6b)$$

$$p^{(n+1)} = u^{(n+1)} - v^{(n+1)}, \quad s_{\tilde{\mathcal{D}}}^{(n+1)}(\mathbf{x}) = S(\mathbf{x}, \Pi_{\tilde{\mathcal{D}}} p^{(n+1)}(\mathbf{x})), \quad (5.6c)$$

$$\delta_{\tilde{\mathcal{D}}}^{(n+\frac{1}{2})} s_{\tilde{\mathcal{D}}}(\mathbf{x}) = \frac{s_{\tilde{\mathcal{D}}}^{(n+1)}(\mathbf{x}) - s_{\tilde{\mathcal{D}}}^{(n)}(\mathbf{x})}{\delta t^{(n+\frac{1}{2})}}, \quad (5.6d)$$

$$\begin{aligned} & \int_{\Omega} \left( \Phi(\mathbf{x}) \delta_{\tilde{\mathcal{D}}}^{(n+\frac{1}{2})} s_{\tilde{\mathcal{D}}}(\mathbf{x}) \Pi_{\tilde{\mathcal{D}}} w(\mathbf{x}) \right. \\ & \quad \left. + k_1(\mathbf{x}, s_{\tilde{\mathcal{D}}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x}) \nabla_{\tilde{\mathcal{D}}} u^{(n+1)}(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} w(\mathbf{x}) \right) d\mathbf{x} = \end{aligned} \quad (5.6e)$$

$$\frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} f_1(\mathbf{x}, t) \Pi_{\tilde{\mathcal{D}}} w(\mathbf{x}) d\mathbf{x} dt, \quad \forall w \in X_{\tilde{\mathcal{D}},0},$$

$$\begin{aligned} & \int_{\Omega} \left( -\Phi(\mathbf{x}) \delta_{\tilde{\mathcal{D}}}^{(n+\frac{1}{2})} s_{\tilde{\mathcal{D}}}(\mathbf{x}) \Pi_{\tilde{\mathcal{D}}} w(\mathbf{x}) \right. \\ & \quad \left. + k_2(\mathbf{x}, s_{\tilde{\mathcal{D}}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x}) \nabla_{\tilde{\mathcal{D}}} v^{(n+1)}(\mathbf{x}) \cdot \nabla_{\tilde{\mathcal{D}}} w(\mathbf{x}) \right) d\mathbf{x} = \end{aligned} \quad (5.6f)$$

$$\frac{1}{\delta t^{(n+\frac{1}{2})}} \int_{t^{(n)}}^{t^{(n+1)}} \int_{\Omega} f_2(\mathbf{x}, t) \Pi_{\bar{\mathcal{D}}} w(\mathbf{x}) d\mathbf{x} dt, \quad \forall w \in X_{\bar{\mathcal{D}}, 0}.$$

In the case of the Richards equation, the scheme is obtained by replacing one of the discrete conservation equations (5.6e) or (5.6f) by the imposed value for the pressure.

Note that this scheme leads to sparse linear systems if Newton's method is used to solve (5.6) and that this scheme is implementable in practice. Indeed, let us consider the equations corresponding to a test function  $w$  given by  $w_i = 1$  for some  $i \in I$ ,  $w_{i'} = 0$  for all  $i' \in I \setminus \{i\}$ ,  $w_j = 0$  for all  $j \in J$ . Let us denote by  $K$  the support of the basis function  $\chi_i$ . Then the discrete equations thus obtained only depend on the unknowns  $u_{i'}^{(n+1)}$  and  $v_{i'}^{(n+1)}$ , where  $i'$  is such that the support of  $\chi_{i'}$  is also  $K$ , and on the unknowns  $u_j^{(n+1)}$  and  $v_j^{(n+1)}$ , where  $j$  is such that the support of  $\xi_j$  is some  $\sigma$ , where  $\sigma$  is a face of  $K$ . Let us now consider the equations corresponding to a test function  $w$  given by  $w_j = 1$  for some  $j \in J$ ,  $w_{j'} = 0$  for all  $j' \in J \setminus \{j\}$ ,  $w_i = 0$  for all  $i \in I$ . Let us denote by  $\sigma$  the support of the basis function  $\xi_j$ , and by  $K$  and  $L$  the two cells having the common face  $\sigma$ . Then the discrete equations thus obtained only depend on the unknowns  $u_{j'}^{(n+1)}$  and  $v_{j'}^{(n+1)}$  where  $j'$  is such that the support of  $\xi_{j'}$  is a face of  $K$  or  $L$ , and on the unknowns  $u_i^{(n+1)}$  and  $v_i^{(n+1)}$  where  $i$  is such that the support of  $\chi_i$  is  $K$  or  $L$ .

Let us remark that the use of the gradient discretization given by (3.1) would not provide an easy discrete problem. Indeed, with this gradient discretization, the discrete gradients involved in (5.6) cannot be reduced to a local equation, using the same computations as that of the proof of Theorem 3.4, since the use of (3.1d) for the test functions gives no way to transform an expression such as  $\int_{\Omega} k_1(\mathbf{x}, s_{\bar{\mathcal{D}}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x}) \nabla_{\bar{\mathcal{D}}} u^{(n+1)}(\mathbf{x}) \cdot \nabla_{\bar{\mathcal{D}}} w(\mathbf{x}) d\mathbf{x}$  in order to get  $\int_{\Omega} \Pi_{\bar{\mathcal{D}}} w(\mathbf{x}) \operatorname{div}(k_1(\mathbf{x}, s_{\bar{\mathcal{D}}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x}) \nabla_{\bar{\mathcal{D}}} u^{(n+1)}(\mathbf{x})) d\mathbf{x}$ : the  $H_{\operatorname{div}}$  conformity of  $k_1(\mathbf{x}, s_{\bar{\mathcal{D}}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x}) \nabla_{\bar{\mathcal{D}}} u^{(n+1)}$  is indeed not ensured by the scheme. Therefore the practical implementation of this gradient discretization appears as impossible, if we do not replace  $\Lambda$  by  $k_1(\mathbf{x}, s_{\bar{\mathcal{D}}}^{(n+1)}(\mathbf{x})) \Lambda(\mathbf{x})$  in (3.1). This would lead to

some complications (different gradient discretizations for  $u$  and  $v$  at least).

The convergence of the scheme (5.6) is proven in [12] for general gradient discretizations which are coercive, consistent, limit-conforming and compact in the sense recalled in this paper, and which are such that  $\Pi_{\mathcal{D}}$  is a piecewise constant reconstruction operator. Therefore, this convergence proof applies to the modified  $RT_0$  scheme (5.6) as stated below.

**Theorem 5.2 (Convergence of the numerical scheme).** *Let  $(\tilde{\mathcal{D}}_m)_{m \in \mathbb{N}}$  be a sequence of gradient discretizations given by (4.1) with  $k = 0$ , and, for all  $m \in \mathbb{N}$ , let  $(t_m^{(n)})_{n=0, \dots, N_m}$  such that  $t_m^{(0)} = 0 < t^{(1)} \dots < t_m^{(N_m)} = T$  be such that  $\lim_{m \rightarrow \infty} \max_n (t_m^{(n+1)} - t_m^{(n)}) = 0$ . Then the scheme (5.6) has at least one solution for any  $m \in \mathbb{N}$ .*

Moreover, we assume that are given  $p_m^{(0)}, \bar{u}_{\tilde{\mathcal{D}}_m}, \bar{v}_{\tilde{\mathcal{D}}_m} \in X_{\tilde{\mathcal{D}}_m}$  such that the sequences  $\|\Pi_{\tilde{\mathcal{D}}_m} p_m^{(0)} - p_{\text{ini}}\|_{L^2(\Omega)}$ ,  $\|\Pi_{\tilde{\mathcal{D}}_m} \bar{u}_{\tilde{\mathcal{D}}_m} - \bar{u}\|_{L^2(\Omega)}$ ,  $\|\Pi_{\tilde{\mathcal{D}}_m} \bar{v}_{\tilde{\mathcal{D}}_m} - \bar{v}\|_{L^2(\Omega)}$ ,  $\|\nabla_{\tilde{\mathcal{D}}_m} \bar{u}_{\tilde{\mathcal{D}}_m} - \nabla \bar{u}\|_{L^2(\Omega)^d}$ ,  $\|\nabla_{\tilde{\mathcal{D}}_m} \bar{v}_{\tilde{\mathcal{D}}_m} - \nabla \bar{v}\|_{L^2(\Omega)^d}$  tend to 0 as  $m \rightarrow \infty$ . Let  $u_m, v_m, s_{\tilde{\mathcal{D}}_m}$  be the piecewise constant time implicit functions equal on  $(t^{(n)}, t^{(n+1)})$  to  $u_m^{(n+1)}, v_m^{(n+1)} \in X_{\tilde{\mathcal{D}}_m, 0}$ , such that the scheme (5.6) holds for  $m \in \mathbb{N}$ .

Then there exist  $u, v \in (L^2(\Omega \times (0, T)))^2$  and  $p = u - v$  such that, for any  $t_0 \in (0, T)$ , up to a subsequence,

1.  $(\Pi_{\mathcal{D}_m} u_m, \Pi_{\mathcal{D}_m} v_m)$  converges in  $L^2(\Omega \times (0, t_0))^2$  to  $(u, v)$  as  $m \rightarrow \infty$ ,
2.  $(\nabla_{\mathcal{D}_m} u_m, \nabla_{\mathcal{D}_m} v_m)$  converges in  $(L^2(\Omega \times (0, t_0))^d)^2$  to  $(\nabla u, \nabla v)$  as  $m \rightarrow \infty$ ,
3.  $s_{\mathcal{D}_m}$  converges in  $L^2(\Omega \times (0, T))$  to  $s$  such that  $s(\mathbf{x}, t) = S(\mathbf{x}, p(\mathbf{x}, t))$ , for a.e.  $(\mathbf{x}, t) \in \Omega \times (0, T)$  as  $m \rightarrow \infty$ ,

and  $(u, v)$  is a weak solution of Problem (5.3).

In conclusion, the gradient discretization concept gives a simple way to adapt the mixed finite element context to the above nonlinear problem, by dropping the requirement of the  $H_{\text{div}}$  conformity for the nonlinear part of the equations.

## References

- [1] B. Amaziane, M. El Ossmani, C. Serres, Numerical modeling of the flow and transport of radionuclides in heterogeneous porous media, *Comput. Geosci.* 12 (2008) 437–449. URL: <http://dx.doi.org/10.1007/s10596-008-9083-0>. doi:10.1007/s10596-008-9083-0.
- [2] T. Arbogast, M.F. Wheeler, N.Y. Zhang, A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media, *SIAM J. Numer. Anal.* 33 (1996) 1669–1687. URL: <http://dx.doi.org/10.1137/S0036142994266728>. doi:10.1137/S0036142994266728.
- [3] D.N. Arnold, F. Brezzi, Mixed and conforming finite element methods; implementation, postprocessing and error estimates, *Modélisation mathématique et analyse numérique* 19 (1985) 7–32.
- [4] G. Chavent, J. Jaffré, *Mathematical models and finite elements for reservoir simulation: single phase, multiphase, and multicomponent flows through porous media*, volume 17, North Holland, 1986.
- [5] J. Douglas, Jr., J.E. Roberts, Global estimates for mixed methods for second order elliptic equations, *Math. Comp.* 44 (1985) 39–52. URL: <http://dx.doi.org/10.2307/2007791>. doi:10.2307/2007791.
- [6] J. Droniou, R. Eymard, T. Gallouët, R. Herbin, A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods, *Math. Models Methods Appl. Sci.* 20 (2010) 265–295. URL: <http://dx.doi.org/10.1142/S0218202510004222>. doi:10.1142/S0218202510004222.
- [7] J. Droniou, R. Eymard, T. Gallouët, R. Herbin, Gradient schemes: a generic framework for the discretisation of linear, nonlinear and nonlocal elliptic and parabolic equations, *Math. Models Methods Appl. Sci.* 23 (2013) 2395–2432.

- [8] R. Durán, Mixed finite elements, in: D. Boffi, L. Gastaldi (Eds.), Mixed finite elements, compatibility conditions, and applications: lectures given at the CIME Summer School held in Cetraro, Italy, June 26-July 1, 2006, volume 1939 of *Lecture Notes in Mathematics*, Springer, 2008, pp. 1–44.
- [9] R. Eymard, P. Féron, T. Gallouët, R. Herbin, C. Guichard, Gradient schemes for the Stefan problem, <http://hal.archives-ouvertes.fr/>, International Journal On Finite Volumes (2013). URL: <http://www.latp.univ-mrs.fr/IJFV/>.
- [10] R. Eymard, T. Gallouët, R. Herbin, Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI: a scheme using stabilization and hybrid interfaces, *IMA J. Numer. Anal.* 30 (2010) 1009–1043. URL: <http://dx.doi.org/10.1093/imanum/drn084>. doi:10.1093/imanum/drn084.
- [11] R. Eymard, C. Guichard, R. Herbin, Small-stencil 3d schemes for diffusive flows in porous media, *ESAIM: Mathematical Modelling and Numerical Analysis* 46 (2012) 265–290.
- [12] R. Eymard, C. Guichard, R. Herbin, R. Masson, Gradient schemes for two-phase flow in heterogeneous porous media and Richards equation, *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 94 (2014) 560–585. URL: <http://dx.doi.org/10.1002/zamm.201200206>. doi:10.1002/zamm.201200206.
- [13] R. Eymard, A. Handlovicova, R. Herbin, K. Mikula, O. Stasová, Applications of approximate gradient schemes for nonlinear parabolic equations, <http://hal.archives-ouvertes.fr/hal-00744767/PDF/perorev.pdf>, accepted for publication in *Applications of Mathematics* (2014).
- [14] R. Eymard, G. Henry, R. Herbin, F. Hubert, R. Klöforn, G. Manzini, 3d benchmark on discretization schemes for anisotropic diffusion problem on general grids, in: *Finite volumes for complex applications VI*, Springer, 2011, pp. 895–930.

- [15] R. Herbin, F. Hubert, Benchmark on discretization schemes for anisotropic diffusion problems on general grids for anisotropic heterogeneous diffusion problems, in: R. Eymard, J.M. Hérard (Eds.), *Finite Volumes for Complex Applications V*, Wiley, 2008, pp. 659–692.
- [16] P.A. Raviart, J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, in: *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*, Springer, Berlin, 1977, pp. 292–315. *Lecture Notes in Math.*, Vol. 606.
- [17] J.E. Roberts, J.M. Thomas, Mixed and hybrid methods, in: *Handbook of numerical analysis, Vol. II, Handb. Numer. Anal., II*, North-Holland, Amsterdam, 1991, pp. 523–639.
- [18] F.J. Sayas, From Raviart-Thomas to HDG, <http://arxiv.org/pdf/1307.2491v1.pdf>, 2013. URL: <http://arxiv.org/abs/1307.2491>.