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Abstract

We are concerned with the discretization of a solution of a Forward-Backward stochastic differential equation (FBSDE) with a jump process depending on the Brownian motion. In this paper, we study the cases of Lipschitz generators and the generators with a quadratic growth w.r.t. the variable z. We propose a recursive scheme based on a general existence result given in the companion paper [15] and we study the error induced by the time discretization. We prove the convergence of the scheme when the number of time steps n goes to infinity. Our approach allows to get a convergence rate similar to that of schemes of Brownian FBSDEs.

Keywords: discrete-time approximation, forward-backward SDE, Lipschitz generator, generator of quadratic growth, progressive enlargement of filtrations, decomposition in the reference filtration.

MSC classification (2010): 65C99, 60J75, 60G57.

1 Introduction

In this paper, we study a discrete-time approximation for the solution of a forward-backward stochastic differential equation (FBSDE) with a jump of the form

$$\left\{\begin{array}{rcl} X_t &=& x+\displaystyle\int_0^t b(s,X_s)ds+\int_0^t \sigma(s,X_s)dW_s+\int_0^t \beta(s,X_{s^-})dH_s\ , \\ Y_t &=& g(X_T)+\displaystyle\int_t^T f(s,X_s,Y_s,Z_s,U_s)ds-\int_t^T Z_s dW_s-\int_t^T U_s dH_s\ , \end{array}\right.$$

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where $H_t = \mathbb{1}_{\tau \leq t}$ and τ is a jump time, which can represent a default time in credit risk or counterparty risk. Such equations naturally appear in finance, see for example Bielecki and Jeanblanc [2], Lim and Quenez [18], Peng and Xu [20], Ankirchner *et al.* [1] for an application to exponential utility maximization problem and Kharroubi and Lim [15] for the hedging problem in a complete market. The approximation of such equation is therefore of important interest for practical applications in finance. In this paper, we study the case where the generator f is Lipschitz or with quadratic growth w.r.t. Z.

In the literature, the problem of discretization of FBSDEs with Lipschitz generator has been widely studied in the Brownian framework, *i.e.* no jump, see e.g. [19, 7, 5, 3, 22, 6]. More recently, the case of quadratic generators w.r.t. Z has been considered by Imkeller *et al.* [8] and Richou [21]. For Lipschitz generators, the discrete-time approximation of FBSDEs with jumps is studied by Bouchard and Elie [4] in the case of Poissonian jumps independent of the Brownian motion. Their approach is based on a regularity result for the process Z, which is given by Malliavin calculus tools. This regularity result for the process Z was first proved by Zhang [22] in a Brownian framework to provide a convergence rate for the discrete-time approximation of FBSDEs. The use of Malliavin calculus to prove regularity on Z is possible in [4] since the authors suppose that the Brownian motion is independent of the jump measure.

In our case, we only assume that the random jump time τ admits a conditional density given W, which is assumed to be absolutely continuous w.r.t. the Lebesgue measure. In particular, we do not specify a particular law for τ and we do not assume that τ is independent of W as for the case of a Poisson random measure.

To the best of our knowledge, no Malliavin calculus theory has been set for such a framework. Thus, the method used in [4] fails to provide a convergence rate for the approximation in this context.

We therefore follow another approach, which consists in using the decomposition result given in the companion paper [15] to write the solution of a FBSDE with a jump as a combination of solutions to a recursive system of FBSDEs without jump. We then prove a regularity result on the Z components of Brownian BSDEs coming from the decomposition of the BSDE with a jump. This regularity result allows to get a rate for the convergence of the discrete-time schemes for these BSDEs as in [22] or [4] for the Lipschitz case and [21] for the quadratic case.

Finally, we recombine the approximations of the solutions to recursive system of Brownian FBSDEs to get a discretization of the solution to the FBSDE with a jump.

We notice that our approach also allows to weaken the assumption on the forward jump coefficient in the Lipschitz case. More precisely, we only assume that β is Lipschitz continuous, unlike [4] supposing that β is regular and the matrix $I_d + \nabla\beta$ is elliptic.

As said above, this kind of FBSDEs with a jump appears in finance. The general assumptions made on the jump time τ allow to modelize general phenomena as a firm default or simpler as a jump of an asset that can be seen as contagion from the default of another firm on the market, see e.g. [13] for some examples. In particular, the approximation of these FBSDEs has its own interest, since it provides approximations of optimal gains and strategies of the studied investment problems.

We choose to present our results in the case of a single jump and a one-dimensional Brownian motion for the sake of simplicity. We notice that they can easily be extended to the case of a d-dimensional Brownian motion and multiple jumps with eventually random marks, as in [15], taking values in a finite space.

The paper is organized as follows. The next section presents the framework of progressive enlargement of a Brownian filtration by a random jump, and the well posedness of FBSDEs in this context. In Section 3, we present the discrete-time schemes for the forward and backward solutions based on the decomposition given in the previous section. Finally, in Section 4, we study the convergence rate of the scheme for the forward solution. In Sections 5 and 6, we study

the convergence rate of the scheme for the backward solution for the Lipschitz case respectively for the quadratic case.

2 Preliminaries

2.1 Notation

Throughout this paper, we let $(\Omega, \mathcal{G}, \mathbb{P})$ a complete probability space on which is defined a standard one dimensional Brownian motion W. We denote $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the natural filtration of W augmented by all the $\mathbb{P}$-null sets. We also consider on this space a random time τ, i.e. a nonnegative $\mathcal{F}$-measurable random variable, and we denote classically the associated jump process by H which is given by

$$H_t \;:=\; \mathbb{1}_{\tau\leq t}\,, \quad t\geq 0\,.$$

We denote by $\mathbb{D} := (\mathcal{D}_t)_{t\geq 0}$ the smallest right-continuous filtration for which τ is a stopping time. The global information is then defined by the progressive enlargement $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ of the initial filtration where

$$\mathcal{G}_t \;:=\; \bigcap_{\varepsilon>0}\Big(\mathcal{F}_{t+\varepsilon}\vee\mathcal{D}_{t+\varepsilon}\Big)$$

for all $t\geq 0$. This kind of enlargement was introduced by Jacod, Jeulin and Yor in the 80s (see e.g. [10], [11] and [9]). We introduce some notations used throughout the paper

- $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable measurable subsets of $\Omega\times\mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes,
- $\mathcal{PM}(\mathbb{F})$ (resp. $\mathcal{PM}(\mathbb{G})$) is the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-progressively measurable subsets of $\Omega\times\mathbb{R}_+$.

We shall make, throughout the sequel, the standing assumption in the progressive enlargement of filtrations known as density assumption (see e.g. [12, 13, 15]).

(DH) There exists a positive and bounded $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable process γ such that

$$\mathbb{P}\big[\tau\in d\theta \;\big|\; \mathcal{F}_t\big] \;=\; \gamma_t(\theta)d\theta\,, \quad t\geq 0\,.$$

Using Proposition 2.1 in [15] we get that **(DH)** ensures that the process H admits an intensity.

Proposition 2.1. *The process H admits a compensator of the form $\lambda_t dt$, where the process λ is defined by*

$$\lambda_t \;:=\; \frac{\gamma_t(t)}{\mathbb{P}\big[\tau>t\;\big|\;\mathcal{F}_t\big]}\mathbb{1}_{t\leq\tau}\,, \quad t\geq 0\,.$$

We impose the following assumption to the process λ.

(HBI) The process λ is bounded.

We also introduce the martingale invariance assumption known as the **(H)**-hypothesis.

(H) Any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

We now introduce the following spaces, where $a, b\in\mathbb{R}_+$ with $a<b$, and $T<\infty$ is the terminal time.

– $\mathcal{S}^\infty_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^\infty_{\mathbb{F}}[a,b]$) is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^\infty[a,b]} \;:=\; \operatorname*{ess\,sup}_{t\in[a,b]} |Y_t| \;<\; \infty\,.$$

– $\mathcal{S}^p_{\mathbb{G}}[a,b]$ (resp. $\mathcal{S}^p_{\mathbb{F}}[a,b]$), with $p \geq 2$, is the set of $\mathcal{PM}(\mathbb{G})$ (resp. $\mathcal{PM}(\mathbb{F})$)-measurable processes $(Y_t)_{t\in[a,b]}$ such that

$$\|Y\|_{\mathcal{S}^p[a,b]} \;:=\; \Big(\mathbb{E}\Big[\sup_{t\in[a,b]} |Y_t|^p\Big]\Big)^{\frac{1}{p}} \;<\; \infty\,.$$

– $H^p_{\mathbb{G}}[a,b]$ (resp. $H^p_{\mathbb{F}}[a,b]$), with $p \geq 2$, is the set of $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_t)_{t\in[a,b]}$ such that

$$\|Z\|_{H^p[a,b]} \;:=\; \mathbb{E}\Big[\Big(\int_a^b |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \;<\; \infty\,.$$

– $L^2(\lambda)$ is the set of $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\mu)} \;:=\; \Big(\mathbb{E}\Big[\int_0^T \lambda_s |U_s|^2 ds\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

2.2 Forward-Backward SDE with a jump

Given measurable functions $b : [0,T]\times\mathbb{R}\to\mathbb{R}$, $\sigma : [0,T]\times\mathbb{R}\to\mathbb{R}$, $\beta : [0,T]\times\mathbb{R}\to\mathbb{R}$, $g : \mathbb{R}\to\mathbb{R}$ and $f : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and an initial condition $x\in\mathbb{R}$, we study the discrete-time approximation of the solution (X,Y,Z,U) in $\mathcal{S}^2_{\mathbb{G}}[0,T]\times\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times H^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$ to the following forward-backward stochastic differential equation

$$\begin{aligned} X_t &= x + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s + \int_0^t \beta(s,X_{s^-})dH_s\,, \quad 0\leq t\leq T\,, \qquad (2.1)\\ Y_t &= g(X_T) + \int_t^T f\big(s,X_s,Y_s,Z_s,(1-H_s)U_s\big)ds \\ &\qquad - \int_t^T Z_s dW_s - \int_t^T U_s dH_s\,, \quad 0\leq t\leq T\,, \qquad (2.2)\end{aligned}$$

when the generator of the BSDE is Lipschitz or has a quadratic growth w.r.t. Z.

Remark 2.1. In the BSDE (2.2), the jump component U of the unknown (Y,Z,U) appears in the generator f with the additional multiplicative term $1-H$. This ensures the equation to be well posed in $\mathcal{S}^\infty_{\mathbb{G}}[0,T]\times H^2_{\mathbb{G}}[0,T]\times L^2(\lambda)$. Indeed, the component U lives in $L^2(\lambda)$, thus its value on $(\tau\wedge T, T]$ is not defined since the intensity λ vanishes on $(\tau\wedge T, T]$. We therefore introduce the term $1-H$ to kill the value of U on $(\tau\wedge T, T]$ and hence to avoid making the equation depending on it.

We first prove that the decoupled system (2.1)-(2.2) admits a solution. To this end, we introduce several assumptions on the coefficients b, σ, β, g and f. We consider the following assumption for the forward coefficients.

(HF) There exists a constant K such that the functions b, σ and β satisfy

$$|b(t,0)| + |\sigma(t,0)| + |\beta(t,0)| \;\leq\; K\,,$$

and

$$|b(t,x)-b(t,x')|+|\sigma(t,x)-\sigma(t,x')|+|\beta(t,x)-\beta(t,x')| \quad \leq \quad K|x-x'|\ ,$$

for all $(t,x,x')\in[0,T]\times\mathbb{R}\times\mathbb{R}$.

For the backward coefficients g and f, we impose the following assumptions for the Lipschitz case.

(HBL) There exists a constant K such that the functions g and f satisfy

$$|f(t,x,0,0,0)|+|g(x)| \quad \leq \quad K\ ,$$

and

$$|f(t,x,y,z,u)-f(t,x,y',z',u')| \quad \leq \quad K\big(|y-y'|+|z-z'|+|u-u'|\big)\ ,$$

for all $(t,x,y,y',z,z',u,u')\in[0,T]\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$.

For the backward coefficients g and f, we consider the following assumptions for the quadratic case.

(HBQ)

- There exist three constants M_g, K_g and K_q such that the functions g and f satisfy

$$\begin{aligned} |g(x)| &\leq M_g\ ,\\ |g(x)-g(x')| &\leq K_g|x-x'|\ ,\\ |f(t,x,y,z,u)-f(t,y',z,u)| &\leq K_q|y-y'|\ ,\\ |f(t,x,y,z,u)| &\leq K_q\big(1+|y|+|z|^2+|u|\big)\ , \end{aligned}$$

  for all $(t,x,x',y,y',z,u)\in[0,T]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}$.

- For any $R>0$ there exists a function mc_R^f such that $\lim_{\epsilon\to 0} mc_R^f(\epsilon)=0$ and

$$|f(t,x,y,z,u-y)-f(t,x,y',z',u-y')| \quad \leq \quad mc_R^f(\epsilon)$$

  for all $(t,x,y,y',z,z',u)\in[0,T]\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}$ s.t. $|y|,\ |z|,\ |y'|,\ |z'|\leq R$ and $|y-y'|+|z-z'|\leq\epsilon$.

- $f(t,.,u)=f(t,.,0)$ for all $u\in\mathbb{R}$ and all $t\in(\tau\wedge T,T]$.

- The function $f(t,x,y,.,u)$ is convexe (or concave) uniformly in $(t,x,y,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

In the sequel K denotes a generic constant appearing in **(HBL)**, **(HBQ)** and **(HF)** and which may vary from line to line.

In the purpose to prove the existence of a solution to the FBSDE (2.1)-(2.2) we follow the decomposition approach initiated by [15] and for that we introduce the recursive system of FBSDEs associated with (2.1)-(2.2).

• Find $(X^1(\theta),Y^1(\theta),Z^1(\theta))\in\mathcal{S}^2_{\mathbb{F}}[0,T]\times\mathcal{S}^\infty_{\mathbb{F}}[\theta,T]\times H^2_{\mathbb{F}}[\theta,T]$ such that

$$X_t^1(\theta) \quad = \quad x+\int_0^t b\big(s,X_s^1(\theta)\big)ds+\int_0^t\sigma\big(s,X_s^1(\theta)\big)dW_s+\beta\big(\theta,X_{\theta^-}^1(\theta)\big)\mathbb{1}_{\theta\leq t}\ ,\ 0\leq t\leq T\ , \quad (2.3)$$

$$Y_t^1\big(\theta\big) \quad = \quad g\big(X_T^1(\theta)\big)+\int_t^T f\big(s,X_s^1(\theta),Y_s^1(\theta),Z_s^1(\theta),0\big)ds-\int_t^T Z_s^1(\theta)dW_s\ ,\ \theta\leq t\leq T\ , \quad (2.4)$$

for all $\theta \in [0,T]$.

• Find $(X^0, Y^0, Z^0) \in \mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times H^2_{\mathbb{F}}[0,T]$ such that

$$X^0_t = x + \int_0^t b(s, X^0_s)ds + \int_0^t \sigma(s, X^0_s)dW_s\ , \quad 0 \le t \le T\ , \tag{2.5}$$

$$Y^0_t = g(X^0_T) + \int_t^T f\big(s, X^0_s, Y^0_s, Z^0_s, Y^1_s(s) - Y^0_s\big)ds - \int_t^T Z^0_s dW_s\ , \quad 0 \le t \le T\ . \tag{2.6}$$

Then, the link between the FBSDE (2.1)-(2.2) and the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6) is given by the following result.

Theorem 2.1. *Assume that **(DH)**, **(HBI)**, **(H)**, **(HF)** and **(HBL)** or **(HBQ)** hold true. Then, the FBSDE* (2.1)-(2.2) *admits a unique solution* $(X, Y, Z, U) \in \mathcal{S}^2_{\mathbb{G}}[0,T] \times \mathcal{S}^\infty_{\mathbb{G}}[0,T] \times H^2_{\mathbb{G}}[0,T] \times L^2(\lambda)$ *given by*

$$\begin{cases} X_t = X^0_t \mathbb{1}_{t<\tau} + X^1_t(\tau)\mathbb{1}_{\tau \le t}\ , \\ Y_t = Y^0_t \mathbb{1}_{t<\tau} + Y^1_t(\tau)\mathbb{1}_{\tau \le t}\ , \\ Z_t = Z^0_t \mathbb{1}_{t \le \tau} + Z^1_t(\tau)\mathbb{1}_{\tau<t}\ , \\ U_t = \big(Y^1_t(t) - Y^0_t\big)\mathbb{1}_{t \le \tau}\ , \end{cases} \tag{2.7}$$

where $(X^1(\theta), Y^1(\theta), Z^1(\theta))$ *is the unique solution to the FBSDE* (2.3)-(2.4) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[\theta,T] \times H^2_{\mathbb{F}}[\theta,T]$, *for* $\theta \in [0,T]$, *and* (X^0, Y^0, Z^0) *is the unique solution to the FBSDE* (2.5)-(2.6) *in* $\mathcal{S}^2_{\mathbb{F}}[0,T] \times \mathcal{S}^\infty_{\mathbb{F}}[0,T] \times H^2_{\mathbb{F}}[0,T]$.

Proof.

Step 1. Solution to (2.1) under **(HF)**.

Under **(HF)** there exist unique processes $X^0 \in \mathcal{S}^2_{\mathbb{F}}[0,T]$ satisfying (2.5), and $X^1(\theta) \in \mathcal{S}^2_{\mathbb{F}}[0,T]$ satisfying (2.3) for all $\theta \in [0,T]$ such that X^1 is $\mathcal{PM}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Then, from the definition of H, we check that the process X defined by

$$X_t = X^0_t \mathbb{1}_{t<\tau} + X^1_t(\tau)\mathbb{1}_{t \ge \tau}\ , \tag{2.8}$$

satisfies (2.1). We now check that $X \in \mathcal{S}^2_{\mathbb{G}}[0,T]$. We first notice that from **(HF)**, there exists a constant K such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X^0_t\big|^2\Big] \le K\ . \tag{2.9}$$

Then, from the definition of X^0 and X^1, we have for all $t \in [\theta, T]$

$$\sup_{s\in[\theta,t]} \big|X^1_s(\theta)\big|^2 \le K\Big(\big|X^0_\theta\big|^2 + \big|\beta(\theta, X^0_\theta)\big|^2 + \int_\theta^t \big|b(u, X^1_u(\theta))\big|^2 du + \sup_{s\in[\theta,t]} \Big|\int_\theta^s \sigma(u, X^1_u(\theta))dW_u\Big|^2\Big)\ .$$

Using **(HF)** and BDG-inequality, we get

$$\mathbb{E}\Big[\sup_{s\in[\theta,t]} \big|X^1_s(\theta)\big|^2\Big] \le K\Big(1 + \int_\theta^t \mathbb{E}\Big[\sup_{u\in[\theta,s]} \big|X^1_u(\theta)\big|^2\Big]du\Big)\ .$$

Applying Gronwall's lemma, we get

$$\sup_{\theta\in[0,T]} \big\|X^1(\theta)\big\|_{\mathcal{S}^2_{\mathbb{F}}[\theta,T]} \le K\ . \tag{2.10}$$

Combining (2.8), (2.9) and (2.10), we get that $X \in \mathcal{S}^2_{\mathbb{G}}[0,T]$. Moreover still using **(HF)** we get the uniqueness of a solution to (2.1) in $\mathcal{S}^2_{\mathbb{G}}[0,T]$.

Step 2. Solution to (2.2) under **(DH)**, **(HBI)**, **(H)** and **(HBL)**.
To follow the decomposition approach initiated by the authors in [15], we need the generator to be predictable. To this end, we notice that in the BSDE (2.2), we can replace the generator $(t,y,z,u) \mapsto f(t,X_t,y,z,(1-H_t)u)$ by the predictable map $(t,y,z,u) \mapsto f(t,X_{t^-},y,z,(1-H_{t^-})u)$.

Using the decomposition (2.8), we are able to write explicitly the decompositions of the $\mathcal{G}_T$-measurable random variable $g(X_T)$ and the $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable map $(\omega,t,y,z,u) \mapsto f(t,X_{t^-}(\omega),y,z,u(1-H_{t^-}(\omega)))$ given by Lemma 2.1 in [15]

$$\begin{aligned} g(X_T) &= g(X_T^0)\mathbb{1}_{T<\tau} + g(X_T^1(\tau))\mathbb{1}_{T\geq\tau} \ , \\ f(t,X_{t^-},y,z,(1-H_{t^-})u) &= f^0(t,y,z,u)\mathbb{1}_{t\leq\tau} + f^1(t,y,z,u,\tau)\mathbb{1}_{t>\tau} \ , \end{aligned}$$

with $f^0(t,y,z,u) = f(t,X_t^0,y,z,u)$ and $f^1(t,y,z,u,\theta) = f(t,X_{t^-}^1(\theta),y,z,0)$, for all $(t,y,z,u,\theta) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$.

Suppose now that **(DH)**, **(HBI)**, **(H)** and **(HBL)** hold true. Then, from Theorem C.1 in [15], the BSDE (2.4) admits a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}([0,T])$-measurable solution (Y^1,Z^1) and the BSDE (2.6) admits a solution (Y^0,Z^0). Using Proposition 2.1 in [17], we obtain

$$\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]} + \|Z^1(\theta)\|_{H^2[\theta,T]} \leq K \ ,$$

for all $\theta \in [0,T]$, and

$$\|Y^0\|_{\mathcal{S}^\infty[0,T]} + \|Z^0\|_{H^2[0,T]} \leq K \ .$$

We can then apply Theorem 3.1 in [15] and we get the existence of a solution to (2.2) in $\mathcal{S}_{\mathbb{G}}^\infty[0,T] \times H_{\mathbb{G}}^2[0,T] \times L^2(\lambda)$.

Let (Y,Z,U) and (Y',Z',U') be two solutions to (2.2) in $\mathcal{S}_{\mathbb{G}}^\infty[0,T] \times H_{\mathbb{G}}^2[0,T] \times L^2(\lambda)$. Since $f(t,x,y,z,(1-H_t)u) = f(t,x,y,z,0)$ for all $t \in (\tau \wedge T, T]$ and λ vanishes on $(\tau \wedge T, T]$, we can assume w.l.o.g. that $U_t = U'_t = 0$ for $t \in (\tau \wedge T, T]$. Then, from **(DH)**, **(HBI)**, **(H)** and **(HBL)**, we can apply Theorem 4.1 in [15] and we get that $Y \leq Y'$. Since Y and Y' play the same role, we obtain $Y = Y'$. Identifying the pure jump parts of Y and Y' gives $U = U'$. Finally, identifying the unbounded variation gives $Z = Z'$.

Case 3. Solution to (2.2) under **(DH)**, **(HBI)**, **(H)** and **(HBQ)**.
The existence of a solution $(Y,Z,U) \in \mathcal{S}_{\mathbb{G}}^2[0,T] \times L_{\mathbb{G}}^2[0,T] \times L^2(\lambda)$ is a direct consequence of Proposition 3.1 in [15]. We then notice that from the definition of H we have $f(t,x,y,z,u(1-H_t)) = f(t,x,y,z,0)$ for all $t \in (\tau \wedge T, T]$. This property and **(DH)**, **(HBI)**, **(H)** and **(HBQ)** allow to apply Theorem 4.2 in [15], which gives the uniqueness of a solution of (2.2). □

Throughout the sequel, we give an approximation of the solution to the FBSDE (2.1)-(2.2) by studying the approximation of the solutions to the recursive system of FBSDEs (2.3)-(2.4) and (2.5)-(2.6).

3 Discrete-time scheme for the FBSDE

In this section, we introduce a discrete-time approximation of the solution (X,Y,Z,U) to the FBSDE (2.1)-(2.2) based on its decomposition given by Theorem 2.1.

Throughout the sequel, we consider a discretization grid $\pi := \{t_0,\ldots,t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \ldots < t_n = T$. For $t \in [0,T]$, we denote by $\pi(t)$ the largest element of π smaller than t

$$\pi(t) := \max\left\{ t_i \ , \ i = 0,\ldots,n \mid t_i \leq t \right\} .$$

We also denote by $|\pi|$ the mesh of π

$$|\pi| \;:=\; \max\big\{\, t_{i+1}-t_i\,,\; i=0,\dots,n-1\,\big\}\,,$$

that we suppose satisfying $|\pi|\le 1$, and by ΔW_i^π (resp. Δt_i^π) the increment of W (resp. the difference) between t_i and t_{i-1}: $\Delta W_i^\pi := W_{t_i}-W_{t_{i-1}}$ (resp. $\Delta t_i^\pi := t_i - t_{i-1}$), for $1\le i\le n$.

3.1 Discrete-time scheme for X

We introduce an approximation of the process X based on the discretization of the processes X^0 and X^1.

• *Euler scheme for X^0.* We consider the scheme $X^{0,\pi}$ defined by

$$\begin{cases} X^{0,\pi}_{t_0} \;=\; x\,, \\ X^{0,\pi}_{t_i} \;=\; X^{0,\pi}_{t_{i-1}} + b(t_{i-1},X^{0,\pi}_{t_{i-1}})\Delta t_i^\pi + \sigma(t_{i-1},X^{0,\pi}_{t_{i-1}})\Delta W_i^\pi\,, \quad 1\le i\le n\,. \end{cases}$$

• *Euler scheme for X^1.* Since the process X^1 depends on two parameters t and θ, we introduce a discretization of X^1 in these two variables. We then consider the following scheme

$$\begin{cases} X^{1,\pi}_{t_0}(\pi(\theta)) \;=\; x+\beta(t_0,x)\mathbb{1}_{\pi(\theta)=0}\,, \\ X^{1,\pi}_{t_i}(\pi(\theta)) \;=\; X^{1,\pi}_{t_{i-1}}(\pi(\theta)) + b(t_{i-1},X^{1,\pi}_{t_{i-1}}(\pi(\theta)))\Delta t_i^\pi + \sigma(t_{i-1},X^{1,\pi}_{t_{i-1}}(\pi(\theta)))\Delta W_i^\pi \\ \qquad\qquad + \beta(t_{i-1},X^{1,\pi}_{t_{i-1}}(\pi(\theta)))\mathbb{1}_{t_i=\pi(\theta)}\,, \quad 1\le i\le n\,, \quad 0\le\theta\le T\,. \end{cases} \tag{3.1}$$

We are now able to provide an approximation of the process X solution to the FSDE (2.1). We consider the scheme X^π defined by

$$X_t^\pi \;=\; X^{0,\pi}_{\pi(t)}\mathbb{1}_{t<\tau} + X^{1,\pi}_{\pi(t)}(\pi(\tau))\mathbb{1}_{t\ge\tau}\,, \quad 0\le t\le T\,. \tag{3.2}$$

We shall denote by $\{\mathcal{F}_i^{0,\pi}\}_{0\le i\le n}$ (resp. $\{\mathcal{F}_i^{1,\pi}(\theta)\}_{0\le i\le n}$) the discrete-time filtration associated with $X^{0,\pi}$ (resp. $X^{1,\pi}$)

$$\begin{aligned} \mathcal{F}_i^{0,\pi} \;&:=\; \sigma(X^{0,\pi}_{t_j},\; j\le i) \\ (\text{resp. } \mathcal{F}_i^{1,\pi}(\theta) \;&:=\; \sigma(X^{1,\pi}_{t_j}(\theta),\; j\le i))\,. \end{aligned}$$

3.2 Discrete-time scheme for (Y,Z,U)

We introduce an approximation of (Y,Z) based on the discretization of (Y^0,Z^0) and (Y^1,Z^1). To this end we introduce the backward implicit schemes on π associated with the BSDEs (2.4) and (2.6). Since the system is recursively coupled, we first introduce the scheme associated with (2.4). We then use it to define the scheme associated with (2.6).

• *Backward Euler scheme for (Y^1,Z^1).* We consider the implicit scheme $(Y^{1,\pi},Z^{1,\pi})$ defined by

$$\begin{cases} Y^{1,\pi}_T(\pi(\theta)) \;=\; g(X^{1,\pi}_T(\pi(\theta)))\,, \\ Y^{1,\pi}_{t_{i-1}}(\pi(\theta)) \;=\; \mathbb{E}^{1,\pi(\theta)}_{i-1}\big[Y^{1,\pi}_{t_i}(\pi(\theta))\big] + f\big(t_{i-1},X^{1,\pi}_{t_{i-1}}(\pi(\theta)),Y^{1,\pi}_{t_{i-1}}(\pi(\theta)),Z^{1,\pi}_{t_{i-1}}(\pi(\theta)),0\big)\Delta t_i^\pi\,, \\ Z^{1,\pi}_{t_{i-1}}(\pi(\theta)) \;=\; \dfrac{1}{\Delta t_i^\pi}\mathbb{E}^{1,\pi(\theta)}_{i-1}\big[Y^{1,\pi}_{t_i}(\pi(\theta))\Delta W_i^\pi\big]\,, \quad \pi(\theta)\le t_{i-1}\,,\; 1\le i\le n\,, \end{cases} \tag{3.3}$$

where $\mathbb{E}^{1,s}_i = \mathbb{E}[\,.\,|\mathcal{F}_i^{1,\pi}(s)]$ for $0\le i\le n$ and $s\in[0,T]$.

• *Backward Euler scheme for (Y^0,Z^0).* Since the generator of (2.6) involves the process $(Y^1_t(t))_{t\in[0,T]}$, we consider a discretization based on $Y^{1,\pi}$. We therefore consider the scheme

$(Y^{0,\pi}, Z^{0,\pi})$ defined by

$$\begin{cases} Y_T^{0,\pi} &= g(X_T^{0,\pi}) \,, \\ Y_{t_{i-1}}^{0,\pi} &= \mathbb{E}_{i-1}^0\big[Y_{t_i}^{0,\pi}\big] + \bar{f}^\pi\big(t_{i-1}, X_{t_{i-1}}^{0,\pi}, Y_{t_{i-1}}^{0,\pi}, Z_{t_{i-1}}^{0,\pi}\big)\Delta t_i^\pi \,, \\ Z_{t_{i-1}}^{0,\pi} &= \dfrac{1}{\Delta t_i^\pi}\mathbb{E}_{i-1}^0\big[Y_{t_i}^{0,\pi}\Delta W_i^\pi\big] \,, \quad 1 \le i \le n \,, \end{cases} \tag{3.4}$$

where $\mathbb{E}_i^0 = \mathbb{E}[\,.\,|\mathcal{F}_i^{0,\pi}]$ for $0 \le i \le n$, and $\bar{f}^\pi$ is defined by

$$\bar{f}^\pi(t,x,y,z) \;=\; f\big(t,x,y,z,Y_{\pi(t)}^{1,\pi}(\pi(t)) - y\big) \,,$$

for all $(t,x,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.
We then consider the following scheme for the solution (Y,Z,U) of the BSDE (2.2)

$$\begin{cases} Y_t^\pi &= Y_{\pi(t)}^{0,\pi}\mathbb{1}_{t<\tau} + Y_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t\ge\tau} \,, \\ Z_t^\pi &= Z_{\pi(t)}^{0,\pi}\mathbb{1}_{t\le\tau} + Z_{\pi(t)}^{1,\pi}(\pi(\tau))\mathbb{1}_{t>\tau} \,, \\ U_t^\pi &= \big(Y_{\pi(t)}^{1,\pi}(\pi(t)) - Y_{\pi(t)}^{0,\pi}\big)\mathbb{1}_{t\le\tau} \,, \end{cases} \tag{3.5}$$

for $t \in [0,T]$.

4 Convergence of the forward scheme

We introduce the following assumption, which will be used to control the error between X and X^π.

(HFD) There exists a constant K such that the functions b, σ and β satisfy

$$\begin{aligned} \big|b(t,x) - b(t',x)\big| + \big|\sigma(t,x) - \sigma(t',x)\big| &\le K|t-t'|^{\frac{1}{2}} \,, \\ \big|\beta(t,x) - \beta(t',x)\big| + \big|\sigma(t,x) - \sigma(t',x)\big| &\le K|t-t'| \,, \end{aligned}$$

for all $(t,t',x) \in [0,T]\times[0,T]\times\mathbb{R}$.

In the following we provide an error estimate of the approximation schemes for X^0 and X^1 which are used to control the error between X and X^π.

4.1 Error estimates for X^0 and X^1

Under **(HF)** and **(HFD)**, the upper bound of the error between X^0 and its Euler scheme $X^{0,\pi}$ is well understood, see e.g. [16], and we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t^0 - X_{\pi(t)}^{0,\pi}\big|^2\Big] \;\le\; K|\pi| \,, \tag{4.1}$$

for some constant K which does not depend on π.

The next result provides an upper bound for the error between X^1 and its Euler scheme $X^{1,\pi}$ defined by (3.1).

Theorem 4.1. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\sup_{\theta\in[0,T]}\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\theta) - X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \;\le\; K|\pi| \,,$$

for a constant K which does not depend on π.

Proof. Fix $\theta \in [0,T]$, we then have

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\theta) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2\Big] \;\leq\; 2\,\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\theta) - X_t^1(\pi(\theta))\big|^2\Big] + 2\,\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta)) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2\Big]\,. \quad (4.2)$$

We study separately the two terms of the right hand side.
Since $\pi(\theta) \leq \theta \leq t$, we have by definition $X_s^1(\pi(\theta)) = X_s^0$ for all $s \in [0,\pi(\theta))$, and $X_s^1(\theta) = X_s^0$ for all $s \in [0,\theta)$, which implies

$$\begin{aligned} X_t^1(\theta) - X_t^1(\pi(\theta)) \;=\; & \int_{\pi(\theta)}^{\theta} b\big(s,X_s^0\big)ds + \int_{\pi(\theta)}^{\theta}\sigma\big(s,X_s^0\big)dW_s + \beta\big(\theta,X_\theta^0\big) + \int_{\theta}^{t} b\big(s,X_s^1(\theta)\big)ds \\ & + \int_{\theta}^{t}\sigma\big(s,X_s^1(\theta)\big)dW_s - \beta\big(\pi(\theta),X^0_{\pi(\theta)}\big) - \int_{\pi(\theta)}^{t} b\big(s,X_s^1(\pi(\theta))\big)ds \\ & - \int_{\pi(\theta)}^{t}\sigma\big(s,X_s^1(\pi(\theta))\big)dW_s\,, \end{aligned}$$

for all $t \in [\theta,T]$.
Hence, there exists a constant K such that

$$\begin{aligned} \big|X_t^1(\theta) - X_t^1(\pi(\theta))\big|^2 \;\leq\; & K\Big\{\Big|\int_{\pi(\theta)}^{\theta} b\big(s,X_s^0\big)ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta} b\big(s,X_s^1(\pi(\theta))\big)ds\Big|^2 \\ & + \int_{\theta}^{t}\Big|b\big(s,X_s^1(\theta)\big) - b\big(s,X_s^1(\pi(\theta))\big)\Big|^2 ds \\ & + \Big|\int_{\pi(\theta)}^{\theta}\sigma\big(s,X_s^0\big)dW_s\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta}\sigma\big(s,X_s^1(\pi(\theta))\big)dW_s\Big|^2 \\ & + \Big|\int_{\theta}^{t}\Big(\sigma\big(s,X_s^1(\theta)\big) - \sigma\big(s,X_s^1(\pi(\theta))\big)\Big)dW_s\Big|^2 \\ & + \big|\beta\big(\theta,X_\theta^0\big) - \beta\big(\pi(\theta),X^0_{\pi(\theta)}\big)\big|^2\Big\}\,. \end{aligned} \quad (4.3)$$

From **(HF)** and **(HFD)**, we have

$$\mathbb{E}\big|\beta\big(\theta,X_\theta^0\big) - \beta\big(\pi(\theta),X^0_{\pi(\theta)}\big)\big|^2 \;\leq\; K\big(|\pi|^2 + \mathbb{E}\big|X_\theta^0 - X^0_{\pi(\theta)}\big|^2\big)\,.$$

We have from **(HF)** and (2.9)

$$\mathbb{E}\Big[\Big|\int_{\pi(\theta)}^{\theta} b(s,X_s^0)ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta}\sigma(s,X_s^0)dW_s\Big|^2\Big] \;\leq\; K|\pi|\,,$$

which implies in particular $\mathbb{E}|X_\theta^0 - X^0_{\pi(\theta)}|^2 \leq K|\pi|$ and hence

$$\mathbb{E}|\beta(\theta,X_\theta^0) - \beta(\pi(\theta),X^0_{\pi(\theta)})|^2 \leq K|\pi|\,.$$

We have also from **(HF)** and (2.10)

$$\mathbb{E}\Big[\Big|\int_{\pi(\theta)}^{\theta} b\big(s,X_s^1(\pi(\theta))\big)ds\Big|^2 + \Big|\int_{\pi(\theta)}^{\theta}\sigma\big(s,X_s^1(\pi(\theta))\big)dW_s\Big|^2\Big] \;\leq\; K|\pi|\,.$$

Combining these inequalities with (4.3), **(HF)** and BDG-inequality, we get

$$\mathbb{E}\Big[\sup_{u\in[\theta,t]}\big|X_u^1(\theta) - X_u^1(\pi(\theta))\big|^2\Big] \;\leq\; K\Big(\int_{\theta}^{t}\mathbb{E}\Big[\sup_{u\in[\theta,s]}\big|X_u^1(\theta) - X_u^1(\pi(\theta))\big|^2\Big]ds + |\pi|\Big)\,.$$

Applying Gronwall's lemma, we get

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\theta)-X_t^1(\pi(\theta))\big|^2\Big] \;\leq\; K|\pi|\,. \tag{4.4}$$

To find an upper bound for the term $\mathbb{E}[\sup_{t\in[\theta,T]}|X_t^1(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))|^2]$ we introduce the scheme $\tilde{X}_\cdot^\pi(\pi(\theta))$ defined by

$$\begin{cases} \tilde{X}_{\pi(\theta)}^\pi(\pi(\theta)) \;=\; X_{\pi(\theta)}^1(\pi(\theta))\,, \\ \tilde{X}_{t_i}^\pi(\pi(\theta)) \;=\; \tilde{X}_{t_{i-1}}^\pi(\pi(\theta)) + b\big(t_{i-1},\tilde{X}_{t_{i-1}}^\pi(\pi(\theta))\big)\Delta t_i^\pi + \sigma\big(t_{i-1},\tilde{X}_{t_{i-1}}^\pi(\pi(\theta))\big)\Delta W_i^\pi\,,\; t_i>\pi(\theta)\,. \end{cases}$$

We have the inequality

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \;\leq\; & 2\,\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta))-\tilde{X}_{\pi(t)}^\pi(\pi(\theta))\big|^2\Big] \\ & + 2\,\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|\tilde{X}_{\pi(t)}^\pi(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\,. \end{aligned}\tag{4.5}$$

Since $\tilde{X}^\pi(\pi(\theta))$ is the Euler scheme of $X^1(\pi(\theta))$ on $[\pi(\theta),T]$, we have under **(HF)** and **(HFD)** (see e.g. [16])

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta))-\tilde{X}_{\pi(t)}^\pi(\pi(\theta))\big|^2\Big] \;\leq\; K\Big(1+\mathbb{E}\Big[\big|X_{\pi(\theta)}^1(\pi(\theta))\big|^2\Big]\Big)|\pi|\,,$$

for some constant K which neither depends on π nor on θ. From (2.10), we get

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X_t^1(\pi(\theta))-\tilde{X}_{\pi(t)}^\pi(\pi(\theta))\big|^2\Big] \;\leq\; K|\pi|\,, \tag{4.6}$$

for all $\theta\in[0,T]$.

We now study the term $\mathbb{E}\big[\sup_{t\in[\theta,T]}\big|\tilde{X}_{\pi(t)}^\pi(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\big]$. We first notice that we have the following identity

$$\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|\tilde{X}_{\pi(t)}^\pi(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \;=\; \mathbb{E}\Big[\sup_{t\in[\pi(\theta),T]}\big|\tilde{X}_{\pi(t)}^\pi(\pi(\theta))-X_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\,.$$

Hence we can work with the second term. From the definition of $\tilde{X}^\pi$ and $X^{1,\pi}$, we get

$$\begin{aligned}&\sup_{u\in[\pi(\theta),t]}\big|\tilde{X}_{\pi(u)}^\pi(\pi(\theta))-X_{\pi(u)}^{1,\pi}(\pi(\theta))\big|^2 \leq \\ &K\Big(\big|X_{\pi(\theta)}^1(\pi(\theta))-X_{\pi(\theta)}^{1,\pi}(\pi(\theta))\big|^2+\int_{\pi(\theta)}^{\pi(t)}\Big|b\big(\pi(s),\tilde{X}_{\pi(s)}^\pi(\pi(\theta))\big)-b\big(\pi(s),X_{\pi(s)}^{1,\pi}(\pi(\theta))\big)\Big|^2 ds \\ &\qquad+\sup_{u\in[\pi(\theta),t]}\Big|\int_{\pi(\theta)}^{\pi(u)}\Big(\sigma\big(\pi(s),\tilde{X}_{\pi(s)}^\pi(\pi(\theta))\big)-\sigma\big(\pi(s),X_{\pi(s)}^{1,\pi}(\pi(\theta))\big)\Big)dW_s\Big|^2\Big)\,.\end{aligned}$$

Then, using **(HF)** and BDG-inequality, we get

$$\begin{aligned}\mathbb{E}\Big[\sup_{u\in[\pi(\theta),t]}\big|\tilde{X}_{\pi(u)}^\pi(\pi(\theta))-X_{\pi(u)}^{1,\pi}(\pi(\theta))\big|^2\Big] \leq K\Big(&\mathbb{E}\big|X_{\pi(\theta)}^1(\pi(\theta))-X_{\pi(\theta)}^{1,\pi}(\pi(\theta))\big|^2 \\ &+\int_{\pi(\theta)}^t\mathbb{E}\Big[\sup_{u\in[\pi(\theta),s]}\big|\tilde{X}_{\pi(u)}^\pi(\pi(\theta))-X_{\pi(u)}^{1,\pi}(\pi(\theta))\big|^2\Big]ds\Big)\,.\end{aligned}$$

From Lipschitz property of β, we have

$$\begin{aligned}\mathbb{E}\big|X_{\pi(\theta)}^1(\pi(\theta))-X_{\pi(\theta)}^{1,\pi}(\pi(\theta))\big|^2 \;&=\; \mathbb{E}\big|X_{\pi(\theta)}^0+\beta\big(\pi(\theta),X_{\pi(\theta)}^0\big)-X_{\pi(\theta)}^{0,\pi}-\beta\big(\pi(\theta),X_{\pi(\theta)}^{0,\pi}\big)\big|^2 \\ &\leq\; K\,\mathbb{E}\big|X_{\pi(\theta)}^0-X_{\pi(\theta)}^{0,\pi}\big|^2\,.\end{aligned}$$

This last inequality with (4.1) gives

$$\mathbb{E}\big|X^1_{\pi(\theta)}(\pi(\theta)) - X^{1,\pi}_{\pi(\theta)}(\pi(\theta))\big|^2 \quad \leq \quad K|\pi|\,.$$

Applying Gronwall's lemma, we get

$$\mathbb{E}\Big[\sup_{t\in[\pi(\theta),T]}\big|\tilde{X}^{\pi}_{\pi(t)}(\pi(\theta)) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2\Big] \quad \leq \quad K|\pi|\,. \tag{4.7}$$

Combining (4.2), (4.4), (4.5), (4.6) and (4.7), we get the result. □

4.2 Error estimate for the FSDE with a jump

We are now able to provide an estimate of the error approximation of the process X by its scheme X^π defined by (3.2).

Theorem 4.2. *Under **(HF)** and **(HFD)**, we have the following estimate*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t - X^{\pi}_t\big|^2\Big] \quad \leq \quad K|\pi|\,,$$

for a constant K which does not depend on π.

Proof. From the definition of X^π, **(DH)** and (4.1) we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t - X^{\pi}_t\big|^2\Big] &\leq \mathbb{E}\Big[\sup_{t\in[0,\tau)}\big|X^0_t - X^{0,\pi}_{\pi(t)}\big|^2\Big] + \mathbb{E}\Big[\sup_{t\in[\tau,T]}\big|X^1_t(\tau) - X^{1,\pi}_{\pi(t)}(\pi(\tau))\big|^2\Big]\\
&\leq \mathbb{E}\Big[\sup_{t\in[0,T]}\big|X^0_t - X^{0,\pi}_{\pi(t)}\big|^2\Big] + \int_0^T \mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X^1_t(\theta) - X^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2\gamma_T(\theta)\Big]d\theta\\
&\leq \quad K\Big(|\pi| + \sup_{\theta\in[0,T]}\mathbb{E}\Big[\sup_{s\in[\theta,T]}\big|X^1_s(\theta) - X^{1,\pi}_{\pi(s)}(\pi(\theta))\big|^2\Big]\Big)\,.
\end{aligned}$$

From Theorem 4.1, we get

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X_t - X^{\pi}_t\big|^2\Big] \quad \leq \quad K|\pi|\,.$$

□

5 Convergence of the backward scheme in the Lipschitz case

To provide error estimates for the Euler scheme of the BSDE, we need an additional regularity property for the coefficients g and f. We then introduce the following assumption.

(HBLD) There exists a constant K such that the functions g and f satisfy

$$\big|g(x) - g(x')\big| + \big|f(t,x,y,z,u) - f(t',x',y,z,u)\big| \quad \leq \quad K\big(|x-x'| + |t-t'|^{\frac{1}{2}}\big)\,,$$

for all $(t,t',x,x',y,z,u) \in [0,T]^2\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

We are now ready to provide error estimates of the approximation schemes for (Y^0,Z^0) and (Y^1,Z^1), and then for (Y,Z).

5.1 Regularity results

In this part, we give some results on the regularity of the processes Z^1 and Z^0. We denote $\mathcal{F}_t^0 := \sigma\{X_s^0\ ,\ 0 \le s \le t\}$ and $\mathcal{F}_t^1(\theta) := \sigma\{X_s^1(\theta)\ ,\ \theta \le s \le t\}$.

Proposition 5.1. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, there exists a constant K such that*

$$\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^1(\theta)\big|^2 dt\Big] \quad \le \quad K\Big(1 + \mathbb{E}\Big[\big|X_\theta^1(\theta)\big|^4\Big]^{\frac{1}{2}}\Big)|\pi|\ , \tag{5.1}$$

for all $\theta \in \pi$.

Proof. We first suppose that b, σ, f and g are in C_b^1. Let us define the processes Λ and M by

$$\Lambda_t \quad := \quad \exp\Big(\int_\theta^t \partial_y f\big(\Theta_r^1(\theta)\big)dr\Big)\ ,$$

and

$$M_t \quad := \quad 1 + \int_\theta^t M_r \partial_z f\big(\Theta_r^1(\theta)\big)dW_r\ ,$$

where $\Theta_r^1(\theta) := (r, X_r^1(\theta), Y_r^1(\theta), Z_r^1(\theta), 0)$. We give classically the link between $\nabla^\theta X_t^1(\theta)(:= \partial X_t^1(\theta)/\partial X_\theta^1(\theta))$ and $(D_s X_t^1(\theta))_{\theta \le s \le t}$ the Malliavin derivative of $X_t^1(\theta)$. Recall that $X^1(\theta)$ satisfies

$$X_t^1(\theta) \quad = \quad X_\theta^1(\theta) + \int_\theta^t b\big(r, X_r^1(\theta)\big)dr + \int_\theta^t \sigma\big(r, X_r^1(\theta)\big)dW_r\ , \quad \theta \le t \le T\ .$$

Therefore, we get

$$\nabla^\theta X_t^1(\theta) \quad = \quad 1 + \int_\theta^t \partial_x b\big(r, X_r^1(\theta)\big)\nabla^\theta X_r^1(\theta)dr + \int_\theta^t \partial_x \sigma\big(r, X_r^1(\theta)\big)\nabla^\theta X_r^1(\theta)dW_r\ , \quad \theta \le t \le T\ ,$$

and for $\theta \le s \le t$

$$D_s X_t^1(\theta) \quad = \quad \sigma\big(s, X_s^1(\theta)\big) + \int_s^t \partial_x b\big(r, X_r^1(\theta)\big)D_s X_r^1(\theta)dr + \int_s^t \partial_x \sigma\big(r, X_r^1(\theta)\big)D_s X_r^1(\theta)dW_r\ .$$

Thus, we have

$$D_s X_t^1(\theta) \quad = \quad \nabla^\theta X_t^1(\theta)\big[\nabla^\theta X_s^1(\theta)\big]^{-1}\sigma\big(s, X_s^1(\theta)\big)\ . \tag{5.2}$$

Using Malliavin calculus we obtain that a version of $Z^1(\theta)$ is given by $(D_t Y_t^1(\theta))_{t\in[\theta,T]}$. By Itô's formula, we get

$$\Lambda_t M_t Z_t^1(\theta) \quad = \quad \mathbb{E}\Big[M_T\Big(\Lambda_T \nabla g\big(X_T^1(\theta)\big)D_t X_T^1(\theta) + \int_t^T \partial_x f\big(\Theta_r^1(\theta)\big)D_t X_r^1(\theta)\Lambda_r dr\Big)\Big|\mathcal{F}_t^1(\theta)\Big]\ ,$$

for $t \in [\theta, T]$. Using (5.2), we get

$$\Lambda_t M_t Z_t^1(\theta) \quad = \quad \mathbb{E}\Big[M_T\Big(\Lambda_T \nabla g\big(X_T^1(\theta)\big)\nabla^\theta X_T^1(\theta) + \int_t^T F_r \Lambda_r dr\Big)\Big|\mathcal{F}_t^1(\theta)\Big]\big[\nabla^\theta X_t^1(\theta)\big]^{-1}\sigma\big(t, X_t^1(\theta)\big)\ ,$$

with $F_r := \partial_x f\big(\Theta_r^1(\theta)\big)\nabla^\theta X_r^1(\theta)$. This implies that

$$\Lambda_t M_t Z_t^1(\theta) \quad = \quad \Big(\mathbb{E}[G|\mathcal{F}_t^1(\theta)] - M_t \int_\theta^t F_r \Lambda_r dr\Big)\big[\nabla^\theta X_t^1(\theta)\big]^{-1}\sigma\big(t, X_t^1(\theta)\big)\ ,$$

with $G := M_T\Big(\Lambda_T \nabla g(X_T^1(\theta))\nabla^\theta X_T^1(\theta) + \int_\theta^T F_r \Lambda_r dr\Big)$. Since b, σ, f and g have bounded derivatives, we have

$$\mathbb{E}\big[|G|^p\big] \quad < \quad \infty\,, \quad p \geq 2\,. \tag{5.3}$$

Define $m_r := \mathbb{E}[G|\mathcal{F}_r^1(\theta)]$ for $r \in [\theta, T]$. From (5.3) and Doob's inequality, we have

$$\|m\|_{S^p[\theta,T]} \quad < \quad \infty\,, \quad p \geq 2\,. \tag{5.4}$$

Hence, there exists a process ϕ such that

$$m_r \quad = \quad \mathbb{E}[G|\mathcal{F}_\theta^1(\theta)] + \int_\theta^r \phi_u dW_u\,, \quad r \in [\theta, T]\,,$$

and

$$\|\phi\|_{H^p[\theta,T]} \quad < \quad \infty\,, \quad p \geq 2\,.$$

We define $\tilde{Z}$ by

$$\tilde{Z}_t(\theta) \quad := \quad (\Lambda_t M_t)^{-1}\Big(m_t - M_t \int_\theta^t F_r \Lambda_r dr\Big)\big[\nabla^\theta X_t^1(\theta)\big]^{-1}\,.$$

By Itô's formula, we can write

$$\tilde{Z}_t(\theta) \quad = \quad \tilde{Z}_\theta(\theta) + \int_\theta^t \alpha_r^1(\theta) dr + \int_\theta^t \alpha_r^2(\theta) dW_r\,, \quad \theta \leq r \leq T\,.$$

Since b, σ, f and g have bounded derivatives, we get from (5.4)

$$\sup_{\theta\in[0,T]} \|\tilde{Z}(\theta)\|^p_{\mathcal{S}^p[\theta,T]} \quad < \quad \infty\,, \quad p \geq 2\,, \tag{5.5}$$

and

$$\sup_{\theta\in[0,T]} \Big(\|\alpha^1(\theta)\|_{H^p[\theta,T]} + \|\alpha^2(\theta)\|_{H^p[\theta,T]}\Big) \quad < \quad \infty\,, \quad p \geq 2\,. \tag{5.6}$$

We now write for $t \in [t_i, t_{i+1})$

$$\mathbb{E}\big[|Z_t(\theta) - Z_{t_i}(\theta)|^2\big] \quad \leq \quad K(I^1_{t_i,t} + I^2_{t_i,t})\,,$$

with

$$\begin{cases} I^1_{t_i,t} := \ \mathbb{E}\big[|\tilde{Z}_t(\theta) - \tilde{Z}_{t_i}(\theta)|^2 \big|\sigma\big(t_i, X^1_{t_i}(\theta)\big)\big|^2\big]\,, \\ I^2_{t_i,t} := \ \mathbb{E}\big[|\tilde{Z}_t(\theta)|^2 \big|\sigma\big(t, X^1_t(\theta)\big) - \sigma\big(t_i, X^1_{t_i}(\theta)\big)\big|^2\big]\,. \end{cases}$$

We give an upper bound for each term.

$$\begin{aligned} I^1_{t_i,t} \quad &= \quad \mathbb{E}\Big[\mathbb{E}\big[|\tilde{Z}_t(\theta) - \tilde{Z}_{t_i}(\theta)|^2\big|\mathcal{F}^1_{t_i}(\theta)\big]\big|\sigma\big(t_i, X^1_{t_i}(\theta)\big)\big|^2\Big] \\ &\leq \quad K\,\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} \big(|\alpha^1_r(\theta)|^2 + |\alpha^2_r(\theta)|^2\big)dr \sup_{t\in[\theta,T]} \big|\sigma\big(t, X^1_t(\theta)\big)\big|^2\Big] \end{aligned}$$

which implies

$$\int_{t_i}^{t_{i+1}} I^1_{t_i,t} dt \quad \leq \quad K|\pi|\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} \big(|\alpha^1_r(\theta)|^2 + |\alpha^2_r(\theta)|^2\big)dr \sup_{t\in[\theta,T]} \big|\sigma\big(t, X^1_t(\theta)\big)\big|^2\Big]\,,$$

therefore we have

$$\sum_{i=0,\ t_i\geq\theta}^{n-1}\int_{t_i}^{t_{i+1}} I^1_{t_i,t}dt \quad\leq\quad K|\pi|\mathbb{E}\Big[\int_\theta^T \big(|\alpha^1_r(\theta)|^2+|\alpha^2_r(\theta)|^2\big)dr \sup_{t\in[\theta,T]}\big|\sigma\big(t,X^1_t(\theta)\big)\big|^2\Big]\ .$$

From Hölder's inequality and **(HFD)** and **(HF)**, we have

$$\sum_{i=0,\ t_i\geq\theta}^{n-1}\int_{t_i}^{t_{i+1}} I^1_{t_i,t}dt \quad\leq\quad K|\pi|\mathbb{E}\Big[\int_\theta^T \big(|\alpha^1_r(\theta)|^4+|\alpha^2_r|^4(\theta)\big)dr\Big]^{\frac12}\Big(1+\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X^1_t(\theta)\big|^4\Big]^{\frac12}\Big)\ .$$

Using (5.6), we get

$$\begin{aligned}\sum_{i=0,\ t_i\geq\theta}^{n-1}\int_{t_i}^{t_{i+1}} I^1_{t_i,t}dt \quad&\leq\quad K|\pi|\Big(1+\mathbb{E}\Big[\sup_{t\in[\theta,T]}\big|X^1_t(\theta)\big|^4\Big]^{\frac12}\Big)\\ &\leq\quad K|\pi|\Big(1+\mathbb{E}\Big[\big|X^1_\theta(\theta)\big|^4\Big]^{\frac12}\Big)\ .\end{aligned}\tag{5.7}$$

We get from (5.5), **(HFD)** and **(HF)**

$$I^2_{t_i,t} \quad\leq\quad K\Big(\mathbb{E}\Big[\big|\tilde Z_t-\tilde Z_{t_i}\big|^2\big|X^1_{t_i}(\theta)\big|^2\Big]+\mathbb{E}\Big[\big|X^1_t(\theta)\tilde Z_t-X^1_{t_i}(\theta)\tilde Z_{t_i}\big|^2\Big]+|\pi|^2\Big)\ .$$

Arguing as above, we obtain

$$\sum_{i=0,\ t_i\geq\theta}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\Big[\big|\tilde Z_t-\tilde Z_{t_i}\big|^2\big|X^1_{t_i}(\theta)\big|^2\Big]dt \quad\leq\quad K|\pi|\mathbb{E}\Big[\sup_{\theta\leq t\leq T}\big(1+|X^1_t(\theta)|^4\big)\Big]^{\frac12}\ .$$

Moreover, from Itô's formula, $X^1(\theta)\tilde Z$ is a semimartingale of the form

$$X^1_t(\theta)\tilde Z_t \quad=\quad X^1_\theta(\theta)\tilde Z_\theta+\int_\theta^t\tilde\alpha^1_r dr+\int_\theta^t\tilde\alpha^2_r dW_r\ ,$$

where $||\tilde\alpha^1||_{H^2[\theta,T]}+||\tilde\alpha^2||_{H^2[\theta,T]}\leq K(1+\mathbb{E}[|X^1_\theta(\theta)|^4]^{\frac14})$. Therefore, we have

$$\mathbb{E}\Big[\big|X^1_t(\theta)\tilde Z_t-X^1_{t_i}(\theta)\tilde Z_{t_i}\big|^2\Big] \quad\leq\quad K\,\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}\big(|\tilde\alpha^1_r|^2+|\tilde\alpha^2_r|^2\big)dr\Big]\ ,$$

which implies

$$\sum_{i=0,\ t_i\geq\theta}^{n-1}\int_{t_i}^{t_{i+1}}\mathbb{E}\Big[\big|X^1_t(\theta)\tilde Z_t-X^1_{t_i}(\theta)\tilde Z_{t_i}\big|^2\Big] \quad\leq\quad K|\pi|\mathbb{E}\Big[\big(1+|X^1_\theta(\theta)|^4\big)\Big]^{\frac12}\ .\tag{5.8}$$

Using (5.7) and (5.8) we get the result.

When b, σ, β, f and g are not in C^1_b, we can also prove the result by regularization. We first suppose that f and g are in C^1_b. We consider a density q which is C^∞_b on $\mathbb{R}$ with a compact support, and we define an approximation $(b^\epsilon,\sigma^\epsilon,\beta^\epsilon)$ of (b,σ,β) in C^1_b by

$$(b^\epsilon,\sigma^\epsilon,\beta^\epsilon)(t,x) \quad=\quad \frac{1}{\varepsilon}\int_{\mathbb{R}}(b,\sigma,\beta)(t,x')q\Big(\frac{x-x'}{\epsilon}\Big)dx'\ ,\quad (t,x)\in[0,T]\times\mathbb{R}\ .$$

We then use the convergence of $(X^{1,\epsilon}(\theta),Y^{1,\epsilon}(\theta),Z^{1,\epsilon}(\theta))$ to $(X^1(\theta),Y^1(\theta),Z^1(\theta))$ and we get the result. Next we assume that f and g are not C^1_b and we consider for that f^ϵ and g^ϵ which are defined as previously and we get the result. □

Using the link between X^0 and $X^1_\theta(\theta)$, we obtain that the bound (5.1) is actually uniform in θ.

Corollary 5.1. *Under* ***(HF)***, ***(HFD)***, ***(HBL)*** *and* ***(HBLD)***, *there exists a constant K such that*

$$\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^1(\theta)\big|^2 dt\Big] \;\leq\; K|\pi|\,, \tag{5.9}$$

for all $\theta \in \pi$.

Proof. Since X^0 is a Brownian diffusion, we have for any $p \geq 2$, from **(HFD)** and **(HF)**, that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^0|^p\Big] < \infty\,.$$

We notice that from the Lipschitz property of β we have

$$\begin{aligned}\mathbb{E}\Big[\big|X_\theta^1(\theta)\big|^4\Big] &= \mathbb{E}\Big[\big|X_\theta^0 + \beta\big(\theta, X_\theta^0\big)\big|^4\Big] \\ &\leq K\Big(1 + \mathbb{E}\Big[\sup_{t\in[0,T]} \big|X_t^0\big|^4\Big]\Big) \;<\; \infty\,.\end{aligned}$$

Combining this result with (5.1), we get (5.9) □

We now study the regularity of Z^0.

Proposition 5.2. *Under* ***(HF)***, ***(HFD)***, ***(HBL)*** *and* ***(HBLD)***, *there exists a constant K such that we have*

$$\mathbb{E}\Big[\int_0^T \big|Z_t^0 - Z_{\pi(t)}^0\big|^2 dt\Big] \;\leq\; K|\pi|\,.$$

Proof. The proof is similar to the previous one. The only difference is that the BSDE (2.6) involves Y^1. We denote $\Theta_r^0 = (r, X_r^0, Y_r^0, Z_r^0, Y_r^1(r) - Y_r^0)$. We first suppose that b, σ, β, f and g are in C_b^1. We recall that

$$Y_t^0 \;=\; g(X_T^0) + \int_t^T f\big(\Theta_s^0\big)ds - \int_t^T Z_s^0 dW_s\,.$$

Therefore, for $0 \leq r \leq t \leq T$, we have

$$\begin{aligned}D_r Y_t^0 &= \nabla g(X_T^0) D_r X_T^0 + \int_t^T \Big(\partial_x f\big(\Theta_s^0\big) D_r X_s^0 + (\partial_y - \partial_u) f\big(\Theta_s^0\big) D_r Y_s^0 \\ &\quad + \partial_z f\big(\Theta_s^0\big) D_r Z_s^0 + \partial_u f\big(\Theta_s^0\big) D_r Y_s^1(s)\Big)dr - \int_t^T D_r Z_s^0 dW_s\,,\end{aligned}$$

where DX_r^0, DY_r^0, DZ_r^0 and $DY_r^1(r)$ denote the Malliavin derivatives of X_r^0, Y_r^0, Z_r^0 and $Y_r^1(r)$ for $r \in [0,T]$. Using Malliavin calculus, we obtain that a version of Z^0 is given by $(D_t Y_t^0)_{t\in[0,T]}$. By Itô's formula, we get

$$\Lambda_t M_t Z_t \;=\; \mathbb{E}\Big[M_T\Big(\Lambda_T \nabla g\big(X_T^0\big) D_t X_T^0 + \int_t^T \big(\partial_x f\big(\Theta_r^0\big) D_t X_r^0 + \partial_u f\big(\Theta_r^0\big) D_t Y_r^1(r)\big)\Lambda_r dr\Big)\Big|\mathcal{F}_t^0\Big]\,,$$

where $\Lambda_t := \exp(\int_0^t (\partial_y - \partial_u) f(\Theta_r^0) dr)$ and $M_t := 1 + \int_0^t M_r \partial_z f(\Theta_r^0) dW_r$. Denote by $\nabla X_t^0 := \frac{\partial X_t^0}{\partial X_0^0}$ and $\nabla X_t^1(\theta) := \frac{\partial X_t^1(\theta)}{\partial X_0^1(\theta)}$ for $0 \leq t \leq \theta \leq T$. We then have for $r \leq s \leq T$

$$D_r X_s^1(s) = (1 + \partial_x \beta(s, X_s^0)) D_r X_s^0 = (1 + \partial_x \beta(s, X_s^0)) \nabla X_s^0 \sigma(r, X_r^0) [\nabla X_r^0]^{-1}\,,$$

thus we can see that $D_r X_s^1(s) = \nabla X_s^1(s) \sigma(r, X_r^0) [\nabla X_r^0]^{-1}$. Therefore, we get by writing the SDEs satisfied by $(D_r X_s^1(\theta))_{s\in[\theta,T]}$ for $r \leq \theta$, and $(\nabla X_s^1(\theta))_{s\in[\theta,T]}$

$$D_r X_s^1(\theta) \;=\; \nabla X_s^1(\theta) \big[\nabla X_r^0\big]^{-1} \sigma\big(r, X_r^0\big)\,, \quad r \leq \theta \leq s\,.$$

Writing the BSDEs satisfied by $(D_rY_s^1(\theta))_{s\in[\theta,T]}$ for $r\leq\theta$ and $(\nabla Y_s^1(\theta))_{s\in[\theta,T]}$, and using the previous equality, we get

$$D_rY_s^1(s) \;=\; \nabla Y_s^1(s)\big[\nabla X_r^0\big]^{-1}\sigma\big(r,X_r^0\big)\;,\quad s\leq\theta\;.$$

This implies

$$\Lambda_tM_tZ_t \;=\; \mathbb{E}\Big[M_T\Big(\Lambda_T\nabla g\big(X_T^0\big)\nabla X_T^0+\int_t^T F_r\Lambda_r dr\Big)\Big]\big[\nabla X_t^0\big]^{-1}\sigma\big(t,X_t^0\big)\;,$$

with $F_r:=\partial_x f(\Theta_r^0)\nabla X_r^0+\partial_u f(\Theta_r^0)\nabla Y_r^1(r)$. We can write

$$\Lambda_tM_tZ_t \;=\; \Big(\mathbb{E}[G|\mathcal{F}_t^0]-\int_0^t M_tF_r\Lambda_r dr\Big)\big[\nabla X_t^0\big]^{-1}\sigma\big(t,X_t^0\big)\;,$$

with $G:=M_T(\Lambda_T\nabla g(X_T^0)\nabla X_T^0+\int_0^T F_r\Lambda_r dr)$. Since b, σ, f and g have bounded derivatives, we have

$$\mathbb{E}\big[|G|^p\big] \;<\; \infty\;,\quad p\geq 2\;. \tag{5.10}$$

Define $m_r:=\mathbb{E}[G|\mathcal{F}_r^0]$ for $r\in[0,T]$. From (5.10) and Doob's inequality, we have

$$\|m\|_{S^p[0,T]} \;<\; \infty\;,\quad p\geq 2\;. \tag{5.11}$$

Hence, there exists a process ϕ such that

$$m_r \;=\; \mathbb{E}[G]+\int_0^r \phi_u dW_u\;,\quad r\in[0,T]\;,$$

and

$$\|\phi\|_{H^p[0,T]} \;<\; \infty\;,\quad p\geq 2\;.$$

We define $\tilde{Z}$ by

$$\tilde{Z}_t \;:=\; (\Lambda_tM_t)^{-1}\Big(m_t-M_t\int_0^t F_r\Lambda_r dr\Big)\big[\nabla X_t^0\big]^{-1}\;,\quad t\in[0,T]\,.$$

By Itô's formula, we can write

$$\tilde{Z}_t \;=\; \tilde{Z}_0+\int_0^t\alpha_r^1 ds+\int_0^t\alpha_r^2 dW_r\;,\quad t\in[0,T]\,.$$

Using the fact that b, σ, f and g have bounded derivatives and (5.11), we get

$$||\tilde{Z}||^p_{\mathcal{S}^p[0,T]} \;<\; \infty\;,\quad p\geq 2\;,$$

and

$$\|\alpha^1\|_{H^p[0,T]}+\|\alpha^2\|_{H^p[0,T]} \;<\; \infty\;,\quad p\geq 2\;. \tag{5.12}$$

We now write for $t\in[t_i,t_{i+1})$

$$\mathbb{E}\big[|Z_t^0-Z_{t_i}^0|^2\big] \;\leq\; K(I_{t_i,t}^1+I_{t_i,t}^2)\;,$$

with

$$\begin{cases} I_{t_i,t}^1 := \mathbb{E}\big[|\tilde{Z}_t-\tilde{Z}_{t_i}|^2|\sigma(t_i,X_{t_i}^0)|^2\big]\;,\\ I_{t_i,t}^2 := \mathbb{E}\big[|\tilde{Z}_t|^2\big|\sigma\big(t,X_t^0\big)-\sigma\big(t_i,X_{t_i}^0\big)\big|^2\big]\;. \end{cases}$$

As previously we give an upper bound for each term.

$$I^1_{t_i,t} \quad \leq \quad K\,\mathbb{E}\Big[\int_{t_i}^{t_{i+1}} \big(|\alpha^1_r|^2+|\alpha^2_r|^2\big)dr \sup_{t\in[0,T]} \big|\sigma\big(t,X^0_t\big)\big|^2\Big]\,.$$

From Hölder's inequality and Lipschitz property of σ, we have

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} I^1_{t_i,t}dt \quad \leq \quad K|\pi|\mathbb{E}\Big[\int_0^T \big(|\alpha^1_r|^4+|\alpha^2_r|^4\big)dr\Big]^{\frac{1}{2}}\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X^0_t\big|^4\Big]^{\frac{1}{2}}\Big)\,.$$

Using (5.12), we get

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} I^1_{t_i,t}dt \quad \leq \quad K|\pi|\,.$$

From **(HFD)** and **(HF)**, we get

$$I^2_{t_i,t} \quad \leq \quad K\Big(\mathbb{E}\Big[\big|\tilde{Z}_t-\tilde{Z}_{t_i}\big|^2\big|X^0_{t_i}\big|^2\Big]+\mathbb{E}\Big[\big|X^0_t\tilde{Z}_t-X^0_{t_i}\tilde{Z}_{t_i}\big|^2\Big]+|\pi|^2\Big)\,.$$

Arguing as above, we obtain

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \mathbb{E}\Big[\big|\tilde{Z}_t-\tilde{Z}_{t_i}\big|^2\big|X^0_{t_i}\big|^2\Big]dt \quad \leq \quad K|\pi|\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T]}\big|X^0_t\big|^4\Big]^{\frac{1}{2}}\Big)\,.$$

Moreover, $X^0\tilde{Z}$ is a semimartingale of the form

$$X^0_t\tilde{Z}_t \quad = \quad X^0_0\tilde{Z}_0+\int_0^t\tilde{\alpha}^1_r dr+\int_0^t\tilde{\alpha}^2_r dW_r$$

where $||\tilde{\alpha}^1||_{H^2[0,T]}+||\tilde{\alpha}^2||_{H^2[0,T]}\leq K$ and we have

$$\mathbb{E}\Big[\big|X^0_t\tilde{Z}_t-X^0_{t_i}\tilde{Z}_{t_i}\big|^2\Big] \quad \leq \quad K\,\mathbb{E}\int_{t_i}^{t_{i+1}}\big(|\tilde{\alpha}^1_r|^2+|\tilde{\alpha}^2_r|^2\big)dr\,,$$

which implies

$$\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \mathbb{E}\Big[\big|X^0_t\tilde{Z}_t-X^0_{t_i}\tilde{Z}_{t_i}\big|^2\Big] \quad \leq \quad K|\pi|\,.$$

When b, σ, f and g are not C^1_b, we can also prove the result by regularization as for Proposition 5.1. □

5.2 Error estimates for the recursive system of BSDEs

We first state an estimate of the approximation error for (Y^1,Z^1).

Proposition 5.3. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following estimate*

$$\sup_{\theta\in[0,T]}\Big\{\sup_{t\in[\theta,T]}\mathbb{E}\Big[\big|Y^1_t(\theta)-Y^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2\Big]+\mathbb{E}\Big[\int_\theta^T\big|Z^1_s(\theta)-Z^{1,\pi}_{\pi(s)}(\pi(\theta))\big|^2ds\Big]\Big\} \quad \leq \quad K|\pi|\,,$$

for some constant K which does not depend on π.

Proof. Fix $\theta \in [0,T]$ and $t \in [\theta, T]$. We then have

$$\begin{aligned}\mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] &\leq 2\,\mathbb{E}\Big[\big|Y_t^1(\theta) - Y_t^1(\pi(\theta))\big|^2\Big] \\ &\quad + 2\,\mathbb{E}\Big[\big|Y_t^1(\pi(\theta)) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\,. \end{aligned} \tag{5.13}$$

We study separately the two terms of right hand side.

Define $\delta X_t^1(\theta) := X_t^1(\theta) - X_t^1(\pi(\theta))$, $\delta Y_t^1(\theta) := Y_t^1(\theta) - Y_t^1(\pi(\theta))$ and $\delta Z_t^1(\theta) := Z_t^1(\theta) - Z_t^1(\pi(\theta))$. Applying Itô's formula, we get

$$\begin{aligned}|\delta Y_T^1(\theta)|^2 - |\delta Y_t^1(\theta)|^2 &= 2\int_t^T \delta Y_s^1(\theta)\Big[f\big(\Theta_s^1(\pi(\theta))\big) - f\big(\Theta_s^1(\theta)\big)\Big]ds \\ &\quad + 2\int_t^T \delta Y_s^1(\theta)\delta Z_s^1(\theta)dW_s + \int_t^T |\delta Z_s^1(\theta)|^2 ds\,,\end{aligned}$$

where $\Theta_s^1(\theta) := (s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)$. From **(HBL)** and **(HBLD)**, we get

$$\begin{aligned}\mathbb{E}\big[|\delta Y_t^1(\theta)|^2\big] &\leq K\Big(\mathbb{E}\big[|\delta X_T^1(\theta)|^2\big] + \mathbb{E}\Big[\int_t^T |\delta Y_s^1(\theta)||\delta X_s^1(\theta)|ds\Big] + \mathbb{E}\Big[\int_t^T |\delta Y_s^1(\theta)|^2 ds\Big] \\ &\quad + \mathbb{E}\Big[\int_t^T |\delta Y_s^1(\theta)||\delta Z_s^1(\theta)|ds\Big)\Big) - \mathbb{E}\Big[\int_t^T |\delta Z_s^1(\theta)|^2 ds\Big]\,.\end{aligned}$$

Using the inequality $2ab \leq a^2/\eta + \eta b^2$ for $a, b \in \mathbb{R}$ and $\eta > 0$, we can see that

$$\begin{aligned}\mathbb{E}\big[|\delta Y_t^1(\theta)|^2\big] + \mathbb{E}\Big[\int_t^T |\delta Z_s^1(\theta)|^2 ds\Big] &\leq K\Big(\mathbb{E}\big[|\delta X_T^1(\theta)|^2\big] + \int_t^T \mathbb{E}\big[|\delta Y_s^1(\theta)|^2\big]ds \\ &\quad + \mathbb{E}\Big[\int_t^T |\delta X_s^1(\theta)|^2 ds\Big]\Big)\,.\end{aligned} \tag{5.14}$$

From (4.4) and Gronwall's lemma, we get

$$\mathbb{E}\big[\big|Y_t^1(\theta) - Y_t^1(\pi(\theta))\big|^2\big] \leq K|\pi|\,. \tag{5.15}$$

We now study the second term of the right hand side of (5.13). Using the same argument as in the proof of Theorem 3.1 in [3], we get from the regularity of Z^1 given by Corollary 5.1

$$\mathbb{E}\Big[\big|Y_t^1(\pi(\theta)) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] \leq K|\pi|\,. \tag{5.16}$$

This last inequality with (5.13) and (5.15) gives

$$\sup_{\theta\in[0,T]}\Big\{\sup_{t\in[\theta,T]}\mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big]\Big\} \leq K|\pi|\,.$$

We now turn to the error on the term $Z^1(\theta)$. We first use the inequality

$$\begin{aligned}\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] &\leq 2\,\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] \\ &\quad + 2\,\mathbb{E}\Big[\int_\theta^T \big|\delta Z_t^1(\theta)\big|^2 dt\Big]\,.\end{aligned} \tag{5.17}$$

Using (5.14) and (5.15) with $t = \theta$, we get

$$\mathbb{E}\Big[\int_\theta^T \big|\delta Z_s^1(\theta)\big|^2 ds\Big] \leq K|\pi|\,. \tag{5.18}$$

The other term in the right hand side of (5.17) is the classical error in an approximation of BSDE. Therefore, using Corollary 5.1 and (5.16), we have

$$\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\pi(\theta)) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] \quad \leq \quad K|\pi| \,. \tag{5.19}$$

Combining (5.17), (5.18) and (5.19), we get

$$\mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 dt\Big] \quad \leq \quad K|\pi| \,.$$

□

We now turn to the estimation of the error between (Y^0, Z^0) and its Euler scheme (3.4). Since this scheme involves the approximation $Y^{1,\pi}$ of Y^1, we first need to introduce an intermediary scheme involving the "true" value of the process Y^1. We therefore consider the scheme $(\tilde{Y}^{0,\pi}, \tilde{Z}^{0,\pi})$ defined by

$$\left\{\begin{array}{rcl} \tilde{Y}_T^{0,\pi} &=& g(X_T^{0,\pi}) \,, \\ \tilde{Y}_{t_{i-1}}^{0,\pi} &=& \mathbb{E}_{i-1}^0\big[\tilde{Y}_{t_i}^{0,\pi}\big] + f\big(t_{i-1}, X_{t_{i-1}}^{0,\pi}, \tilde{Y}_{t_{i-1}}^{0,\pi}, \tilde{Z}_{t_{i-1}}^{0,\pi}, Y_{t_{i-1}}^1(t_{i-1}) - \tilde{Y}_{t_{i-1}}^{0,\pi}\big)\Delta t_i^\pi \,, \\ \tilde{Z}_{t_{i-1}}^{0,\pi} &=& \dfrac{1}{\Delta t_i^\pi}\mathbb{E}_{i-1}^0\big[\tilde{Y}_{t_i}^{0,\pi}\Delta W_i^\pi\big] \,, \quad 1 \leq i \leq n \,. \end{array}\right. \tag{5.20}$$

Using the regularity result of Proposition 5.2 and the same arguments as in the proof of Theorem 3.1 in [3], we get under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - \tilde{Y}_{\pi(t)}^{0,\pi}\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t^0 - \tilde{Z}_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \quad \leq \quad K|\pi| \,. \tag{5.21}$$

With this inequality, we get the following estimate for the error between (Y^0, Z^0) and the Euler scheme (3.4).

Proposition 5.4. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following estimate*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \quad \leq \quad K|\pi| \,,$$

for some constant K which does not depend on π.

Proof. We first remark that

$$\left\{\begin{array}{ll} \sup\limits_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big] \leq & 2\sup\limits_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - \tilde{Y}_{\pi(t)}^{0,\pi}\big|^2\Big] + 2\sup\limits_{t\in[0,T]} \mathbb{E}\Big[\big|Y_{\pi(t)}^{0,\pi} - \tilde{Y}_{\pi(t)}^{0,\pi}\big|^2\Big] \,, \\ \mathbb{E}\Big[\int_0^T \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \leq & 2\,\mathbb{E}\Big[\int_0^T \big|Z_t^0 - \tilde{Z}_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + 2\,\mathbb{E}\Big[\int_0^T \big|Z_{\pi(t)}^{0,\pi} - \tilde{Z}_{\pi(t)}^{0,\pi}\big|^2 dt\Big] \,. \end{array}\right.$$

Using (5.21), we only need to study $\sup_{t\in[0,T]} \mathbb{E}[|Y_{\pi(t)}^{0,\pi} - \tilde{Y}_{\pi(t)}^{0,\pi}|^2]$ and $\mathbb{E}[\int_0^T |Z_{\pi(t)}^{0,\pi} - \tilde{Z}_{\pi(t)}^{0,\pi}|^2 dt]$. To this end, we need to introduce continuous schemes for all $0 \leq i \leq n-1$. Since $\mathbb{E}[|Y_{t_i}^{0,\pi}|^2] < \infty$ and $\mathbb{E}[|\tilde{Y}_{t_i}^{0,\pi}|^2] < \infty$ for all $1 \leq i \leq n$, we deduce, from the martingale representation theorem, that there exist square integrable processes $\underline{Z}^{0,\pi}$ and $\underline{\tilde{Z}}^{0,\pi}$ such that

$$\begin{array}{rcl} Y_{t_i}^{0,\pi} &=& \mathbb{E}\big[Y_{t_{i+1}}^{0,\pi}\big|\mathcal{F}_{t_i}\big] + \displaystyle\int_{t_i}^{t_{i+1}} \underline{Z}_s^{0,\pi} dW_s \,, \\ \tilde{Y}_{t_i}^{0,\pi} &=& \mathbb{E}\big[\tilde{Y}_{t_{i+1}}^{0,\pi}\big|\mathcal{F}_{t_i}\big] + \displaystyle\int_{t_i}^{t_{i+1}} \underline{\tilde{Z}}_s^{0,\pi} dW_s \,. \end{array}$$

We then define

$$\begin{cases} Y_t^{0,\pi} = \; Y_{t_i}^{0,\pi} - (t-t_i) f\big(t_i, X_{t_i}^{0,\pi}, Y_{t_i}^{0,\pi}, Z_{t_i}^{0,\pi}, Y_{t_i}^{1,\pi}(t_i) - Y_{t_i}^{0,\pi}\big) + \displaystyle\int_{t_i}^t \underline{Z}_s^{0,\pi} dW_s \;, \\ \tilde{Y}_t^{0,\pi} = \; \tilde{Y}_{t_i}^{0,\pi} - (t-t_i) f\big(t_i, X_{t_i}^{0,\pi}, \tilde{Y}_{t_i}^{0,\pi}, \tilde{Z}_{t_i}^{0,\pi}, Y_{t_i}^{1}(t_i) - \tilde{Y}_{t_i}^{0,\pi}\big) + \displaystyle\int_{t_i}^t \underline{\tilde{Z}}_s^{0,\pi} dW_s \;, \end{cases}$$

for $t \in [t_i, t_{i+1})$. Let $i \in \{0, \dots, n-1\}$ be fixed, and set $\delta Y_t := Y_t^{0,\pi} - \tilde{Y}_t^{0,\pi}$, $\delta Z_i := Z_{t_i}^{0,\pi} - \tilde{Z}_{t_i}^{0,\pi}$, $\delta \underline{Z}_t := \underline{Z}_t^{0,\pi} - \underline{\tilde{Z}}_t^{0,\pi}$ and $\delta f_t := f(t_i, X_{t_i}^{0,\pi}, Y_{t_i}^{0,\pi}, Z_{t_i}^{0,\pi}, Y_{t_i}^{1,\pi}(t_i) - Y_{t_i}^{0,\pi}) - f(t_i, X_{t_i}^{0,\pi}, \tilde{Y}_{t_i}^{0,\pi}, \tilde{Z}_{t_i}^{0,\pi}, Y_{t_i}^{1}(t_i) - \tilde{Y}_{t_i}^{0,\pi})$ for $t \in [t_i, t_{i+1})$. By Itô's formula, we compute that

$$A_t \;\; := \;\; \mathbb{E}|\delta Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds - \mathbb{E}|\delta Y_{t_{i+1}}|^2 \;=\; 2\int_t^{t_{i+1}} \mathbb{E}[\delta Y_s \delta f_s] ds \;, \quad t_i \le t \le t_{i+1} \;.$$

Let $\alpha > 0$ be a constant to be chosen later on. From the Lipschitz property of f and the inequality $2ab \le \alpha a^2 + b^2/\alpha$, we get

$$A_t \;\; \le \;\; \alpha \int_t^{t_{i+1}} \mathbb{E}|\delta Y_s|^2 ds + \frac{K}{\alpha} \int_t^{t_{i+1}} \mathbb{E}\Big[|\delta Y_{t_i}|^2 + |\delta Z_i|^2 + |Y_{t_i}^1(t_i) - Y_{t_i}^{1,\pi}(t_i)|^2\Big] ds \;.$$

Using Proposition 5.3, we get

$$A_t \;\; \le \;\; \alpha \int_t^{t_{i+1}} \mathbb{E}|\delta Y_s|^2 ds + \frac{K}{\alpha}|\pi|\, \mathbb{E}|\delta Y_{t_i}|^2 + \frac{K}{\alpha} \int_t^{t_{i+1}} \mathbb{E}|\delta Z_i|^2 ds + \frac{K}{\alpha}|\pi|^2 \;.$$

We can write

$$\mathbb{E}|\delta Y_t|^2 \;\; \le \;\; \mathbb{E}|\delta Y_{t_{i+1}}|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \;\le\; \alpha \int_t^{t_{i+1}} \mathbb{E}|\delta Y_s|^2 ds + B_i \;, \tag{5.22}$$

where

$$B_i \;\; := \;\; \mathbb{E}|\delta Y_{t_{i+1}}|^2 + \frac{K}{\alpha}|\pi|\, \mathbb{E}|\delta Z_i|^2 + \frac{K}{\alpha}|\pi|\, \mathbb{E}|\delta Y_{t_i}|^2 + \frac{K}{\alpha}|\pi|^2 \;.$$

By Gronwall's lemma, this shows that $\mathbb{E}|\delta Y_t|^2 \le B_i e^{\alpha|\pi|}$ for $t_i \le t < t_{i+1}$, which plugged in the second inequality of (5.22) provides

$$\mathbb{E}|\delta Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \;\; \le \;\; B_i\Big(1 + \alpha|\pi| e^{\alpha|\pi|}\Big) \;. \tag{5.23}$$

Interpreting $Z_{t_i}^{0,\pi}$ (resp. $\tilde{Z}_{t_i}^{0,\pi}$) as the projection of $\underline{Z}^{0,\pi}$ (resp. $\underline{\tilde{Z}}^{0,\pi}$) in $H^2_{\mathbb{F}}[t_i, t_{i+1}]$ on the set of constant processes, we have

$$\int_{t_i}^{t_{i+1}} \mathbb{E}|\delta Z_i|^2 ds \;\; \le \;\; \int_{t_i}^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \;. \tag{5.24}$$

Applying (5.23) for $t = t_i$ and $\alpha = 2K$, and using the previous inequality, we get

$$\mathbb{E}|\delta Y_{t_i}|^2 + k_1(\pi) \int_{t_i}^{t_{i+1}} \mathbb{E}|\delta \underline{Z}_s|^2 ds \;\; \le \;\; k_2(\pi)\mathbb{E}|\delta Y_{t_{i+1}}|^2 + k_3(\pi)|\pi|^2 \;, \quad 0 \le i \le n-1 \;,$$

where $k_1(\pi) = \frac{\frac{1}{2} - K|\pi| e^{2K|\pi|}}{1 - \frac{|\pi|}{2} - K|\pi|^2 e^{2K|\pi|}}$, $k_2(\pi) = \frac{1 + 2K|\pi| e^{2K|\pi|}}{1 - \frac{|\pi|}{2} - K|\pi|^2 e^{2K|\pi|}}$ and $k_3(\pi) = \frac{\frac{1}{2} + K|\pi| e^{2K|\pi|}}{1 - \frac{|\pi|}{2} - K|\pi|^2 e^{2K|\pi|}}$. Since for small $|\pi|$ we have $k_1(\pi) \ge 0$, we get

$$\mathbb{E}|\delta Y_{t_i}|^2 \;\; \le \;\; k_2(\pi)\mathbb{E}|\delta Y_{t_{i+1}}|^2 + k_3(\pi)|\pi|^2 \;, \quad 0 \le i \le n-1 \;,$$

for $|\pi|$ small enough.
Iterating this inequality, we get

$$\mathbb{E}|\delta Y_{t_i}|^2 \quad \leq \quad k_2(\pi)^{\frac{1}{|\pi|}}\mathbb{E}|\delta Y_{t_n}|^2 + |\pi|^2 k_3(\pi)\sum_{j=i}^{n} k_2(\pi)^{j-i}\ .$$

Since $k_2(\pi) \geq 1$ and $\delta Y_{t_n} = 0$, we get for small $|\pi|$

$$\mathbb{E}|\delta Y_{t_i}|^2 \quad \leq \quad |\pi| k_3(\pi) k_2(\pi)^{\frac{1}{|\pi|}} \ \leq \ K|\pi|\ , \quad 0 \leq i \leq n\ , \tag{5.25}$$

which gives

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y^{0,\pi}_{\pi(t)} - \tilde{Y}^{0,\pi}_{\pi(t)}\big|^2\Big] \quad \leq \quad K|\pi|\ .$$

Summing up the inequality (5.23) with $t = t_i$ and $\alpha = 2K$ and using (5.24), we get

$$\begin{aligned}\Big(\frac{1}{2} - K|\pi|e^{2K|\pi|}\Big)\int_0^T \mathbb{E}|Z^{0,\pi}_{\pi(s)} - \tilde{Z}^{0,\pi}_{\pi(s)}|^2 ds \quad &\leq \quad 2K|\pi|e^{2K|\pi|}\sum_{i=1}^{n-1}\mathbb{E}|\delta Y_{t_i}|^2 + (1+2K|\pi|)\mathbb{E}|\delta Y_{t_n}|^2 \\ &\quad + \Big(\frac{1}{2} + K|\pi|e^{2K|\pi|}\Big)\Big(|\pi| + |\pi|\sum_{i=0}^{n-1}\mathbb{E}|\delta Y_{t_i}|^2\Big)\ .\end{aligned}$$

Using (5.25), we get for $|\pi|$ small enough

$$\int_0^T \mathbb{E}|Z^{0,\pi}_{\pi(s)} - \tilde{Z}^{0,\pi}_{\pi(s)}|^2 ds \quad \leq \quad K|\pi|\ .$$

□

5.3 Error estimate for the BSDE with a jump

We now give an error estimate of the approximation scheme for the BSDE with a jump.

Theorem 5.1. *Under **(HF)**, **(HFD)**, **(HBL)** and **(HBLD)**, we have the following error estimate for the approximation scheme*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t - Y^\pi_t\big|^2\Big] + \mathbb{E}\Big[\int_0^T \big|Z_t - Z^\pi_t\big|^2 dt\Big] + \mathbb{E}\Big[\int_0^T \lambda_t\big|U_t - U^\pi_t\big|^2 dt\Big] \quad \leq \quad K|\pi|\ ,$$

for some constant K which does not depend on π.

Proof.
Step 1. *Error for the variable Y.* Fix $t \in [0,T]$. From Theorem 2.1 and (3.5), we have

$$\mathbb{E}\Big[\big|Y_t - Y^\pi_t\big|^2\Big] \quad = \quad \mathbb{E}\Big[\big|Y^0_t - Y^{0,\pi}_{\pi(t)}\big|^2 \mathbb{1}_{t<\tau}\Big] + \mathbb{E}\Big[\big|Y^1_t(\tau) - Y^{1,\pi}_{\pi(t)}(\pi(\tau))\big|^2 \mathbb{1}_{t\geq\tau}\Big]\ .$$

Using **(DH)**, we get

$$\begin{aligned}\mathbb{E}\Big[\big|Y_t - Y^\pi_t\big|^2\Big] \quad &\leq \quad \mathbb{E}\Big[\big|Y^0_t - Y^{0,\pi}_{\pi(t)}\big|^2\Big] + \int_0^T \mathbb{E}\Big[\big|Y^1_t(\theta) - Y^{1,\pi}_{\pi(t)}(\pi(\theta))\big|^2 \mathbb{1}_{t\geq\theta}\gamma_T(\theta)\Big]d\theta \\ &\leq \quad K\Big(\mathbb{E}\Big[\big|Y^0_t - Y^{0,\pi}_{\pi(t)}\big|^2\Big] + \sup_{\theta\in[0,T]}\sup_{s\in[\theta,T]}\mathbb{E}\Big[\big|Y^1_s(\theta) - Y^{1,\pi}_{\pi(s)}(\pi(\theta))\big|^2\Big]\Big)\ .\end{aligned}$$

Using Propositions 5.3 and 5.4, and since t is arbitrary chosen in $[0,T]$, we get

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t - Y^\pi_t\big|^2\Big] \quad \leq \quad K|\pi|\ .$$

Step 2. *Error estimate for the variable Z.* From Theorem 2.1 and (3.5), we have

$$\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] = \mathbb{E}\Big[\int_0^{T\wedge\tau} \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + \mathbb{E}\Big[\int_{T\wedge\tau}^T \big|Z_t^1(\tau) - Z_{\pi(t)}^{1,\pi}(\pi(\tau))\big|^2 dt\Big] .$$

Using **(DH)**, we get

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] &= \int_0^T\int_0^\theta \mathbb{E}\Big[\big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 \gamma_T(\theta)\Big] dt d\theta \\ &\quad + \int_0^T\int_\theta^T \mathbb{E}\Big[\big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2 \gamma_T(\theta)\Big] dt d\theta . \\ &\le K\Big(\mathbb{E}\Big[\int_0^T \big|Z_t^0 - Z_{\pi(t)}^{0,\pi}\big|^2 dt\Big] + \sup_{\theta\in[0,T]} \mathbb{E}\Big[\int_\theta^T \big|Z_t^1(\theta) - Z_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] dt\Big) .\end{aligned}$$

From Propositions 5.3 and 5.4, we get

$$\mathbb{E}\Big[\int_0^T \big|Z_t - Z_t^\pi\big|^2 dt\Big] \le K|\pi| .$$

Step 3. *Error estimate for the variable U.* From Theorem 2.1 and (3.5), we have

$$\mathbb{E}\Big[\int_0^T \big|U_t - U_t^\pi\big|^2 \lambda_t dt\Big] \le K\,\mathbb{E}\Big[\int_0^T \Big(|Y_t^1(t) - Y_{\pi(t)}^{1,\pi}(\pi(t))|^2 + |Y_t^0 - Y_{\pi(t)}^{0,\pi}|^2\Big)\lambda_t dt\Big] .$$

Using **(HBI)**, we get

$$\mathbb{E}\Big[\int_0^T \big|U_t - U_t^\pi\big|^2 \lambda_t dt\Big] \le K\Big(\sup_{\theta\in[0,T]}\sup_{t\in[\theta,T]} \mathbb{E}\Big[\big|Y_t^1(\theta) - Y_{\pi(t)}^{1,\pi}(\pi(\theta))\big|^2\Big] + \sup_{t\in[0,T]} \mathbb{E}\Big[\big|Y_t^0 - Y_{\pi(t)}^{0,\pi}\big|^2\Big]\Big) .$$

Combining this last inequality with Propositions 5.3 and 5.4, we get the result. □

Remark 5.1. Our decomposition approach allows us to suppose that the jump coefficient β is only Lipschitz continuous. We do not need to impose any regularity condition on β and any ellipticity assumption on $I_d + \nabla\beta$ as done in [4] in the case of Poissonian jumps independent of W.

6 Convergence of the backward scheme for the quadratic case

In this section we assume that **(HBQ)** holds and that $\sigma(t,x) = \sigma(t,0) = \sigma(t)$ for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

Before giving the error of the scheme we give a uniform bound for the processes Z^0 and Z^1 which allows to prove that the BSDE (2.2) is Lipschitz and thus we can use Theorem 5.1. For that we introduce the BMO-martingales class, and we also give some bounds for the processes X^0, X^1, Y^0 and Y^1.

6.1 BMO property for the solution of the BSDE

To obtain a uniform bound for the processes Z^0 and Z^1 we need the following assumption.

(HBQD) There exists a constant K_f such that the function f satisfies

$$\begin{aligned}|f(t,x,y,z,u) - f(t',x',y',z',u')| &\le K_f\big[|x-x'| + |y-y'| + |u-u'| + |t-t'|^{\frac{1}{2}}\big] \\ &\quad + L_{f,z}(1 + |z| + |z'|)|z - z'| ,\end{aligned}$$

for all $(t, x, x', y, y', z, z', u, u') \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$.

In the sequel of this section, the space of BMO martingales plays a key role for the a priori estimates of processes Z^0 and Z^1. We refer to [14] for the theory of BMO martingales. Here, we just give the definition of a BMO martingale and recall a property that we use in the sequel.

Definition 6.1. *A process M is said to be a $BMO_{\mathbb{F}}[0,T]$-martingale if M is a square integrable $\mathbb{F}$-martingale s.t.*

$$\|M\|_{BMO_{\mathbb{F}}[0,T]} \;:=\; \sup_{\tau \in \mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\Big[\big| M_T - M_\tau \big|^2 \Big| \mathcal{F}_\tau \Big]^{1/2} \;<\; \infty \,,$$

where $\mathcal{T}_{\mathbb{F}}[0,T]$ denotes the set of $\mathbb{F}$-stopping times valued in $[0,T]$.

The BMO condition provides a property on the Dolean-Dade exponential of the process M.

Lemma 6.1. *Let M be a $BMO_{\mathbb{F}}[0,T]$-martingale. Then the stochastic exponential $\mathcal{E}(M)$ defined by*

$$\mathcal{E}(M)_t \;=\; \exp\Big(M_t - \frac{1}{2}\langle M, M\rangle \Big)_t \,, \quad 0 \le t \le T \,,$$

is a uniformly integrable $\mathbb{F}$-martingale.

We refer to [14] for the proof of this result.

We first state a BMO property for the processes Z^0 and Z^1, which will be used in the sequel to provide an estimate for these processes.

Lemma 6.2. *Under **(HF)**, **(HBQ)** and **(HBQD)**, the martingales $\int_0^\cdot Z_s^0 dW_s$ and $\int_0^\cdot Z_s^1(\theta) \mathbb{1}_{s \ge \theta} dW_s$, $\theta \in [0,T]$ are $BMO_{\mathbb{F}}[0,T]$-martingales and there exists a constant K which is independent from θ such that*

$$\begin{aligned} \Big\| \int_0^\cdot Z_s^0 dW_s \Big\|_{BMO_{\mathbb{F}}[0,T]} &\le K \,, \\ \sup_{\theta \in [0,T]} \Big\| \int_0^\cdot Z_s^1(\theta) \mathbb{1}_{s \ge \theta} dW_s \Big\|_{BMO_{\mathbb{F}}[0,T]} &\le K \,. \end{aligned}$$

Proof. Define the function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) \;=\; (e^{2K_q x} - 2K_q x - 1)/2K_q^2 \,, \quad x \in \mathbb{R} \,. \tag{6.1}$$

We notice that ϕ satisfies

$$\phi'(x) \;\ge\; 0 \;\text{ and }\; \frac{1}{2}\phi''(x) - K_q \phi'(x) \;=\; 1 \,,$$

for $x \ge 0$. Since Y^0 and $Y^1(.)$ are solutions to quadratic BSDEs with bounded terminal conditions, we get from Proposition 2.1 in [17] the existence of a constant m such that

$$\|Y^0\|_{\mathcal{S}^\infty[0,T]} \;\le\; m \quad \text{and} \quad \sup_{\theta \in [0,T]} \|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]} \;\le\; m \,. \tag{6.2}$$

Applying Itô's formula we get

$$\begin{aligned} \phi(Y_\nu^0 + m) + \mathbb{E}\Big(\int_\nu^T \frac{1}{2}\phi''(Y_s^0 + m) |Z_s^0|^2 ds \Big| \mathcal{F}_\nu \Big) = \\ \mathbb{E}(\phi(Y_T^0 + m)|\mathcal{F}_\nu) + \mathbb{E}\Big(\int_\nu^T \phi'(Y_s^0 + m) f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0) ds \Big| \mathcal{F}_\nu \Big) \,, \end{aligned}$$

for any $\mathbb{F}$-stopping time ν valued in $[0,T]$. From the growth assumption on the generator f in **(HBD)**, (6.1) and (6.2), we obtain

$$\phi(Y_\nu^0+m)+\mathbb{E}\Big(\int_\nu^T|Z_s^0|^2ds\Big|\mathcal{F}_\nu\Big)\leq$$
$$\mathbb{E}(\phi(Y_T^0+m)|\mathcal{F}_\nu)+\mathbb{E}\Big(\int_\nu^T\phi'(Y_s^0+m)K_q(1+2\|Y^0\|_{\mathcal{S}^\infty}+\sup_{\theta\in[0,T]}\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]})ds\Big|\mathcal{F}_\nu\Big)\,.$$

This last inequality and (6.2) imply that there exists a constant K which depends only on m, T and K_q such that for all $\mathbb{F}$-stopping times $\nu\in[0,T]$

$$\mathbb{E}\Big(\int_\nu^T|Z_s^0|^2ds\Big|\mathcal{F}_\nu\Big)\quad\leq\quad K\,.$$

For the process Z^1, we use the same technics. Let us fix $\theta\in[0,T]$. Applying Itô's fomula we get

$$\phi(Y^1_{\nu\vee\theta}(\theta)+m)+\mathbb{E}\Big(\int_{\nu\vee\theta}^T\frac{1}{2}\phi''(Y_s^1(\theta)+m)|Z_s^1(\theta)|^2ds\Big|\mathcal{F}_{\nu\vee\theta}\Big)=$$
$$\mathbb{E}(\phi(Y_T^1(\theta)+m)|\mathcal{F}_\nu)+\mathbb{E}\Big(\int_{\nu\vee\theta}^T\phi'(Y_s^1(\theta)+m)f(s,X_s^1(\theta),Y_s^1(\theta),Z_s^1(\theta),0)ds\Big|\mathcal{F}_{\nu\vee\theta}\Big)\,,$$

for any $\mathbb{F}$-stopping time ν valued in $[0,T]$. From the growth assumption on the generator f in **(HBQ)**, (6.1) and (6.2), we obtain

$$\phi(Y^1_{\nu\vee\theta}(\theta)+m)+\mathbb{E}\Big(\int_{\nu\vee\theta}^T|Z_s^1(\theta)|^2ds\Big|\mathcal{F}_\nu\Big)\leq\mathbb{E}(\phi(Y_T^1(\theta)+m)|\mathcal{F}_\nu)$$
$$+\mathbb{E}\Big(\int_{\nu\vee\theta}^T\phi'(Y_s^1(\theta)+m)K_q(1+\|Y^1(\theta)\|_{\mathcal{S}^\infty[\theta,T]})ds\Big|\mathcal{F}_\nu\Big)\,.$$

This last inequality and (6.2) imply that there exists a constant K which depends only on m, T and K_q, such that for all $\mathbb{F}$-stopping times ν valued in $[0,T]$

$$\mathbb{E}\Big(\int_\nu^T|Z_s^1(\theta)|^2\mathbb{1}_{s\geq\theta}ds\Big|\mathcal{F}_\nu\Big)\quad\leq\quad K\,.$$

□

6.2 Some bounds about X^0 and X^1

In this part, we give some bounds about the processes X^0 and X^1 which are used to get a uniform bound for the processes Z^0 and Z^1.

Proposition 6.1. *Suppose that **(HF)** holds. Then, we have*

$$|\nabla X_t^0|\;:=\;\Big|\frac{\partial X_t^0}{\partial x}\Big|\quad\leq\quad e^{L_aT}\,,\quad 0\leq t\leq T\,,\tag{6.3}$$

and for any $\theta\in[0,T]$ we have

$$|\nabla^\theta X_t^1(\theta)|\;:=\;\Big|\frac{\partial X_t^1(\theta)}{\partial X_\theta^1(\theta)}\Big|\quad\leq\quad e^{L_aT}\,,\quad \theta\leq t\leq T\,,\tag{6.4}$$

$$|\nabla X_t^1(\theta)|\;:=\;\Big|\frac{\partial X_t^1(\theta)}{\partial x}\Big|\quad\leq\quad(1+L_ae^{L_aT})e^{L_aT}\,,\quad \theta\leq t\leq T\,.\tag{6.5}$$

Proof. We first suppose that b and β are C_b^1 w.r.t. x. By definition we have

$$\nabla X_t^0 \;=\; 1+\int_0^t \partial_x b(s,X_s^0)\nabla X_s^0 ds\;,\quad 0\le t\le T\;.$$

We get from Gronwall's lemma

$$|\nabla X_t^0| \;\le\; e^{L_a T}\;,\qquad 0\le t\le T\;.$$

In the same way, we have

$$\nabla^\theta X_t^1(\theta) \;=\; 1+\int_\theta^t \partial_x b(s,X_s^1(\theta))\nabla^\theta X_s^1(\theta) ds\;,\qquad \theta\le t\le T\;,$$

and from Gronwall's lemma we get

$$|\nabla^\theta X_t^1| \;\le\; e^{L_a T}\;,\quad \theta\le t\le T\;.$$

Finally we prove the last inequality. By definition

$$\nabla X_t^1(\theta) \;=\; 1+\int_0^t \partial_x b(s,X_s^1(\theta))\nabla X_s^1(\theta) ds+\partial_x\beta(\theta,X_\theta^0)\nabla X_\theta^0\;,\qquad \theta\le t\le T\;.$$

Using the inequality (6.3), we get

$$|\nabla X_t^1(\theta)| \;\le\; 1+L_a e^{L_a T}+\int_0^t L_a|\nabla X_s^1(\theta)| ds\;,\qquad \theta\le t\le T\;,$$

from Gronwall's lemma we get

$$|\nabla X_t^1(\theta)| \;\le\; (1+L_a e^{L_a T})e^{L_a T}\;,\quad \theta\le t\le T\;.$$

When b and β are not differentiable, we can also prove the result by regularization. We consider a density q which is C_b^∞ on $\mathbb{R}$ with a compact support, and we define an approximation $(b^\epsilon,\beta^\epsilon)$ of (b,β) in C_b^1 by

$$(b^\epsilon,\beta^\epsilon)(t,x) \;=\; \frac{1}{\varepsilon}\int_{\mathbb{R}}(b,\beta)(t,x')q\Big(\frac{x-x'}{\epsilon}\Big)dx'\;,\quad (t,x)\in[0,T]\times\mathbb{R}\;.$$

We then use the convergence of $(X^{0,\epsilon},X^{1,\epsilon}(\theta))$ to $(X^0,X^1(\theta))$ and we get the result. □

6.3 Some bounds about Y^0 and Y^1

In this part, we give some bounds about the processes Y^0 and Y^1 which are used to get a uniform bound for the processes Z^0 and Z^1.

Lemma 6.3. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then, for any $\theta\in[0,T]$*

$$|\nabla^\theta Y_t^1(\theta)| \;:=\; \Big|\frac{\partial Y_t^1(\theta)}{\partial X_\theta^1(\theta)}\Big| \;\le\; e^{(L_a+K_f)T}(K_g+TK_f)\;,\quad \theta\le t\le T\;. \tag{6.6}$$

Proof. We first suppose that b, f and g are C_b^1 w.r.t. x, y and z. In this case $(X^1(\theta),Y^1(\theta),Z^1(\theta))$ is also differentiable w.r.t. $X_\theta^1(\theta)$ and we have

$$\begin{aligned}\nabla^\theta Y_t^1(\theta) \;=\;& \nabla g(X_T^1(\theta))\nabla^\theta X_T^1(\theta)-\int_t^T \nabla^\theta Z_s^1(\theta)dW_s \\ &+\int_t^T \nabla f(s,X_s^1(\theta),Y_s^1(\theta),Z_s^1(\theta),0)\big(\nabla^\theta X_s^1(\theta),\nabla^\theta Y_s^1(\theta),\nabla^\theta Z_s^1(\theta)\big)ds\;,\end{aligned} \tag{6.7}$$

for $t \in [\theta, T]$. Define the process $R(\theta)$ by

$$R_t(\theta) \;:=\; \exp\Big(\int_0^t \partial_y f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta} ds\Big)\;, \quad 0 \leq t \leq T\;.$$

Applying Itô's formula, we get

$$\begin{aligned} R_t(\theta)\nabla^\theta Y_t^1(\theta) &= R_T(\theta)\nabla g(X_T^1(\theta))\nabla^\theta X_T^1(\theta) \\ &\quad + \int_t^T R_s(\theta)\partial_x f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\nabla^\theta X_s^1(\theta) ds \\ &\quad - \int_t^T R_s(\theta)\nabla^\theta Z_s^1(\theta) dW_s^1(\theta)\;, \quad \theta \leq t \leq T\;, \end{aligned} \tag{6.8}$$

where the process $W^1(\theta)$ is defined by

$$W_t^1(\theta) := W_t - \int_0^t \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta} ds \tag{6.9}$$

for $t \in [0, T]$. From **(HBQD)**, there exists a constant $K > 0$ such that we have

$$\begin{aligned} \Big\| \int_0^\cdot \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta} dW_s \Big\|^2_{BMO_{\mathbb{F}}[0,T]} &\leq \\ K\Big(1 + \sup_{\vartheta\in\mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\Big[\int_\vartheta^T |Z_s^1(\theta)|^2 \mathbb{1}_{s\geq\theta} ds \Big| \mathcal{F}_\vartheta\Big]\Big) &\leq \\ K\Big(1 + \Big\| \int_0^\cdot Z_s^1(\theta)\mathbb{1}_{s\geq\theta} dW_s \Big\|^2_{BMO_{\mathbb{F}}[0,T]}\Big) &< \infty\;, \end{aligned}$$

where the last inequality comes from Lemma 6.2.

Hence by Lemma 6.1 the process $\mathcal{E}(\int_0^\cdot \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta} dW_s)$ is a uniformly integrable martingale. Therefore, under the probability measure $\mathbb{Q}^1(\theta)$ defined by

$$\frac{d\mathbb{Q}^1(\theta)}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \;:=\; \mathcal{E}\Big(\int_0^\cdot \partial_z f(s, X_s^1(\theta), Y_s^1(\theta), Z_s^1(\theta), 0)\mathbb{1}_{s\geq\theta} dW_s\Big)_t\;, \quad 0 \leq t \leq T\;,$$

we can apply Girsanov's theorem and $W^1(\theta)$ is a Brownian motion under the probability measure $\mathbb{Q}^1(\theta)$. We then get from (6.8)

$$R_t(\theta)\nabla^\theta Y_t^1 \;=\; \mathbb{E}_{\mathbb{Q}^1(\theta)}\Big[R_T(\theta)\nabla g(X_T^1)\nabla^\theta X_T^1 + \int_t^T R_s(\theta)\partial_x f(s, X_s^1, Y_s^1, Z_s^1, 0)\nabla^\theta X_s^1 ds \Big| \mathcal{F}_t\Big]\;.$$

This last equality, **(HBQD)** and (6.4) give

$$|\nabla^\theta Y_t^1(\theta)| \;\leq\; e^{(L_a+K_f)T}(K_g + TK_f)\;, \quad \theta \leq t \leq T\;. \tag{6.10}$$

When b, f and g are not C_b^1, we can also prove the result by regularization as for Proposition 6.1. □

Lemma 6.4. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then,*

$$|\nabla Y_t^1(t)| \;\leq\; (1 + L_a e^{L_a T})e^{(L_a+K_f)T}(K_g + TK_f)\;, \quad 0 \leq t \leq T\;. \tag{6.11}$$

Proof. Firstly, we suppose that b, β, g and f are C_b^1 w.r.t. x, y and z. Then, for any $t \in [0, T]$

$$\begin{aligned} \nabla Y_t^1(t) \;:=\; \frac{\partial Y_t^1(t)}{\partial x} &= \nabla g(X_T^1(t))\nabla X_T^1(t) \\ &\quad + \int_t^T \nabla f(s, X_s^1(t), Y_s^1(t), Z_s^1(t), 0)(\nabla X_s^1(t), \nabla Y_s^1(t), \nabla Z_s^1(t)) ds \\ &\quad - \int_t^T \nabla Z_s^1(t) dW_s\;. \end{aligned} \tag{6.12}$$

Applying Itô's formula, we get

$$\begin{aligned} R_t(t)\nabla Y_t^1(t) &= R_T(t)\nabla g(X_T^1(t))\nabla X_T^1(t) \\ &\quad + \int_t^T R_s(t)\partial_x f(s, X_s^1(t), Y_s^1(t), Z_s^1(t), 0)\nabla X_s^1(t)ds \\ &\quad - \int_t^T R_s(t)\nabla Z_s^1(t)dW_s^1(t)\ , \quad 0 \le t \le T\ , \end{aligned}$$

where the process $W^1(.)$ is defined in (6.9). We have proved previously that $W^1(t)$ is a Brownian motion under the probability measure $\mathbb{Q}^1(t)$. We then get

$$R_t(t)\nabla Y_t^1(t) = \mathbb{E}_{\mathbb{Q}^1(t)}\Big[R_T(t)\nabla g(X_T^1(t))\nabla X_T^1(t) + \int_t^T R_s(t)\partial_x f(s, X_s^1(t), Y_s^1(t), Z_s^1(t), 0)\nabla X_s^1(t)ds\Big|\mathcal{F}_t\Big]\ .$$

This last inequality, **(HBQD)** and (6.4) give

$$|\nabla Y_t^1(t)| \le (1 + L_a e^{L_a T})e^{(L_a+K_f)T}(K_g + TK_f)\ , \quad 0 \le t \le T\ .$$

When b, f and g are not C_b^1, we can also prove the result by regularization as for Proposition 6.1. □

Lemma 6.5. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then,*

$$|\nabla Y_t^0| \le e^{(2K_f+L_a)T}(K_g + K_f T)\big(1 + TK_f e^{K_f T}(1 + L_a e^{L_a T})\big)\ , \quad 0 \le t \le T\ .$$

Proof. We first suppose that b, β, g and f are C_b^1 w.r.t. x, y, z and u, then (X^0, Y^0, Z^0) is differentiable w.r.t. x and we have

$$\begin{aligned} \nabla Y_t^0 &= \nabla g(X_T^0)\nabla X_T^0 \\ &\quad + \int_t^T \Big(\nabla f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)(\nabla X_s^0, \nabla Y_s^0, \nabla Z_s^0, \nabla Y_s^1(s) - \nabla Y_s^0)ds \\ &\quad - \int_t^T \nabla Z_s^0 dW_s\ , \quad 0 \le t \le T\ . \end{aligned}$$

Define the process R^0 by

$$R_t^0 := \exp\Big(\int_0^t (\partial_y - \partial_u)f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)ds\Big)\ , \quad 0 \le t \le T\ .$$

Applying Itô's fomula we have

$$\begin{aligned} R_t^0\nabla Y_t^0 &= R_T^0\nabla g(X_T^0)\nabla X_T^0 \\ &\quad + \int_t^T R_s^0\partial_x f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla X_s^0 ds \\ &\quad + \int_t^T R_s^0\partial_u f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla Y_s^1(s) ds \\ &\quad - \int_t^T R_s^0\nabla Z_s^0 dW_s^0 \end{aligned}$$

where $dW_s^0 := dW_s - \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)ds$. From **(HBQD)**, there exists a constant $K > 0$ such that we have

$$\begin{aligned} \Big\|\int_0^\cdot \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)dW_s\Big\|^2_{BMO_{\mathbb{F}}[0,T]} &\le \\ K\Big(1 + \sup_{\vartheta \in \mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\Big[\int_\vartheta^T |Z_s^0|^2 ds\Big|\mathcal{F}_\vartheta\Big]\Big) &\le \\ K\Big(1 + \Big\|\int_0^\cdot Z_s^0 dW_s\Big\|^2_{BMO_{\mathbb{F}}[0,T]}\Big) &< \infty\ , \end{aligned}$$

where the last inequality comes from Lemma 6.2.

Hence by Lemma 6.1 the process $\mathcal{E}(\int_0^\cdot \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)dW_s)$ is a uniformly integrable martingale. Therefore, under the probability measure $\mathbb{Q}^0$ defined by

$$\frac{d\mathbb{Q}^0}{d\mathbb{P}}\Big|_{\mathcal{F}_t} := \mathcal{E}\Big(\int_0^\cdot \partial_z f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)dW_s\Big)_t$$

we can apply Girsanov's theorem and W^0 is a Brownian motion under the probability measure $\mathbb{Q}^0$. Then, we get

$$\begin{aligned} R_t^0 \nabla Y_t^0 &= \mathbb{E}_{\mathbb{Q}^0}\Big[R_T^0 \nabla g(X_T^0)\nabla X_T^0 \\ &\quad + \int_t^T R_s^0 \partial_x f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla X_s^0 ds \\ &\quad + \int_t^T R_s^0 \partial_u f(s, X_s^0, Y_s^0, Z_s^0, Y_s^1(s) - Y_s^0)\nabla Y_s^1(s) ds \Big| \mathcal{F}_t\Big] . \end{aligned}$$

Using inequalities (6.3) and (6.11) we get

$$|\nabla Y_t^0| \leq e^{(2K_f + L_a)T}(K_g + K_f T)\big(1 + K_f T e^{K_f T}(1 + L_a e^{L_a T})\big) , \quad 0 \leq t \leq T .$$

When b, β, f and g are not C_b^1, we can also prove the result by regularization as for Proposition 6.1. □

6.4 A uniform bound for Z^0 and Z^1

Using the previous bounds, we obtain a uniform bound for the processes Z^0 and Z^1.

Proposition 6.2. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then, for any $\theta \in [0, T]$, there exists a version of $Z^1(\theta)$ such that*

$$|Z_t^1(\theta)| \leq e^{(2L_a + K_f)T}(K_g + TK_f)K_a , \quad \theta \leq t \leq T .$$

Proof. Using Malliavin calculus, we have the classical representation of the process $Z^1(\theta)$ given by $\nabla^\theta Y^1(\theta)(\nabla^\theta X^1(\theta))^{-1}\sigma(.)$ (see Section 5). In the case where b, f and g are C_b^1 w.r.t. x, y and z, we obtain from (6.10)

$$|Z_t^1(\theta)| \leq e^{(2L_a + K_f)T}(K_g + TK_f)K_a \quad a.s.$$

since $|(\nabla^\theta X^1(\theta))^{-1}| \leq e^{L_a T}$ (the proof of this inequality is similar to the one of (6.4)).
When b, f and g are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

Proposition 6.3. *Suppose that **(HF)**, **(HBQ)** and **(HBQD)** hold. Then, there exists a version of Z^0 such that*

$$|Z_t^0| \leq e^{2(K_f + L_a)T}(K_g + K_f T)\big(1 + TK_f e^{K_f T}(1 + L_a e^{L_a T})\big)K_a , \quad 0 \leq t \leq T .$$

Proof. Thanks to the Malliavin calculus, it is classical to show that a version of Z^0 is given by $\nabla Y^0(\nabla X^0)^{-1}\sigma(.)$ (see Section 5). So, in the case where b, β, g and f are C_b^1 w.r.t. x, y, z and u, we obtain from (6.3) and Lemma 6.5

$$|Z_t^0| \leq e^{2(K_f + L_a)T}(K_g + K_f T)\big(1 + TK_f e^{K_f T}(1 + L_a e^{L_a T})\big)M_\sigma \quad a.s.$$

since $|(\nabla X_t^0)^{-1}| \leq e^{L_a T}$ (the proof of this inequality is similar to the one of (6.3)).

When b, β, g and f are not differentiable, we can also prove the result by a standard approximation and stability results for BSDEs with linear growth. □

6.5 Convergence of the scheme for the BSDE

Theorem 6.1. *Under* ***(HF)***, ***(HFD)***, ***(HBQ)*** *and* ***(HBQD)*** *we have the following estimate*

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\big|Y_t-Y_t^\pi\big|^2\Big]+\mathbb{E}\Big[\int_0^T\big|Z_t-Z_t^\pi\big|^2dt\Big]+\mathbb{E}\Big[\int_0^T\lambda_t\big|U_t-U_t^\pi\big|^2dt\Big] \quad\leq\quad K|\pi|\,,$$

for a constant K which does not depend on π.

Proof. Fix $M\in\mathbb{R}$ such that

$$\begin{aligned} M \quad\geq\quad & \max\Big\{e^{(2L_a+K_f)T}(K_g+TK_f)K_a\;; \\ & \qquad e^{2(K_f+L_a)T}(K_g+K_fT)\big(1+TK_fe^{K_fT}(1+L_ae^{L_aT})\big)K_a\Big\}\,, \end{aligned}$$

and define the function $\tilde f$ by

$$\tilde f(t,x,y,z,u) \quad=\quad f(t,x,y,\varphi_M(z),u)\,,\quad (t,x,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\,,$$

where

$$\varphi_M(z):=\left\{\begin{array}{ccc} z & \text{if} & |z|\leq M \\ M\frac{z}{|z|} & \text{if} & |z|>M \end{array}\right.\,,\quad z\in\mathbb{R}\,.$$

We notice that φ_M is Lipschitz continuous and bounded. Therefore we obtain from **(HBQD)** that $\tilde f$ is Lipschitz continuous.

Moreover, using Propositions 6.2 and 6.3, we get that under **(HF)**, **(HBQ)** and **(HBQD)**, (X,Y,Z) is also solution to the Lipschitz FBSDE

$$\begin{aligned} X_t \quad=\quad & x+\int_0^t b(s,X_s)ds+\int_0^t\sigma(s,X_s)dW_s+\int_0^t\beta(s,X_{s^-})dH_s\,,\quad 0\leq t\leq T\,, \\ Y_t \quad=\quad & g(X_T)+\int_t^T\tilde f\big(s,X_s,Y_s,Z_s,U_s(1-H_s)\big)ds \\ & \qquad -\int_t^T Z_sdW_s-\int_t^T U_sdH_s\,,\quad 0\leq t\leq T\,. \end{aligned}$$

Applying Theorem 5.1, we get the result. □

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