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Pseudo-Orbits, Stationary Measures and Metastability

W. Bahsoun, H. Hu and S. Vaienti

Abstract We study random perturbations of multidimensional piecewise expanding maps. We characterize absolutely continuous stationary measures (acsm) of randomly perturbed dynamical systems in terms of pseudo-orbits linking the ergodic components of absolutely continuous invariant measures (acim) of the unperturbed system. We focus on those components, called *least-elements*, which attract pseudo-orbits. We show that each least element is in a one-to-one correspondence with an ergodic acsm of the random system. Moreover our result permits to identify random perturbations that exhibit a metastable behavior.

1 Introduction

In this chapter we study statistical aspects of random perturbations of a multidimensional piecewise expanding map[1] $T : M \to M$, $M \subset \mathbb{R}^q$. A random orbit $\{x_n^\varepsilon\}_n$ is a random process where, for all n, x_{n+1}^ε is a random variable whose possible values are obtained in an ε-neighbourhood of Tx_n^ε according to a transition proba-

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[1] See [16] and [22] for recent results and for an exhaustive list of references on deterministic multidimensional expanding maps.

bility $P_\varepsilon(x_n,.)$. We consider the case where $P_\varepsilon(x_n,.)$ is absolutely continuous with respect to Lebesgue measure m. Denoting the density of the transition probability by $p_\varepsilon(x,.)$, we define a perturbed transfer operator , $\mathscr{L}_\varepsilon$, by:

$$\mathscr{L}_\varepsilon f(x) = \int_M p_\varepsilon(y,x) f(y) dy$$

where $f \in L^1_m(M)$. We focus on non-invertible dynamical systems whose transfer operators, perturbed and non-perturbed[2], satisfy a uniform Lasota-Yorke inequality. Among other things, such an inequality implies the existence of a finite number of ergodic acim for the initial system T, and a finite number of ergodic acsm for the random system.

We then define an equivalence relation between the ergodic components of acim of T using *pseudo-orbits*. Using this equivalence relation, we consequently introduce equivalence classes of ergodic acim. Among the latter, we identify those which attract pseudo-orbits and call them *least elements*. We show that each least element admits a neighbourhood which supports exactly one ergodic acsm of the random system, and the converse is also true, namely the support of any ergodic acsm contains only one least element. This result allows us to identify the *attractors* of the random orbits, namely the least elements, and we give a nice illustration in the example 2 in Section 5. Moreover, we use our result to identify random perturbations that exhibit a *metastable behavior.* Such a phenomenon has recently been a very active topic of research in both ergodic theory [3, 4, 11, 13, 14, 19] and applied dynamical systems [9, 25].

Section 2 contains the setup of the problem, our assumptions, the notion of a least element and the statement of our main result (Theorem 1). Section 3 contains the proof of Theorem 1. In Section 4 we use the results of the previous sections to identify random systems which exhibit a metastable behavior. In Section 5 we apply our results to random transformations, in particular, we provide examples to illustrate the results of Sections 2 and 4.

2 Setup and statement of the main result

2.1 The initial system

Let $M \subset \mathbb{R}^q$ be compact[3] with $\overline{intM} = M$. We denote by d the Euclidean distance on $\mathbb{R}^q$. Let $(M, \mathscr{A}, m)$ be the measure space, where $\mathscr{A}$ is Borel σ-algebra, and m is the normalized Lebesgue measure on M; i.e, $m(M) = 1$.

[2] In this chapter, the non-perturbed operator, $\mathscr{L}$, is the traditional transfer operator (Perron-Frobenius) associated with the map T [5].

[3] All our results can be carried to the case where M is a compact Riemannian manifold.

Let $T : M \to M$ be a measurable map, $\mathscr{D}_T$ be the set of discontinuities of T[4]. We assume that $m(\mathscr{D}_T) = 0$, and T is non-singular with respect to m. The transfer operator (Perron-Frobenius) [5] associated with T, $\mathscr{L} : L^1_m \hookrightarrow L^1_m$ is defined by duality: for $g_1 \in L^1_m$ and $g_2 \in L^\infty_m$

$$\int_M g_1 g_2 \circ T dm = \int_M \mathscr{L} g_1 g_2 dm.$$

2.2 The perturbed system

We perturb the map T by introducing a family of Markov chains $(\mathscr{X}^\varepsilon_n)$, $\varepsilon > 0, n \geq 0$ with state space M and transition probabilities $\{P_\varepsilon(x,.)\}_{x\in M}$; i.e, $P_\varepsilon(x,A)$ is the probability that a point x is mapped into a measurable set A. At time $n = 0$, $(\mathscr{X}^\varepsilon_0)$ can have any probability distribution. We assume that:

(P1) For all $x \in M$, $P_\varepsilon(x,.)$ is absolutely continuous with respect to the Lebesgue measure. We will denote its density by $p_\varepsilon(x,.)$. Therefore $P_\varepsilon(\mathscr{X}^\varepsilon_{n+1} \in A | \mathscr{X}^\varepsilon_n = x) = \int_A p_\varepsilon(x,y)dm(y)$.

(P2) We have: $\text{supp} P_\varepsilon(x,.) = B_\varepsilon(Tx)$ for all x.

Assumption (P2) can be weakened by supposing that $\text{supp} P_\varepsilon(x,.)$ is a subset of a slightly larger ball around Tx. Our proofs can be easily adapted to show that the results of this chapter will still hold under such a slightly weakened assumption.

2.2.1 Random orbits and stationary measures

The perturbed evolution of a state $x \in M$ will be represented by a *random orbit*:

Definition 1. A sequence $\{x^\varepsilon_n\}_{n\geq 0} \subset M$ is an ε*-random orbit* if each x^ε_{n+1} is a random variable whose distribution is $P_\varepsilon(x^\varepsilon_n,.)$, namely $\{x^\varepsilon_n\}_{n\geq 0}$ coincides with the Markov chains $(\mathscr{X}^\varepsilon_n)$, $\varepsilon > 0, n \geq 0$.

The counterpart of an invariant measure in the case of randomly perturbed dynamical systems is called a stationary measure:

Definition 2. A probability measures μ_ε is called a *stationary measure* if for any $A \in \mathscr{A}$

$$\mu_\varepsilon(A) = \int_M P_\varepsilon(x,A) d\mu_\varepsilon(x).$$

We call it an absolutely continuous stationary measure (*acsm*), if it has a density with respect to the Lebesgue measure (see below).

[4] From now on with the notation $\mathscr{D}_g$ we mean the set of discontinuities of the function g.

2.2.2 The transfer operator of the random system

To study Markov processes it is useful to define the transition operator $\mathscr{T}_\varepsilon$ acting on bounded real-valued measurable functions g defined on M:

$$(\mathscr{T}_\varepsilon g)(x) = \int_M g(y) P_\varepsilon(x,dy).$$

Its adjoint $\mathscr{T}^*$ is defined on the space $\mathscr{M}(M)$ of all finite signed measures and is given by:

$$\mathscr{T}_\varepsilon^* \mu(A) = \int_M P_\varepsilon(x,A) d\mu(x).$$

The measure μ_ε is stationary if and only if $\mathscr{T}_\varepsilon^* \mu_\varepsilon = \mu_\varepsilon$. Moreover, a stationary measure is ergodic if, any $A \in \mathscr{A}$ with $\mathscr{T}_\varepsilon 1_A = 1_A$ implies that $\mu_\varepsilon(A) = 0$ or $\mu_\varepsilon(A) = 1$.

Since condition (P1) implies the absolute continuity of any stationary measure, it will be convenient to define an operator, $\mathscr{L}_\varepsilon$, acting on densities. That is to say, if μ is an absolutely continuous measure with respect to m, whose density is a function $g \in L^1_m$, then $\mathscr{T}_\varepsilon^* \mu$ is an absolutely continuous measure whose density is $\mathscr{L}_\varepsilon g$, where $\mathscr{L}_\varepsilon g$ is given by:

$$\mathscr{L}_\varepsilon g(x) = \int_M p_\varepsilon(z,x) g(z) dz. \tag{1}$$

Thus, densities of acsm are fixed points of $\mathscr{L}_\varepsilon$. We will comment again about this definition of the random transfer operator in Remark 5, Section 5.

2.3 A Banach space and quasi-compactness of $\mathscr{L}$ and $\mathscr{L}_\varepsilon$

We now introduce a Banach space $\mathscr{B}(M) \subset \mathscr{L}^1_m$. We assume that

(B1) The constant function 1 belongs to $\mathscr{B}(M)$.
(B2) The set of discontinuities, $\mathscr{D}_f$, of any function $f \in \mathscr{B}(M)$ has Lebesgue measure zero.
(B3) There is a semi-norm $|.|$ on $\mathscr{B}(M)$ such that the unit ball of $\mathscr{B}(M)$ is compact in $\mathscr{L}^1_m$ with respect to the complete norm $||\cdot||_B \equiv |.| + \|.\|_1$, where $\|.\|_1$ denotes the L^1_m norm.

We also assume that the transfer operators $\mathscr{L}$ and $\mathscr{L}_\varepsilon$ satisfy a uniform Lasota-Yorke inequality: there exist an $\eta \in (0,1)$ and a $D \in (0,\infty)$ such that for all $f \in \mathscr{B}(M)$ and $\varepsilon > 0$ small enough:

$$|\mathscr{L} f| \leq \eta\, |f| + D \|f\|_1\,; \tag{LY}$$

$$|\mathscr{L}_\varepsilon f| \leq \eta\, |f| + D \|f\|_1\,. \tag{RLY}$$

Assumptions (LY) and (RLY) ensure the quasi-compactness of both $\mathscr{L}$ and $\mathscr{L}_\varepsilon$, see [5] and [2] for non-invertible systems and [10] and reference therein for invertible

ones in higher dimensions. In particular, among other things, (LY) implies the existence of a finite number of T-ergodic acim, and (RLY) implies the existence of a finite number of ergodic acsm for the Markov process $(\mathscr{X}_n^\varepsilon)$. More precisely, we have for the operator $\mathscr{L}$ (see [7, 15]):

- The subspace of non-negative fixed points w of $\mathscr{L}$, is a convex set with a finite number of extreme points $w_1, \cdots, w_l$ with supports $\Lambda_k, k = 1, \cdots, l$. The supports $\Lambda_k, k = 1, \cdots, l$, are mutually disjoint Lebesgue a.e..
- The measures $\mu_1 = w_1 m, \cdots, \mu_l = w_l m$ are ergodic and they give the ergodic decomposition of any acim $\mu = hm$, $h \in \mathscr{L}_m^1$. We also call them the extreme (ergodic) points (measures) decomposing μ.

The Lasota-Yorke inequality (RLY) of $\mathscr{L}_\varepsilon$ ensures that the random system admits finitely many ergodic acsm $\mu_1^\varepsilon = h_{\varepsilon,1} m, \ldots, \mu_K^\varepsilon = h_{\varepsilon,K} m$. Note that in general the number K of extreme points for μ_ε is different from the number l of ergodic components of μ. In our setting, (see Corollary 1), we show that the number of ergodic ascm is bounded above by the number of ergodic acim.

Remark 1. We point out that for certain perturbations one can prove that (RLY) follows from (LY). See [6, 17] for precise examples.

A well known consequence of assumption (LY) and (RLY) is the following proposition, see for instance [8].

Proposition 1. . *Let $\{h_\varepsilon\}_{\varepsilon>0}$ be a family of densities of absolutely continuous stationary measures of $(\mathscr{X}_\varepsilon^n)$. Then any limit point, as $\varepsilon \to 0$, of $\{h_\varepsilon\}_{\varepsilon>0}$ in the L_m^1-norm is a density of T-acim.*

2.4 Pseudo-orbits, least elements and the statement of the main result

We now introduce the notion of a pseudo orbit which will be our main tool to characterize ergodic acsm. Pseudo-orbits were previously used by Ruelle [23], followed by Kifer [20], to study attractors of randomly perturbed smooth maps.

Definition 3 (pseudo-orbit). For $\varepsilon > 0$, an ε-*pseudo-orbit* is a finite set $\{x_i\}_{i=0}^n \subset M$ such that $d(Tx_i, x_{i+1}) < \varepsilon$ for $i = 0 \ldots n-1$.

Using pseudo-orbits, we define a pre-order (reflexive and transitive) "$\to$" among the supports $\{\Lambda_i\}$ of T-ergodic acim by writing $\Lambda_i \to \Lambda_j$ if for any $\varepsilon > 0$ there is an ε-pseudo-orbit $\{x_i\}_{i=0}^k$ such that $x_0 \in \Lambda_i$ and $x_k \in \Lambda_j$. Then we define a relation "$\sim$" among $\{\Lambda_i\}$ by writing $\Lambda_i \sim \Lambda_j$ if both $\Lambda_i \to \Lambda_j$ and $\Lambda_j \to \Lambda_i$. By ergodicity, given a point $y \in \Lambda_i \equiv \mathrm{supp}\mu_i$ and $\varepsilon > 0$, μ_i-almost any point $x \in \Lambda_i$ will enter the ball $B_\varepsilon(y)$ of positive μ_i measure, and therefore all the pairs $(x, y) \in \Lambda_i$ can be connected with a (finite) ε-pseudo-orbit. Hence we get an equivalence relation among those ergodic components and we define $\tilde{\Lambda}_i$ the equivalence class which contains Λ_i. We write $\tilde{\Lambda}_i \to \tilde{\Lambda}_j$ if $\Lambda_k \to \Lambda_l$ for any $\Lambda_k \in \tilde{\Lambda}_i$ and $\Lambda_l \in \tilde{\Lambda}_j$.

Definition 4. $\tilde{\Lambda}_i$ is said to be a *least element*, if there is no $\tilde{\Lambda}_j$, $j \neq i$, such that $\tilde{\Lambda}_i \to \tilde{\Lambda}_j$.

This in particular means that for all ε small enough no ε-pseudo-orbit can travel from the least element to other equivalence classes. We point out that, in general, a dynamical system may have more than one least-element[5]. We now state our first result:

Theorem 1. *Under Assumption (P1) and (P2), if (LY) and (RLY) hold for functions in a Banach space $\mathscr{B}(M)$ satisfying (B1-B3), then we have:*

1. *If $\tilde{\Lambda}$ is a least element, then for ε small enough there exists an open neighborhood $\mathscr{U}_\varepsilon \supset \tilde{\Lambda}$ which supports a unique ergodic acsm $\mu_\varepsilon^{\tilde{\Lambda}}$.*
2. *If $\tilde{\Lambda}$ is not a least element, then for ε small enough, $\mu_\varepsilon(\tilde{\Lambda}) = 0$ for any acsm μ_ε. Therefore, for any weak-limit of μ_ε as $\varepsilon \to 0$, $\tilde{\Lambda}$ is a set of measure 0.*

Theorem 1 implies the following three corollaries:

Corollary 1. *The number of acsm of the random system $(\mathscr{X}_\varepsilon^n)$ is bounded above by the number of acim of the map T. In particular, if T has a unique acim, then $(\mathscr{X}_\varepsilon^n)$ has a unique acsm.*

We also have the converse of part 1 of Theorem 1, namely:

Corollary 2. *The support of any ergodic acsm contains exactly one least element.*

Therefore, for ε small enough, we can uniquely associate to each least element $\tilde{\Lambda}$ the family of densities $\{h_\varepsilon^{\tilde{\Lambda}}\}_{\varepsilon>0}$.

Definition 5. We say that the system $(M, \mathscr{A}, T)$ is *strongly stochastically stable* if any L_m^1 limit point of the densities of the ergodic absolutely continuous stationary measures $\{\mu_j^\varepsilon\}_{\varepsilon>0}$, $j = 1, \ldots, K$ and as ε goes to zero, is a convex combination of the densities of the absolutely continuous ergodic extreme measures of μ.

In our setting, by Proposition 1 and Theorem 1, any limit point of the family $\{h_\varepsilon^{\tilde{\Lambda}}\}_{\varepsilon>0}$ as ε goes to 0, is a convex combination of the densities of the ergodic measures spanning $\tilde{\Lambda}$[6]. Hence we proved that

Corollary 3. *The system $(M, \mathscr{A}, T)$ is strongly stochastically stable.*

Remark 2. Our definition of stochastic stability is inspired by the definition in Sect. 1.1 of [1]. Whenever there are only one absolutely continuous ergodic invariant measure μ and only one absolutely continuous ergodic stationary measure μ_ε, it makes sense to speak of strongly stochastic stability of the *measure* μ if the density

[5] This will be illustrated in Example 1.

[6] Note that within the general setting of this paper, we do not claim that the values of the weights in the convex combination that determine the limiting density can be easily identified. However, for certain perturbations of one dimensional maps, using insights form open dynamical systems, such weights can be determined. See [14, 3, 4].

of μ converges in L^1_m to the density of μ_ε, see for instance [8]. Adapting this point of view we can restate the previous corollary by saiyng that: *an absolutely continuous invariant measure for the original system $(M,\mathscr{A},T)$ whose support is the union of least elements, is strongly stochastically stable*.

3 Proofs

We first prove a key lemma.

Lemma 1. *Let $f \in \mathscr{B} \equiv \mathscr{B}(M)$ and $\{x_i\}_{i=0}^N \subset M$ be an ε-pseudo-orbit such that $x_j \in M \setminus (\mathscr{D}_{\mathscr{L}_\varepsilon^j f} \cap \mathscr{D}_T)$, $0 \le j \le N$ and $\mathscr{L}_\varepsilon^i f(x_i) > 0$ for some $0 \le i < N$. Then for all $i < k \le N$ we have $\mathscr{L}_\varepsilon^k f(x_k) > 0$.*

Proof. Let $f \in \mathscr{B}$ and $\{x_i\}_{i=0}^N$ be an ε-pseudo-orbit, satisfying the assumptions of the lemma. In particular, suppose that for some fixed $0 \le i < N$, $\mathscr{L}_\varepsilon^i f(x_i) > 0$. Then

$$\mathscr{L}_\varepsilon^{i+1} f(x_{i+1}) = \int_M \mathscr{L}_\varepsilon^i f(y) p_\varepsilon(y, x_{i+1})\, dy.$$

By the hypothesis (P2) we have $x_{i+1} \in B_\varepsilon(Ty) \Rightarrow p_\varepsilon(y, x_{i+1}) > 0$ and by the preceding continuity assumptions there exists $\delta > 0$ such that $y \in B_\delta(x_i) \Rightarrow \mathscr{L}_\varepsilon^i f(y) > 0$. But this δ-neighborhood of x_i can be made smaller in such a way that when y belongs to it, $d(Ty, Tx_i) \le \frac{\varepsilon - d(Tx_i, x_{i+1})}{2}$ which implies that x_{i+1} is ε-close to Ty. Therefore for all the y in this δ-neighborhood (which is of positive Lebesgue measure), the integrand above is strictly positive and this finishes the proof of the Lemma. □

Proof (of Theorem 1). We first show that for every least element $\tilde{\Lambda}$ there exists a neighborhood $\mathscr{U}_\varepsilon$ such that $T(\mathscr{U}_\varepsilon) \subset \mathscr{U}_\varepsilon$. We denote by $B_\varepsilon(A)$ the (open) ε-neighborhood of a set A, that is, $B_\varepsilon(A) = \{x \in M : d(x,A) < \varepsilon\}$. We observe that even though a least element $\tilde{\Lambda}$ is forward invariant, the image of a ball of radius ε centred at a point in $\tilde{\Lambda}$ may not be necessarily contained in $\tilde{\Lambda}$. However, this ball will surely be a subset of the open set $\mathscr{U}_{\varepsilon,1} := B_\varepsilon(T\tilde{\Lambda})$. We define inductively a family of nested open sets $\mathscr{U}_{\varepsilon,n} := B_\varepsilon(T\mathscr{U}_{\varepsilon,n-1})$ and consider the open neighborhood $\mathscr{U}_\varepsilon$ of $\tilde{\Lambda}$ defined by $\mathscr{U}_\varepsilon := \cup_{n=1}^\infty \mathscr{U}_{\varepsilon,n}$. This set is clearly forward invariant under T, and its closure is disjoint, for ε small enough, from the supports of the other ergodic acim; otherwise, we can construct an ε pseudo-orbit linking the least element to them.

Let $\mathscr{B}(\mathscr{U}_\varepsilon) := \{f \in \mathscr{B} | f \text{ is supported on } \mathscr{U}_\varepsilon\}$. Since $\mathscr{L}_\varepsilon f(x) = \int_M p_\varepsilon(y,x) f(y) dy$, the forward invariance of $\mathscr{U}_\varepsilon$ together with $x \in B_\varepsilon(Ty)$, insure that $\mathscr{L}_\varepsilon$ leaves the Banach space $\mathscr{B}(\mathscr{U}_\varepsilon)$ invariant. Then by applying on $\mathscr{B}(\mathscr{U}_\varepsilon)$ the Lasota-Yorke inequality and successively the Ionescu-Tulcea-Marinescu spectral theorem, (see for instance [7, 15],) we obtain a fixed point $\mathscr{L}_\varepsilon h_\varepsilon = h_\varepsilon \in \mathscr{B}(\mathscr{U}_\varepsilon)$: the measure $\mu_\varepsilon = h_\varepsilon m$, $\int_{\mathscr{U}_\varepsilon} h_\varepsilon dm = 1$ is therefore stationary.

We now prove that h_ε is the only fixed point of $\mathscr{L}_\varepsilon$ in $\mathscr{B}(\mathscr{U}_\varepsilon)$. Suppose there is another function $h'_\varepsilon \in \mathscr{B}(\mathscr{U}_\varepsilon))$ with the same property, and let us define the

function $\hat{h} = \min(h_\varepsilon, h'_\varepsilon)$. Then clearly: $\min(h_\varepsilon, h'_\varepsilon) \le h_\varepsilon$ and $\min(h_\varepsilon, h'_\varepsilon) \le h'_\varepsilon$ and thus $\mathscr{L}_\varepsilon(\min(h_\varepsilon, h'_\varepsilon)) \le \mathscr{L}_\varepsilon h_\varepsilon$ and $\mathscr{L}_\varepsilon(\min(h_\varepsilon, h'_\varepsilon)) \le \mathscr{L}_\varepsilon h'_\varepsilon$ which implies $\mathscr{L}_\varepsilon(\min(h_\varepsilon, h'_\varepsilon)) \le \min(\mathscr{L}_\varepsilon h_\varepsilon, \mathscr{L}_\varepsilon h'_\varepsilon)$. But $\min(\mathscr{L}_\varepsilon h_\varepsilon, \mathscr{L}_\varepsilon h'_\varepsilon) = \min(h_\varepsilon, h'_\varepsilon) = \hat{h}$, so $\mathscr{L}_\varepsilon \hat{h} \le \hat{h}$ and therefore $\mathscr{L}_\varepsilon \hat{h} = \hat{h}$. Let us consider $h_\varepsilon - \hat{h}$. It is a nonnegative function and it satisfies $\mathscr{L}_\varepsilon(h_\varepsilon - \hat{h}) = h_\varepsilon - \hat{h}$. By Proposition 1 and by taking ε small enough, we insure that the supports of h_ε and h'_ε will intersect the least element in a Borel set B of positive Lebesgue measure. Starting from almost any point in this set we can attain any other point in $\mathscr{U}_\varepsilon$ with a (finite) ε-pseudo-obit [7]. Take any $x_0 \in B \setminus (\mathscr{D}_{h_\varepsilon - h} \cap \mathscr{D}_T)$ such that $(h_\varepsilon - \hat{h})(x_0) > 0$. For any point $x \in \mathscr{U}_\varepsilon \setminus (\mathscr{D}_{h_\varepsilon - h} \cap \mathscr{D}_T)$ there is an ε-pseudo-orbit $\{x_i\}_{i=0}^N \subset M \setminus (\mathscr{D}_{h_\varepsilon - h} \cap \mathscr{D}_T)$ which starts from x_0 and lands at $x = x_N$. Hence we can apply Lemma 1 with $f = h_\varepsilon - \hat{h}$ to get $\mathscr{L}_\varepsilon^k(h_\varepsilon - \hat{h})(x_k) = (h_\varepsilon - \hat{h})(x_k) > 0$, $k = 1, \cdots, N$. This implies that $(h_\varepsilon - \hat{h})(x) > 0$ for all $x \in \mathscr{U}_\varepsilon \setminus (\mathscr{D}_{h_\varepsilon - h} \cap \mathscr{D}_T)$, that is $h_\varepsilon > h'_\varepsilon$ almost everywhere on $\mathscr{U}_\varepsilon$, contradicting the fact that $\int h_\varepsilon dm = \int h'_\varepsilon dm = 1$. Therefore $h_\varepsilon = h'_\varepsilon$ almost everywhere on $\mathscr{U}_\varepsilon$. We get part 1 of the theorem.

Now we prove part 2. Suppose $\tilde{\Lambda}_0$ is not a least element. We will show that $h'_\varepsilon = 0$ on $\tilde{\Lambda}_0$ for the density h'_ε be the density of any acsm. We proceed by contradiction. In this case we have $\tilde{\Lambda}_0 \to \tilde{\Lambda}$ for some least element $\tilde{\Lambda}$, if otherwise $\tilde{\Lambda}_0$ would be a least element itself. So there is an ε-pseudo-orbit which starts from $\tilde{\Lambda}_0$ and ends up in $\tilde{\Lambda}$. Let h_ε denote the density of the unique acsm supported on $\tilde{\Lambda}$. If $h'_\varepsilon \neq 0$, we can invoke again the arguments of Lemma 1 and part 1 of this theorem to conclude that $h_\varepsilon > 0$ on $\tilde{\Lambda}$ too and also that $\min(h_\varepsilon, h'_\varepsilon)$ is a fixed point of $\mathscr{L}_\varepsilon$. But the support of such a minimum is a subset of $\mathscr{U}_\varepsilon$ since for ε small enough, $h_\varepsilon = 0$ outside $\mathscr{U}_\varepsilon$. Then by uniqueness of the density h_ε over $\mathscr{U}_\varepsilon$ we get that $h_\varepsilon = \min(h_\varepsilon, h'_\varepsilon)$. This implies that $\int_M h'_\varepsilon dm > 1$, which is false. □

Proof (of Corollary 3). By Proposition 1 and part 2 of Theorem 1, for ε small enough, the support of any ergodic acsm μ_1^ε intersects the support of a least element. By part 1 of Theorem 1 a small neighborhood of this least element supports an ergodic acsm μ_2^ε. By repeating the arguments of the proof of part 2 of Theorem 1 we obtain that those two ergodic acsm must coincide. Finally the unicity of the least element inside the support of μ_1^ε follows from the fact that two disjoints least elements are at strictly positive distance and therefore they cannot share the same ergodic acsm. □

4 Pseudo-orbits and metastability

An ergodic dynamical system is said to be *metastable* if it possesses regions in its phase space that remain close to invariant for long periods of time. A well-known approach for detecting such a behaviour is by proving that the corresponding Perron-

[7] The ergodicity insures this possibility for points in the same ergodic component; the equivalence relation allows to pass from one representative to the other in the least element, and finally the recursive construction of $\mathscr{U}_\varepsilon$ allows to get the external points $\mathscr{U}_\varepsilon / \Lambda$.

Frobenius operator has a large real eigenvalue, ξ_ε, very close to 1. Then the positive and negative parts of the eigenfunction corresponding to ξ_ε can be used to identify sets which remain close to invariant for long periods of time. Such sets are often called *almost invariant sets*. For more information on almost invariant sets we refer the reader to [12] and references therein. An analogous theory also exists in the framework of non-autonomous dynamical systems, where the analogous sets are called *coherent structures* (see for instance [13] and references therein).

In this section we assume that:
(M1) As an operator on $\mathscr{B}(M)$, $\mathscr{L}$ has 1 as an eigenvalue of multiplicity two. Moreover, if $\lambda \neq 1$ is an eigenvalue of $\mathscr{L}$, then $|\lambda| < 1$.
(M2) The map T has a unique least element.

Under conditions (M1) and (M2), we will show that random perturbations of T exhibit a *metastable behavior*. In particular, we will show that $\mathscr{L}_\varepsilon$, as an operator on $\mathscr{B}$, will have 1 as a *simple* eigenvalue and will have another real eigenvalue ξ_ε close to 1. Moreover ξ_ε has the second largest modulus among eigenvalues of $\mathscr{L}_\varepsilon$. Such a ξ_ε determines the rate of mixing [5] of the random system $(\mathscr{X}_n^\varepsilon)$.

For this purpose, we first introduce some notation and recall the Keller-Liverani perturbation theorem [18]. We adapt it to our situation which deals with the two adapted norms $||\cdot||_{\mathscr{B}} = |\cdot| + ||\cdot||_1$ and $||\cdot||_1$. For the unperturbed Perron-Frobenius operator $\mathscr{L}$ let us consider the set

$$V_{\delta,r}(\mathscr{L}) = \{z \in C : |z| \leq r \text{ or } \operatorname{dist}(z, \sigma(\mathscr{L})) \leq \delta\},$$

where $\sigma(\mathscr{L})$ is the spectrum of $\mathscr{L}$ as an operator on $\mathscr{B}$. Further, for $\varepsilon \geq 0$, we define the following operator norm

$$|||\mathscr{L}_\varepsilon||| = \sup_{||f||_{\mathscr{B}} \leq 1} ||\mathscr{L}_\varepsilon f||_1. \tag{2}$$

Conditions (LY) and (RLY) are necessary for the operators $\mathscr{L}$, $\mathscr{L}_\varepsilon$ to satisfy the assumptions [18]. Thus, we are ready to state and use the following important result of [18]:

Theorem 2. *[18] If* $\lim_{\varepsilon\to 0} |||\mathscr{L}_\varepsilon - \mathscr{L}||| = 0$ *then for sufficiently small* $\varepsilon > 0$, $\sigma(\mathscr{L}_\varepsilon) \subset V_{\delta,r}(\mathscr{L})$. *Moreover, in each connected component of* $V_{\delta,r}(\mathscr{L})$ *that does not contain* 0 *both* $\sigma(\mathscr{L})$ *and* $\sigma(\mathscr{L}_\varepsilon)$ *have the same multiplicity; i.e., the associated spectral projections have the same rank.*

Using Theorem 2, we show that our random system $(\mathscr{X}_\varepsilon^n)$ exhibits metastable behavior:

Proposition 2. *Suppose that*

- *T satisfies assumptions (M1), (M2);*
- $\lim_{\varepsilon\to 0} |||\mathscr{L} - \mathscr{L}_\varepsilon||| = 0$.

Then, as an operator on $\mathscr{B}$, $\mathscr{L}_\varepsilon$ has 1 *as a simple eigenvalue. Moreover, $\mathscr{L}_\varepsilon$ has a real eigenvalue ξ_ε very close to* 1*. In particular, ξ_ε has the second largest modulus among eigenvalues of $\mathscr{L}_\varepsilon$.*

Proof. Assumption (M1) states that the spectrum of $\mathscr{L}$, as an operator on $\mathscr{B}$, satisfies the following: $\exists$ an $r \in (\eta, 1)$[8] and a $\delta > 0$ such that:

1. The eigenvalue 1 of $\mathscr{L}$ is of multiplicity two;
2. if $\lambda_i \neq 1$ is an eigenvalue of $\mathscr{L}$, then $\lambda_i \in B(0,r)$;
3. $B(0,r) \cap B(1,\delta) = \emptyset$.

Moreover, under assumptions (M2), using Theorem 1, the random map $(\mathscr{X}_\varepsilon^n)$ has exactly one ergodic acsm; i.e., as an operator on $\mathscr{B}$, $\mathscr{L}_\varepsilon$ has 1 as a *simple* eigenvalue. Consequently, by Theorem 2, for sufficiently small ε, the spectrum of $\mathscr{L}_\varepsilon$ satisfies the following:

1. $\mathscr{L}_\varepsilon$ has a real eigenvalue $\xi_\varepsilon < 1$, with $\xi_\varepsilon \in B(1,\delta)$;
2. if $\lambda_{i,\varepsilon} \notin \{1, \xi_\varepsilon\}$ is an eigenvalue of $\mathscr{L}_\varepsilon$, then $\lambda_{i,\varepsilon} \in B(0,r)$. $\square$

Remark 3. The condition $\lim_{\varepsilon\to 0} |||\mathscr{L} - \mathscr{L}_\varepsilon||| = 0$ of Proposition 2 can be checked in several cases. A general theorem is presented in Lemma 8 of [6] for piecewise expanding maps of the interval endowed with our pair of adapted spaces where the noise is represented by a convolution kernel. In the multidimensional case, using quasi-Hölder spaces, the proof is given in Proposition 4.3 of [2]. It should be noted that the previous result implies that the non-essential spectrum of $\mathscr{L}$ is stable [18].

Remark 4. The technique followed to prove Proposition 2 does not work when the number of ergodic T-acim is $l > 2$. This is due to the fact that
a) If l is odd, then $l-1$ is even. Therefore, the transfer operator $\mathscr{L}_\varepsilon$ may have $l-1$ complex eigenvalues of modulus one sitting in $B(1,\delta)$.
b) If l is even, then $l-2$ is even. Therefore, the transfer operator $\mathscr{L}_\varepsilon$ may have $l-2$ complex eigenvalues of modulus one sitting in $B(1,\delta)$.

Whether Proposition 2 is true or not for $l > 2$ is an interesting question.

5 Random Transformations

In sections 2, 3 and 4 we studied random perturbations of a dynamical system in the framework of general Markov processes. Nevertheless, it is often useful to deal with the case when the Markov process is generated by *random transformations* [21]. In

[8] By (LY) and (RLY), η is an upper bound on the essential spectral radius of $\mathscr{L}$ and the essential spectral radius of $\mathscr{L}_\varepsilon$.

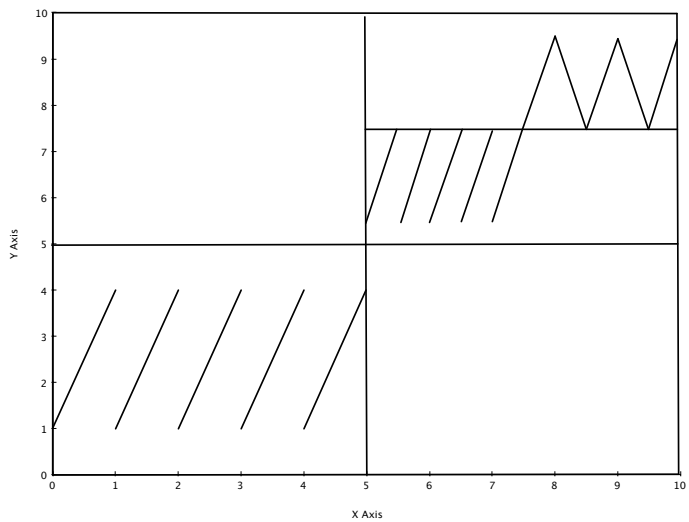

Fig. 1 An example of a $1-$dimensional map T with two least elements

this setting, we consider an i.i.d. stochastic process $(\omega_k)_{k\in N}$ with values in Ω_ε and with probability distribution θ_ε. We associate with each $\omega \in \Omega_\varepsilon$ a map $T_\omega : M \to M$ and we consider the random orbit starting from the point x and generated by the realization $\underline{\omega}_n = (\omega_1, \omega_2, \cdots, \omega_n)$, defined as : $T_{\underline{\omega}_n} := T_{\omega_n} \circ \cdots \circ T_{\omega_1}(x)$. This defines a Markov process $\mathscr{X}_\varepsilon$ with transition function

$$P(x,A) = \int_{\Omega_\varepsilon} \mathbf{1}_A(T_\omega(x))d\theta_\varepsilon(\omega), \tag{3}$$

where $A \in \mathscr{B}(M)$, $x \in M$ and $\mathbf{1}_A$ is the indicator function of a set A. The transition function induces an operator $\mathscr{U}_\varepsilon^*$ which acts on measures μ on $(I, \mathscr{B}(M))$ as:

$$\mathscr{U}_\varepsilon^* \mu(A) = \int_M \int_{\Omega_\varepsilon} \mathbf{1}_A(T_\omega(x))d\theta_\varepsilon(\omega)d\mu(x) = \int_I \mathscr{U}_\varepsilon \mathbf{1}_A(x)d\mu_\varepsilon(x),$$

where $\mathscr{U}_\varepsilon$ is the random evolution operator acting on L_m^∞ functions g:

$$\mathscr{U}_\varepsilon g = \int_{\Omega_\varepsilon} g \circ T_\omega d\theta_\varepsilon(\omega). \tag{4}$$

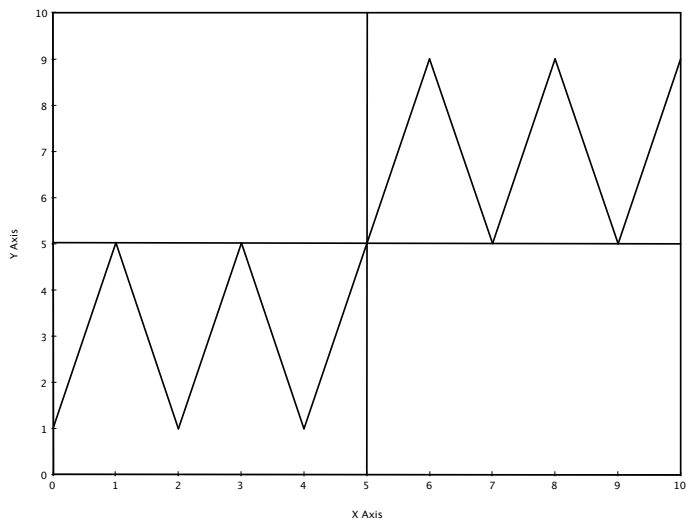

Fig. 2 An example of a $1-$dimensional map T with one least element

A measure μ_ε on $(M,\mathscr{B}(M))$ is called a $\mathscr{X}_\varepsilon$-stationary measure if and only if, for any $A \in \mathscr{B}(M)$,

$$\mathscr{U}_\varepsilon^* \mu_\varepsilon(A) = \mu_\varepsilon(A). \tag{5}$$

We are interested in studying $\mathscr{X}_\varepsilon$-acsm. By (4), one can define the transfer operator $\mathscr{L}_\varepsilon$ (Perron-Frobenius) acting on $L^1(M,\mathscr{B}(M),m)$ by:

$$(\mathscr{L}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} \mathscr{L}_\omega f(x) d\theta_\varepsilon(\omega), \tag{6}$$

which satisfies the duality condition

$$\int_M g\mathscr{L}_\varepsilon f dm = \int_M \mathscr{U}_\varepsilon g f dm \tag{7}$$

where g is in L_m^∞ and $\mathscr{L}_\omega$ is the transfer operator associated with T_ω. It is well know that $\mu_\varepsilon := \rho_\varepsilon m$ is a $\mathscr{X}_\varepsilon$-acsm if and only if $\mathscr{L}_\varepsilon \rho_\varepsilon = \rho_\varepsilon$; i.e., ρ_ε is a $\mathscr{X}_\varepsilon$-invariant density. In order to use the results of sections 2, 3 and 4, we also assume

that assumptions (P1), (P2)[9] (B1)-(B3), (LY) and (RLY) hold. Moreover, we assume that (6) reduces to

$$(\mathscr{L}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} \mathscr{L}_\omega f(x) d\theta_\varepsilon(\omega) = \int_M p_\varepsilon(z,x) f(z) dz. \tag{8}$$

In fact, an important example of a random perturbation where $\mathscr{L}_\varepsilon$ can be reduced as in (8) is the case of *additive* noise. For instance if $M = S^q$, the q-dimensional torus [10], let define $T_\omega = T(x) - \omega \bmod S^q$, where $\omega \in S^q$. Let the density of θ_ε with respect to the Lebesgue measure $d\omega$ on S^m, h_ε, be continuously differentiable with support contained in the square $\Omega_\varepsilon \equiv [-\varepsilon, \varepsilon]^q$: $\int d\theta_\varepsilon = \int h_\varepsilon(\omega) d\omega = 1$. It is then straightforward to check that $p_\varepsilon(x,y) = h_\varepsilon(Tx - y)$.

Remark 5. The preceding example illustrates very well the relation between the two approaches used in this paper to deal with randomness. Namely the *Markov chain* approach, which was used to prove Theorem 1, and the *random transformations* approach which permits to follow the orbit of a point under the concatenation of the randomly chosen maps. Consequently, the latter allows for more explicit representation of objects like the evolution operator and the transfer operator. In fact , relation (8) is a general fact whenever the transition function for the Markov chain is given by the integral (see (3)), $P(x,A) = \int_{\Omega_\varepsilon} \mathbf{1}_A(T_\omega(x)) d\theta_\varepsilon(\omega)$, and the noise is *absolutely continuos*, namely $\theta_\varepsilon(\omega \in \Omega_\varepsilon;\ T_\omega x \in A) = \int_A p_\varepsilon(x,z) dz$, where A is a measurable set in M. This in particular means that we can always construct a Markov chain starting with a random transformation; the converse is also true but the random maps are very often defined in an implicit way, we defer to ([21]) for the construction. Finally, We stress that Theorem 1 has been proved for an absolutely continuous noise. This means that it cannot be applied, in its actual form, to random perturbations on a finite noise space, for which the probability θ_ε is an atomic measure and the random transfer operator $\mathscr{L}_\varepsilon$ applied to the function f becomes a weighted operator of the form $\sum_{j\geq 1} \mathscr{L}_{\omega_j}\ f\ p_{\omega_j}$, where the weights p_{ω_j}, $\sum_{j\geq 1} p_{\omega_j} = 1$, are associated to the values of the random variables ω_j.

5.1 1-dimensional examples

To illustrate our results, we first present two simple examples of 1-dimensional maps[11]. In Example 1 we present a map that has three ergodic components and

[9] (P1) and (P2) in this setting are analogous to:
(P1) For all $x \in I$ the measure $P(x,\cdot)$ defined above on the Borel subsets of I by $P(x,A) = \theta_\varepsilon\{\omega \in \Omega_\varepsilon\ ,\ T_\omega(x) \in A\}$ is absolutely continuous with respect to Lebesgue, namely we have a summable density $p_\varepsilon(x,\cdot)$ such that: $P(x,A) = \int_A p_\varepsilon(x,y) dy$;
(P2) We have: support of $P(x,\cdot)$ coincides with $B_\varepsilon(Tx), \forall x \in I$.

[10] If M is not the torus we assume that, for all $\omega \in \Omega_\varepsilon$, $T_\omega(M) \subseteq M$.

[11] In both examples the Banach space $\mathscr{B}$ is considered to be the space of functions of bounded variation.

two least elements. In Example 2 we present a map which has two ergodic components and exactly one least element. Moreover, we show that when T of Example 2 is perturbed by an additive noise, it exhibits a metastable behavior.

Example 1. In this example $T : [0,10] \to [0,10]$. The graph of T is shown in Fig. 1. T is piecewise linear and Markov with respect to the partition:

$$[0,1),\dots,[4,5),[5,5.5)\dots,[9.5,10].$$

One can easily check that T has exactly three ergodic acim whose supports $\Lambda_1,\Lambda_2,\Lambda_3$ are, respectively, equal to $[1,4]$, $[5.5,7.5]$, and $[7.5,9.5]$. Moreover, one can easily check that T admits two least elements. Namely, $\{\Lambda_1\}$ and $\{\Lambda_2,\Lambda_3\}$.

Example 2. In this example $T : [0,10] \to [0,10]$. The graph of T is shown in Fig. 2. T is piecewise linear and Markov with respect to the partition:

$$[0,1),\dots,[9,10].$$

One can easily check that that T has exactly two ergodic acim whose supports Λ_1,Λ_2 are, respectively, equal to $[1,5]$, and $[5,9]$. Moreover, one can easily check that T admits exactly one least elements. Namely, $\{\Lambda_1,\Lambda_2\}$. When T is perturbed by an additive noise, by using Lemma 8 of [6], we obtain that $\lim_{\varepsilon\to 0}|||\mathscr{L}_\varepsilon-\mathscr{L}|||=0$. Consequently, by Proposition 2, the random system exhibits a metastable behavior.

5.2 A 2-dimensional example

We now present an example in higher dimensions, in particular the two-dimensional skew system $\Phi_\omega : [0,1]\times S \to [0,1]\times S$ defined as $(x',y') = \Phi_\omega(x,y)$ where

$$x' = T(x) + \omega y$$

$$y' = 2y \ \ \mathrm{mod}1,$$

where S denotes the unit circle, and $T : [0,1] \to [0,1]$ is given by

- for $x \in [0,a),\ T(x) = (\frac{1}{2a}-3)x + a$;
- for $x \in [a,\frac{1}{4}),\ T(x) = -2x + (a+\frac{1}{2})$;
- for $x \in [\frac{1}{4},\frac{1}{2}+a),\ T(x) = 2x + (a-\frac{1}{2})$;
- for $x \in [\frac{1}{2}+a,\frac{3}{4}), T(x) = -2x + (2+a)$;
- for $x \in [\frac{3}{4},1-a), T(x) = 2x + (a-1)$;
- for $x \in [1-a,1], T(x) = (3-\frac{1}{2a})x + (-\frac{5}{2}+\frac{1}{2a}+2a)$.

The graph of T is depicted in Fig.3

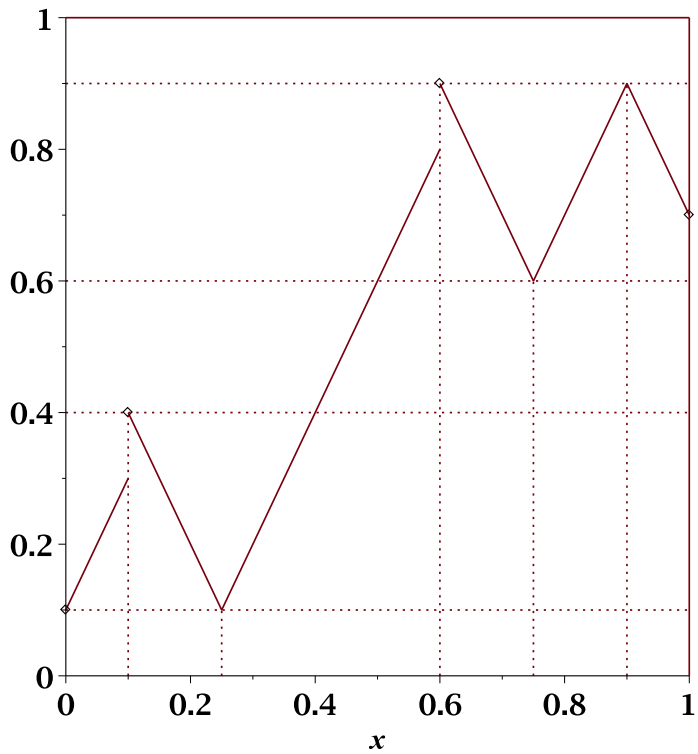

Fig. 3 The graph of the map T for $a = 0.1$. The least element for T is the closed interval $[\frac{1}{2} + a, 1-a]$. This induces the least element $[\frac{1}{2} + a, 1-a] \times S$ for Φ_0 on $[0,1] \times S$.

We considered a piecewise linear map to simply the exposition. However, this is not really needed to apply our results. In fact, a map with nonlinear branches can be used in the example as long as we keep uniform dilatation, bounded distortion, and a C^{1+} smoothness. For each ω we have a different random map and we compose them by taking ω uniformly distributed, for instance, between $(-\varepsilon, \varepsilon)$. In this case $\theta_\varepsilon(\omega) = \frac{1}{2\varepsilon} d\omega$. The positive parameter a can be chosen equal to $\frac{1}{10}$, in such a way that the image of the unit interval remains in $[0,1]$ when $\varepsilon < a$. It is very easy to check that the unperturbed map Φ_0 has two ergodic components which are, respectively, subsets of $[a, \frac{1}{2} - a] \times S$ and $[\frac{1}{2} + a, 1-a] \times S$ and the latter is a least element. These ergodic components are with respect to absolutely continuous invariant measures

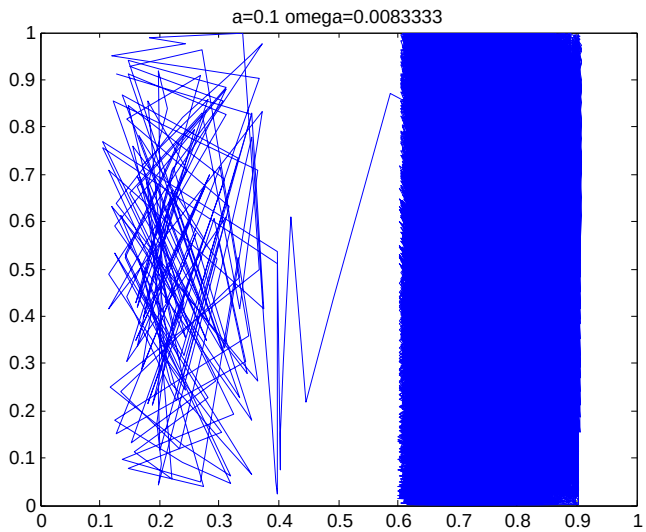

Fig. 4 The least element on the right side $x > \frac{1}{2}$ for the map Φ_ω; notice that ω on the top of the figure is just $\varepsilon = 1/120$.

whose densities are the fixed points of the Perron-Frobenius operator associated to Φ_0. The existence of such fixed points follow easily by obtaining a Lasota-Yorke inequality on a suitable function space, such as the space of functions of bounded variation [7] or quasi-Hölder functions, [24, 16], which satisfy all the assumptions required in this paper. Notice that a Lasota-Yorke inequality can be obtained as well for the random Perron-Frobenius operator associated to the random system, by using the closeness of the perturbed maps Φ_ω, $|\omega| \leq \varepsilon$ for small ε (this means that the constants η and D in (LY) and (RLY) can be chosen to be the same for the unperturbed and the perturbed systems[12]). According to our main theorem, there must be only one ergodic absolutely continuous stationary measure and this must be supported in a neighborhood of the least element. In Fig. 4 we show the limit set of several random orbits taken with $\varepsilon = 1/120$ and $a = 0.1$. All these random orbits accumulate in the right hand side of $[0,1] \times S$ as predicted by our theory.

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[12] For these kind of uniformly expanding maps those factors are basically related to the multiplicity of the intersection of the discontinuous lines, which is 2 in this example, and to the norm and the determinant of the jacobian matrix of Φ_ω, which is $D\Phi_\omega = \begin{pmatrix} T'(x) & \omega \\ 0 & 2 \end{pmatrix}$, where $|T'| \in \{|3 - \frac{1}{2a}|, 2\}$. The determinant does not depend on the noise and (any) norm of the matrix can be chosen uniformly bounded for ε small enough.

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