# Rare Events for the Manneville-Pomeau Map 

Ana Cristina Freitas, Jorge Milhazes Freitas, Mike Todd, Sandro Vaienti

## To cite this version:

Ana Cristina Freitas, Jorge Milhazes Freitas, Mike Todd, Sandro Vaienti. Rare Events for the Manneville-Pomeau Map . Stochastic Processes and their Applications, 2016, 126 (11), pp.3463-3479 10.1016/j.spa.2016.05.001 . hal-01127757

## HAL Id: hal-01127757

## https://hal.science/hal-01127757

Submitted on 8 Mar 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# RARE EVENTS FOR THE MANNEVILLE-POMEAU MAP 

ANA CRISTINA MOREIRA FREITAS, JORGE MILHAZES FREITAS, MIKE TODD, AND SANDRO VAIENTI


#### Abstract

We prove a dichotomy for Manneville-Pomeau maps $f:[0,1] \rightarrow[0,1]$ : given any point $\zeta \in[0,1]$, either the Rare Events Point Processes (REPP), counting the number of exceedances, which correspond to entrances in balls around $\zeta$, converge in distribution to a Poisson process; or the point $\zeta$ is periodic and the REPP converge in distribution to a compound Poisson process. Our method is to use inducing techniques for all points except 0 and its preimages, extending a recent result [HWZ14, and then to deal with the remaining points separately. The preimages of 0 are dealt with applying recent results in AFV14. The point $\zeta=0$ is studied separately because the tangency with the identity map at this point creates too much dependence, which causes severe clustering of exceedances. The Extremal Index, which measures the intensity of clustering, is equal to 0 at $\zeta=0$, which ultimately leads to a degenerate limit distribution for the partial maxima of stochastic processes arising from the dynamics and for the usual normalising sequences. We prove that using adapted normalising sequences we can still obtain non-degenerate limit distributions at $\zeta=0$.


## 1. Introduction

One of the standard ways to investigate the statistical properties of a dynamical system $f: \mathcal{X} \rightarrow \mathcal{X}$ with respect to some measure $\mathbb{P}$ is to look at its recurrence to certain points $\zeta$ in the system. This can be connected to Extreme Value theory: by taking a suitable observable $\varphi: \mathcal{X} \rightarrow \mathbb{R}$ taking its unique maximum $u_{F}$ at $\zeta$, one can look at the behaviour of the iterates $x, f(x), f^{2}(x), \ldots$ via the observations

$$
X_{i}=X_{i}(x)=\varphi \circ f^{n}(x)
$$

If $\mathbb{P}$ is an $f$-invariant probability measure then $X_{0}, X_{1}, \ldots$ is a stationary stochastic process. We furthermore assume that $\mathbb{P}$ is ergodic in order to isolate specific statistical behaviour. So Birkhoff's Ergodic Theorem implies that these random variables will satisfy the law of large numbers. We can now consider the random variable given by the maximum of this process:

$$
M_{n}=M_{n}(x):=\max \left\{X_{0}(x), \ldots, X_{n-1}(x)\right\}
$$

Date: March 5, 2015.
2000 Mathematics Subject Classification. 37A50, 60G55, 60G70, 37B20, 60G10, 37C25.
ACMF was partially supported by FCT grant SFRH/BPD/66174/2009. JMF was partially supported by FCT grant SFRH/BPD/66040/2009. Both these grants are financially supported by the program POPH/FSE. ACMF, JMF and SV are partially supported by FCT (Portugal) project PTDC/MAT/120346/2010, which is funded by national and European structural funds through the programs FEDER and COMPETE. ACMF, JMF and MT are also supported by CMUP (UID/MAT/00144/2013), which is funded by FCT (Portugal) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020. SV has been supported by the ANR grant PERTURBATIONS and by the project MOD TER COM of the french Region PACA.

Again by the ergodic theorem, if $\zeta$ is in the support of $\mathbb{P}$, we expect $M_{n} \rightarrow u_{F}$, so to obtain a non-trivial limit law, we need to rescale $\left\{M_{n}\right\}_{n}$. Indeed, we say that we have an Extreme Value Law (EVL) for $M_{n}$ if there is a non-degenerate distribution function $H: \mathbb{R} \rightarrow[0,1]$ with $H(0)=0$ and for every $\tau>0$ there exists a sequence of levels $u_{n}=u_{n}(\tau)$ such that

$$
\begin{equation*}
n \mathbb{P}\left(X_{0}>u_{n}\right) \rightarrow \tau \text { as } n \rightarrow \infty, \tag{1}
\end{equation*}
$$

and for which the following holds:

$$
\mathbb{P}\left(M_{n} \leqslant u_{n}\right) \rightarrow \bar{H}(\tau)=1-H(\tau) \text { as } n \rightarrow \infty
$$

where the convergence is meant at the continuity points of $H(\tau)$.
In recent years, there has been a great deal of work on EVLs in the context of dynamical systems (see for example Col01, FF08a, VHF09, FFT10, GHN11, HNT12, LFW12, Kel12, FHN14, AFV14]), the standard form of the observable $\varphi$ being a function of the distance to $\zeta$, for example $\varphi(x)=-d(x, \zeta)$ for $d$ a metric on $\mathcal{X}$. In many cases it has been shown that for $\mathbb{P}$-a.e. $\zeta \in \mathcal{X}$, this setup gives an EVL with $\bar{H}=e^{-\tau}$. More recently it has been shown that if $\zeta$ is a periodic point of period $p$ then $\bar{H}=e^{-\theta \tau}$ where $\theta \in(0,1)$ depends on the Jacobean of the measure for $f^{p}$, and is referred to as the Extremal Index (EI). The EI is known to measure the intensity of clustering of exceedances of the levels $u_{n}$. In fact, in many cases, the EI is equal to the inverse of the average cluster size, so that the EI is equal to 1 when there is no clustering. In the case of a class of uniformly hyperbolic dynamical systems, a stronger property, a dichotomy, has been shown: either $\zeta$ is periodic and we have an EVL with some extremal index $\theta \in(0,1)$, or there is an EVL $\bar{H}=e^{-\tau}$. This was shown for $f$ some uniformly expanding interval maps with a finite number of branches in [FP12] (see also FFT12, Section 6]) and with a countable number of branches in AFV14; here, depending on the precise form of the map, the measure can be absolutely continuous with respect to Lebesgue (acip), or an equilibrium state for some Hölder potential. Inducing methods have been used to extend some of these results to non-uniformly hyperbolic dynamical systems (see FFT13] which built on [BSTV03]), but the results have not thus far extended to such a complete dichotomy.

We can further enrich our process by considering the point process formed by entries into the regions $\left\{X>u_{n}\right\}$, which in good cases gives rise to a Poisson process. An analogous dichotomy can often be shown there also: in the case of a periodic point $\zeta$, we obtain a compound Poisson process. We leave the details of this construction to later.

In this note, we extend the dichotomy to a simple non-uniformly hyperbolic dynamical system, the Manneville-Pomeau (MP) map equipped with an absolutely continuous invariant probability measure. The form for such maps given in [LSV99, BSTV03] is, for $\alpha \in(0,1)$, the corresponding MP map is

$$
f=f_{\alpha}(x)= \begin{cases}x\left(1+2^{\alpha} x^{\alpha}\right) & \text { for } x \in[0,1 / 2) \\ 2 x-1 & \text { for } x \in[1 / 2,1]\end{cases}
$$

Members of this family of maps are often referred to as Liverani-Saussol-Vaienti maps since their actual equation was first introduced in [LSV99. As it can be seen for example in LSVV99, You99, Hu04, these maps have polynomial decay of correlations:

$$
\begin{equation*}
\left|\int \varphi \cdot\left(\psi \circ f^{t}\right) d \mu_{\alpha}-\int \varphi d \mu_{\alpha} \int \psi d \mu_{\alpha}\right| \leq C\|\varphi\|_{\mathcal{H}_{\beta}}\|\psi\|_{\infty} \frac{1}{t^{\frac{1}{\alpha}-1}}, \tag{2}
\end{equation*}
$$

where $\mathcal{H}_{\beta}$ denotes the space of Hölder continuous functions $\varphi$ with exponent $\beta$ equipped with the norm $\|\varphi\|_{\mathcal{H}_{\beta}}=\|\varphi\|_{\infty}+|\varphi|_{\mathcal{H}_{\beta}}$, where

$$
|\varphi|_{\mathcal{H}_{\beta}}=\sup _{x \neq y} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{\beta}} .
$$

Let $h_{\alpha}=\frac{d \mu_{\alpha}}{d x}$. From Hu04 we know that $\lim _{x \rightarrow 0} \frac{h(x)}{x^{-\alpha}}=C_{0}>0$. Hence, for small $s>0$ we have that

$$
\begin{equation*}
\mu_{\alpha}([0, s)) \sim_{c} s^{1-\alpha}, \tag{3}
\end{equation*}
$$

where the notation $\sim_{c}$ is used in the sense that there is $c>0$ such that $\lim _{s \rightarrow \infty} \frac{\mu_{\alpha}([0, s))}{s^{1-\alpha}}=c$.
In this case there are canonical induced maps which capture all but a countable number of points in the phase space, so with some extra consideration for those points not captured, we can prove the full dichotomy, where for $\zeta$ a periodic point of period $p$, the extremal index is $\theta=1-1 /\left|D f^{p}(\zeta)\right|$.

For the special case in which $\zeta$ is the indifferent fixed point, we prove that there exists an EI equal to zero, which corresponds to a degenerate limit law, when the usual normalising sequences are used. Moreover, we show that by changing the definition of $\left(u_{n}\right)_{n}$ given by (1) in a suitable way, we recover a non-degenerate EVL. This latter result relies on information on the transfer operator in HSV99 as well as a refinement of the techniques for proving EVLs at periodic points developed in [FFT15].
1.1. Point process of hitting times. We will use our observations on our dynamical system to generate point processes. Here we adopt the approach and notation of [Zwe07]. Let $\mathcal{M}_{p}([0, \infty))$ be the space of counting measures on $\left([0, \infty), \mathcal{B}_{[0, \infty)}\right)$. We equip this space with the vague topology, i.e., $\nu_{n} \rightarrow \nu$ in $\mathcal{M}_{p}([0, \infty))$ whenever $\nu_{n}(\psi) \rightarrow \nu(\psi)$ for any continuous function $\psi:[0, \infty) \rightarrow \mathbb{R}$ with compact support. A point process $N$ on $[0, \infty)$ is a random element of $\mathcal{M}_{p}([0, \infty))$. We will be interested in point processes $N_{n}: X \rightarrow \mathcal{M}_{p}([0, \infty))$. If we have a fixed measure $\mu$ on $X$, we say that $\left(N_{n}\right)_{n}$ converges in distribution to $N$ if $\mu \circ N_{n}^{-1}$ converges weakly to $\mu \circ N^{-1}$. We write $N_{n} \xrightarrow{\mu} N$.

So given $X_{0}, X_{1}, X_{2}, \ldots$ and some $u \in \mathbb{R}$, we begin the construction of our point process $\mathbb{R} \rightarrow \mathcal{M}_{p}([0, \infty))$ as follows. Given $A \subset \mathbb{R}$ we define

$$
\mathscr{N}_{u}(A):=\sum_{i \in A \cap \mathbb{N}_{0}} \mathbb{1}_{X_{i}>u} .
$$

So $\mathscr{N}_{u}[0, n)$ counts the number of exceedances of the parameter $u$ among the first $n$ observations of the process $X_{0}, X_{1}, \ldots, X_{n-1}$ or, in other words, the number of entrances in $U(u)$ up to time $n$.

We next re-scale time using the factor $v:=1 / \mathbb{P}(X>u)$ given by Kac's Theorem. However, before we give the definition, we need some formalism. Let $\mathcal{S}$ denote the semi-ring of subsets of $\mathbb{R}_{0}^{+}$whose elements are intervals of the type $[a, b)$, for $a, b \in \mathbb{R}_{0}^{+}$. Let $\mathcal{R}$ denote the ring generated by $\mathcal{S}$. Recall that for every $J \in \mathcal{R}$ there are $k \in \mathbb{N}$ and $k$ intervals $I_{1}, \ldots, I_{k} \in \mathcal{S}$ such that $J=\cup_{i=1}^{k} I_{j}$. In order to fix notation, let $a_{j}, b_{j} \in \mathbb{R}_{0}^{+}$be such that $I_{j}=\left[a_{j}, b_{j}\right) \in \mathcal{S}$.

For $I=[a, b) \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, we denote $\alpha I:=[\alpha a, \alpha b)$ and $I+\alpha:=[a+\alpha, b+\alpha)$. Similarly, for $J \in \mathcal{R}$ define $\alpha J:=\alpha I_{1} \cup \cdots \cup \alpha I_{k}$ and $J+\alpha:=\left(I_{1}+\alpha\right) \cup \cdots \cup\left(I_{k}+\alpha\right)$.

We suppose that we are given $\tau>0$ such that $\mathbb{P}\left(X>u_{n}\right)=\frac{\tau}{n}$. We let $U\left(u_{n}\right)=\left\{X>u_{n}\right\}$ and let $v_{n}$ be the corresponding scaling factor defined above.

Definition. We define the rare event point process (REPP) by counting the number of exceedances (or hits to $U\left(u_{n}\right)$ ) during the (re-scaled) time period $v_{n} J \in \mathcal{R}$, where $J \in \mathcal{R}$. To be more precise, for every $J \in \mathcal{R}$, set

$$
N_{n}(J):=\mathscr{N}_{u_{n}}\left(v_{n} J\right)=\sum_{j \in v_{n} J \cap \mathbb{N}_{0}} \mathbb{1}_{X_{j}>u_{n}} .
$$

### 1.2. Main results.

Theorem 1. Given $\zeta \in(0,1]$, consider the $R E P P N_{n}$ defined above. Then either
(1) $\zeta$ is not periodic and $N_{n}$ converges in distribution to a Poisson process $N$ with intensity 1.
(2) $\zeta$ is periodic with period $p$ and $N_{n}$ converges in distribution to a compound Poisson process $N$ with intensity $\theta=1-\left|\operatorname{det} D\left(f^{-p}\right)(\zeta)\right|$ and multiplicity distribution function $\pi$ given by $\pi_{\kappa}=\theta(1-\theta)^{\kappa-1}$, for every $\kappa \in \mathbb{N}_{0}$.

Theorem 2. For $\zeta=0$, consider the maximum function $M_{n}=M_{n}(x)$ defined above.
(1) Let $\left(u_{n}\right)_{n}=\left(u_{n}(\tau)\right)_{n}$ be chosen as in (1), then $\mathbb{P}\left(M_{n} \leqslant u_{n}\right) \rightarrow 1$ as $n \rightarrow \infty$ for any $\tau>0$.
(2) If $\alpha \in(0, \sqrt{5}-2)$, then for each $\tau>0$, there exists a sequence of thresholds $\left(u_{n}\right)_{n}=$ $\left(u_{n}(\tau)\right)_{n}$ so that $\mathbb{P}\left(M_{n} \leqslant u_{n}\right) \rightarrow e^{-\tau}$ as $n \rightarrow \infty$.

Remark 1. We note that in the case $\zeta=0$, while it is possible to rescale the thresholds to recover an EVL as in Theorem圆 (2), the corresponding REPP remains degenerate. This result will form part of a forthcoming work [FFR].
1.3. Comments on history and strategy. Before discussing our approach we introduce some notation. For a dynamical system $f: \mathcal{X} \rightarrow \mathcal{X}$ and a subset $A \subset \mathcal{X}$, for $x \in \mathcal{X}$ define

$$
r_{A}(x):=\inf \left\{n \in \mathbb{N}: f^{n}(x) \in A\right\},
$$

the first hitting time to $A$. Note that there is a connection with the behaviour of the variable $r_{U\left(u_{n}\right)}$ and our REPP since we can break that process down into a sequence of first hits to $U\left(u_{n}\right)$. This gives a connection with our REPP and the asymptotics of $r_{U\left(u_{n}\right)}$, the Hitting Time Statistics (HTS). One basic difference is that here we are concerned with all hits to $U\left(u_{n}\right)$, not just the first.

Our main result for the case of $\mathbb{P}$-typical points and for periodic points in $(0,1)$ follows quickly from previous works, including works already mentioned above, and indeed in some of these papers mention MP explicitly. We also remark that some of the earliest works on HTS for dynamical systems considered the case of MP maps with $\alpha \geqslant 1$, see for example CG93, CGS92, CI95, with a focus on the behaviour at 0 . In these cases, the sets $A_{n}$ considered were
formed from dynamically defined cylinder sets and the analysis was done at $1 / 2$, the preimage of 0 , so that finite measure sets could be used. In this paper we consider the case $\alpha \in(0,1)$, so $f$ has an acip, and we also consider more general points and sets $A_{n}$.

We will first consider all points in ( 0,1 ), using inducing methods. This will require us to generalise the already very flexible result of [HWZ14] to point processes. Finally we use the approach which goes back to Leadbetter [Lea74] of proving some short range and long range recurrence conditions to prove that we have a degenerate law at 0 (the extremal index is 0 ).

## 2. Induced point processes

Here we aim to generalise HWZ14] to point processes. In that paper, they use [Zwe07, Corollary 5] as one of their key tools. In our, fairly analogous, setting we use Zwe07, Corollary 6] instead. Note that previous results here include [BSTV03, Theorem 2.1], where they proved that for balls around typical points, the HTS of first return maps are the same as that for the original map - they also remarked, without details, that this can be extended to successive return times. Also in [FFT13, we extended this idea to periodic points. The strengths of the approach in HWZ14 to HTS are that it covers all points, and that the proof is rather short.

We will give our result comparing the point process of the induced system to that coming from the original system in a general setting and then later apply this to our MP example. In this section, we take a dynamical system $f: \mathcal{X} \rightarrow \mathcal{X}$ with an ergodic $f$-invariant probability measure $\mu$, choose a subset $Y \subset \mathcal{X}$ and consider $F_{Y}: Y \rightarrow Y$ to be the first return map $f^{r_{Y}}$ to $Y$ (note that $F$ may be undefined at a zero Lebesgue measure set of points which do not return to $Y$, but most of these points are not important, so we will abuse notation here). Let $\mu_{Y}(\cdot)=\frac{\mu(\cdot \cap Y)}{\mu(Y)}$ be the conditional measure on $Y$. By Kac's Theorem $\mu_{Y}$ is $F_{Y}$-invariant.

Setting $v_{n}^{Y}=1 / \mu_{Y}\left(X>u_{n}\right)$, for the induced process $X_{i}^{Y}$,

$$
N_{n}^{Y}(J):=\mathscr{N}_{u_{n}}^{Y}\left(v_{n}^{Y} J\right)=\sum_{j \in v_{n}^{Y} J \cap \mathbb{N}_{0}} \mathbb{1}_{X_{j}^{Y}>u_{n}} .
$$

In keeping with HWZ14, we denote our inducing domain by $Y$. Denote the speeded up return time $r_{A}$ by $r_{A, Y}$ and the induced measure on $Y$ by $\mu_{Y}$.

Theorem 3. For $\eta>0$, setting $J_{\eta}:=\cup_{s \in J} B_{\eta}(s)$, we assume that $N\left(J_{\eta}\right)$ is continuous in $\eta$, for all small $\eta$.

$$
N_{n}^{Y} \xrightarrow{\mu_{Y}} N \text { as } n \rightarrow \infty \text { implies } N_{n} \xrightarrow{\mu} N \text { as } n \rightarrow \infty .
$$

Proof. By Zwe07, Corollary 6], for a general sequence of point processes $\left(N_{n}\right)_{n}$ and an ergodic reference measure $m$, if $P \ll m$ then $N_{n} \xlongequal{P} N$ in $\mathcal{M}_{p}([0, \infty))$ implies $N_{n} \xlongequal{Q} N$ in $\mathcal{M}_{p}([0, \infty))$ for any $Q \ll m$. So replacing both $m$ and $Q$ with $\mu$ and replacing $P$ with $\mu_{Y}$ we see that for our sequence of processes, if $N_{n} \xlongequal{\mu_{Y}} N$ in $\mathcal{M}_{p}([0, \infty))$, then $N_{n} \xrightarrow{\mu} N$ in $\mathcal{M}_{p}([0, \infty))$. Thus it suffices to show that for every $J \in \mathcal{R}$ and all $k \in \mathbb{N}$,

$$
\mu_{Y}\left(N_{n}^{Y}(J) \geqslant k\right) \xrightarrow{n \rightarrow \infty} \mu_{Y}(N(J) \geqslant k) \text { implies } \mu_{Y}\left(N_{n}(J) \geqslant k\right) \xrightarrow{n \rightarrow \infty} \mu_{Y}(N(J) \geqslant k) .
$$

For $\delta>0$, let

$$
E_{M}=E_{M}^{\delta}:=\left\{\left(\frac{1-\delta}{\mu(Y)}\right) j \leqslant r_{Y}^{j} \leqslant\left(\frac{1+\delta}{\mu(Y)}\right) j \text { for all } j \geqslant M\right\} \text { and } F_{N}:=\left\{r_{U\left(u_{n}\right), Y} \geqslant N\right\} .
$$

As in HWZ14], $\mu_{Y}\left(F_{N}^{c}\right) \leqslant N \mu_{Y}\left(U\left(u_{n}\right)\right) \rightarrow 0$ as $n \rightarrow \infty$. Also the ergodic theorem says that $\mu_{Y}\left(\left(E_{M}^{\delta}\right)^{c}\right) \rightarrow 0$ as $M \rightarrow \infty$.

For $x \in E_{M}^{\delta}$,
$r_{U\left(u_{n}\right), Y}^{k}(x)\left(\frac{1-\delta}{\mu(Y)}\right) \leqslant r_{U\left(u_{n}\right)}^{k}(x)=\sum_{j=0}^{r_{U\left(u_{n}\right), Y}^{k}(x)-1} r_{Y} \circ F_{Y}^{j}(x)=r_{Y\left(u_{n}\right), Y}^{r^{k}(x)}(x) \leqslant r_{U\left(u_{n}\right), Y}^{k}(x)\left(\frac{1+\delta}{\mu(Y)}\right)$
We can deduce that for $x \in F_{N} \cap E_{M}^{\delta}$,

$$
\mu(Y) r_{U\left(u_{n}\right)}^{k}(x) \in B_{\delta r_{U\left(u_{n}\right), Y}^{k}(x)}\left(r_{U\left(u_{n}\right), Y}^{k}(x)\right)
$$

So if $r_{U\left(u_{n}\right), Y}^{k}(x) \in v_{n}^{Y} J$ then $\mu(Y) r_{U\left(u_{n}\right)}^{k}(x) \in v_{n}^{Y} J_{\delta}$ and so $r_{U\left(u_{n}\right)}^{k}(x) \in v_{n} J_{\delta}$. Therefore,

$$
\mu_{Y}\left(\left\{N_{n}\left(J_{\delta}\right) \geqslant k\right\} \cap\left(E_{N} \cap F_{M}\right)\right) \geqslant \mu_{Y}\left(\left\{N_{n}^{Y}(J) \geqslant k\right\} \cap\left(E_{N} \cap F_{M}\right)\right)
$$

Setting $\delta^{\prime}:=\frac{\delta}{1+\delta}$, we also obtain that

$$
\frac{1}{\mu(Y)} r_{U\left(u_{n}\right), Y}^{k}(x) \in B_{\delta^{\prime} r_{U\left(u_{n}\right)}^{k}(x)}\left(r_{U\left(u_{n}\right)}^{k}(x)\right)
$$

for $x \in F_{N} \cap E_{M}$. Analogously to above, this leads us to

$$
\mu_{Y}\left(\left\{N_{n}^{Y}\left(J_{\delta^{\prime}}\right) \geqslant k\right\} \cap\left(E_{N} \cap F_{M}\right)\right) \geqslant \mu_{Y}\left(\left\{N_{n}(J) \geqslant k\right\} \cap\left(E_{N} \cap F_{M}\right)\right) .
$$

So since $\varepsilon, \delta>0$ were arbitrary, we are finished.

## 3. Application of inducing to Manneville-Pomeau

In this section we prove our main theorem for all points $\zeta \in(0,1)$.
Let $\mathcal{P}$ be the renewal partition, that is the partition defined inductively by $Z \in \mathcal{P}$ if $Z=[1 / 2,1)$ or $f(Z) \in \mathcal{P}$. Now let $Y \in \mathcal{P}$ and let $F_{Y}$ be the first return map to $Y$ and $\mu_{Y}$ be the conditional measure on $Y$. It is well-known that $\left(Y, F_{Y}, \mu_{Y}\right)$ is a Bernoulli map and hence, in particular, a Rychlik system (see Ryc83] or AFV14, Section 3.2.1] for the essential information about such systems) and so the REPP is understood as in [FFT13, Corollary 3]. Hence by AFV14 we have the following theorem.
Theorem 4. Given $\zeta \in Y$, consider the $R E P P N_{n}^{Y}$ defined above. Then either
(1) $\zeta$ is not periodic and $N_{n}^{Y}$ converges in distribution to a Poisson process $N$ with intensity 1.
(2) $\zeta$ is periodic with period $p$ and $N_{n}^{Y}$ converges in distribution to a compound Poisson process $N$ with intensity $\theta=1-\left|D\left(F_{Y}^{-p}\right)(\zeta)\right|$ and multiplicity d.f. $\pi$ given by 1 $\pi_{\kappa}=\theta(1-\theta)^{\kappa-1}$, for every $\kappa \in \mathbb{N}_{0}$.

For points in $Y \backslash \cup_{n \geqslant 1} f^{-n}(0)$, this theorem is Proposition 3.2 of AFV14. For the boundary points $\cup_{n \geqslant 1} f^{-n}(0)$, in the language of AFV14, any such point is called aperiodic non-simple. Hence by Proposition 3.4(1) of that paper, we have a standard extremal index of 1 at all such points. Varying $Y$ means that we have considered all points in $(0,1)$. So combining Theorems 3 and 4 completes the proof of Theorem 1 for $\zeta \neq 0$.

## 4. Analysis of the indifferent fixed point

The tangency of the graph of the MP map with the identity map, creates an intensive clustering of exceedances of levels $\left(u_{n}\right)_{n \in \mathbb{N}}$, when they are chosen as in (1), that leads to the existence of an EI equal to 0, which leads to a degenerate limit distribution for $M_{n}$. However, if we choose the levels $\left(u_{n}\right)_{n \in \mathbb{N}}$ not in the classical way, but rather a sequence of lower thresholds, so that the exceedances that escape the clustering effect have more weight, then we can recover the existence of a non-degenerate distribution for the maxima.

The proof of an EI equal to 0 for the usual normalising sequences follows easily from the existing connections between Return Times Statistics (RTS), Hitting Times Statistics (HTS) and EVL, which we will recall in the next subsection. The proof of the existence of a non degenerate limit, under a different normalising sequence of thresholds, is more complicated and requires some new results from [FFT15, which we will recall below.
4.1. The usual normalising sequences case. For any $\zeta \in[0,1]$, let $B_{\varepsilon}(\zeta)=(\zeta-\varepsilon, \zeta+$ $\varepsilon) \cap[0,1]$. If there exists a non degenerate d.f. G such that for all $t \geqslant 0$,

$$
\lim _{\varepsilon \rightarrow 0} \mu_{\alpha}\left(r_{B_{\varepsilon}(\zeta)} \leq \frac{t}{\mu_{\alpha}\left(B_{\varepsilon}(\zeta)\right)}\right)=G(t)
$$

then we say we have Hitting Time Statistics (HTS) $G$ for balls, at $\zeta$. Similarly, we can restrict our observations to $U(u)$ : if there exists a non degenerate (d.f.) $\tilde{G}$ such that for all $t \geqslant 0$,

$$
\lim _{\varepsilon \rightarrow 0} \mu_{\alpha}\left(\left.r_{B_{\varepsilon}(\zeta)} \leq \frac{t}{\mu_{\alpha}\left(B_{\varepsilon}(\zeta)\right)} \right\rvert\, B_{\varepsilon}(\zeta)\right)=\tilde{G}(t)
$$

then we say we have Return Time Statistics (RTS) $\tilde{G}$ for balls, at $\zeta$.
The normalising term in the definition of HTS/RTS is inspired by Kac's Theorem which states that the expected amount of time you have to wait before you return to $B_{\varepsilon}(\zeta)$ is exactly $\frac{1}{\mu_{\alpha}\left(B_{\varepsilon}(\zeta)\right)}$.

The existence of exponential HTS $\left(G(t)=1-\mathrm{e}^{-t}\right)$ is equivalent to the existence of exponential RTS $\left(\tilde{G}(t)=1-\mathrm{e}^{-t}\right)$. In fact, according to the Main Theorem in HLV05, a system has HTS

[^0]$G$ if and only if it has RTS $\tilde{G}$ and
\[

$$
\begin{equation*}
G(t)=\int_{0}^{t}(1-\tilde{G}(s)) d s \tag{4}
\end{equation*}
$$

\]

This formula was later generalised to obtain a relation between the distributional properties of Hitting Times and Return Times Points Processes in HLV07.

Moreover, according to the main theorem in [FFT10 the existence of HTS $G$ for balls at $\zeta$ is equivalent to the existence of an EVL $H$ for the process generated dynamically by an observable with a maximum at $\zeta$ and, in fact, $G=H$.

Let $U=[0, b)$ and $A=[a, b)$, where $a$ is such that $f(a)=b$, i.e., $b=a+2^{\alpha} a^{1+\alpha}$. Using (3) we easily get $\mu_{\alpha}(U) \sim_{c} a^{1-\alpha}+(1-\alpha) 2^{\alpha} a+o(a)$ and $\mu_{\alpha}([0, a)) \sim_{c} a^{1-\alpha}$.

Next we compute the RTS distribution, which we denote by $\tilde{G}(s)$. For $s \leq 0$, we easily have that $\tilde{G}(s)=0$, since $r_{U} \geq 1$, by definition of hitting time. Let $s>0$ then

$$
\begin{aligned}
\tilde{G}(s) & =\lim _{b \rightarrow 0} \mu_{U}\left(r_{U} \leq \frac{s}{\mu(U)}\right)=\lim _{b \rightarrow 0} \frac{1}{\mu(U)} \mu\left(\left\{r_{U} \leq \frac{s}{\mu(U)}\right\} \cap U\right) \\
& \geq \lim _{b \rightarrow 0} \frac{\mu(U \backslash A)}{\mu(U)}=\lim _{b \rightarrow 0} \frac{\mu([0, a))}{\mu([0, b))}=1
\end{aligned}
$$

Using the HLV05 formula, we get $G(t)=\int_{0}^{t} 1-\tilde{F}(s) d s=0$, which, by FFT10, corresponds to an EI equal to 0 . Recall that $\bar{H}(\tau)=\mathrm{e}^{-\theta \tau}=1$, which means that, in this case, $H(\tau)=0$.
4.2. Adjusted choice of thresholds. In order to prove the existence of EVLs in a dynamical systems context, there are a couple of conditions on the dependence structure of the stochastic process that if verified allow us to obtain such distributional limits. These conditions are motivated by the conditions $D\left(u_{n}\right)$ and $D^{\prime}\left(u_{n}\right)$ of Leadbetter but were adapted to the dynamical setting and further developed both in the absence of clustering, such as in Col01, FF08b, HNT12, and in the presence of clustering in FFT12. Very recently, in [FFT15], the authors provided some more general conditions, called $Д\left(u_{n}\right)$ and $Д_{q}^{\prime}\left(u_{n}\right)$, which subsumed the previous ones and allowed them to address both the presence ( $q \geq 1$ ) and the absence ( $q=0$ ) of clustering. To distinguish these conditions the authors used a Cyrillic D to denote them. We recall these conditions here

Given a sequence $\left(u_{n}\right)_{n \in \mathbb{N}}$ of real numbers satisfying (1I) and $q \in \mathbb{N}_{0}$, set

$$
A_{n}^{(q)}:=\left\{X_{0}>u_{n}, X_{1} \leq u_{n}, \ldots, X_{q} \leq u_{n}\right\} .
$$

For $s, \ell \in \mathbb{N}$ and an event $B$, let

$$
\begin{equation*}
\mathscr{W}_{s, \ell}(B)=\bigcap_{i=s}^{s+\ell-1} f^{-i}\left(B^{c}\right) \tag{5}
\end{equation*}
$$

Condition $\left(Д_{q}\left(u_{n}\right)\right)$. We say that $Д\left(u_{n}\right)$ holds for the sequence $X_{0}, X_{1}, \ldots$ if, for every $\ell, t, n \in \mathbb{N}$

$$
\begin{equation*}
\left|\mathbb{P}\left(A_{n}^{(q)} \cap \mathscr{W}_{t, \ell}\left(A_{n}^{(q)}\right)\right)-\mathbb{P}\left(A_{n}^{(q)}\right) \mathbb{P}\left(\mathscr{W}_{0, \ell}\left(A_{n}^{(q)}\right)\right)\right| \leq \gamma(q, n, t), \tag{6}
\end{equation*}
$$

where $\gamma(q, n, t)$ is decreasing in $t$ and there exists a sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ such that $t_{n}=o(n)$ and $n \gamma\left(q, n, t_{n}\right) \rightarrow 0$ when $n \rightarrow \infty$.

For some fixed $q \in \mathbb{N}_{0}$, consider the sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ given by condition $Д_{q}\left(u_{n}\right)$ and let $\left(k_{n}\right)_{n \in \mathbb{N}}$ be another sequence of integers such that

$$
\begin{equation*}
k_{n} \rightarrow \infty \quad \text { and } \quad k_{n} t_{n}=o(n) \tag{7}
\end{equation*}
$$

Condition $\left(\mathcal{Z}_{q}^{\prime}\left(u_{n}\right)\right)$. We say that $\mathcal{Z}_{q}^{\prime}\left(u_{n}\right)$ holds for the sequence $X_{0}, X_{1}, \ldots$ if there exists a sequence $\left(k_{n}\right)_{n \in \mathbb{N}}$ satisfying (7) and such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n \sum_{j=1}^{\left\lfloor n / k_{n}\right\rfloor} \mathbb{P}\left(A_{n}^{(q)} \cap f^{-j}\left(A_{n}^{(q)}\right)\right)=0 \tag{8}
\end{equation*}
$$

We note that, when $q=0$, condition $Д_{q}^{\prime}\left(u_{n}\right)$ corresponds to condition $D^{\prime}\left(u_{n}\right)$ from [Lea74].
Now let

$$
\begin{equation*}
\vartheta=\lim _{n \rightarrow \infty} \vartheta_{n}=\lim _{n \rightarrow \infty} \frac{\mathbb{P}\left(A_{n}^{(q)}\right)}{\mathbb{P}\left(U_{n}\right)} \tag{9}
\end{equation*}
$$

From [FFT15, Corollary 2.4], it follows that if the stochastic process $X_{0}, X_{1}, \ldots$ satisfies both conditions $Д_{q}\left(u_{n}\right)$ and $Д_{q}^{\prime}\left(u_{n}\right)$ and the limit in (9) exists then

$$
\lim _{n \rightarrow \infty} \mathbb{P}\left(M_{n} \leq u_{n}\right)=\mathrm{e}^{-\vartheta \tau}
$$

Now, we consider the fixed point $\zeta=0$. For every $n \in \mathbb{N}$, let $b_{n}$ be such that $U_{n}=\left\{X_{0}>\right.$ $\left.u_{n}\right\}=\left[0, b_{n}\right)$ and $\mu_{\alpha}\left(U_{n}\right) \sim \tau / n$. Also set $a_{n} \in U_{n}$ so that $f_{\alpha}\left(a_{n}\right)=b_{n}$, i.e., $b_{n}=a_{n}+2^{\alpha} a_{n}^{1+\alpha}$. Using (3) we easily get

$$
\begin{align*}
\mu_{\alpha}\left(U_{n}\right) & \sim_{c} a_{n}^{1-\alpha}+(1-\alpha) 2^{\alpha} a_{n}+o\left(a_{n}\right)  \tag{10}\\
\mu_{\alpha}\left(\left[0, a_{n}\right)\right) & \sim_{c} a_{n}^{1-\alpha}  \tag{11}\\
\mu_{\alpha}\left(\left[a_{n}, b_{n}\right)\right) & \sim_{c}(1-\alpha) 2^{\alpha} a_{n}+o\left(a_{n}\right) \tag{12}
\end{align*}
$$

Now, since we are assuming that $\mu_{\alpha}\left(U_{n}\right) \sim \tau / n$, then $a_{n} \sim_{c} 1 / n^{1 /(1-\alpha)}$. Observe that $\mu_{\alpha}\left(U_{n} \cap\right.$ $\left.f_{\alpha}^{-1}\left(U_{n}\right)\right)=\mu_{\alpha}\left(\left[0, a_{n}\right)\right) \sim_{c} a_{n}^{1-\alpha} \sim_{c} 1 / n$. Hence, if we consider $q=0$, the periodicity of $\zeta$ implies that $\boldsymbol{X}_{q}^{\prime}\left(u_{n}\right)$ does not hold since

$$
n \sum_{j=1}^{\left\lfloor n / k_{n}\right\rfloor} \mathbb{P}\left(U_{n} \cap f^{-j}\left(U_{n}\right)\right) \geq n \mu_{\alpha}\left(U_{n} \cap f_{\alpha}^{-1}\left(U_{n}\right)\right)>0
$$

for all $n \in \mathbb{N}$. Hence, here, given that $\zeta$ is a periodic point of period 1 the natural candidate for $q$ is $q=1$. From here on we always assume that $q=1$.

In this case, $A_{n}^{(q)}=\left[a_{n}, b_{n}\right)=: Q_{n}$. However, if we plug (12) and (10) into (9), we obtain that $\vartheta=0$, which means that the natural candidate for a limit distribution for $\mu_{\alpha}\left(M_{n} \leq u_{n}\right)$ is degenerate.

The problem is that the indifferent fixed point creates too much dependence. In [FFT12], under a condition called $S P$, we have seen that when $\zeta$ is periodic, the probability of having
no entrances in $U_{n}$, among the first $n$ observations, is asymptotically equal to the probability of having no entrances in $Q_{n}$, among the first $n$ observations, i.e.,

$$
\lim _{n \rightarrow \infty} \mathbb{P}\left(M_{n} \leq u_{n}\right)=\lim _{n \rightarrow \infty} \mathbb{P}\left(\mathscr{W}_{0, n}\left(U_{n}\right)\right)=\lim _{n \rightarrow \infty} \mathbb{P}\left(\mathscr{W}_{0, n}\left(Q_{n}\right)\right) .
$$

In [FFT15, it was shown that it is possible to replace $U_{n}$ by $Q_{n}$ even without the $S P$ condition (see [FFT15, Proposition 2.7]). Making use of this upgraded result, we can now change the normalising sequence of levels $\left(u_{n}\right)_{n \in \mathbb{N}}$ so that we can still obtain a non-degenerate limit for $\mathbb{P}\left(M_{n} \leq u_{n}\right)$. To understand the need to change the normalising sequence in order to obtain a non-degenerate limit, recall that condition (1) guaranteed that $M_{n}$ was normalised by a sequence of levels that kept the average of exceedances among the first $n$ observations at an (almost) constant value $\tau>0$. When $\vartheta>0$, condition (1) also guarantees that the average number of entrances in $Q_{n}$ among the first $n$ observations is kept at an (almost) constant value $\theta \tau>0$. Here, since $\vartheta=0$, we need to change $u_{n}$ so that the average number of entrances in $Q_{n}$ is controlled, i.e.,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n \mathbb{P}\left(A_{n}^{(q)}\right)=\tau>0 \tag{13}
\end{equation*}
$$

From equations (2.15) and (2.16) from FFT15 one gets:

$$
\begin{align*}
\left|\mathbb{P}\left(\mathscr{W}_{0, n}\left(A_{n}^{(q)}\right)\right)-\left(1-\left\lfloor\frac{n}{k_{n}}\right\rfloor \mathbb{P}\left(A_{n}^{(q)}\right)\right)^{k_{n}}\right| \leq & 2 k_{n} t_{n} \mathbb{P}\left(U_{n}\right)+2 n \sum_{j=1}^{\left\lfloor n / k_{n}\right\rfloor-1} \mathbb{P}\left(A_{n}^{(q)} \cap f^{-j} A_{n}^{(q)}\right) \\
& +\gamma\left(q, n, t_{n}\right) \tag{14}
\end{align*}
$$

Note that since by (13) we have $\lim _{n \rightarrow \infty}\left(1-\left\lfloor\frac{n}{k_{n}}\right\rfloor \mathbb{P}\left(A_{n}^{(q)}\right)\right)^{k_{n}}=\mathrm{e}^{-\tau}$, then if both conditions $Д_{q}\left(u_{n}\right)$ and $Д_{q}^{\prime}\left(u_{n}\right)$ hold, then all the terms on the left of (14) vanish, as $n \rightarrow \infty$, and consequently:

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}\left(M_{n} \leq u_{n}\right)=\lim _{n \rightarrow \infty} \mathbb{P}\left(\mathscr{W}_{0, n}\left(A_{n}^{(q)}\right)\right)=\mathrm{e}^{-\tau} . \tag{15}
\end{equation*}
$$

Hence, in order to show that we can still obtain a non-degenerate limiting law for the distribution of $M_{n}$ when $\zeta=0$, we start by taking a sequence $\left(u_{n}\right)_{n \in \mathbb{N}}$ so that (13) holds. Note that this implies that by (12) and (10) we have that $a_{n} \sim_{c} 1 / n$ and $\mu_{\alpha}\left(U_{n}\right) \sim_{c} 1 / n^{1-\alpha}$. In particular, this means that $\lim _{n \rightarrow \infty} n \mu_{\alpha}\left(U_{n}\right)=\infty$, which contrasts with the usual case where condition (1) holds.

To prove the existence of the limit in (15) we need to verify conditions $Д_{q}\left(u_{n}\right)$ and $Д_{q}^{\prime}\left(u_{n}\right)$, where $q=1$. We start by the latter, which is more complicated.
4.2.1. Proof of $\boldsymbol{Z}_{q}^{\prime}\left(u_{n}\right)$. We will next focus on the proof of $\boldsymbol{Д}_{q}^{\prime}\left(u_{n}\right)$ in the case of part (2) of Theorem 2. That is, $\left(a_{n}\right)_{n}$ will be chosen so that $a_{n} \sim_{c} 1 / n$, as described above. Later we will note that we can change $\left(a_{n}\right)_{n}$ to recover a degenerate law as in part (1) of that theorem.

We have to estimate the quantity

$$
\Delta_{n}^{\prime}:=n \sum_{j=1}^{\left[n / k_{n}\right]} \mu_{\alpha}\left(Q_{n} \cap f^{-j} Q_{n}\right)
$$

where $Q_{n}=\left[a_{n}, b_{n}\right)$, for $a_{n} \sim_{c} \frac{1}{n}$ and $b_{n}=f\left(a_{n}\right)$. We follow the proof of HSV99, Lemma 3.5]. By denoting by $P$ the transfer operator and by $\tau_{n} \in \mathbb{N}$ the first return time of the set $Q_{n}$ into itself, we have:

$$
\Delta_{n}^{\prime} \leqslant n\left[n / k_{n}\right] \mu_{a}\left(Q_{n}\right) \sup _{j=\tau_{n}, \ldots,\left[n / k_{n}\right]} \sup _{Q_{n}} \frac{P^{j}\left(\mathbf{1}_{Q_{n}} h\right)}{h}
$$

where $h$ is the density of $\mu_{\alpha}$. In order to compute $P^{\tau_{n}}\left(\mathbf{1}_{Q_{n}} h\right)$ we need to know how many branches of $f^{\tau_{n}}$ have their domain intersecting $Q_{n}$. If $\xi_{0}$ is the original partition into the sets $[0,1 / 2),[1 / 2,1]$, we denote with $\xi_{k}$ the join $\xi_{k}:=\xi_{0} \vee f^{-1} \xi_{0}, \ldots, \vee f^{k-1} \xi_{0}$.

We begin to observe that $Q_{n}$ contains at most one boundary point of the partition $\xi_{\tau_{n}-1}$, otherwise one point of $Q_{n}$ should be sent into the same set, being $f^{\tau_{n}-1}$ onto on each domain of injectivity. Then when we move to $\xi_{\tau_{n}}$, the interval $Q_{n}$ will be crossed by at most 4 cylinders of monotonicity of the partition $\xi_{\tau_{n}}$. By denoting them with $C_{\tau_{n}, 1}, \ldots C_{\tau_{n}, 4}$, we have

$$
P^{\tau_{n}}\left(\mathbf{1}_{Q_{n}} h\right)=\sum_{i=1,4} \frac{h \circ f_{i}^{-\tau_{n}} \mathbf{1}_{f_{i}^{\tau_{n}} Q_{n}}}{D f^{\tau_{n}} \circ f_{i}^{-\tau_{n}}}
$$

where $f_{i}^{\tau_{n}}$ denotes the branch of $f^{\tau_{n}}$ restricted to $C_{\tau_{n}, i}$. Notice that the density is computed in $Q_{n}$ whose left boundary point is $1 / n$, so $h$ is bounded from above by a constant times $n^{\alpha}$. We have now to estimate the derivative $D f^{\tau_{n}} \circ f_{i}^{-\tau_{n}}$ on the sets $Q_{n} \cap C_{\tau_{n}, i}$. Let us define $r_{m}$ as the $m$-left preimage of $1, r_{m}:=f_{1}^{-m}(1)$ and define $m(n)$ as $r_{m(n)} \leqslant a_{n} \leqslant r_{m(n)-1}$. Then the interval $\left[a_{n}, b_{n}\right)$ will intersect the two cylinders $\left(r_{m(n)}, r_{m(n)-1}\right)$ and $\left(r_{m(n)-1}, r_{m(n)-2}\right)$ and the first return of $Q_{n}$ will be larger than the first returns of those two cylinders; on the other hand the first return of $\left(r_{m(n)}, r_{m(n)-1}\right)$ is $m(n)$. The derivative $D f^{\tau_{n}}$ will be computed at a point $\iota_{n}$ which will be in one of those two cylinders; suppose without any restriction that $\iota_{n} \in\left(r_{m(n)}, r_{m(n)-1}\right)$. Since we need to bound from below the derivatives, we begin to replace $D f^{\tau_{n}}\left(\iota_{n}\right)$ with $D f^{m(n)}\left(\iota_{n}\right)$; then we observe that the map $f^{m(n)}:\left[r_{m(n)}, r_{m(n)-1}\right] \rightarrow[0,1]$ is onto and we use the distortion bound given, for instance, in [LSY, Lemma 5] which states that there exists a constant $C$ such that for any $m \geqslant 1$ and any $x, y \in\left[r_{m}, r_{m-1}\right]$ we have $\left|\frac{D f^{m}(x)}{D f^{m}(y)}\right| \leqslant C$. We finally note that $m(n) \sim_{c} n^{\alpha}$. This implies immediately that

$$
\frac{1}{D f^{m(n)}\left(\iota_{n}\right)} \leqslant C\left|r_{m(n)-1}-r_{m(n)}\right| \sim_{c} C \frac{1}{m(n)^{\frac{1}{\alpha}+1}} \sim_{c} C \frac{1}{n^{1+\alpha}} .
$$

Consequently ( $C$ will continue to denote a constant which could vary from one bound to another)

$$
P^{\tau_{n}}\left(\mathbf{1}_{Q_{n}} h\right) \sim_{c} \frac{1}{n}
$$

We now continue as in [HSV] by getting for the other powers of the transfer operator:

$$
\frac{P^{j}\left(\mathbf{1}_{Q_{n}} h\right)}{h} \leqslant \frac{P^{j-\tau_{n}} \mathbf{1}}{h} \sup P^{\tau_{n}}\left(\mathbf{1}_{Q_{n}} h\right) \leqslant \frac{P^{j-\tau_{n}} \frac{h}{\inf h}}{h} \sup P^{\tau_{n}}\left(\mathbf{1}_{Q_{n}} h\right) \leqslant \frac{C}{\inf h} \frac{1}{n}
$$

and finally

$$
\Delta_{n}^{\prime} \leqslant n\left[n / k_{n}\right] \mu_{a}\left(Q_{n}\right) \frac{C}{\inf h} \frac{1}{n}
$$

We now know that $\mu_{a}\left(Q_{n}\right) \sim_{c} \frac{1}{n}$; hence

$$
\Delta_{n}^{\prime} \leqslant C n\left[n / k_{n}\right] \mu_{a}\left(Q_{n}\right)^{2} \frac{1}{\mu_{a}\left(Q_{n}\right)} \frac{1}{n} \sim_{c}\left[n^{2} \mu_{a}\left(Q_{n}\right)^{2}\right] \frac{1}{k_{n}} .
$$

So letting $n \rightarrow \infty$, we see that $\mathbb{Z}_{q}^{\prime}\left(u_{n}\right)$ holds.
4.2.2. Proof of $Д_{q}\left(u_{n}\right)$. This follows since, as in (2), we have decay of correlations of Hölder functions against bounded measurable functions and condition $Д_{q}\left(u_{n}\right)$ was designed to follow from sufficiently fast decay of correlations, as shown in Fre13, Proposition 5.2]. In order to compute the required rate of decay of correlations, which will impose a restriction on the domain of the parameter $\alpha$, we recall here the above-mentioned result so that we can follow the computations closely.
Proposition 1 (Fre13, Proposition 5.2]). Assume that $\mathcal{X}$ is a compact subset of $\mathbb{R}^{d}$ and $f: \mathcal{X} \rightarrow \mathcal{X}$ is a system with an acip $\mathbb{P}$, such that $\frac{d \mathbb{P}}{\text { Leb }} \in L^{1+\epsilon}$. Assume, moreover, that the system has decay of correlations for all $\varphi \in \mathcal{H}_{\beta}$ against any $\psi \in L^{\infty}$ so that there exists some $C>0$ independent of $\varphi, \psi$ and $t$, and a rate function $\varrho: \mathbb{N} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\left|\int \varphi \cdot\left(\psi \circ f^{t}\right) d \mathbb{P}-\int \varphi d \mathbb{P} \int \psi d \mathbb{P}\right| \leq C\|\varphi\|_{\mathcal{H}_{\beta}}\|\psi\|_{\infty} \varrho(t), \tag{16}
\end{equation*}
$$

and $n^{1+\beta(1+\max \{0,(\epsilon+1) / \epsilon-d\}+\delta)} \varrho\left(t_{n}\right) \rightarrow 0$, as $n \rightarrow \infty$ for some $\delta>0$ and $t_{n}=o(n)$. Then condition $Д_{q}\left(u_{n}\right)$ holds.

Remark 2. We note that during the proof, in order to obtain the condition on the rate of decay of correlations, it is assumed that $\mathbb{P}\left(A_{n}^{(q)}\right) \sim_{c} 1 / n$.

Observe that since we are working in dimension 1, which means $d=1$, then $\max \{0,(\epsilon+$ $1) / \epsilon-d\}=1 / \epsilon$. Also, from Hu04, we may assume that the decay of correlations is written for Lipschitz functions, which allows us to take $\beta=1$. Hence, for condition $Д_{q}\left(u_{n}\right)$ hold, we need that the rate of decay of correlations $\varrho$ is sufficiently fast so that there exists some $\delta>0$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} n^{2+1 / \epsilon+\delta} \varrho\left(t_{n}\right)=0, \tag{17}
\end{equation*}
$$

where $t_{n}=o(n)$. From (3), in order that the density $h_{\alpha} \in L^{1+\epsilon}$, we need that $\epsilon<1 / \alpha-1$. Since by (2), we have that $\varrho(t)=t^{-(1 / \alpha-1)}$, then by (17) it is obvious that we must have $\alpha<1 / 2$, which implies that $1 / \epsilon<\alpha+2 \alpha^{2}$. Taking $t_{n}=n^{1-\alpha}$, we obtain:

$$
n^{2+1 / \epsilon+\delta}\left(n^{1-\alpha}\right)^{-(1 / \alpha-1)}=n^{2+1 / \epsilon+\delta} n^{-1 / \alpha+2-\alpha}<n^{4+2 \alpha^{2}+\delta-1 / \alpha} .
$$

Hence, if $\alpha<\sqrt{5}-2$ we can always find $\delta>0$ so that (17) holds and consequently condition $Д_{q}\left(u_{n}\right)$ is verified.

## References

[AFV14] Hale Aytaç, Jorge Milhazes Freitas, and Sandro Vaienti, Laws of rare events for deterministic and random dynamical systems, Trans. Amer. Math. Soc. (2014), Published online with DOI: 10.1090/S0002-9947-2014-06300-9.
[BSTV03] H. Bruin, B. Saussol, S. Troubetzkoy, and S. Vaienti, Return time statistics via inducing, Ergodic Theory Dynam. Systems 23 (2003), no. 4, 991-1013. MR 1997964 (2005a:37004)
[CG93] P. Collet and A. Galves, Statistics of close visits to the indifferent fixed point of an interval map, J. Statist. Phys. 72 (1993), no. 3-4, 459-478. MR 1239564 (94i:58058)
[CGS92] P. Collet, A. Galves, and B. Schmitt, Unpredictability of the occurrence time of a long laminar period in a model of temporal intermittency, Ann. Inst. H. Poincaré Phys. Théor. 57 (1992), no. 3, 319-331. MR 1185337 (94a:58064)
[CI55] Massimo Campanino and Stefano Isola, Statistical properties of long return times in type I intermittency, Forum Math. 7 (1995), no. 3, 331-348. MR 1325560 (96c:60050)
[Col01] P. Collet, Statistics of closest return for some non-uniformly hyperbolic systems, Ergodic Theory Dynam. Systems 21 (2001), no. 2, 401-420. MR MR1827111 (2002a:37038)
[FF08a] Ana Cristina Moreira Freitas and Jorge Milhazes Freitas, Extreme values for Benedicks-Carleson quadratic maps, Ergodic Theory Dynam. Systems 28 (2008), no. 4, 1117-1133. MR MR2437222 (2010a:37013)
[FF08b] _ On the link between dependence and independence in extreme value theory for dynamical systems, Statist. Probab. Lett. 78 (2008), no. 9, 1088-1093. MR MR2422964 (2009e:37006)
[FFR] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Fagner Bernardini Rodrigues, The speed of convergence of rare events point processes in non-uniformly hyperbolic systems, in preparation.
[FFT10] Ana Cristina Moreira Freitas, Jorge Milhazes Freitas, and Mike Todd, Hitting time statistics and extreme value theory, Probab. Theory Related Fields 147 (2010), no. 3-4, 675-710. MR 2639719 (2011g:37015)
[FFT12] , The extremal index, hitting time statistics and periodicity, Adv. Math. 231 (2012), no. 5, 2626-2665. MR 2970462
[FFT13] , The compound Poisson limit ruling periodic extreme behaviour of non-uniformly hyperbolic dynamics, Comm. Math. Phys. 321 (2013), 483-527.
[FFT15] , Speed of convergence for laws of rare events and escape rates, Stochastic Process. Appl. 125 (2015), no. 4, 1653-1687.
[FHN14] Jorge Milhazes Freitas, Nicolai Haydn, and Matthew Nicol, Convergence of rare event point processes to the Poisson process for planar billiards, Nonlinearity 27 (2014), no. 7, 1669-1687. MR 3232197
[FP12] Andrew Ferguson and Mark Pollicott, Escape rates for gibbs measures, Ergodic Theory Dynam. Systems 32 (2012), no. 3, 961-988.
[Fre13] Jorge Milhazes Freitas, Extremal behaviour of chaotic dynamics, Dyn. Syst. 28 (2013), no. 3, 302332. MR 3170619
[GHN11] Chinmaya Gupta, Mark Holland, and Matthew Nicol, Extreme value theory and return time statistics for dispersing billiard maps and flows, Lozi maps and Lorenz-like maps, Ergodic Theory Dynam. Systems 31 (2011), no. 5, 1363-1390. MR 2832250
[HLV05] N. Haydn, Y. Lacroix, and S. Vaienti, Hitting and return times in ergodic dynamical systems, Ann. Probab. 33 (2005), no. 5, 2043-2050. MR MR2165587 (2006i:37006)
[HLV07] N. Haydn, E. Lunedei, and S. Vaienti, Averaged number of visits, Chaos 17 (2007), no. 3, 033119, 13. MR 2356973 (2008k:37005a)
[HNT12] Mark Holland, Matthew Nicol, and Andrei Török, Extreme value theory for non-uniformly expanding dynamical systems, Trans. Amer. Math. Soc. 364 (2012), no. 2, 661-688. MR 2846347 (2012k:37064)
[HSV99] Masaki Hirata, Benoît Saussol, and Sandro Vaienti, Statistics of return times: a general framework and new applications, Comm. Math. Phys. 206 (1999), no. 1, 33-55. MR 1736991 (2001c:37007)
[Hu04] Huyi Hu, Decay of correlations for piecewise smooth maps with indifferent fixed points, Ergodic Theory Dynam. Systems 24 (2004), no. 2, 495-524. MR 2054191 (2005a:37064)
[HWZ14] Nicolai Haydn, Nicole Winterberg, and Roland Zweimüller, Return-time statistics, hitting-time statistics and inducing, Ergodic theory, open dynamics, and coherent structures, Springer Proc. Math. Stat., vol. 70, Springer, New York, 2014, pp. 217-227. MR 3213501
[Kel12] Gerhard Keller, Rare events, exponential hitting times and extremal indices via spectral perturbationâÂĂ, Dynamical Systems 27 (2012), no. 1, 11-27.
[Lea74] M. R. Leadbetter, On extreme values in stationary sequences, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 28 (1973/74), 289-303. MR 0362465 (50 \#14906)
[LFW12] Valerio Lucarini, Davide Faranda, and Jeroen Wouters, Universal behavior of extreme value statistics for selected observables of dynamical systems, J. Stat. Phys. 147 (2012), no. 1, 63-73.
[LSV99] Carlangelo Liverani, Benoît Saussol, and Sandro Vaienti, A probabilistic approach to intermittency, Ergodic Theory Dynam. Systems 19 (1999), no. 3, 671-685. MR MR1695915 (2000d:37029)
[Ryc83] Marek Rychlik, Bounded variation and invariant measures, Studia Math. 76 (1983), no. 1, 69-80. MR MR728198 (85h:28019)
[VHF09] Renato Vitolo, Mark P. Holland, and Christopher A. T. Ferro, Robust extremes in chaotic deterministic systems, Chaos 19 (2009), no. 4, 043127.
[You99] Lai-Sang Young, Recurrence times and rates of mixing, Israel J. Math. 110 (1999), 153-188. MR MR1750438 (2001j:37062)
[Zwe07] Roland Zweimüller, Mixing limit theorems for ergodic transformations, J. Theoret. Probab. 20 (2007), no. 4, 1059-1071. MR 2359068 (2008h:60119)

Ana Cristina Moreira Freitas, Centro de Matemática \& Faculdade de Economia da Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal

E-mail address: amoreira@fep.up.pt

Jorge Milhazes Freitas, Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: jmfreita@fc.up.pt
URL: http://www.fc.up.pt/pessoas/jmfreita

Mike Todd, Mathematical Institute, University of St Andrews, North Haugh, St Andrews, KY16 9SS, Scotland,

E-mail address: m.todd@st-andrews.ac.uk
URL: http://www.mcs.st-and.ac.uk/~miket/

Sandro Vaienti, Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France and Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France

E-mail address: vaienti@cpt.univ-mrs.fr


[^0]:    ${ }^{1}$ We note that there is an error in FFT13. Theorem 1], propagated throughout the main results there: the $\kappa$ should be replaced by $\kappa-1$.

