Capturing equilibrium models in modal logic

Luis Fariñas del Cerro, Andreas Herzig, Ezgi Iraz Su

▶ **To cite this version:**

Luis Fariñas del Cerro, Andreas Herzig, Ezgi Iraz Su. Capturing equilibrium models in modal logic. Journal of Applied Logic, 2014, vol. 12 (n° 2), pp. 192-207. 10.1016/j.jal.2013.12.003 . hal-01127960

HAL Id: hal-01127960

https://hal.science/hal-01127960

Submitted on 9 Mar 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Open Archive TOULOUSE Archive Ouverte (OATAO)
OATAO is an open access repository that collects the work of Toulouse researchers and makes it freely available over the web where possible.

This is an author-deposited version published in : http://oatao.univ-toulouse.fr/
Eprints ID : 12614

To link to this article : DOI :10.1016/j.jal.2013.12.003
URL : http://dx.doi.org/10.1016/j.jal.2013.12.003

To cite this version : Fariñas del Cerro, Luis and Herzig, Andreas and Su, Ezgi Iraz *Capturing equilibrium models in modal logic.* (2014) Journal of Applied Logic, vol. 12 (n° 2). pp. 192-207. ISSN 1570-8683

Any correspondance concerning this service should be sent to the repository administrator: staff-oatao@listes-diff.inp-toulouse.fr

Capturing equilibrium models in modal logic

Luis Fariñas del Cerro, Andreas Herzig, Ezgi Iraz Su [*,1]

Université de Toulouse, CNRS, IRIT, 118 Route de Narbonne, F-31062 Toulouse, Cedex 9, France

ABSTRACT

Here-and-there models and equilibrium models were investigated as a semantical framework for answer-set programming by Pearce, Valverde, Cabalar, Lifschitz, Ferraris and others. The semantics of equilibrium logic is given in an indirect way: the notion of an equilibrium model is defined in terms of quantification over here-and-there models. We here give a direct semantics of equilibrium logic, stated for a modal language embedding the language of equilibrium logic.

1. Introduction

A here-and-there (HT) model (H, T) is a couple of sets of propositional variables, H ('here') and T ('there') such that $H \subseteq T$. We understand the inclusion informally as H being *weaker* than T. The logical language to talk about HT models has connectives $\bot$, $\wedge$, $\vee$, and $\rightarrow$. The latter is interpreted in a non-classical way and is therefore different from material implication $\supset$. Its truth condition is:

$$H, T \models \varphi \rightarrow \psi \quad \text{iff } H, T \models \varphi \supset \psi \quad \text{and} \quad T, T \models \varphi \supset \psi,$$

where $\supset$ is interpreted just as in classical propositional logic.[2] HT models give semantics to an implication with strength between intuitionistic and material implication. They were investigated by Pearce, Valverde, Cabalar, Lifschitz, Ferraris, and others as the basis of equilibrium logic, the latter providing a semantical framework for answer-set programming [20,19,22,5,6,14,18].

Equilibrium models of a formula, φ, are defined in an indirect way that is based on HT models: an equilibrium model of φ is a set of propositional variables T such that

* Corresponding author.
E-mail addresses: farinas@irit.fr (L. Fariñas del Cerro), herzig@irit.fr (A. Herzig), Ezgi-Iraz.Su@irit.fr (E.I. Su).

[1] Ezgi Iraz Su is not a member of CNRS.

[2] Material implication '$\supset$' here is just a shorthand enabling a concise formulation. To spell it out, its truth condition is: $H, T \models \varphi \supset \psi$ iff $H, T \not\models \varphi$ or $H, T \models \psi$.

1. $T \models \varphi$ in propositional logic, and
2. there is no HT model (H, T) such that H is strictly weaker than T and $H, T \models \varphi$.

Observe that the condition '$T \models \varphi$ in propositional logic' can be replaced by '$T, T \models \varphi$ in the logic of here-and-there'. To give an example, $T = \emptyset$ is an equilibrium model of $p \to \bot$ because (1) for the HT model $(\emptyset, \emptyset)$ we have $\emptyset, \emptyset \models p \to \bot$, and (2) there is no set H that is strictly included in the empty set. Moreover, $T = \emptyset$ is the only equilibrium model of $p \to \bot$. To see this, suppose T is an equilibrium model for $p \to \bot$ for some $T \neq \emptyset$. Then T cannot contain p, otherwise condition (1) would be violated. Therefore T contains q for some $q \neq p$, but then condition (2) is violated since $\emptyset, T \models p \to \bot$.

In the present paper we give a direct semantics of equilibrium logic in terms of a modal language extending that of propositional logic by two unary modal operators, [T] and [S]. Roughly speaking, [T] allows to talk about valuations[3] that are at least as strong as the actual valuation; and [S] allows to talk about valuations that are weaker than the actual valuation. Our modal language can be interpreted on HT models. However, we also give a semantics in terms of Kripke models. We call our logic **MEM**: the Modal Logic of Equilibrium Models.

We relate the language of equilibrium logic to our bimodal language by means of the Gödel translation, tr, whose main clause is:

$$tr(\varphi \to \psi) = [\mathrm{T}]\big(tr(\varphi) \supset tr(\psi)\big).$$

A first attempt to relate equilibrium logic to modal logic in the style of the present approach was presented in [12]. We here extend and improve that paper by simplifying the translation.

The paper is organised as follows. In Section 2 we introduce our modal logic of equilibrium models, **MEM**,[4] syntactically, semantically and also axiomatically. In Section 3 we recall both the logic of here-and-there and equilibrium logic. In Section 4 we define the Gödel translation, tr, from the language of the logic of here-and-there to the language of **MEM** and prove its correctness: for every formula φ, φ is HT valid if and only if $tr(\varphi)$ is **MEM** valid. This theorem paves the way for the proof of the grand finale given in Section 5: φ is a logical consequence of χ in equilibrium logic if and only if the modal formula

$$\big(tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi)\big) \supset tr(\varphi)$$

is valid in **MEM**. It follows that φ has an equilibrium model if and only if $tr(\varphi) \wedge [\mathrm{S}]\neg tr(\varphi)$ is satisfiable in the corresponding Kripke model. Section 6 makes a brief overview of our past, present and future interests. They all appear in a line of work that aims to reexamine the logical foundations of equilibrium logic and answer-set programming.

2. The modal logic of equilibrium models: MEM

We introduce the modal logic of equilibrium models, **MEM**, in the classical way: we start by defining its bimodal language and its semantics. Then we axiomatise its validities.

2.1. Language

Throughout the paper we suppose $\mathbb{P}$ is a countably infinite set of propositional variables. The elements of $\mathbb{P}$ are noted p, q, etc. Our language $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ is bimodal: it has two modal operators, [T] and [S]. Precisely, $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ is defined by the following grammar:

[3] Here, and in general in this paragraph, we use the term 'valuation' in the sense of a set of proposition variables.

[4] To avoid confusion we could have used another name instead of **MEM** again. It should however be clear to the reader that the modal logic we are talking about, here, is just slightly different from the one we introduced in [12].

$$\varphi ::= p \mid \bot \mid \varphi \supset \varphi \mid [\mathrm{T}]\varphi \mid [\mathrm{S}]\varphi,$$

where p ranges over $\mathbb{P}$. The formula $[\mathrm{T}]\varphi$ may be read "φ holds at every possible there-world at least as strong as the current world", and $[\mathrm{S}]\varphi$ may be read "φ holds at every possible weaker or equal here-world".

The set of propositional variables occurring in a formula φ is noted $\mathbb{P}_\varphi$.

The language $\mathcal{L}_{[\mathrm{T}]}$ is the set of $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$-formulas without the modal operator $[\mathrm{S}]$. So the $\mathcal{L}_{[\mathrm{T}]}$-formulas are built from $[\mathrm{T}]$ and the Boolean connectives only.

We use the following standard abbreviations: $\top \stackrel{\text{def}}{=} \bot \supset \bot$, $\neg\varphi \stackrel{\text{def}}{=} \varphi \supset \bot$, $\varphi \vee \psi \stackrel{\text{def}}{=} \neg\varphi \supset \psi$, $\varphi \wedge \psi \stackrel{\text{def}}{=} \neg(\varphi \supset \neg\psi)$, and $\varphi \equiv \psi \stackrel{\text{def}}{=} (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. Moreover, $\langle \mathrm{T}\rangle\varphi$ and $\langle \mathrm{S}\rangle\varphi$ respectively abbreviate $\neg[\mathrm{T}]\neg\varphi$ and $\neg[\mathrm{S}]\neg\varphi$.

2.2. *Kripke frames*

Consider the class of Kripke frames $\langle W, \mathcal{T}, \mathcal{S}\rangle$ such that

- W is a non-empty set of possible worlds;
- $\mathcal{T}, \mathcal{S} \subseteq W \times W$ are (binary) relations on W such that:

$\mathrm{refl}(\mathcal{T})$	for every w, $w\mathcal{T}w$;
$\mathrm{alt}_2(\mathcal{T})$	for every w, u, u', u'', if $w\mathcal{T}u, w\mathcal{T}u'$ and $w\mathcal{T}u''$ then $u = u'$ or $u = u''$ or $u' = u''$;
$\mathrm{trans}(\mathcal{T})$	for every w, u, v, if $w\mathcal{T}u$ and $u\mathcal{T}v$ then $w\mathcal{T}v$;
$\mathrm{refl}_2(\mathcal{S})$	for every w, u, if $w\mathcal{S}u$ then $u\mathcal{S}u$;
$\mathrm{wtriv}_2(\mathcal{S})$	for every w, u, v, if $w\mathcal{S}u$ and $u\mathcal{S}v$ then $u = v$;
$\mathrm{wmconv}(\mathcal{T}, \mathcal{S})$	for every w, u, if $w\mathcal{T}u$ then $w = u$ or $u\mathcal{S}w$;
$\mathrm{mconv}(\mathcal{S}, \mathcal{T})$	for every w, u, if $w\mathcal{S}u$ then $u\mathcal{T}w$.

We call a frame $\langle W, \mathcal{T}, \mathcal{S}\rangle$ satisfying the above-mentioned constraints a **MEM** frame. Let us explain these constraints informally.

To begin with, the first three constraints are about the relation $\mathcal{T}$. The constraints $\mathrm{refl}(\mathcal{T})$ and $\mathrm{alt}_2(\mathcal{T})$ say respectively that a world w is $\mathcal{T}$-reflexive and has at most two $\mathcal{T}$-successors. To sum it up, a world w is either a single $\mathcal{T}$-loop or has an accompanying $\mathcal{T}$-accessible world. Then the transitivity constraint, $\mathrm{trans}(\mathcal{T})$, makes that the neighbouring $\mathcal{T}$-accessible world is a single $\mathcal{T}$-loop. Briefly, these constraints together imply the following constraint about the relation $\mathcal{T}$:

$$\mathrm{depth}_1(\mathcal{T}): \quad \text{for every } w, u, v, \text{ if } w\mathcal{T}u \text{ and } u\mathcal{T}v \text{ then } w = u \text{ or } u = v.$$

In words, every world can be reached in at most one $\mathcal{T}$-step.

The next two constraints are about the relation $\mathcal{S}$. Let $\mathcal{S}(u) = \{v\colon\ u\mathcal{S}v\}$. For any w, u, if $w\mathcal{S}u$ then the constraint $\mathrm{refl}_2(\mathcal{S})$ gives us $u \in \mathcal{S}(u)$. The constraint $\mathrm{wtriv}_2(\mathcal{S})$ tells us that when $w\mathcal{S}u$ then we must have $\mathcal{S}(u) = \emptyset$ or $\mathcal{S}(u) = \{u\}$. Together, they say that if $w\mathcal{S}u$ then $\mathcal{S}(u) = \{u\}$: any world we access by the relation $\mathcal{S}$ can see itself through $\mathcal{S}$, but none of the others. At this point, it is worth noting that $\mathcal{S}$ is trivially transitive due to $\mathrm{wtriv}_2(\mathcal{S})$. It then also follows from this constraint that every world can be reached in at most one $\mathcal{S}$-step. In other words, the relation $\mathcal{S}$ is of depth 1.

The next two constraints involve both $\mathcal{T}$ and $\mathcal{S}$. We obtain from the weak mixed conversion constraint, $\mathrm{wmconv}(\mathcal{T}, \mathcal{S})$, that $\mathcal{T}$ is contained in $\mathcal{S}^{-1} \cup \Delta_W$, where $\Delta_W = \{(w, w)\colon\ w \in W\}$ is the diagonal of $W \times W$. Moreover, the mixed conversion constraint, $\mathrm{mconv}(\mathcal{S}, \mathcal{T})$, says that $\mathcal{S}$ is contained in $\mathcal{T}^{-1}$. As a result, together with $\mathrm{refl}(\mathcal{T})$ these two constraints give us $\mathcal{T} = \mathcal{S}^{-1} \cup \Delta_W$.

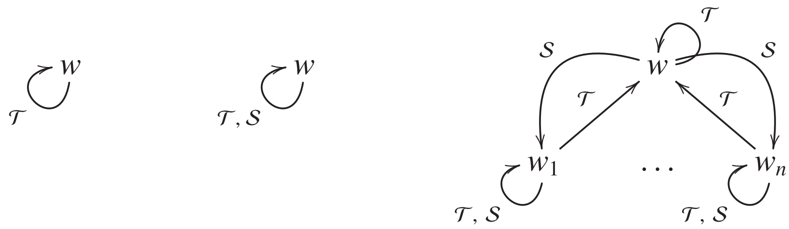

Fig. 1. Graphical representation of **MEM**-frames, for $n \geqslant 0$. The two singleton graphs are for $n = 0$. The rightmost graph is for $n \geqslant 1$, where $f(n) = 2n$ represents the number of $\mathcal{S}$-arrows in the diagram.

Let us sum up the constraints that we have introduced so far: the $\mathcal{T}$ relation is a tree of height 0 or 1, and $\mathcal{S}$ is the converse of $\mathcal{T}$, except for the root. In our frames, any root w is characterised by the fact that $\mathcal{T}(w) \setminus \{w\}$ is empty. **MEM** frames basically have the form of one of the diagrams depicted in Fig. 1. The constraints wmconv$(\mathcal{T}, \mathcal{S})$, refl$_2(\mathcal{S})$, wtriv$_2(\mathcal{S})$, refl$(\mathcal{T})$ and alt$_2(\mathcal{T})$ imply that for every w, $\mathcal{T}(w) \cap \mathcal{S}(w)$ is equal to either the empty-set or the singleton $\{w\}$.

The following properties hold for every **MEM** frame $\langle W, \mathcal{T}, \mathcal{S}\rangle$. First, the relation $\mathcal{T}$ is serial, i.e., for all w there is a u such that $w\mathcal{T}u$. Formally, this property is guaranteed by the constraint refl$(\mathcal{T})$. Moreover, $\mathcal{T}$ is directed, i.e., for every w, u, v, if $w\mathcal{T}u$ and $w\mathcal{T}v$ then there exists z such that $u\mathcal{T}z$ and $v\mathcal{T}z$. This follows from the constraints refl$(\mathcal{T})$ and alt$_2(\mathcal{T})$. Besides, $\mathcal{T}$ is also anti-symmetric, that is to say, for every w, u, if $w\mathcal{T}u$ and $u\mathcal{T}w$ then $w = u$. This follows from the constraints wmconv$(\mathcal{T}, \mathcal{S})$ and wtriv$_2(\mathcal{S})$. Together with mconv$(\mathcal{S}, \mathcal{T})$, this implies that $\mathcal{S}$ is anti-symmetric, too. However, $\mathcal{T}$ is not euclidean: we may have $w\mathcal{T}u$ and $w\mathcal{T}w$ without $u\mathcal{T}w$, and therefore the condition 'for every w, u, v, if $w\mathcal{T}u$ and $w\mathcal{T}v$ then $u\mathcal{T}v$' does not hold in general. Finally, the relations $\mathcal{T}$ and $\mathcal{S}$ are trivially idempotent.[5] We obtain the idempotence property of $\mathcal{T}$ from depth$_1(\mathcal{T})$, while we get that of $\mathcal{S}$ through wtriv$_2(\mathcal{S})$. As a last word, all of the properties above can be visualised from the diagram above; in addition, we can also see that the properties of seriality, euclideanity, and directedness don't hold for the relation $\mathcal{S}$.

2.3. Kripke models

We interpret the formulas of our language $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ in a class of Kripke models that has to satisfy some particular constraints.

Consider the class of Kripke models $M = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ such that:

- $\langle W, \mathcal{T}, \mathcal{S}\rangle$ is an **MEM** frame;
- V is a valuation on W mapping all possible worlds $w \in W$ to sets of propositional variables $V_w \subseteq \mathbb{P}$ such that:

$$\begin{array}{ll} \text{heredity}(\mathcal{S}) & \text{for every } w, u, \text{ if } w\mathcal{S}u \text{ then } V_u \subseteq V_w; \\ \text{neg}(\mathcal{S}, \mathcal{T}) & \text{for every } w, \text{ there exists } u \text{ such that: } w\mathcal{T}u \text{ and if } V_u \neq \emptyset \\ & \text{then for every non-empty } P \subseteq V_u, \text{ there is } v \text{ satisfying} \\ & u\mathcal{S}v \text{ and } V_v = V_u \setminus P. \end{array}$$

A quadruple $M = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ satisfying all the conditions above is called an **MEM** model.

Now let us comment a bit on the constraints heredity$(\mathcal{S})$ and neg$(\mathcal{S}, \mathcal{T})$. They involve not only the relations $\mathcal{S}$ and $\mathcal{T}$, but also the valuation V. The constraint heredity$(\mathcal{S})$ is just as the heredity constraint of intuitionistic logic, except that $\mathcal{S}$ is the inverse of the intuitionistic relation. The neg$(\mathcal{S}, \mathcal{T})$ constraint basically says that if w is the root of a tree and has a non-empty valuation then the set of worlds that are

[5] A relation r is idempotent if $r \circ r = r$, where $\circ$ is the relation composition operation.

accessible from w via the relation $\mathcal{S}$ contains all those worlds u whose valuations V_u are strictly included in V_w. In every **MEM** model, if singleton points appear (such as in the leftmost two graphs in Fig. 1) then they should certainly have an empty valuation.

The following properties include the valuation as well.

Proposition 1. *Let $M = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ be an* **MEM** *model.*

1. *For every w, u, if $w\mathcal{T}u$ then $V_w \subseteq V_u$.*
2. *For every w, if $\mathcal{T}(w)\setminus\{w\}$ is empty, then the set $\{V_u\colon w\mathcal{S}u\}$ equals either $\{V\colon V \subseteq V_w\}$ or $\{V\colon V \subset V_w\}$.*

The first property can be proved from wmconv$(\mathcal{T}, \mathcal{S})$ and heredity$(\mathcal{S})$. The second property is due to heredity$(\mathcal{S})$, neg$(\mathcal{S}, \mathcal{T})$ and refl$(\mathcal{T})$. In words: for a root point w,[6] the set of valuations associated to the worlds that are accessible from w via $\mathcal{S}$ is either the set of subsets of V_w, 2^{V_w}, or the set of strict subsets of V_w, $2^{V_w} \setminus \{V_w\}$. This property will be used later in the paper in the proof of Theorem 12.

2.4. Truth conditions

The semantics of our bimodal logic is fairly standard, the relation $\mathcal{T}$ interpreting the modal operator [T] and the relation $\mathcal{S}$ interpreting the modal operator [S]. The truth conditions are:

$$\begin{array}{ll} M, w \models p & \text{iff } p \in V_w; \\ M, w \not\models \bot; & \\ M, w \models \varphi \supset \psi & \text{iff } M, w \not\models \varphi \text{ or } M, w \models \psi; \\ M, w \models [\mathrm{T}]\varphi & \text{iff } M, u \models \varphi \text{ for every } u \text{ such that } w\mathcal{T}u; \\ M, w \models [\mathrm{S}]\varphi & \text{iff } M, u \models \varphi \text{ for every } u \text{ such that } w\mathcal{S}u. \end{array}$$

We say that φ has an **MEM** model when $M, w \models \varphi$ for some model M and world w in M. We also say that φ is **MEM** *satisfiable*. Furthermore, φ is **MEM** *valid* if and only if $M, w \models \varphi$ for every model M and possible world w in M.

The next proposition says that to check satisfiability it suffices to just consider models with finite valuations.

Proposition 2. *Let φ be an $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$-formula. Let $M = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ be an* **MEM** *model. Let the valuation V^φ be defined as follows*:

$$V^\varphi_w = V_w \cap \mathbb{P}_\varphi, \quad \text{for every } w \in W.$$

Then $M^\varphi = \langle W, \mathcal{T}, \mathcal{S}, V^\varphi\rangle$ is also an **MEM** *model and*

$$M, w \models \varphi \quad \text{if and only if } M^\varphi, \quad w \models \varphi, \quad \text{for every } w \in W.$$

Proof. First, we prove that if φ is a subformula of χ then $M, w \models \varphi$ if and only if $M^\chi, w \models \varphi$, by induction on the form of φ. The base case and the Boolean cases are routine. As for the modalities, we only give the proof for the case where φ is of the form $[\mathrm{T}]\psi$, the case $[\mathrm{S}]\psi$ being similar. We have:

[6] Remember that in an **MEM** frame, the property $\mathcal{T}(w) \setminus \{w\} = \emptyset$ characterises that w is the root of a tree. Moreover, when $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$ then the singleton $\mathcal{T}(w) \setminus \{w\}$ contains the root.

Table 1
Axiomatisation of **MEM**.

$\mathbf{K}([\mathrm{T}])$	the axioms and the inference rules of modal logic $\mathbf{K}$ for $[\mathrm{T}]$
$\mathbf{K}([\mathrm{S}])$	the axioms and the inference rules of modal logic $\mathbf{K}$ for $[\mathrm{S}]$
$\mathrm{T}([\mathrm{T}])$	$[\mathrm{T}]\varphi \supset \varphi$
$\mathrm{Alt}_2([\mathrm{T}])$	$[\mathrm{T}]\varphi \vee [\mathrm{T}](\varphi \supset \psi) \vee [\mathrm{T}](\varphi \wedge \psi \supset \bot)$
$4([\mathrm{T}])$	$[\mathrm{T}]\varphi \supset [\mathrm{T}][\mathrm{T}]\varphi$
$\mathrm{T}_2([\mathrm{S}])$	$[\mathrm{S}]([\mathrm{S}]\varphi \supset \varphi)$
$\mathrm{WTriv}_2([\mathrm{S}])$	$[\mathrm{S}](\varphi \supset [\mathrm{S}]\varphi)$
$\mathrm{WMConv}([\mathrm{T}],[\mathrm{S}])$	$\varphi \supset [\mathrm{T}](\varphi \vee \langle \mathrm{S}\rangle\varphi)$
$\mathrm{MConv}([\mathrm{S}],[\mathrm{T}])$	$\varphi \supset [\mathrm{S}]\langle \mathrm{T}\rangle\varphi$
$\mathrm{Heredity}([\mathrm{S}])$	$\langle \mathrm{S}\rangle\varphi^+ \supset \varphi^+$ for φ^+ a positive Boolean formula
$\mathrm{Neg}([\mathrm{S}],[\mathrm{T}])$	$\langle \mathrm{T}\rangle(\varphi^+ \wedge \psi) \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle(\neg\varphi^+ \wedge \psi)$ for φ^+ a positive Boolean formula such that $\mathbb{P}_{\varphi^+} \cap \mathbb{P}_\psi = \emptyset$

$$\begin{aligned} M, w \models [\mathrm{T}]\psi \quad & \text{iff } M, u \models \psi \text{ for every } u \text{ such that } w\mathcal{T}u \\ & \text{iff } M^{[\mathrm{T}]\psi}, u \models \psi \text{ for every } u \text{ such that } w\mathcal{T}u \text{ (by I.H.)} \\ & \text{iff } M^{[\mathrm{T}]\varphi}, w \models [\mathrm{T}]\psi. \end{aligned}$$

Let us show that M^φ is also an **MEM** model. The frame constraints are only about the accessibility relations and are clearly preserved because we just modify the valuation. As for the constraints involving the valuation, the model M^φ satisfies heredity($\mathcal{S}$) constraint: suppose $w\mathcal{S}u$; as M satisfies heredity($\mathcal{S}$) we have $V_u \subseteq V_w$; hence $V_u^\varphi \subseteq V_w^\varphi$ as well. Finally, the model M^φ also satisfies the constraint neg($\mathcal{S}, \mathcal{T}$): for every $w \in W$, by the constraints refl($\mathcal{T}$) and alt$_2$($\mathcal{T}$) there exists either one or two u such that $w\mathcal{T}u$; in the former case, $u = w$ whereas in the latter, we choose u different from w; for such u's, let $V_u^\varphi = V_u \cap \mathbb{P}_\varphi$ and $P \subseteq V_u^\varphi$ be non-empty; then since M satisfies the neg($\mathcal{S}, \mathcal{T}$) constraint there is v with $u\mathcal{S}v$ satisfying $V_v = V_u \setminus P$; clearly, for that v we also have $V_v^\varphi = V_u^\varphi \setminus P$ since $V_v^\varphi = V_v \cap \mathbb{P}_\varphi = (V_u \setminus P) \cap \mathbb{P}_\varphi = (V_u \cap \mathbb{P}_\varphi) \setminus P = V_u^\varphi \setminus P$. $\square$

Remark 1. Observe that Proposition 2 should not be confused with the finite model property (f.m.p.) of modal logics: the f.m.p. is about finiteness of the set of possible worlds, while Proposition 2 is about finiteness of valuations. We might call the latter finite valuation property (f.v.p.).

2.5. Axiomatics, provability, and completeness

The main purpose of this section is to give an axiomatisation of the **MEM** validities and to prove its completeness.

We start by defining the fragment of *positive Boolean formulas* of $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ by the following grammar:

$$\varphi^+ ::= p \mid \varphi^+ \wedge \varphi^+ \mid \varphi^+ \vee \varphi^+.$$

We immediately observe that every positive Boolean formula is falsifiable. (Note that this holds because $\top$ is not a positive Boolean formula.)

Now we are ready to give our axiomatisation of **MEM**. The axiom schemas and the inference rules are listed in Table 1.

The axiom schemas $\mathrm{T}([\mathrm{T}])$, $\mathrm{Alt}_2([\mathrm{T}])$ and $4([\mathrm{T}])$ are well-known from modal logic textbooks. We observe that $\mathrm{Alt}_2([\mathrm{T}])$ could be replaced by the axiom schema $\langle \mathrm{T}\rangle(\varphi \wedge \psi) \wedge \langle \mathrm{T}\rangle(\varphi \wedge \neg\psi) \supset [\mathrm{T}]\varphi$, or even $(\varphi \wedge \psi) \wedge \langle \mathrm{T}\rangle(\varphi \wedge \neg\psi) \supset [\mathrm{T}]\varphi$.

The axiom schema $\mathrm{WTriv}_2([\mathrm{S}])$ is a weakening of the triviality axiom for [S], i.e., $[\mathrm{S}]\varphi \equiv \varphi$, yet the axiom schemas $\mathrm{T}_2([\mathrm{S}])$ and $\mathrm{WTriv}_2([\mathrm{S}])$ can be combined into the single axiom, $[\mathrm{S}]([\mathrm{S}]\varphi \equiv \varphi)$ that we call 'triviality in the second step' axiom, and symbolise with $\mathrm{Triv}_2([\mathrm{S}])$.

The weak mixed conversion axiom, WMConv([T], [S]), and the mixed conversion axiom, MConv([S], [T]), are familiar from tense logics.

Finally, the schema Heredity([S]) captures the heredity constraint, heredity($\mathcal{S}$). Note that it could be replaced by the axiom $\langle \mathrm{S} \rangle p \supset p$, where p is a propositional variable. It could also be replaced by the axiom schema $\neg\varphi^+ \supset [\mathrm{S}]\neg\varphi^+$, for φ^+ a positive Boolean formula. The schema Neg([S], [T]) ensures that the modal operator [S] quantifies over all strict subsets of the actual valuation under some restrictions.

Remark 2. T([T]) and Alt_2([T]) give us that for every w, $\mathcal{T}(w)$ contains w, and has at least 1 and at most 2 elements. If $\mathcal{T}(w)$ contains two elements, say $\mathcal{T}(w) = \{w, u\}$ where $w \neq u$, then 4([T]) implies that $\mathcal{T}(u) = \{u\}$. Moreover, WMConv([T], [S]) guarantees that u is always $\mathcal{S}$-related to w when it exists.

Finally, the notions of *proof* and of *provability of a formula* are defined as it is in any modal logic.

For example, it is possible to prove the schema Depth_1([T]) (corresponding to the constraint $\mathrm{depth}_1(\mathcal{T})$), i.e., $\neg\varphi \supset [\mathrm{T}](\varphi \supset [\mathrm{T}]\varphi)$. The proof uses the axiom schemas T([T]), Alt_2([T]), and 4([T]).

As another example, we give the proof of the schema that corresponds to the heredity condition for $\mathcal{T}$, i.e., Heredity([T]): $\varphi^+ \supset [\mathrm{T}]\varphi^+$, for φ^+ a positive Boolean formula. This will be helpful in the proof of the grand finale, Theorem 12.

Proposition 3. *The schema Heredity*([T])*, i.e.,* $\varphi^+ \supset [\mathrm{T}]\varphi^+$*, for* φ^+ *a positive Boolean formula, is provable.*

Proof.

1. $\langle \mathrm{S} \rangle \varphi^+ \supset \varphi^+$ (Heredity([S]))
2. $\varphi^+ \supset [\mathrm{T}](\varphi^+ \vee \langle \mathrm{S} \rangle \varphi^+)$ (WMConv([T], [S]))
3. $\varphi^+ \supset [\mathrm{T}]\varphi^+$ (from 1 and 2 by K[S]).

□

Heredity([T]) ensures that $w\mathcal{T}u$ implies $V_w \subseteq V_u$, i.e., the heredity condition for $\mathcal{T}$.

Here is one more schema just concerning the $\mathcal{T}$ relation.

Proposition 4. *The schema,* 2([T])*, i.e.,* $\langle \mathrm{T} \rangle [\mathrm{T}]\varphi \supset [\mathrm{T}]\langle \mathrm{T} \rangle \varphi$ *is provable.*

Proof.

1. $\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \supset \langle \mathrm{T} \rangle \neg\varphi$ (from T([T]) by K([T]))
2. $\langle \mathrm{T} \rangle [\mathrm{T}]\varphi \supset \langle \mathrm{T} \rangle ([\mathrm{T}]\varphi \wedge \varphi)$ (from T([T]) by K([T]))
3. $[\mathrm{T}]\varphi \vee [\mathrm{T}](\varphi \supset \langle \mathrm{T} \rangle \neg\varphi) \vee [\mathrm{T}]((\varphi \wedge \langle \mathrm{T} \rangle \neg\varphi) \supset \bot)$ (Alt_2([T]))
4. $(\langle \mathrm{T} \rangle \neg\varphi \wedge \langle \mathrm{T} \rangle (\varphi \wedge [\mathrm{T}]\varphi)) \supset [\mathrm{T}](\varphi \supset [\mathrm{T}]\varphi)$ (from 3 by K([T]))
5. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}]\varphi) \supset [\mathrm{T}](\varphi \supset [\mathrm{T}]\varphi)$ (from 1, 2 and 4 by K([T]))
6. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}]\varphi) \supset [\mathrm{T}](\neg\varphi \supset [\mathrm{T}]\neg\varphi)$ (from 5 by K([T]))
7. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}]\varphi) \supset [\mathrm{T}]([\mathrm{T}]\varphi \vee [\mathrm{T}]\neg\varphi)$ (from 5 and 6 by K([T]))
8. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}]\varphi) \supset ([\mathrm{T}][\mathrm{T}]\varphi \vee [\mathrm{T}][\mathrm{T}]\neg\varphi)$ (from 7 by T([T]) and 4([T]))
9. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge [\mathrm{T}][\mathrm{T}]\varphi) \supset \langle \mathrm{T} \rangle [\mathrm{T}]\bot$ (by K([T]))
10. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge [\mathrm{T}][\mathrm{T}]\neg\varphi) \supset [\mathrm{T}][\mathrm{T}]\neg\varphi$ (by K([T]))
11. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge ([\mathrm{T}][\mathrm{T}]\varphi \vee [\mathrm{T}][\mathrm{T}]\neg\varphi)) \supset (\langle \mathrm{T} \rangle [\mathrm{T}]\bot \vee [\mathrm{T}][\mathrm{T}]\neg\varphi)$ (from 9 and 10)
12. $(\langle \mathrm{T} \rangle [\mathrm{T}]\varphi \wedge ([\mathrm{T}][\mathrm{T}]\varphi \vee [\mathrm{T}][\mathrm{T}]\neg\varphi)) \supset (\langle \mathrm{T} \rangle [\mathrm{T}]\bot \vee [\mathrm{T}][\mathrm{T}]\varphi)$ (from 11 by K([T]))
13. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}]\varphi \wedge ([\mathrm{T}][\mathrm{T}]\varphi \vee [\mathrm{T}][\mathrm{T}]\neg\varphi)) \supset (\langle \mathrm{T} \rangle [\mathrm{T}]\bot \vee ([\mathrm{T}][\mathrm{T}]\varphi \wedge [\mathrm{T}][\mathrm{T}]\neg\varphi))$ (from 11 and 12 by K([T]))
14. $(\langle \mathrm{T} \rangle [\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T} \rangle [\mathrm{T}]\varphi) \supset (\langle \mathrm{T} \rangle [\mathrm{T}]\bot \vee ([\mathrm{T}][\mathrm{T}]\varphi \wedge [\mathrm{T}][\mathrm{T}]\neg\varphi))$ (from 8 and 13)

15. $(\langle \mathrm{T}\rangle[\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T}\rangle[\mathrm{T}]\varphi) \supset (\langle \mathrm{T}\rangle[\mathrm{T}]\bot \vee \bot)$ (from 14 by K([T]) and T([T]))
16. $(\langle \mathrm{T}\rangle[\mathrm{T}]\neg\varphi \wedge \langle \mathrm{T}\rangle[\mathrm{T}]\varphi) \supset \langle \mathrm{T}\rangle\bot$ (from 15 by T([T]) and K([T]))
17. $(\langle \mathrm{T}\rangle[\mathrm{T}]\varphi \wedge \langle \mathrm{T}\rangle[\mathrm{T}]\neg\varphi) \supset \bot$ (from 16 by K([T]))
18. $\langle \mathrm{T}\rangle[\mathrm{T}]\varphi \supset [\mathrm{T}]\langle \mathrm{T}\rangle\varphi$ (from 17 by K([T])).

□

Let us turn to schemas about [S]. For example, 4([S]): $[\mathrm{S}]\varphi \supset [\mathrm{S}][\mathrm{S}]\varphi$ is a direct consequence of WTriv_2([S]). The proof is in one step by the K([S]) axiom and modus ponens (MP).

Finally, we state and prove a schema regarding both operators [T] and [S] that will be useful in the completeness proof.

Lemma 5. *The following formula schema is provable*:

$$\mathrm{Neg}'([\mathrm{S}],[\mathrm{T}]) \qquad \langle \mathrm{T}\rangle\Big(\Big(\bigwedge_{p\in P} p\Big) \wedge \Big(\bigwedge_{q\in Q} q\Big)\Big) \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\Big(\Big(\bigwedge_{p\in P} \neg p\Big) \wedge \Big(\bigwedge_{q\in Q} q\Big)\Big)$$

for $P, Q \subseteq \mathbb{P}$ finite, $P \neq \emptyset$, and $P \cap Q = \emptyset$.

Proof. $\mathrm{Neg}'([\mathrm{S}],[\mathrm{T}])$ can be proved using the axiom schema Neg([S], [T]) by standard modal logic principles, i.e., by **K**([T]). Suppose P and Q are finite subsets of $\mathbb{P}$ such that $P \neq \emptyset$ and $P \cap Q = \emptyset$. The implication

$$\Big(\Big(\bigwedge_{p\in P} p\Big) \wedge \Big(\bigwedge_{q\in Q} q\Big)\Big) \supset \Big(\Big(\bigvee_{p\in P} p\Big) \wedge \Big(\bigwedge_{q\in Q} q\Big)\Big)$$

is valid in classical propositional logic. Then $\mathrm{Neg}'([\mathrm{S}],[\mathrm{T}])$ follows through the argument below:

1. $(\bigwedge_{p\in P} p) \wedge (\bigwedge_{q\in Q} q) \supset (\bigvee_{p\in P} p) \wedge (\bigwedge_{q\in Q} q)$ (tautology)
2. $\langle \mathrm{T}\rangle((\bigwedge_{p\in P} p) \wedge (\bigwedge_{q\in Q} q)) \supset \langle \mathrm{T}\rangle((\bigvee_{p\in P} p) \wedge (\bigwedge_{q\in Q} q))$ (from 1 by K[T])
3. $\langle \mathrm{T}\rangle((\bigvee_{p\in P} p) \wedge (\bigwedge_{q\in Q} q)) \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle((\neg \bigvee_{p\in P} p) \wedge (\bigwedge_{q\in Q} q))$ (Neg([S], [T]))
4. $\langle \mathrm{T}\rangle((\bigwedge_{p\in P} p) \wedge (\bigwedge_{q\in Q} q)) \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle((\bigwedge_{p\in P} \neg p) \wedge (\bigwedge_{q\in Q} q))$ (from 2–3 by K[T]).

□

Our axiomatisation is sound and complete.

Theorem 6. *Let φ be an $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$-formula. Then φ is* **MEM** *valid if and only if φ is provable from the axioms and the inference rules of* **MEM**.

Proof. Soundness is proved as usual. We just consider the proof of axiom schema Neg([S], [T]). Let φ^+ be a positive Boolean formula such that $\mathbb{P}_{\varphi^+} \cap \mathbb{P}_\psi = \emptyset$. Suppose

$$M, w \models \langle \mathrm{T}\rangle(\varphi^+ \wedge \psi) \quad (*).$$

Put φ^+ in conjunctive normal form (CNF), and let $\kappa = (\bigvee P)$ be a clause of this CNF, for some $P \subseteq \mathbb{P}_{\varphi^+}$. Observe that $P \neq \emptyset$ by the definition of positive Boolean formulas and CNF. Now, we need to consider two cases (according to Remark 2 above).

Case (1). Let $\mathcal{T}(w) \setminus \{w\} = \emptyset$. Hence $\mathcal{T}(w) = \{w\}$ by the reflexivity of $\mathcal{T}$. Then from $(*)$ we obtain that $M, w \models \varphi^+ \wedge \psi$ $(**)$. Moreover, $M, w \models \varphi^+$ implies that $V_w \neq \emptyset$. In addition, non-emptiness of V_w yields that w is not a singleton point (because singleton points always have an empty valuation; otherwise that would contradict $\mathrm{neg}(\mathcal{S},\mathcal{T})$). Now, take $P_w = P \cap V_w$. We have $P_w \neq \emptyset$ because $M, w \models \kappa$ (since we have

$M, w \models \varphi^+$). As M satisfies the constraint neg($\mathcal{S}, \mathcal{T}$), there exists u with $w\mathcal{T}u$, but since $\mathcal{T}(w) = \{w\}$ we have $u = w$. Since $V_w \neq \emptyset$, according to the negatable constraint, for non-empty $P_w \subseteq V_w$, there is v such that $w\mathcal{S}v$ and $V_v = V_w \setminus P_w$. Since $P_w \cap V_v = \emptyset$, we also have $P \cap V_v = \emptyset$ (because $V_v \subseteq V_w$, but $P \setminus P_w \nsubseteq V_w$). Hence, $M, v \not\models \kappa$. As a result, $M, v \not\models \varphi^+$ either. So $M, v \models \neg\varphi^+$. In addition, $M, v \models \psi$ because $M, w \models \psi$ by $(**)$, $\mathbb{P}_{\varphi^+} \cap \mathbb{P}_\psi = \emptyset$, and $V_v = V_w \setminus P_w$. Hence, we deduce that $M, v \models \neg\varphi^+ \wedge \psi$, but $w\mathcal{S}v$, so we also have $M, w \models \langle \mathrm{S} \rangle(\neg\varphi^+ \wedge \psi)$. Finally, it is trivial to conclude that $M, w \models \langle \mathrm{T} \rangle\langle \mathrm{S} \rangle(\neg\varphi^+ \wedge \psi)$ since $\mathcal{T}(w) = \{w\}$.
Case (2). Let $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$. So there exists u with $u \neq w$ and $w\mathcal{T}u$. Moreover, u is uniquely determined (see Remark 2). Then we choose u, but not w, as a candidate to satisfy the formula $\varphi^+ \wedge \psi$ (see $(*)$ above). Hence, we obtain from $(*)$ that $M, u \models \varphi^+ \wedge \psi$. The proof follows almost the same reasoning as in the previous case and we leave it to the reader. Following the same steps for u, we obtain $M, u \models \langle \mathrm{S} \rangle(\neg\varphi^+ \wedge \psi)$. Then $M, w \models \langle \mathrm{T} \rangle\langle \mathrm{S} \rangle(\neg\varphi^+ \wedge \psi)$ results automatically.

To prove completeness w.r.t. **MEM** models we use canonical models [3,7]. Let φ be a consistent $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ formula. We define the canonical model $M^\varphi = \langle W, \mathcal{T}, \mathcal{S}, V \rangle$ as follows. W is the set of maximal consistent sets of **MEM**. The accessibility relations $\mathcal{T}$ and $\mathcal{S}$ are such that:

$$\begin{array}{ll} w\mathcal{T}u & \text{iff } \big\{\psi\colon\ [\mathrm{T}]\psi \in w\big\} \subseteq u \\ w\mathcal{S}u & \text{iff } \big\{\psi\colon\ [\mathrm{S}]\psi \in w\big\} \subseteq u; \end{array}$$

The valuation V is defined by $V_w = w \cap \mathbb{P}_\varphi$, for every $w \in W$. Let us prove that the canonical model $M^\varphi = \langle W, \mathcal{T}, \mathcal{S}, V \rangle$ is a legal **MEM** model. It is standard to prove that M^φ satisfies the constraints associated to the axioms T([T]), Alt_2([T]), 4([T]), T_2([S]), and WTriv_2([S]).

- The weak mixed conversion axiom WMConv([T], [S]) implies that the constraint wmconv($\mathcal{T}, \mathcal{S}$) is satisfied in the canonical model: suppose that $w\mathcal{T}u$ and $w \neq u$; we want to show $u\mathcal{S}w$; assume for a contradiction that u isn't $\mathcal{S}$-related to w $(*)$; then there exists φ such that $[\mathrm{S}]\varphi \in u$ and $\neg\varphi \in w$; next, since $w \neq u$, there exists ψ with $\psi \in w$ and $\neg\psi \in u$; as w is maximal consistent $\neg\varphi \wedge \psi \in w$, but so is any instance of WMConv([T], [S]) as well; hence $(\neg\varphi \wedge \psi) \supset [\mathrm{T}]((\neg\varphi \wedge \psi) \vee \langle \mathrm{S} \rangle(\neg\varphi \wedge \psi)) \in w$, and then $(\neg\varphi \wedge \psi) \vee \langle \mathrm{S} \rangle(\neg\varphi \wedge \psi) \in u$ first through (MP) and then using our initial assumption $w\mathcal{T}u$; $\neg\psi \in u$ implies $\neg\psi \vee \varphi \in u$, but then so must $\langle \mathrm{S} \rangle(\neg\varphi \wedge \psi) \in u$; using maximal consistency of u we assert $\langle \mathrm{S} \rangle(\neg\varphi \wedge \psi) \supset (\langle \mathrm{S} \rangle\neg\varphi \wedge \langle \mathrm{S} \rangle\psi) \in u$ as well, but then so is $\langle \mathrm{S} \rangle\neg\varphi \wedge \langle \mathrm{S} \rangle\psi \in u$, which gives us the desired contradiction; eventually $u\mathcal{S}w$.
- The mixed conversion axiom MConv([S], [T])[7] guarantees that the constraint mconv($\mathcal{S}, \mathcal{T}$) holds in the canonical model: let $w\mathcal{S}u$ and assume φ is such that $[\mathrm{T}]\varphi \in u$; then by the definition of $\mathcal{S}$, $\langle \mathrm{S} \rangle[\mathrm{T}]\varphi \in w$; since w is maximal consistent, any instance of MConv([S], [T]) is in w, so is $\langle \mathrm{S} \rangle[\mathrm{T}]\varphi \supset \varphi$; therefore through (MP) we get $\varphi \in w$ and this completes the proof.
- The axiom schema Heredity([S]) ensures that the canonical model satisfies the constraint heredity($\mathcal{S}$), viz. that for every w, u, $w\mathcal{S}u$ implies $V_u \subseteq V_w$: indeed, suppose $w\mathcal{S}u$ and $p \in V_u = u \cap \mathbb{P}_\varphi$; as w is a maximal consistent set, it contains all instances of Heredity([S]), in particular, $\langle \mathrm{S} \rangle p \supset p$; since $w\mathcal{S}u$ we also obtain $\langle \mathrm{S} \rangle p \in w$ from $p \in u$. (Otherwise, w being maximal consistent, it includes $\neg\langle \mathrm{S} \rangle p = [\mathrm{S}]\neg p$; since $w\mathcal{S}u$ by assumption, we get $\neg p \in u$, contradicting the fact that u is consistent since $p \in u$ as well.) Hence, $p \in w$, and so $p \in w \cap \mathbb{P}_\varphi = V_w$.
- The negatable axiom Neg([S], [T]) guarantees that neg($\mathcal{S}, \mathcal{T}$) holds in the canonical model: to see this take an arbitrary $w \in W$; since the canonical model satisfies the constraints refl($\mathcal{T}$) and alt_2($\mathcal{T}$) (as the reader can easily check), we go through the following two cases:

[7] It is a bit handier to work with the contrapositive of the axiom schema MConv([S], [T]) here, i.e., with $\langle \mathrm{S} \rangle[\mathrm{T}]\varphi \supset \varphi$.

Case (i). Let $\mathcal{T}(w) \setminus \{w\} = \emptyset$. Hence $\mathcal{T}(w) = \{w\}$ by the reflexivity of $\mathcal{T}$. (Then it is trivial to conclude that there exists u such that $w\mathcal{T}u$, and moreover $u = w$.) If $V_w = w \cap \mathbb{P}_\varphi = \emptyset$ (i.e., if w contains the negations of the propositional variables of φ) then the constraint trivially holds. Let $V_w \neq \emptyset$. Suppose $P \subseteq V_w = w \cap \mathbb{P}_\varphi$ is such that $P \neq \emptyset$. Then we choose $Q = V_w \setminus P$. Since $P, Q \subseteq \mathbb{P}$ are finite with $P \neq \emptyset$ and $P \cap Q = \emptyset$, now we can use Lemma 5. As w is a maximal consistent set it includes $(\bigwedge_{p\in P} p) \wedge (\bigwedge_{q\in Q} q)$, but then also $\langle \mathrm{T}\rangle((\bigwedge_{p\in P} p) \wedge (\bigwedge_{q\in Q} q))$ since $\mathcal{T}(w) = \{w\}$. Next, again since w is maximal consistent, by Lemma 5 it also has every instance of $\mathrm{Neg}'([\mathrm{S}], [\mathrm{T}])$, so it must contain $\langle \mathrm{T}\rangle\langle \mathrm{S}\rangle((\bigwedge_{p\in P} \neg p) \wedge (\bigwedge_{q\in Q} q))$ as well. By our initial assumption, $\langle \mathrm{S}\rangle((\bigwedge_{p\in P} \neg p) \wedge (\bigwedge_{q\in Q} q)) \in w$. Thus we can conclude that there is $v \in W$ such that $w\mathcal{S}v$. Furthermore v contains $(\bigwedge_{p\in P} \neg p) \wedge (\bigwedge_{q\in Q} q)$. Therefore, $P \cap v = \emptyset$ and $Q \subseteq v$, but the canonical model satisfies the heredity($\mathcal{S}$) constraint (see above), so $V_v \subseteq V_w$. We know that $P, Q \subseteq \mathbb{P}_\varphi$ are mutually exclusive and cover V_w. Also, $Q \subseteq v$ and $Q \subseteq \mathbb{P}_\varphi$ implies $Q \subseteq v \cap \mathbb{P}_\varphi = V_v$. On the other hand, $V_v \cap P = (v \cap \mathbb{P}_\varphi) \cap P = v \cap (\mathbb{P}_\varphi \cap P) = v \cap P = \emptyset$. (Alternatively, note that $V_v = v \cap \mathbb{P}_\varphi = Q$, so apparently, $V_v \cap P = Q \cap P = \emptyset$.) It follows that $V_v = Q = V_w \setminus P$ and we are done.
Case (ii). Now suppose $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$. It is obvious from our assumption that there exists u such that $u \neq w$ and $w\mathcal{T}u$. Moreover, since the canonical model satisfies the constraints refl($\mathcal{T}$) and alt$_2$($\mathcal{T}$) (which is easy to prove), we claim that $\mathcal{T}(w) = \{w, u\}$. Additionally, trans($\mathcal{T}$) also holds in the canonical model (again easily verified), so we further have $\mathcal{T}(u) = \{u\}$. Therefore the rest of the proof can basically be done in the same way as before.

To sum it up, the canonical model M^φ satisfies all constraints, and is therefore a legal **MEM** model. Moreover, as φ is a consistent **MEM** formula, there must exist a maximal **MEM** consistent set $w \subseteq W$ containing φ. It can then be proved in the standard way that $M^\varphi, w \models \varphi$. □

3. HT logic and equilibrium logic

In this section we recall HT logic and equilibrium logic.

3.1. The language $\mathcal{L}_\rightarrow$

The language $\mathcal{L}_\rightarrow$ is common to HT logic and equilibrium logic. It is defined by the following grammar:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi,$$

where p ranges over $\mathbb{P}$. The other Boolean connectives are defined as abbreviations in the same way as in our bimodal language: negation $\neg\varphi$ is defined as $\varphi \rightarrow \bot$, and $\top$ is defined as $\bot \rightarrow \bot$.

3.2. Here-and-there logic

A *HT model* is a couple (H, T) such that $H \subseteq T \subseteq \mathbb{P}$. The sets H and T are respectively called 'here' and 'there'.

Let (H, T) be an HT model. The truth conditions are as follows:

$$\begin{array}{ll}
H, T \models p & \text{iff } p \in H; \\
H, T \not\models \bot; & \\
H, T \models \varphi \wedge \psi & \text{iff } H, T \models \varphi \text{ and } H, T \models \psi; \\
H, T \models \varphi \vee \psi & \text{iff } H, T \models \varphi \text{ or } H, T \models \psi; \\
H, T \models \varphi \rightarrow \psi & \text{iff } (H, T \not\models \varphi \text{ or } H, T \models \psi) \text{ and } (T, T \not\models \varphi \text{ or } T, T \models \psi).
\end{array}$$

When $H, T \models \varphi$ we say that (H, T) is an HT model of φ. A formula φ is *HT valid* if and only if every HT model is also an HT model of φ.

We can claim as a consequence of the following lemma that the finite model property (perhaps better called a finite valuation property) holds for HT logic: if an $\mathcal{L}_{\rightarrow}$-formula φ has an HT model then there also exists a pair of finite here and there sets (H, T) such that $H, T \models \varphi$. This is the counterpart of Proposition 2.

Lemma 7. *Let φ be an $\mathcal{L}_{\rightarrow}$-formula and let q be a propositional variable such that $q \notin \mathbb{P}_{\varphi}$. Then $H, T \models \varphi$ iff $H, T \cup \{q\} \models \varphi$ iff $H \cup \{q\}, T \cup \{q\} \models \varphi$.*

3.3. Equilibrium logic

An *equilibrium model* of an $\mathcal{L}_{\rightarrow}$-formula φ is a set of propositional variables $T \subseteq \mathbb{P}$ such that:

1. (T, T) is an HT model of φ;
2. no (H, T) with $H \subset T$ is an HT model of φ.

Here are three examples. First, the empty set is the only equilibrium model of both $\top$ and $\neg p$: for any $q \in \mathbb{P}$, $\{q\}$ is neither an equilibrium model of $\top$ nor of $\neg p$. Second, as $\emptyset, \{p\} \models \neg p \rightarrow q$, the set $\{p\}$ is *not* an equilibrium model of $\neg p \rightarrow q$. Third, $\{q\}$ is an equilibrium model of $\neg p \rightarrow q$ because $\{q\}, \{q\} \models \neg p \rightarrow q$ and $\emptyset, \{q\} \not\models \neg p \rightarrow q$.

For two $\mathcal{L}_{\rightarrow}$-formulas φ and χ, we say that φ is a *consequence of χ in equilibrium models*, written $\chi \mid\!\approx \varphi$, if and only if for every equilibrium model T of χ, (T, T) is an HT model of φ. For example, $\top \mid\!\approx \neg p$ and $\neg p \mid\!\approx \top$. We also have $q \mid\!\approx \neg p \rightarrow q$ and $\neg p \rightarrow q \mid\!\approx q$.

4. From HT logic and equilibrium logic to the modal logic, MEM

In this section we are going to translate HT logic and equilibrium logic into our logic **MEM**.

4.1. Translating $\mathcal{L}_{\rightarrow}$ to $\mathcal{L}_{[\mathrm{T}]}$

To warm up, let us translate the language $\mathcal{L}_{\rightarrow}$ of both HT logic and equilibrium logic into the sub-language $\mathcal{L}_{[\mathrm{T}]}$ of **MEM**. We recursively define the mapping tr as follows:

$$
\begin{aligned}
tr(p) &= p \quad \text{for } p \in \mathbb{P};\\
tr(\bot) &= \bot;\\
tr(\varphi \wedge \psi) &= tr(\varphi) \wedge tr(\psi);\\
tr(\varphi \vee \psi) &= tr(\varphi) \vee tr(\psi);\\
tr(\varphi \rightarrow \psi) &= [\mathrm{T}](tr(\varphi) \supset tr(\psi)).
\end{aligned}
$$

This translation is similar to the Gödel translation from intuitionistic logic to modal logic **S4** whose main clause is $tr(\varphi \rightarrow \psi) = \Box(tr(\varphi) \supset tr(\psi))$, where $\Box$ is an **S4** operator (just as the $[\mathrm{T}]$ operator of our bimodal logic).

Here are some examples.

$$tr(\top) = tr(\bot \rightarrow \bot) = [\mathrm{T}](\bot \supset \bot).$$

This is equivalent to $\top$ in any normal modal logic.

$$tr(\neg p) = tr(p \rightarrow \bot) = [\mathrm{T}](p \supset \bot).$$

This is equivalent to $[\mathrm{T}]\neg p$ in any normal modal logic.

$$tr(p \vee \neg p) = tr(p) \vee tr(p \to \bot) = p \vee [\mathrm{T}](p \supset \bot).$$

This is equivalent to $p \vee [\mathrm{T}]\neg p$ in any normal modal logic.

4.2. From HT logic to **MEM**

On HT models, the fragment $\mathcal{L}_{[\mathrm{T}]}$ of the language $\mathcal{L}_{[\mathrm{T}],[\mathrm{S}]}$ is at least as expressive as $\mathcal{L}_{\to}$, modulo the translation tr.

Proposition 8. *Let T be a set of propositional variables, and let $M_T = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ be a quadruple such that*:

$$\begin{aligned}
&W = 2^T;\\
&V_h = h \quad \textit{for every } h \in W;\\
&\mathcal{T} = \Delta_W \cup \big(W \times \{T\}\big) = \big\{(x,y) \in W \times W\colon\ x = y \textit{ or } y = T\big\};\\
&\mathcal{S} = \Delta_{(W \setminus \{T\})} \cup \big(\{T\} \times \big(W \setminus \{T\}\big)\big) = \mathcal{T}^{-1} \setminus \big\{(T,T)\big\}.
\end{aligned}$$

Then M_T is an **MEM** *model, and $H, T \models \varphi$ if and only if $M_T, H \models tr(\varphi)$, for every $H \subseteq T$ and $\mathcal{L}_{\to}$-formula φ.*

In the last line, $\mathcal{S}$ is defined as the (relative) difference between the inverse of $\mathcal{T}$ and $\{(T,T)\}$. For example, for $T = \emptyset$ we obtain $M_\emptyset = (W, \mathcal{T}, \mathcal{S}, V)$ with $W = \{\emptyset\}$, $\mathcal{T} = \{(\emptyset,\emptyset)\}$, and $\mathcal{S} = \emptyset$; and for $T = \{p\}$ we obtain $M_{\{p\}} = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ with $W = \{\emptyset, \{p\}\}$, $\mathcal{T} = \{(\emptyset,\emptyset), (\emptyset,\{p\}), (\{p\},\{p\})\}$, and $\mathcal{S} = \{(\emptyset,\emptyset), (\{p\},\emptyset)\}$.

Proof. First, M_T is a legal **MEM** model: M_T satisfies all constraints by construction, i.e., refl($\mathcal{T}$), alt$_2$($\mathcal{T}$), trans($\mathcal{T}$), refl$_2$($\mathcal{S}$), wtriv$_2$($\mathcal{S}$), wmconv($\mathcal{T},\mathcal{S}$), mconv($\mathcal{S},\mathcal{T}$), heredity($\mathcal{S}$), and neg($\mathcal{S},\mathcal{T}$). Second, one can prove by a straightforward induction on the form of φ that $H, T \models \varphi$ iff $M_T, H \models tr(\varphi)$, for every $H \subseteq T$. □

Proposition 9. *Let $M = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ be an* **MEM** *model. Then for every $w \in W$ and every $\mathcal{L}_{\to}$-formula φ we have*:

1. *If $\mathcal{T}(w) \setminus \{w\} = \emptyset$ then $M, w \models tr(\varphi)$ if and only if $V_w, V_w \models \varphi$;*
2. *If $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$ then $M, w \models tr(\varphi)$ if and only if $V_w, V_u \models \varphi$ for the uniquely determined $u \in \mathcal{T}(w) \setminus \{w\}$.*

Proof. As expected we go through induction on the form of φ in both cases. For the first case $\mathcal{T}(w)\setminus\{w\} = \emptyset$, the base, the Boolean and even the intuitionistic implication steps are straightforward. For the second case, suppose $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$. Then $\mathcal{T}(w) \setminus \{w\}$ contains exactly one element, say u (see Remark 2). The base and the Boolean cases are still easy, and only the case of intuitionistic implication is worth analysing. We sketch of the argument and leave the gaps to the reader. We have:

$$\begin{aligned}
M, w \models tr(\psi_1 \to \psi_2) \quad &\text{iff } M, w \models [\mathrm{T}](tr(\psi_1) \supset tr(\psi_2))\\
&\text{iff } M, w \models tr(\psi_1) \supset tr(\psi_2) \text{ and } M, u \models tr(\psi_1) \supset tr(\psi_2)\\
&\text{iff } V_w, V_u \models \psi_1 \supset \psi_2 \quad \text{and} \quad V_u, V_u \models \psi_1 \supset \psi_2 \quad \text{(by I.H. and using Remark 2)}\\
&\text{iff } V_w, V_u \models \psi_1 \to \psi_2. \qquad \square
\end{aligned}$$

Theorem 10. *Let φ be an $\mathcal{L}_{\to}$-formula. Then φ is HT valid if and only if $tr(\varphi)$ is* **MEM** *valid.*

Proof. It follows from Proposition 8 and Proposition 9 in the way given below:

($\Longleftarrow$): Let (H,T) be an HT model, then $H \subseteq T \subseteq \mathbb{P}$. Now, construct a quadruple $M_T = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ as in Proposition 8. By that proposition, M_T is an **MEM** model, and since $tr(\varphi)$ is **MEM** valid by assumption, we have $M_T, H \models tr(\varphi)$. Finally, again by Proposition 8, $H, T \models \varphi$, i.e., (H,T) is an HT model of φ.

($\Longrightarrow$): Let $M = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ be an **MEM** model and let $w \in W$. We have to go through two cases.

Case 1. Assume that $\mathcal{T}(w) \setminus \{w\} = \emptyset$. By assumption, we know that (V_w, V_w) is an HT model of φ. Therefore, by Proposition 9 we have $M, w \models tr(\varphi)$.

Case 2. Suppose that $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$. Then there exists a unique u such that $\mathcal{T}(w) = \{w, u\}$ is of cardinality 2 (see Remark 2). Hence, Proposition 1.1 gives us $V_w \subseteq V_u$. Next, by hypothesis (V_w, V_u) is an HT model of φ. Therefore, by Proposition 9, $M, w \models tr(\varphi)$. $\square$

5. From equilibrium logic to MEM

The same construction as for HT logic allows us to turn equilibrium models into **MEM** models.

Proposition 11. *Let $T \subseteq \mathbb{P}$, and let $M_T = \langle W, \mathcal{T}, \mathcal{S}, V\rangle$ be a quadruple such that*:

$$\begin{aligned}
&W = 2^T;\\
&V_h = h, \quad \textit{for every } h \in W;\\
&\mathcal{T} = \Delta_W \cup \big(W \times \{T\}\big);\\
&\mathcal{S} = \Delta_{(W\setminus\{T\})} \cup \big(\{T\} \times \big(W \setminus \{T\}\big)\big).
\end{aligned}$$

*Then M_T is an **MEM** model, and T is an equilibrium model of φ if and only if $M_T, T \models tr(\varphi) \wedge [\mathrm{S}]\neg tr(\varphi)$, for every $\mathcal{L}_{\rightarrow}$-formula φ.*

Proof. As we have already seen in Proposition 8, M_T is a legal **MEM** model. So it remains to prove that T is an equilibrium model of φ iff $M_T, T \models tr(\varphi) \wedge [\mathrm{S}]\neg tr(\varphi)$ for every $\mathcal{L}_{\rightarrow}$-formula φ. We indeed have:

$$\begin{aligned}
&\quad T \text{ is an equilibrium model of } \varphi\\
\text{iff}\quad &T, T \models \varphi \text{ and } H, T \not\models \varphi \text{ for every } H \subset T\\
\text{iff}\quad &M_T, T \models tr(\varphi) \text{ and } M_T, H \not\models tr(\varphi) \text{ for every } H \subset T \quad \text{(by Proposition 8)}\\
\text{iff}\quad &M_T, T \models tr(\varphi) \text{ and } M_T, H \models \neg tr(\varphi) \text{ for every } H \text{ such that } T\mathcal{S}H \quad (\text{because } T\mathcal{S}H \text{ iff } H \subset T)\\
\text{iff}\quad &M_T, T \models tr(\varphi) \text{ and } M_T, T \models [\mathrm{S}]\neg tr(\varphi)\\
\text{iff}\quad &M_T, T \models tr(\varphi) \wedge [\mathrm{S}]\neg tr(\varphi). \quad \square
\end{aligned}$$

For example, consider the set $T = \emptyset$ and the $\mathcal{L}_{\rightarrow}$-formula $\varphi = \top$. We have seen before that $\emptyset$ is the only equilibrium model of $\top$. Let M_T be the **MEM** model as constructed in Propositions 8 and 11. We know that M_T is a legal **MEM** model. We also deduce $M_T, T \models [\mathrm{T}](tr(\top) \wedge [\mathrm{S}]\neg tr(\top))$ following the conclusion of Proposition 11 and the structure of the model. This can also be seen by simplifying the latter:

$$\begin{aligned}
[\mathrm{T}]\big(tr(\top) \wedge [\mathrm{S}]\neg tr(\top)\big) \quad &\text{iff } [\mathrm{T}](\top \wedge [\mathrm{S}]\neg\top)\\
&\text{iff } [\mathrm{T}][\mathrm{S}]\bot.
\end{aligned}$$

We are now ready for the grand finale where we capture equilibrium logic in our bimodal logic, **MEM**.

Theorem 12. *Let χ and φ be $\mathcal{L}_{\rightarrow}$-formulas. Then $\chi \approx\!\!\!\!\!\mid\;\; \varphi$ if and only if*

$$\big(tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi)\big) \supset tr(\varphi)$$

*is **MEM** valid.*

Proof. We use Propositions 9 and 11. Let us abbreviate $(tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi)) \supset tr(\varphi)$ by ξ.

($\Longrightarrow$): Assume ξ isn't **MEM** valid. Hence there exists an **MEM** model M and a world w in M such that:

$$M, w \models tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi) \wedge \neg tr(\varphi) \quad (*).$$

If w were such a point satisfying $\mathcal{T}(w) \setminus \{w\} \neq \emptyset$, then from $(*)$ we would immediately obtain a contradiction, namely that $M, w \models tr(\chi)$ and also $M, w \not\models tr(\chi)$ (by Remark 2 and the constraints $\mathrm{wmconv}(\mathcal{T}, \mathcal{S})$ and $\mathrm{refl}_2(\mathcal{S})$). Hence we conclude that $\mathcal{T}(w) = \{w\}$. Using $(*)$, it also turns out that $M, w \models tr(\chi)$ and $M, w \models \neg tr(\varphi)$. Then Proposition 9 yields $V_w, V_w \models \chi$ and $V_w, V_w \not\models \varphi$. If $V_w = \emptyset$ then we can easily deduce that V_w is an equilibrium model of χ, so the result we are looking for trivially follows. Now consider the case where $V_w \neq \emptyset$. The point w cannot be a singleton point (otherwise and as noted before, it would then have an empty valuation). First of all, we recall $M, w \models [\mathrm{S}]\neg tr(\chi)$ $(**)$, which is an immediate consequence of $(*)$. Then, from $(**)$ we obtain $M, u \not\models tr(\chi)$ for every u with $w\mathcal{S}u$. Next, through the constraints $\mathrm{neg}(\mathcal{S}, \mathcal{T})$, $\mathrm{refl}_2(\mathcal{S})$ and $\mathrm{mconv}(\mathcal{S}, \mathcal{T})$ we get $\mathcal{T}(u) = \{u, w\}$ with $u \neq w$, for any u with $w\mathcal{S}u$. Hence, $\mathcal{T}(u) \setminus \{u\} \neq \emptyset$, and moreover $w \in \mathcal{T}(u) \setminus \{u\}$ is uniquely determined. Thus, using Proposition 9 we can assert that $V_u, V_w \not\models \chi$ for any u such that $w\mathcal{S}u$. As a next step, by Proposition 2 we suppose w.l.o.g. that V_w is finite. Finally, it remains to conclude that $H, V_w \not\models \chi$ for any H with $H \subset V_w$: this is nothing but a consequence of Proposition 1.2. We also know that $V_w, V_w \models \chi$ (see above). Hence, we again show the existence of an equilibrium model of χ, namely V_w. Moreover V_w is such that $V_w, V_w \not\models \varphi$ (see above). Thus, we get $\chi \not\approx \varphi$. So, we are done.

($\Longleftarrow$): Let ξ be **MEM** valid, and let $T \subseteq \mathbb{P}$ be an equilibrium model of χ. Now we construct the **MEM** model, M_T, as it is done in Proposition 11. Then we conclude that M_T is a legal **MEM** model and that $M_T, T \models tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi)$ again by Proposition 11. Since ξ is **MEM** valid, $M_T, T \models \xi$, but as M_T is an **MEM** model, we also have $M_T, T \models tr(\varphi)$. Finally, from Proposition 9 we obtain that $T, T \models \varphi$. Thus we conclude that $\chi \approx \varphi$, and this ends the proof. □

Corollary 13. *For every $\mathcal{L}_{\rightarrow}$-formula χ, χ has an equilibrium model if and only if*

$$tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi)$$

is **MEM** *satisfiable.*

Here is an example. We have seen before that $\top \approx \neg p$ for every p, i.e., $\neg p$ is a consequence of $\top$ in equilibrium models. Hence Theorem 12 tells us that the formula $\xi = tr(\top) \wedge [\mathrm{S}]\neg tr(\top) \supset tr(\neg p)$ must be provable from the axioms and the inference rules of **MEM**. This can be established by the following sequence of equivalent formulas. Before, we recall that in any normal modal logic, $tr(\top)$ is equivalent to $\top$ and $tr(\neg p)$ is equivalent to $[\mathrm{T}]\neg p$ (see Section 4.1).

1. $tr(\top) \wedge [\mathrm{S}]\neg tr(\top) \supset tr(\neg p)$
2. $\top \wedge [\mathrm{S}]\neg\top \supset [\mathrm{T}]\neg p$ (see above)
3. $[\mathrm{S}]\bot \supset [\mathrm{T}]\neg p$.

The last line is provable in our logic: indeed, we see below that $[\mathrm{S}]\bot \supset [\mathrm{T}]\neg p$ can be proved in our logic **MEM** by standard principles of modal logic.

1. $\langle\mathrm{T}\rangle(p \wedge \top) \supset \langle\mathrm{T}\rangle\langle\mathrm{S}\rangle(\neg p \wedge \top)$ (axiom $\mathrm{Neg}([\mathrm{S}], [\mathrm{T}])$)
2. $\langle\mathrm{T}\rangle p \supset \langle\mathrm{T}\rangle\langle\mathrm{S}\rangle\neg p$ (from 1 by classical logic)
3. $\neg p \supset \top$ (tautology)

4. $\langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\neg p \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top$ (from 3 by $\mathbf{K}([\mathrm{T}])$ and $\mathbf{K}([\mathrm{S}])$)
5. $\langle \mathrm{T}\rangle p \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top$ (from 2 & 4)
6. $p \supset \langle \mathrm{T}\rangle p$ (axiom T ($[\mathrm{T}]$))
7. $p \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top$ (from 5 & 6)
8. $\langle \mathrm{T}\rangle p \supset \langle \mathrm{T}\rangle\langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top$ (from 7 by $\mathbf{K}([\mathrm{T}])$)
9. $\langle \mathrm{T}\rangle\langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top$ (axiom 4 ($[\mathrm{T}]$))
10. $\langle \mathrm{T}\rangle p \supset \langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top$ (from 8 & 9)
11. $\langle \mathrm{T}\rangle(\langle \mathrm{S}\rangle\top \wedge [\mathrm{S}]\langle \mathrm{S}\rangle\top) \supset \langle \mathrm{S}\rangle\top$ (axiom WMConv($[\mathrm{T}], [\mathrm{S}]$))
12. $[\mathrm{S}](\top \supset \langle \mathrm{S}\rangle\top)$ (axiom $\mathrm{T}_2([\mathrm{S}])$)
13. $[\mathrm{S}](\langle \mathrm{S}\rangle\top \supset \top)$ (axiom $\mathrm{WTriv}_2([\mathrm{S}])$)
14. $[\mathrm{S}]\langle \mathrm{S}\rangle\top \equiv [\mathrm{S}]\top$ (from 12 & 13 by $\mathbf{K}([\mathrm{S}])$)
15. $[\mathrm{S}]\langle \mathrm{S}\rangle\top \equiv \top$ (from 14 by $\mathbf{K}([\mathrm{S}])$)
16. $\langle \mathrm{T}\rangle\langle \mathrm{S}\rangle\top \supset \langle \mathrm{S}\rangle\top$ (from 11 & 15 by $\mathbf{K}([\mathrm{T}])$ and $\mathbf{K}([\mathrm{S}])$)
17. $\langle \mathrm{T}\rangle p \supset \langle \mathrm{S}\rangle\top$ (from 10 & 16)
18. $[\mathrm{S}]\bot \supset [\mathrm{T}]\neg p$ (from 17 by $\mathbf{K}([\mathrm{T}])$ and $\mathbf{K}([\mathrm{S}])$).

Therefore the original formula ξ is also provable in our logic.

6. Conclusion and further research

In this paper we have proposed a monotonic modal logic **MEM** that is powerful enough to characterise the existence of an equilibrium model as well as consequence in equilibrium models. Its modal operators [T] and [S] are interpreted in a fairly standard class of Kripke models. We have given a sound and complete axiomatisation of our logic and we have shown that it can be checked whether $\chi \models\!\!\approx \varphi$, i.e., whether φ is a logical consequence of χ in equilibrium logic, by checking if the modal formula

$$tr(\chi) \wedge [\mathrm{S}]\neg tr(\chi) \supset tr(\varphi)$$

is valid in **MEM** or not, where tr is a polynomial translation from the language of here-and-there logic into **MEM**. The logic **MEM** thus captures the minimisation that is central in the definition of equilibrium models and which is only formulated in the metalanguage there.

We had started the investigation of modal logics underlying equilibrium logic in two previous papers [11,12]. Although we were also able to capture consequence in equilibrium models in the logic of [12], we were not able to avoid the exponential growth of formula length when translating formulas of equilibrium logic into our bimodal logic. The present logic **MEM** avoids that undesired growth. In both papers, what we did for equilibrium logic parallels what Levesque did for autoepistemic logic: he also designed a monotonic modal logic that was able to capture nonmonotonic autoepistemic reasoning [16,17].

Besides embedding a nonmonotonic logic into a monotonic logic, our logic has a further interesting feature that may be exploited in future work: we can now apply well-known automated deduction methods for modal logics [10,8,15,23,24] to equilibrium logic. We may use in particular our LoTREC tableau proving platform [9]. The implementation of a tableau procedure for **MEM** requires a specific tableau rule that does the following: for each subset of the set of propositional variables appearing in some node, create an $\mathcal{S}$-accessible node where all these variables are false.

In future work we plan to extend our approach in two ways. First, we would like to extend our language with modal operators allowing to talk about belief, knowledge, action, and time, providing thus a comprehensive framework for extensions of answer-set programming by modal concepts. Only few approaches exist up to now, essentially temporal extensions of equilibrium logic [1,4]. In particular, we plan to integrate into **MEM** logic the modal operators of dynamic logic of propositional assignments **DL-PA** that we have recently studied [2]. Propositional assignments set the truth values of propositional variables to either true or false and update the current model in the style of dynamic epistemic logics. Our second research avenue is to capture other nonmonotonic logics such as the nonmonotonic extension of **S4F** [21]. However, the very next step is to study how **DL-PA** and **MEM** have to be combined. First results are in [13].

Acknowledgements

We would like to thank the two reviewers for their careful reading and the helpful comments they provided.

This work was partially supported by the French–Spanish *Laboratoire Européen Associé* (*LEA*) *"French–Spanish Lab of Advanced Studies in Information Representation and Processing"*.

References

[1] F. Aguado, P. Cabalar, G. Pérez, C. Vidal, Strongly equivalent temporal logic programs, in: S. Hölldobler, C. Lutz, H. Wansing (Eds.), JELIA, in: Lect. Notes Comput. Sci., vol. 5293, Springer, 2008, pp. 8–20.

[2] P. Balbiani, A. Herzig, N. Troquard, Dynamic logic of propositional assignments: a well-behaved variant of PDL, in: O. Kupferman (Ed.), Logic in Computer Science (LICS), New Orleans, 25–28 June, 2013, IEEE, 2013.

[3] P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, Camb. Tracts Theor. Comput. Sci., University Press, 2001.

[4] P. Cabalar, S. Demri, Automata-based computation of temporal equilibrium models, in: G. Vidal (Ed.), LOPSTR, in: Lect. Notes Comput. Sci., vol. 7225, Springer, 2011, pp. 57–72.

[5] P. Cabalar, P. Ferraris, Propositional theories are strongly equivalent to logic programs, Theory Pract. Log. Program. (TPLP) 7 (2007) 745–759.

[6] P. Cabalar, D. Pearce, A. Valverde, Minimal logic programs, in: V. Dahl, I. Niemelä (Eds.), Proc. ICLP, in: Lect. Notes Comput. Sci., vol. 4670, Springer-Verlag, 2007, pp. 104–118.

[7] W.A. Carnielli, C. Pizzi, J. Bueno-Soler, Modalities and Multimodalities, Logic, Epistemology, and the Unity of Science, Springer-Verlag, 2009.

[8] P. Enjalbert, L. Fariñas del Cerro, Modal resolution in clausal form, Theor. Comput. Sci. 65 (1989) 1–33.

[9] L. Fariñas del Cerro, D. Fauthoux, O. Gasquet, A. Herzig, D. Longin, F. Massacci, Lotrec: The generic tableau prover for modal and description logics, in: R. Goré, A. Leitsch, T. Nipkow (Eds.), Proceedings of the First International Joint Conference on Automated Reasoning (IJCAR 2001), Siena, Italy, June 18–23, 2001, in: Lect. Notes Comput. Sci., vol. 2083, Springer-Verlag, 2001, pp. 453–458.

[10] L. Fariñas del Cerro, A. Herzig, Modal deduction with applications in epistemic and temporal logic, in: D. Gabbay, J. Chris, J.A. Robinson (Eds.), Handbook of Logic and Artificial Intelligence, in: Epistemic and Temporal Reasoning, vol. 4, Oxford University Press, 1995, pp. 499–594.

[11] L. Fariñas del Cerro, A. Herzig, Contingency-based equilibrium logic, in: Logic Programming and Nonmonotonic Reasoning, Springer-Verlag, 2011, pp. 223–228.

[12] L. Fariñas del Cerro, A. Herzig, The modal logic of equilibrium models, in: Frontiers of Combining Systems (FroCoS), Springer-Verlag, 2011, pp. 135–146.

[13] L. Fariñas del Cerro, A. Herzig, E.I. Su, Combining equilibrium logic and dynamic logic, in: P. Cabalar, T.C. Son (Eds.), Logic Programming and Nonmonotonic Reasoning (LPNMR), in: Lect. Notes Comput. Sci., vol. 8148, Springer, 2013, pp. 304–316.

[14] P. Ferraris, J. Lee, V. Lifschitz, A new perspective on stable models, in: M.M. Veloso (Ed.), IJCAI, 2007, pp. 372–379.

[15] U. Hustadt, R.A. Schmidt, MSPASS: Modal reasoning by translation and first-order resolution, in: R. Dyckhoff (Ed.), Automated Reasoning with Analytic Tableaux and Related Methods, International Conference (TABLEAUX 2000), in: Lecture Notes in Artificial Intelligence, vol. 1847, Springer, 2000, pp. 67–71.

[16] G. Lakemeyer, H.J. Levesque, Only-knowing meets nonmonotonic modal logic, in: G. Brewka, T. Eiter, S.A. McIlraith (Eds.), KR, AAAI Press, 2012.

[17] H.J. Levesque, All I know: A study in autoepistemic logic, Artif. Intell. J. 42 (1990) 263–309.

[18] V. Lifschitz, Thirteen definitions of a stable model, in: A. Blass, N. Dershowitz, W. Reisig (Eds.), Fields of Logic and Computation, in: Lect. Notes Comput. Sci., vol. 6300, Springer-Verlag, 2010, pp. 488–503.

[19] V. Lifschitz, D. Pearce, A. Valverde, Strongly equivalent logic programs, ACM Trans. Comput. Log. 2 (2001) 526–541.

[20] D. Pearce, A new logical characterisation of stable models and answer sets, in: J. Dix, L.M. Pereira, T.C. Przymusinski (Eds.), NMELP, in: Lect. Notes Comput. Sci., vol. 1216, Springer Verlag, 1996, pp. 57–70.

[21] D. Pearce, L. Uridia, An approach to minimal belief via objective belief, in: T. Walsh (Ed.), IJCAI, AAAI Press, 2011, pp. 1045–1050.

[22] D. Pearce, I.P. de Guzmán, A. Valverde, A tableau calculus for equilibrium entailment, in: R. Dyckhoff (Ed.), TABLEAUX, in: Lect. Notes Comput. Sci., vol. 1847, Springer-Verlag, 2000, pp. 352–367.

[23] R.A. Schmidt, D. Tishkovsky, A general tableau method for deciding description logics, modal logics and related first-order fragments, in: A. Armando, P. Baumgartner, G. Dowek (Eds.), Automated Reasoning (IJCAR, 2008), in: Lect. Notes Comput. Sci., vol. 5195, Springer, 2008, pp. 194–209.

[24] R. Sebastiani, A. Tacchella, Sat techniques for modal and description logics, in: A. Biere, M. Heule, H. van Maaren, T. Walsh (Eds.), Handbook of Satisfiability, in: Frontiers in Artificial Intelligence and Applications, vol. 185, IOS Press, 2009, pp. 781–824.