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ESCAPE RATES FORMULAE AND METASTABLILTY FOR RANDOMLY PERTURBED MAPS

WAEL BAHSOUN AND SANDRO VAIENTI

Abstract. We provide escape rates formulae for piecewise expanding interval maps with 'random holes'. Then we obtain rigorous approximations of invariant densities of randomly perturbed metabstable interval maps. We show that our escape rates formulae can be used to approximate limits of invariant densities of randomly perturbed metastable systems.

1. Introduction

A dynamical system is called open if there is a subset in the phase space, called a *hole*, such that whenever an orbit lands in it, the dynamics of this obit is terminated. In open dynamical systems, long-term statistics are described by a conditionally invariant measure and its related escape rate, measuring the mass lost from the system per unit time [13, 15]. For the past three years there has been a considerable interest in describing the escape rate of an open system as a function of the hole's position and size. In [8] it was observed that for the doubling map, given two holes, H_1 and H_2 of the same fixed size, each centred around a periodic point, say x_1, x_2 respectively, with the period of x_1 smaller than that of x_2, then the escape rate through H_1 is bigger than that through H_2. Later in [27] it was shown that, by shrinking the hole to a point, say x_0, the escape rate depends on two things: i) whether x_0 is periodic or not; ii) the invariant density of the corresponding closed system. Following the success of [8, 27], other researchers studied this phenomenon for different types of systems [1, 12, 17][1]. These results have lead to insights in studying metastable dynamical systems which behave approximately like a collection of open systems: the infrequent transitions between almost invariant regions in a metastable system are similar to infrequent escapes from associated open systems [22, 16, 19, 27].

This research topic is currently very active in ergodic theory and dynamical systems. In part, this is due to the interesting applications of open and metastable dynamical systems in physical sciences. For instance, open dynamical systems are

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[1][17, 12] are in the direction of [27], while [1] is in the spirit of [8].

used to study transport in heat conduction [21]. They also play an important in astronomy [33]. In molecular dynamics, almost invariant regions of metastable systems are used to identify sets where stable molecular conformations occur [31]. In astrodynamics, metastable systems are used to recognize regions from which asteroids escape is infrequent [11]. Moreover, metastable dynamical systems have been recently used to develop realistic models of atmospheric and ocean circulation [10, 18, 30]. In these models, it is believed that metastable states[2] lie behind long term global circulation patterns, and form large scale barriers to transport.

In this paper we first study piecewise smooth and expanding interval maps with 'random holes'. We prove that the random open system admits an absolutely continuous conditionally stationary measure (accsm). Moreover we obtain escape rates formulae (first order approximations) depending on the position of the holes. An older escape rate formula for randomly perturbed maps with specific holes positions and a specific distribution on the noise space was obtained in [9]. Our result is in the spirt of the recent trend of describing the escape rate as a function of the position of the hole [1, 8, 17, 27]. In particular, our result generalizes Keller-Liverani escape rates formulae [27] which were obtained for deterministic perturbations.

In the second part of the paper we study random perturbations of interval maps that initially admit exactly two invariant ergodic densities. However, under random perturbations that generate *random holes* and allow leakage of mass between the two initially ergodic subsystems, the random system admits a unique stationary density. We show that such a density for the metastable random system can be approximated in the L^1 norm (with respect to the ambient Lebesgue measure m), by a particular convex combination of the two invariant ergodic densities of the initial system. In particular, we show that the ratio of the weights in the convex combination is equal to the ratio of the averages of the measures of the left and right random holes. Moreover, as a by product of our escape rates formulae, we show that these weights can be also identified as the ratio of the escape rates from the left and right random open systems.

We finish the introduction by addressing possible generalizations of our work. We foresee two natural generalizations. The first generalization concerns the extension to piecewise expanding maps in higher dimensions. Two difficulties arise in this setting. First, obtaining a Lasota-Yorke inequality for the operator associated with the open system with random holes, see Lemma 2.6 would require a more careful control of the variation, whatever the Banach space $\mathbf{B}$ that operator acts on is, of the characteristic function along the boundaries of the partitions, which are now codimension-1 piecewise smooth submanifolds and not points as in the setting of this paper. The second difficulty is to get local smoothness of elements of $\mathbf{B}$ around the holes similar to Lemma 1 of [22]. We believe that the latter property is fundamental to obtain results on approximating invariant densities of metastable systems, even in the one-dimensional case.

The second generalization is the extension to non-uniformly expanding maps. Metastability for deterministic maps with indifferent fixed points was treated by

[2]Metastable states correspond to eigenvalues of a transfer operator which are very close to 1, a phenomenon which is very transparent in the metastable systems of this paper.

us in [3]. One of the main tools we used there was to *reduce* the study of metastability to an induced subsystem where the map is uniformly expanding, and then pullback the result to the original system. It is not clear how this technique can be implemented in the presence of noise. A few recent results [2, 32] used induction to achieve stochastic stability in some classes of one-dimensional maps. It would be interesting to explore the possibility of apply those techniques to metastability. Supposing the analogous of our Theorem 3.4 can be proved in the random case, it would be also interesting to explore if something similar to Corollary 3.5 (see section 3.3) would still hold. In this paper, the latter is related to the *exponential* escape rate which is in turn associated to the existence of a spectral gap of the transfer operator. This is a 'luxury' that is not available in a non-uniformly hyperbolic setting.

In section 2 we obtain escape rates formulae for expanding interval maps with random holes. Our main result in this section is Theorem 2.1. In section 3.3 we study rigorous approximations in the L^1-norm of invariant densities of randomly perturbed metastable maps. Our main result in this section is Theorem 3.4.

2. Escape Rate Formulae For Randomly Perturbed Maps

2.1. **Notation.** Throughout the paper we use the following notation: $(I, \mathfrak{B}, m)$ is the measure space where $I = [0, 1]$, $\mathfrak{B}$ is the Borel σ-algebra and m is Lebesgue measure. For $f \in L^1(I, \mathfrak{B}, m)$, we define

$$Vf = \inf\{\mathrm{var}\bar{f} : f = \bar{f} \text{ a.e.}\},$$

where

$$\mathrm{var}\bar{f} = \sup\{\sum_{j=0}^{t-1} |f(x_{j+1}) - f(x_j)| : 0 = x_0 < x_1 < \cdots < x_t = 1\}.$$

We denote by $BV(I)$ the space of functions of bounded variation on I equipped with the norm $||\cdot||_{BV} = V(\cdot) + ||\cdot||_1$, where $||\cdot||_1$ is the L^1 norm with respect to m.

2.2. **Piecewise expanding maps.** Let $T : I \to I$ be a map which satisfies the following conditions:
(O1) There exists a partition of I, which consists of intervals $\{I_i\}_{i=1}^q$, $I_i \cap I_j = \emptyset$ for $i \neq j$, $\bar{I}_i := [c_{i,0}, c_{i+1,0}]$ and there exists $\delta > 0$ such that $T_{i,0} := T|_{(c_{i,0}, c_{i+1,0})}$ is C^2 which extends to a C^2 function $\bar{T}_{i,0}$ on a neighbourhood $[c_{i,0} - \delta, c_{i+1,0} + \delta]$ of $\bar{I}_i$;
(O2) $\inf_{x \in I \setminus \mathcal{C}_0} |T'(x)| \geq \kappa^{-1} > 1$, where $\mathcal{C}_0 = \{c_{i,0}\}_{i=1}^q$.
(O3) T preserves a unique acim[3] μ which is equivalent to m. Moreover, the system (I, μ, T) is mixing.

2.3. **Random Holes.** Let $(\omega_k)_{k \in \mathbb{N}}$ be an i.i.d. stochastic process with values in the interval $\Omega_\varepsilon = [0, \varepsilon]$, $\varepsilon > 0$, and with probability distribution θ_ε. We fix $z \in (0, 1)$ and we associate with each $\omega \in \Omega_\varepsilon$ an interval H_ω such that $z \in H_\omega \subseteq H_\varepsilon$, and we assume that $H_\varepsilon \subseteq H_{\varepsilon'}$ for $\varepsilon \leq \varepsilon'$. Further,
(O4) We assume that T is continuous at z; this assumption will be explicitly used in the proof of Theorem 2.1. In order to apply the results of this section to section

[3]The existence of μ follows from the well known result of [28].

3.3, we will also assume that the density ρ is continuous at z.

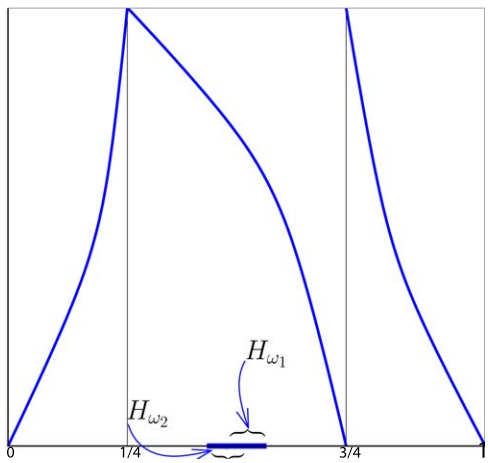

FIGURE 1. A map with random holes H_{ω_1} and H_{ω_2} sitting inside H_ε. Note that H_{ω_1} and H_{ω_2} are not nested.

2.4. **Transfer operator of the random open system.** Our goal in this section is to study the existence of absolutely continuous conditionally stationary measures (accsm) and their associated escape rates through the random holes H_ω around a given point z under the dynamics of T. An example of an interval map with random holes is shown in Figure 1. We set

$$X_\omega := I \setminus H_\omega,$$

and define for $f \in L^1(I, \mathfrak{B}, m)$

$$\hat{\mathcal{L}}_\varepsilon f(x) := \int_{\Omega_\varepsilon} \mathcal{L}_0(f \mathbf{1}_{X_\omega})(x) d\theta_\varepsilon, \tag{2.1}$$

where $\mathcal{L}_0$ is the transfer operator (Perron-Frobenius) [4, 7] associated with T; i.e., for $f \in L^1(I, \mathfrak{B}, m)$ we have

$$\mathcal{L}_0 f(x) = \sum_{y := T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

The transfer operator $\hat{\mathcal{L}}_\varepsilon$ will be used to prove that the random open system admits an accsm with exponential escape rate. By using the fact that for any measurable set A and integrable function f

$$\mathbf{1}_A \mathcal{L}_0 f = \mathcal{L}_0(\mathbf{1}_{T^{-1}A} f),$$

we obtain

(2.2)

$$\begin{aligned}\hat{\mathcal{L}}_\varepsilon^n f(x) &= \int_{\Omega_\varepsilon} \cdots \int_{\Omega_\varepsilon} \mathcal{L}_0^n(f \mathbf{1}_{X_{\omega_1} \cap T^{-1} X_{\omega_2} \cap \cdots \cap T^{-(n-1)} X_{\omega_n}})(x) d\theta_\varepsilon(\omega_1) \cdots d\theta_\varepsilon(\omega_n) \\ &\overset{\text{def}}{:=} \int_{\bar{\Omega}_\varepsilon} \mathcal{L}_0^n(f \mathbf{1}_{X_{\omega_1} \cap T^{-1} X_{\omega_2} \cap \cdots \cap T^{-(n-1)} X_{\omega_n}})(x) d\theta_\varepsilon^\infty(\bar{\omega}).\end{aligned}$$

where $\theta_\varepsilon^\infty := \Pi_{i=1}^\infty \theta_\varepsilon$ and $\bar{\omega} := (\omega_1, \omega_2, \dots)$.

2.5. **Statement of the main result of section 2.** The following theorem provides random versions of Keller-Liverani escape rates formulae [27].

Theorem 2.1. *For sufficiently small $\varepsilon > 0$, there exists an e_ε, $0 < e_\varepsilon < 1$, and a $g \in BV(I)$, $g_\varepsilon(x) > 0$, with $\int_I g_\varepsilon, dm = 1$, such that*

$$\hat{\mathcal{L}}_\varepsilon g_\varepsilon = e_\varepsilon g_\varepsilon.$$

Moreover,

(1) *If z is a non-periodic point of T, then*

$$\lim_{\varepsilon \to 0} \frac{1 - e_\varepsilon}{\int_{\Omega_\varepsilon} \mu(H_\omega) d\theta_\varepsilon(\omega)} = 1.$$

(2) *If z is a periodic point of T of minimal period p, T^p is C^1 in a neighbourhood of z, and the probability distribution θ_ε on the noise space Ω_ε satisfies the following condition: $\exists\ \upsilon > 1$ such that*

(C) $\theta_\epsilon\{\omega;\ m(H_\omega) \in (\epsilon - \epsilon^\upsilon, \epsilon)\} > 1 - \epsilon^\upsilon$;

then

$$\lim_{\varepsilon \to 0} \frac{1 - e_\varepsilon}{\int_{\Omega_\varepsilon} \mu(H_\omega) d\theta_\varepsilon(\omega)} = 1 - \frac{1}{|(T^p)'(z)|}.$$

Remark 2.2.

(1) In (2) of Theorem 2.1, condition **(C)** means that for a fixed $\varepsilon > 0$ most of the random holes H_ω are not too small when compared to the size of H_ε. Notice that the two "υ" in the condition can be different, provided they remain larger than 1. We considered the same υ for simplicity [4]. Condition **(C)** is needed to insure a lower bound on the escape rate in case (2) of Theorem 2.1.

(2) For the existence of the accsm we required ε to be sufficiently small since we use the perturbation result of [26] in the proof our Lemma 2.7. The strict positivity of the density g_ε also follows from [26] since μ is equivalent to m.

Remark 2.3. Since $\hat{\mathcal{L}}_\varepsilon g_\varepsilon = e_\varepsilon g_\varepsilon$, with $\int_I g_\varepsilon dm = 1$, by using the definition of $\hat{\mathcal{L}}_\varepsilon$, it follows that:

$$e_\varepsilon = \int_{\Omega_\varepsilon} d\theta_\varepsilon(\omega) \int_I g_\varepsilon \mathbf{1}_{X_\omega} dm.$$

Set ν_ε to be the Borel probability measure:

$$\nu_\varepsilon(A) \overset{\text{def}}{:=} \frac{1}{e_\varepsilon} \int_I \int_{\Omega_\varepsilon} \mathbf{1}_A \mathbf{1}_{X_\omega} g_\varepsilon d\theta_\varepsilon dm,$$

[4]It is very easy to construct examples that satisfy condition **(C)**. For instance, suppose we associate to any $\omega \in \Omega_\varepsilon$ a symmetric hole around the point z. Then: (i) $H_\varepsilon = [z - \frac{\varepsilon}{2};\ z + \frac{\varepsilon}{2}]$; $H_\omega = [z - \frac{\omega}{2};\ z + \frac{\omega}{2}]$; (ii) $m(H_\varepsilon) = \varepsilon$; $m(H_\omega) = \omega$. Let us choose an absolutely continuous θ_ε with density d_ε. Our condition **(C)** will be satisfied whenever $\int_{\varepsilon - \varepsilon^\upsilon}^{\varepsilon} d_\varepsilon(\omega) d\omega > 1 - \varepsilon^\upsilon$ which can be obtained, for instance, by taking $d_\varepsilon = \varepsilon^{-\upsilon}$ on the interval $(\varepsilon - \varepsilon^\upsilon, \varepsilon)$ and 0 everywhere else.

where $A \subseteq I$ is a measurable set. By using the fact that $\hat{\mathcal{L}}_\varepsilon g_\varepsilon = e_\varepsilon g_\varepsilon$ and the definition of $\hat{\mathcal{L}}_\varepsilon$, we obtain that ν_ε satisfies the following:

$$\nu_\varepsilon(A) = \frac{1}{e_\varepsilon}\int_{\Omega_\varepsilon} d\theta_\epsilon(\omega)\nu_\epsilon(T^{-1}A \cap T^{-1}X_\omega); \tag{2.3}$$

$$e_\varepsilon = \int_{\Omega_\varepsilon} d\theta_\varepsilon(\omega)\nu_\varepsilon(T^{-1}X_\omega); \tag{2.4}$$

and

$$e_\varepsilon^n \nu_\varepsilon(A) = \int_{\bar{\Omega}_\varepsilon} d\theta_\varepsilon^\infty(\bar{\omega})\nu_\varepsilon(T^{-n}A \cap T^{-1}X_{\omega_1} \cap T^{-2}X_{\omega_2} \cap \cdots \cap T^{-n}X_{\omega_n}). \tag{2.5}$$

Definition 2.4. We will call $-\ln e_\epsilon$ the *escape rate for the random system*, and ν_ϵ the *absolutely continuous conditionally stationary measure.*

Remark 2.5. It is interesting to remark that the items (1) and (2) of Theorem 2.1 provide explicit perturbation formulae for the leading eigenvalue of the operator $\hat{\mathcal{L}}_\varepsilon$. This operator enjoys all the properties of what Keller [25] recently called *rare events Perron-Frobenius operators.*

2.6. **Proof of Theorem 2.1.** We start this subsection by proving a uniform Lasota-Yorke inequality for $\hat{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_0$. A deterministic version of Lemma 2.6 can be found in [29].

Lemma 2.6. *There exists a $\gamma \in (0,1)$ and constants $A, B > 0$ such that for any $n \geq 1$ and $f \in BV(I)$ we have*

$$||\mathcal{L}_0^n f||_{BV} \leq A\gamma^n ||f||_{BV} + B||f||_1;$$

$$||\hat{\mathcal{L}}_\varepsilon^n f||_{BV} \leq A\gamma^n ||f||_{BV} + B||f||_1.$$

Proof. We begin to compute the total variation, leaving the L^1 estimate at the end. Let $\mathcal{Z}^n = \mathcal{Z} \vee T^{-1}\mathcal{Z} \cdots \vee T^{-(n-1)}\mathcal{Z}$, where $\mathcal{Z} = \{I_i\}_{i=1}^q$. Let $J_n = \frac{1}{|(T^n)'|}$. For $A \in \mathcal{Z}^n$ and $f \in BV(I)$: there is a $\kappa \in (0,1)$, such that

$$V\mathbf{1}_{T^nA}(J_n f) \circ T_{|A}^{-n} \leq 2\kappa^n V_A f + \frac{1}{\min_{A\in\mathcal{Z}^n} m(A)}\int_A |f|dm. \tag{2.6}$$

In particular,

$$V_I\mathcal{L}_0^n(f) \leq \sum_{A\in\mathcal{Z}^n} V\mathbf{1}_{T^nA}(J_n f) \circ T_{|A}^{-n} \leq 2\kappa^n V_I f + \frac{1}{\min_{A\in\mathcal{Z}^n} m(A)}\int_I |f|dm. \tag{2.7}$$

Now, let us consider a fixed random path of length n, $(\omega_1, \omega_2, \ldots, \omega_n)$, $f \in BV(I)$ and define

$$g := f\mathbf{1}_{X_{\omega_1}\cap T^{-1}X_{\omega_2}\cap\cdots\cap T^{-(n-1)}X_{\omega_n}}$$

and observe that $g \in BV(I)$. Then using (2.6) and noticing that

$$\left(X_{\omega_1} \cap T^{-1}X_{\omega_2} \cap \cdots \cap T^{-(n-1)}X_{\omega_n}\right) \cap A$$

consists of at most $n+1$ connected components, we obtain

$$
\begin{aligned}
&V\mathbf{1}_{T^nA}(J_nf\mathbf{1}_{X_{\omega_1}\cap T^{-1}X_{\omega_2}\cap\cdots\cap T^{-(n-1)}X_{\omega_n}})\circ T_{|A}^{-n}\\
&=V\mathbf{1}_{T^nA}(J_ng)\circ T_{|A}^{-n}\le 2\kappa^nV_Ag+\frac{1}{\min_{A\in\mathcal{Z}^n}m(A)}\int_A|g|dm\\
&\le 2\kappa^n(V_Af+2(n+1)\sup_{x\in A}|f|)+\frac{1}{\min_{A\in\mathcal{Z}^n}m(A)}\int_A|f|dm\\
&\le 2\kappa^n(2n+3)V_Af+\frac{1}{\min_{A\in\mathcal{Z}^n}m(A)}(2\kappa^n(2n+2)+1)\int_A|f|dm.
\end{aligned}
\tag{2.8}
$$

Summing over $A\in\mathcal{Z}^n$ in (2.8) we obtain

$$
\begin{aligned}
&V_I\mathcal{L}_0^n(f\mathbf{1}_{X_{\omega_1}\cap T^{-1}X_{\omega_2}\cap\cdots\cap T^{-(n-1)}X_{\omega_n}})\\
&\qquad\le 2\kappa^n(2n+3)V_If+\frac{1}{\min_{A\in\mathcal{Z}^n}m(A)}(2\kappa^n(2n+2)+1)\int_I|f|dm
\end{aligned}
\tag{2.9}
$$

Since the inequality in (2.9) does not depend on ω and

$$\hat{\mathcal{L}}_\varepsilon^nf(x)=\int_{\Omega_\varepsilon}\cdots\int_{\Omega_\varepsilon}\mathcal{L}_0^n(f\mathbf{1}_{X_{\omega_1}\cap T^{-1}X_{\omega_2}\cap\cdots\cap T^{-(n-1)}X_{\omega_n}})(x)d\theta_\varepsilon(\omega_1)\cdots d\theta_\varepsilon(\omega_n),$$

we also have

$$V_I\hat{\mathcal{L}}_\varepsilon^nf\le 2\kappa^n(2n+3)V_If+\frac{1}{\min_{A\in\mathcal{Z}^n}m(A)}(2\kappa^n(2n+2)+1)\int_I|f|dm. \tag{2.10}$$

The estimate on $||\hat{\mathcal{L}}_\varepsilon^nf||_1$ is easy. Indeed, this can be done by splitting f into its positive and negative parts and by using the linearity of the transfer operator. Therefore, we may suppose that f is non negative. This allows us to interchange the integrals w.r.t. the Lebesgue measure and $\theta_\varepsilon^\infty$ and to use duality. In conclusion we get

$$||\hat{\mathcal{L}}_\varepsilon^nf||_1\le\int|f|\mathbf{1}_{X_{\omega_1}\cap T^{-1}X_{\omega_2}\cap\cdots\cap T^{-(n-1)}X_{\omega_n}})dm\le||f||_1.$$

Since there exists n_0 and $\gamma\in(\kappa,1)$ such that $2\kappa^{n_0}(2n_0+3)\le\gamma^{n_0}$, we can choose $A:=2n_0+3$, $B:=\sup_{n\le n_0}\frac{1}{\min_{A\in\mathcal{Z}^n}m(A)}2(n+1)\frac{2}{1-\gamma^{n_0}}$ and use (2.10), (2.7) to obtain a uniform the Lasota-Yorke inequality for $\hat{\mathcal{L}}_\varepsilon$ and $\mathcal{L}_0$.

□

Lemma 2.7. *Consider $\hat{\mathcal{L}}_\varepsilon:BV(I)\to BV(I)$. Then there are $e_\varepsilon\in(0,1)$, $\varphi_\varepsilon\in BV(I)$, a probability Borel measure ν_ε and linear operators $\mathcal{Q}_\varepsilon:BV(I)\to BV(I)$ such that*

1. $e_\varepsilon^{-1}\hat{\mathcal{L}}_\varepsilon=\varphi_\varepsilon\otimes\nu_\varepsilon+\mathcal{Q}_\varepsilon$;
2. $\hat{\mathcal{L}}_\varepsilon\varphi_\varepsilon=e_\varepsilon\varphi_\varepsilon$, $\nu_\varepsilon\hat{\mathcal{L}}_\varepsilon=e_\varepsilon\nu_\varepsilon$, $\mathcal{Q}_\varepsilon\varphi_\varepsilon=0$, $\nu_\varepsilon\mathcal{Q}_\varepsilon=0$;
3. $\sum_{n=0}^\infty\sup_\varepsilon||\mathcal{Q}_\varepsilon^n||_{BV(I)}<\infty$;
4. $m(\varphi_\varepsilon)=1$ *and* $\sup_\varepsilon||\varphi_\varepsilon||_{BV(I)}<\infty$;
5. $\exists\,C>0$ *such that*

$$\eta_\varepsilon:=\sup_{||\psi||_{BV(I)}\le1}|\int_I(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)\psi dm|\to0\ \textit{as}\ \varepsilon\to0,$$

and

$$\eta_\varepsilon\cdot||(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)\rho||_{BV(I)}\le C|\Delta_\varepsilon|,$$

where $\Delta_\varepsilon := m((\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)\rho)$.

Proof. We have for any $\psi \in BV(I)$:

$$(\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)\psi = \int_{\Omega_\varepsilon} \mathcal{L}_0(\psi \mathbf{1}_{H_\omega}) d\theta_\varepsilon. \tag{2.11}$$

Since ψ is also in $L^\infty(m)$ we have

$$\eta_\varepsilon = \sup_{||\psi||_{BV(I)} \leq 1} |\int_I (\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)\psi dm| \leq \int_{\Omega_\varepsilon} m(H_\omega) d\theta_\varepsilon \leq m(H_\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \tag{2.12}$$

Thus, by Lemma 2.6 and (2.12), using the abstract perturbation result on the stability of spectrum of transfer operators [26], we obtain (1)-(4) of the lemma. Now by (2.11), notice that

$$\eta_\varepsilon \leq A_\varepsilon := \int_{\Omega_\varepsilon} m(H_\omega) d\theta_\varepsilon, \tag{2.13}$$

and

$$\Delta_\varepsilon = \int_{\Omega_\varepsilon} \mu(H_\omega) d\theta_\varepsilon. \tag{2.14}$$

Moreover, using (2.6), condition **(O3)**, and calling: (i) $K = \max(A\gamma, B)$; (ii) $\tilde{C} = 3||\rho||_{BV}$; (iii) $\rho_m := \inf_{x \in I} \rho(x), x - m$ *a.e.*, we obtain

$$||(\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)\rho||_{BV(I)} = ||\int_{\Omega_\varepsilon} \mathcal{L}_0(\rho \mathbf{1}_{H_\omega}) d\theta_\varepsilon||_{BV(I)}$$

$$\leq K \int_{\Omega_\varepsilon} (V(\rho \mathbf{1}_{H_\omega}) + ||\rho \mathbf{1}_{H_\omega}||_1)\, d\theta_\varepsilon \leq \frac{K\tilde{C}}{\rho_m} \frac{\int_{\Omega_\varepsilon} \mu(H_\omega) d\theta_\varepsilon}{A_\varepsilon}. \tag{2.15}$$

Thus, by (2.13), (2.14) and (2.15),

$$\eta_\varepsilon \cdot ||(\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)\rho||_{BV(I)} \leq \frac{K\tilde{C}}{\rho_m} \cdot \Delta_\varepsilon.$$

□

We are now ready to apply the abstract perturbation result of [27]. Indeed, Lemma 2.7 shows that all the conditions imposed in [27] are satisfied by our random systems.

Lemma 2.8.

(1) *For sufficiently small* ε, $\Delta_\varepsilon \neq 0$.
(2) *If for each integer* $k \geq 0$ *the following limit:*

$$q_k := \lim_{\varepsilon \to 0} q_{k,\varepsilon} := \lim_{\varepsilon \to 0} \frac{m((\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)\hat{\mathcal{L}}_\varepsilon^k(\mathcal{L}_0 - \hat{\mathcal{L}}_\varepsilon)(\rho))}{\Delta_\varepsilon}$$

exists, then

$$\lim_{\varepsilon \to 0} \frac{1 - e_\varepsilon}{\Delta_\varepsilon} = 1 - \sum_{k=0}^{\infty} q_k.$$

Proof. By condition **(O3)**, for sufficiently small $\varepsilon > 0$, $\Delta_\varepsilon > 0$. Thus, Part (2) of the lemma follows from the abstract perturbation result of [27], since we will show in the proof of Theorem 2.1 that for each $k \geq 0$ the limit in (2) exists. □

Proof. (Proof of Theorem 2.1) We start by computing $q_{k,\varepsilon}$ and show that the limit in (2) of Lemma 2.8 holds. Throughout the computation in (2.16) below, we will use repeatedly the fact that for any measurable set A and integrable function f we have $\mathbf{1}_A\mathcal{L}_0 f = \mathcal{L}_0(\mathbf{1}_{T^{-1}A}f)$.

Consequently, by (2.11) and (2.2), we obtain

(2.16)

$$
\begin{aligned}
&m((\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)\hat{\mathcal{L}}_\varepsilon^k(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))\\
&=\int_{\Omega_\varepsilon}\int_I((\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)\hat{\mathcal{L}}_\varepsilon^k(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))dmd\theta_\varepsilon(\omega)\\
&=\int_{\Omega_\varepsilon}\int_I\mathcal{L}_0(\mathbf{1}_{H_\omega}\cdot\hat{\mathcal{L}}_\varepsilon^k(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))dmd\theta_\varepsilon(\omega)\\
&=\int_{\Omega_\varepsilon}\int_I\mathbf{1}_{H_\omega}\cdot\hat{\mathcal{L}}_\varepsilon^k(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))dmd\theta_\varepsilon(\omega)\\
&=\int_{\Omega_\varepsilon}\int_I\int_{\bar{\Omega}_\varepsilon}\mathbf{1}_{H_\omega}\cdot\mathcal{L}_0^k\mathbf{1}_{X_{\omega_1}\cap\cdots\cap T^{-(k-1)}X_{\omega_k}}\cdot(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))d\theta_\varepsilon(\bar{\omega})dmd\theta_\varepsilon(\omega)\\
&=\int_{\Omega_\varepsilon}\int_I\int_{\bar{\Omega}_\varepsilon}\mathcal{L}_0^k(\mathbf{1}_{T^{-k}H_\omega}\cdot\mathbf{1}_{X_{\omega_1}\cap\cdots\cap T^{-(k-1)}X_{\omega_k}}\cdot(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))d\theta_\varepsilon(\bar{\omega})dmd\theta_\varepsilon(\omega)\\
&=\int_{\Omega_\varepsilon}\int_{\bar{\Omega}_\varepsilon}\int_I\mathbf{1}_{T^{-k}H_\omega}\cdot\mathbf{1}_{X_{\omega_1}\cap\cdots\cap T^{-(k-1)}X_{\omega_k}}\cdot(\mathcal{L}_0-\hat{\mathcal{L}}_\varepsilon)(\rho))dmd\theta_\varepsilon(\bar{\omega})d\theta_\varepsilon(\omega)\\
&=\int_{\Omega_\varepsilon}\int_{\bar{\Omega}_\varepsilon}\int_I\int_{\Omega_\varepsilon}\mathbf{1}_{T^{-k}H_\omega}\cdot\mathbf{1}_{X_{\omega_1}\cap\cdots\cap T^{-(k-1)}X_{\omega_k}}\cdot(\mathcal{L}_0\mathbf{1}_{H_{\omega'}}\rho)dmd\theta_\varepsilon(\bar{\omega})d\theta_\varepsilon(\omega)d\theta_\varepsilon(\omega')\\
&=\int_{\Omega_\varepsilon}\int_{\bar{\Omega}_\varepsilon}\int_I\int_{\Omega_\varepsilon}\mathcal{L}_0(\mathbf{1}_{T^{-(k+1)}H_\omega}\cdot\mathbf{1}_{T^{-1}X_{\omega_1}\cap\cdots\cap T^{-k}X_{\omega_k}}\cdot\mathbf{1}_{H_{\omega'}}\rho)dmd\theta_\varepsilon(\bar{\omega})d\theta_\varepsilon(\omega)d\theta_\varepsilon(\omega')\\
&=\int_{\Omega_\varepsilon}\int_{\bar{\Omega}_\varepsilon}\int_{\Omega_\varepsilon}\mu\left([T^{-(k+1)}H_\omega\cap T^{-1}X_{\omega_1}\cap\cdots\cap T^{-k}X_{\omega_k}]\cap H_{\omega'}\right)d\theta_\varepsilon(\omega')d\theta_\varepsilon(\bar{\omega})d\theta_\varepsilon(\omega).
\end{aligned}
$$

Now, if z is not a periodic point of T, since T is continuous at z, for sufficiently small $\varepsilon>0$,

$$[T^{-1}X_{\omega_1}\cap\cdots\cap T^{-k}X_{\omega_k}\cap T^{-(k+1)}H_\omega]\cap H_{\omega'}=\emptyset.$$

Therefore, (2.16) implies $q_{k,\varepsilon}=q_k=0$. This proves (1) of the theorem.

To prove (2) of the theorem, since z is a periodic point of minimal period p and the density ρ is essentially bounded from below, the quantity we have to compute reduces to

$$\frac{\int\int m(T^{-p}H_\omega\cap H_{\omega'})d\theta_\epsilon(\omega)d\theta_\epsilon(\omega')}{\int m(H_\omega)d\theta_\epsilon(\omega)}. \tag{2.17}$$

Since T^p is continuous around z we can suppose that, by taking ϵ small enough, all the H_ω, $|\omega|\le\epsilon$, will be contained in a neighborhood of z where T^p is continuous and monotone. This in particular implies the following fact that we will use later on: if we call $H_{\omega,p}$ the unique connected component of $T^{-p}H_\omega$ which contains z, it will be the only connected component of $T^{-p}H_\omega$ which intersects any other $H_{\omega'}$.

By the differentiability assumption on T^p and the local change of variable we have

$$m(T^{-p}H_\omega \cap H_{\omega'}) = \int_{H_{\omega,p}\cap H_{\omega'}} dm = \int_{T^p(H_{\omega,p}\cap H_{\omega'})} |DT^p(y)|^{-1} dm(y) = \int_{H_\omega \cap T^p H_{\omega'}} |DT^p(y)|^{-1} dm(y). \tag{2.18}$$

Equation (2.18) gives immediately the upper bound

$$m(T^{-p}H_\omega \cap H_{\omega'}) \leq \sup_{H_\omega} |DT^p|^{-1}\ m(H_\omega) \leq \sup_{H_\varepsilon} |DT^p|^{-1}\ m(H_\omega). \tag{2.19}$$

By (2.19) and the continuity of DT^p at z we obtain

$$\lim_{\varepsilon\to 0} \frac{\int\int m(T^{-p}H_\omega \cap H_{\omega'}) d\theta_\epsilon(\omega) d\theta_\epsilon(\omega')}{\int m(H_\omega) d\theta_\epsilon(\omega)} \leq |DT^p|^{-1}(z). \tag{2.20}$$

For the lower bound, since T^p is expanding, we have

$$\begin{aligned} m(T^{-p}H_\omega \cap H_{\omega'}) &\geq \int_{H_\omega \cap H_{\omega'}} |DT^p(y)|^{-1} dm(y) \\ &= \int_{H_\omega} |DT^p(y)|^{-1} dm(y) - \int_{H_\omega/H_{\omega'}} |DT^p(y)|^{-1} dm(y) \\ &\geq \inf_{H_\varepsilon} |DT^p|^{-1}\ m(H_\omega) - \int_{H_\omega/H_{\omega'}} |DT^p(y)|^{-1} dm(y). \end{aligned} \tag{2.21}$$

Therefore,

$$\frac{\int\int m(T^{-p}H_\omega \cap H_{\omega'}) d\theta_\epsilon(\omega) d\theta_\epsilon(\omega')}{\int m(H_\omega) d\theta_\epsilon(\omega)} \geq \int\int \inf_{H_\varepsilon} |DT^p|^{-1} d\theta_\epsilon(\omega) d\theta_\epsilon(\omega') - \frac{\int\int\int_{H_\omega/H_{\omega'}} |DT^p(y)|^{-1} dm(y) d\theta_\epsilon(\omega) d\theta_\epsilon(\omega')}{\int m(H_\omega) d\theta_\epsilon(\omega)}. \tag{2.22}$$

Let us consider the second expression on the right hand side of (2.22) and show that it vanishes when ε goes to zero. First, we have

$$\int_{H_\omega/H_{\omega'}} |DT^p(y)|^{-1} dm(y) \leq \sup_{H_\omega} |DT^p|^{-1}\ m(H_\omega \Delta H_{\omega'}).$$

Now, let us call G_ω the complement of H_ω in H_ε; we immediately have by the condition **(C)**: $m(H_\omega \cap H_{\omega'}) = m(H_{\omega'}) - m(H_{\omega'} \cap G_\omega) \geq (\varepsilon - \varepsilon^\upsilon) - \varepsilon^\upsilon \geq \varepsilon - 2\varepsilon^\upsilon$. This implies that $m(H_\omega/H_{\omega'}) = m(H_\omega) - m(H_\omega \cap H_{\omega'}) \leq 2\varepsilon^\upsilon$ and therefore $m(H_\omega \Delta H_{\omega'}) \leq 4\varepsilon^\upsilon$. Let us call $F_\varepsilon := \{\omega \in \Omega_\varepsilon;\ m(H_\omega) \in (\varepsilon - \varepsilon^\upsilon, \varepsilon)\}$; $Q_\varepsilon := \{(\omega, \omega') \in \Omega_\varepsilon \times \Omega_\varepsilon;\ m(H_\omega \Delta H_{\omega'})\} \leq 4\varepsilon^\upsilon$. Notice that $Q_\varepsilon \supset F_\varepsilon \times F_\varepsilon$ and recall that $\theta_\varepsilon(F_\varepsilon) \geq 1 - \varepsilon^\upsilon$.

Then

$$\begin{aligned}
&\frac{\int\int\int_{H_\omega/H_{\omega'}}|DT^p(y)|^{-1}dm(y)d\theta_\epsilon(\omega)d\theta_\epsilon(\omega')}{\int m(H_\omega)d\theta_\epsilon(\omega)}\\
&\leq \frac{\int\int m(H_\omega\Delta H_{\omega'})d\theta_\epsilon(\omega)d\theta_\epsilon(\omega')}{\int m(H_\omega)d\theta_\epsilon(\omega)}\\
&\leq \frac{1}{\int m(H_\omega)d\theta_\epsilon(\omega)}[\int\int_{Q_\epsilon} m(H_\omega\Delta H_{\omega'})d\theta_\epsilon(\omega)d\theta_\epsilon(\omega')\\
&+\int\int_{Q_\varepsilon^c} m(H_\omega\Delta H_{\omega'})d\theta_\epsilon(\omega)d\theta_\epsilon(\omega')]\\
&\leq \frac{1}{\int m(H_\omega)d\theta_\epsilon(\omega)}[4\epsilon^\upsilon+\theta_\varepsilon^2(F_\varepsilon^c))]\\
&\leq \frac{1}{\int_{F_\varepsilon} m(H_\omega)d\theta_\epsilon(\omega)}[4\epsilon^\upsilon+\varepsilon^{2\upsilon}]\leq \frac{4\epsilon^\upsilon+\varepsilon^{2\upsilon}}{(\epsilon-\epsilon^\upsilon)(1-\epsilon^\upsilon)}
\end{aligned}$$

which goes to zero when ε tends to 0. Consequently, taking the limit as ε goes to 0 in (2.22), by the continuity of DT^p at z, we obtain

$$\lim_{\varepsilon\to 0}\frac{\int\int m(T^{-p}H_\omega\cap H_{\omega'})d\theta_\epsilon(\omega)d\theta_\epsilon(\omega')}{\int m(H_\omega)d\theta_\epsilon(\omega)}\geq |DT^p|^{-1}(z). \tag{2.23}$$

Thus (2) of the theorem follows by (2.20) and (2.23). □

3. Metastability of randomly perturbed maps

In this section we study random perturbations of Lasota-Yorke maps. We assume that the initial system admits exactly two ergodic invariant densities. Then under random perturbations which allow leakage of mass through random holes, we will show that the system admits a unique stationary density. Our goal is to show that such a density can be approximated in the L^1-norm by a particular convex combination of the two invariant densities of the initial system. A deterministic Lasota-Yorke system of this type was studied in [22]. Deterministic intermittent systems of this type were studied in [3]. Other recent results on metastable random dynamical systems have been obtained in [20, 24]. We will establish a close link between the escape rate formulae which we obtained in the previous section and invariant densities of randomly perturbed metastable systems. We first introduce the class of maps of this section.

3.1. **The initial system.** Let $T: I\to I$ be a map which satisfies the following conditions:
(A1) There exists a partition of I, which consists of intervals $\{I_i\}_{i=1}^q$, $I_i\cap I_j=\emptyset$ for $i\neq j$, $\bar{I}_i:=[c_{i,0},c_{i+1,0}]$ and there exists $\delta>0$ such that $T_{i,0}:=T|_{(c_{i,0},c_{i+1,0})}$ is C^2 which extends to a C^2 function $\bar{T}_{i,0}$ on a neighbourhood $[c_{i,0}-\delta,c_{i+1,0}+\delta]$ of $\bar{I}_i$;
(A2) $\inf_{x\in I\setminus\mathcal{C}_0}|T'(x)|\geq\beta_0^{-1}>2$, where $\mathcal{C}_0=\{c_{i,0}\}_{i=1}^q$.
(A3) $\exists\ b$ in the interior of I such that $T|_{I_*}\subseteq I_*$, where $*\in\{l,r\}$, I_* is an interval such that $I_l\cup I_r=I$ and $I_l\cap I_r=\{b\}$.
(A4) Let $H_0:=T^{-1}\{b\}\setminus\{b\}$. We call H_0 the set of *infinitesimal holes* and we assume that for every $n\geq 1$, $(T^n\mathcal{C}_0)\cap H_0=\emptyset$.

(A5) T admits exactly two ergodic a.c.i.ms μ_*, such that each μ_* is supported on I_* and the corresponding density ρ_* is positive at each of the points of $H_0 \cap I_*$

Remark 3.1. As shown in [22], assumption **(A4)** guarantees that the invariant densities of the two ergodic measures are continuous at each of the infinitesimal holes.
Assumption **(A2)**, more precisely the fact that β_0^{-1} is strictly bigger than 2 instead of 1, is sufficient to get the uniform Lasota-Yorke inequality of Proposition 3.2, as is explained in section 4.2 of [22]. Finally, we consider T to be bi-valued at the points $c_{i,0}$ of discontinuity in $\mathcal{C}_0$ and we take $T(c_{i,0})$ be both values obtained as x tends to $c_{i,0}$ from either side, and $T(c_{i,0}\pm)$ will be the corresponding right and left limits.

An example of a map T_0 satisfying the above assumption is shown in Figure 2. We denote the transfer operator (Perron-Frobenius) [4, 7] associated with T by $\mathcal{L}_0$.

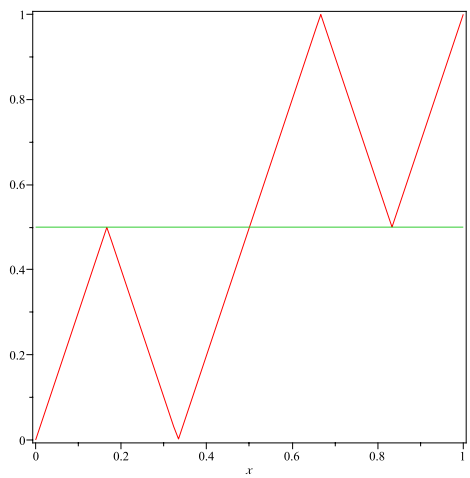

FIGURE 2. A typical example of the initial system T.

3.2. **Random perturbations on a finite noise space.** As in the previous section, let $(\omega_k)_{k\in\mathbb{N}}$ be an i.i.d. stochastic process with values in Ω_ε and with probability distribution θ_ε. However, we assume in this section that $\Omega_\varepsilon = \{\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_L\}$ is a finite set, and θ_ε is an atomic measure on Ω_ε[5]. We associate with each $\omega \in \Omega_\varepsilon$ a map $T_\omega : I \to I$ with $T_0 = T_{\mathbf{s}_0}$, $T_\varepsilon = T_{\mathbf{s}_L}$, and we consider the random orbit starting from the point x and generated by the realization $\underline{\omega}_n = (\omega_1, \omega_2, \cdots, \omega_n)$,

[5]In fact, the assumption that the set Ω_ε is a finite set is not need in some of the proofs of this section. In particular in Lemmas 3.7 and 3.8. However, this assumption is needed to be able to identify the jumps in the stationary density ρ_ε of the random system. It seems that the assumption of this section that Ω_ε is a finite set is the best one can do when dealing with approximating the invariant density of the random metastable system. See Remark 3.10 where we explain why the problem becomes intractable if one relaxes this assumption even to the case where Ω_ε is countable but not finite.

defined as : $T_{\underline{\omega}_n} := T_{\omega_n} \circ \cdots \circ T_{\omega_1}(x)$ (*random transformations*). This defines a Markov process $\mathfrak{T}_\varepsilon$ with transition function

$$\mathbb{P}(x, A) = \int_{\Omega_\varepsilon} \mathbf{1}_A(T_\omega(x)) d\theta_\varepsilon(\omega),$$

where $A \in \mathfrak{B}(I)$, $x \in I$ and $\mathbf{1}_A$ is the indicator function of a set A. The transition function induces an operator $\mathcal{U}_\varepsilon^*$ which acts on measures Λ on $(I, \mathfrak{B}(I))$ as:

$$\mathcal{U}_\varepsilon^* \Lambda(A) = \int_I \int_{\Omega_\varepsilon} \mathbf{1}_A(T_\omega(x)) d\theta_\varepsilon(\omega) d\Lambda(x) = \int_I \mathcal{U}_\varepsilon \mathbf{1}_A(x) d\mu_\varepsilon(x),$$

where $\mathcal{U}_\varepsilon$ is the random evolution operator acting on L_m^∞ functions g:

$$\mathcal{U}_\varepsilon g = \int_{\Omega_\varepsilon} g \circ T_\omega d\theta_\varepsilon(\omega). \tag{3.1}$$

A measure μ_ε on $(I, \mathfrak{B}(I))$ is called a $\mathfrak{T}_\varepsilon$-stationary measure if and only if, for any $A \in \mathfrak{B}(I)$,

$$\mathcal{U}_\varepsilon^* \mu_\varepsilon(A) = \mu_\varepsilon(A). \tag{3.2}$$

We are interested in studying the metastability of $\mathfrak{T}_\varepsilon$-stationary measures which are absolutely continuous with respect to m: let us call them *acsm*. By (3.1), one can define the transfer operator $\mathcal{L}_\varepsilon$ (Perron-Frobenius) acting on $L^1(I, \mathfrak{B}(I), m)$ by:

$$(\mathcal{L}_\varepsilon f)(x) = \int_{\Omega_\varepsilon} \mathcal{L}_\omega f(x) d\theta_\varepsilon(\omega), \tag{3.3}$$

which satisfies the duality condition

$$\int_I g \mathcal{L}_\varepsilon f dm = \int_I \mathcal{U}_\varepsilon g f dm \tag{3.4}$$

where g is in some subset of L_m^∞ and $\mathcal{L}_\omega$ is the transfer operator associated with T_ω. In the present setting $g \in BV(I)$; i.e., a function of bounded variation. It is well know that $\mu_\varepsilon := \rho_\varepsilon m$ is a $\mathfrak{T}_\varepsilon$-acsm if and only if $\mathcal{L}_\varepsilon \rho_\varepsilon = \rho_\varepsilon$; i.e., ρ_ε is a $\mathfrak{T}_\varepsilon$-invariant density. For each $\omega \in \Omega_\varepsilon$, we assume that T_ω satisfies the following conditions:

(B1) There exists a partition of I, which consists of intervals $\{I_{i,\omega}\}_{i=1}^q$, $I_{i,\omega} \cap I_{j,\omega} = \emptyset$ for $i \neq j$, $\bar{I}_{i,\omega} := [c_{i,\omega}, c_{i+1,\omega}]$ such that
(i) for each i and for all $0 \leq \omega \leq \varepsilon$, ε small enough, we have that (the quantity δ was introduced in the assumption **(A1)** above):
$[c_{i,0} + \delta, c_{i+1,0} - \delta] \subset [c_{i,\omega}, c_{i+1,\omega}] \subset [c_{i,0} - \delta, c_{i+1,0} + \delta]$; whenever $c_{i,0} \neq 0$ and $c_{i+1}, 0 \neq 1$. In this way we have established a one-to-one correspondence between the unperturbed and the perturbed boundary points of I_i and $I_{i,\omega}$. If $c_{i,0}$ and $c_{i,\omega}$ are two such (left or right) *corresponding points* we will ask that $\forall i$ and $\forall \omega \in \Omega_\varepsilon$:

$$\lim_{\varepsilon \to 0} |c_{i,0} - c_{i,\omega}| = 0 \quad (\textit{uniform collapsing of boundary points}) \tag{3.5}$$

(ii) The map T_ω is locally injective over the closed intervals $\overline{I}_{i,\omega}$, of class C^2 in their interiors, and expanding with $\inf_{\omega,x} |T'_\omega x| \geq \beta > 2$. Moreover, $\forall \omega \in \Omega$, $\forall i = 1, \cdots, q$ and $\forall x \in [c_{i,0} - \delta, c_{i+1,0} + \delta]$ we have

$$\lim_{\varepsilon \to 0} |\bar{T}_{i,0}(x) - T_{i,\omega}(x)| = 0 \quad (\textit{uniform convergence of maps}) \tag{3.6}$$

(B2) Boundary condition:
(i) if $b \notin \mathcal{C}_0$, then $T(b) = b$ and for all $\omega > 0$, $T_\omega(b) = b$;
(ii) if $b \in \mathcal{C}_0$, then $T(b-) < b < T(b+)$ and for all $\omega > 0$, $b \in \mathcal{C}_\omega$, where $\mathcal{C}_\omega = \{c_{i,\omega}\}_{i=1}^q$.

The last condition, as explained in section 2.4 of [22] does not allow the appearance of other holes, besides those around the infinitesimal holes, in the neighborhood of b. These conditions are satisfied when the noise is, for instance, additive: $T_\omega(x) = T(x) + \omega$. It is always well defined on the circle by taking the mod-1 unfolding; however, on the interval we should consider maps T for which the images $|T_\omega(x)|$ lie in the unit interval for all $\omega \in \Omega_\varepsilon$. In both cases the intervals of local monotonicity will be always the same and the differences between the perturbed and unperturbed images are uniformly bounded by ε. Another example of a T_ω is shown in Figure 3.

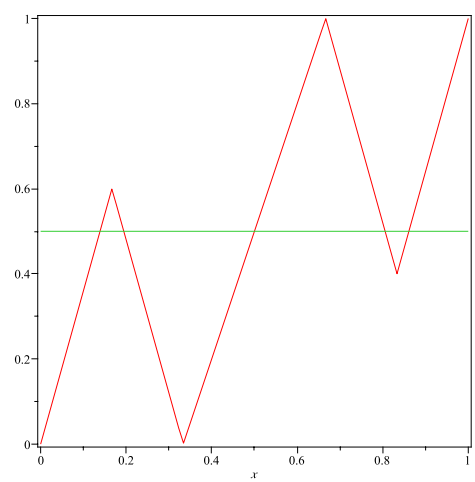

FIGURE 3. A typical graph of one of the constituent maps T_ω of the metastable random map $\mathfrak{T}_\varepsilon$ corresponding to the initial system shown in Figure 2.

Under the above assumptions it is well known that (see for instance [6])

Proposition 3.2. *For sufficiently small $\varepsilon > 0$, there exists a $\eta \in (0,1)$ and a $B \in (0,\infty)$, such that for any $f \in BV(I)$ and $n \geq 1$, we have*

$$\|\mathcal{L}_\bullet^n f\|_{BV} \leq \eta^n \|f\|_{BV} + B||f||_1, \tag{3.7}$$

where $\bullet \in \{0, \varepsilon\}$; i.e., both $\mathcal{L}_0$ and $\mathcal{L}_\varepsilon$ satisfy a uniform Lasota-Yorke inequality as operators on $BV(I)$.

Among other things, Proposition 3.2 implies that 1 is an eigenvalue of $\mathcal{L}_\varepsilon$ and $\mathfrak{T}_\varepsilon$ admits a stationary density ρ_ε. We further assume assume that:

(R1) ρ_ε is the unique invariant density of $\mathfrak{T}_\varepsilon$ (*ergodicity*).

Remark 3.3. Since in this section we assumed Ω_ε to be finite, then condition **(R1)** is satisfied if one of the maps T_ω has a unique acim. See Corollary 2 of [23]. For instance, condition **(R1)** is satisfied whenever the deterministic map T_ε is topologically mixing.

3.3. **Random holes in the perturbed system.** We are interested in perturbations of T which produce "leakage" of mass from I_l to I_r and vice versa. For this purpose, for each $\omega \in \Omega_\varepsilon$ we define the following sets:

$$H_{l,\omega} := I_l \cap T_\omega^{-1}(I_r)$$

and

$$H_{r,\omega} := I_r \cap T_\omega^{-1}(I_l).$$

The sets $H_{l,\omega}$ and $H_{r,\omega}$ are called the "left hole" and the "right hole", respectively, of the map T_ω . Thus, when T_ω allows leakage of mass from I_l to I_r, this leakage occurs when orbits of T_ω fall in the set $H_{l,\omega}$. Similarly, when T_ω allows leakage of mass from I_r to I_l, this leakage occurs when orbits of T_ω fall in the set $H_{r,\omega}$.

For each of the left and right 'random open systems', we define a transfer operator which will be used to find the exponential escape rates from I_l and I_r. For this purpose we set

$$X_{l,\omega} := I_l \setminus H_{l,\omega},$$

and define for $f \in L^1(I_l, \mathfrak{B}(I_l), \frac{m}{b})$

$$\hat{\mathcal{L}}_{l,\varepsilon} f(x) := \int_{\Omega_\varepsilon} \mathcal{L}_{l,0}(f \mathbf{1}_{X_{l,\omega}})(x) d\theta_\varepsilon, \tag{3.8}$$

where $\mathcal{L}_{l,0}$ is the transfer operator associated with the map $T_{|_{I_l}} := T_l : I_l \to I_l$. In a similar way, we can define a transfer operator $\hat{\mathcal{L}}_{r,\varepsilon}$ associated with the right 'random open system'.

3.4. **Statement of the main result of section 3.3.** Using Proposition 3.2 and compactness arguments, it is well known that ρ_ε converges in the L^1-norm to a convex combination of ρ_l and ρ_r. Our main goal in this part of the paper is to explicitly identify the weights in the convex combination. Since the acsm is ergodic, at least one of the two holes will have positive Lebsegue measure; we will suppose without loss of generality that the left hole will have this property. It will also carry positive mass for the left unperturbed invariant measure μ_l since we assumed that the densities of μ_l and μ_r are positive in the neighborhood of the respective holes. We therefore define the *limiting averaged holes ratio (l.a.h.r.)* by

$$l.a.h.r. := \lim_{\varepsilon \to 0} \frac{\int_{\Omega_\varepsilon} \mu_r(H_{r,\omega}) d\theta_\varepsilon(\omega)}{\int_{\Omega_\varepsilon} \mu_l(H_{l,\omega}) d\theta_\varepsilon(\omega)} \tag{3.9}$$

whenever the limit exists. We obtain the following theorem:

Theorem 3.4. *Let ρ_ε be the unique $\mathfrak{T}_\varepsilon$-invariant density. Suppose the l.a.h.r exists then*

$$\lim_{\varepsilon \to 0} ||\rho_\varepsilon - \rho_0||_1 = 0,$$

where $\rho_0 = \alpha\rho_l + (1-\alpha)\rho_r$ and l.a.h.r $= \frac{\alpha}{1-\alpha}$.

Theorem 2.1 will give us additional information about the ratio $\frac{\alpha}{1-\alpha}$ which appears in Theorem 3.4 under the additional assumption:

(B4) The map T is continuous on h_* and $\forall\omega,\ h_* \in H_{*,\omega}$. Moreover, (I_*, μ_*, T_*), $* \in \{l, r\}$, is mixing.

Corollary 3.5. *Assume condition* **(B4)**. *Then*

(1) *For sufficiently small $\varepsilon > 0$, there exists an $0 < e_{*,\varepsilon} < 1$, and a $g_{*,\varepsilon}(x) > 0$, with $\int_{I_*} g_{*,\varepsilon} dm = 1$, such that*

$$\hat{\mathcal{L}}_{*,\varepsilon} g_{*,\varepsilon} = e_{*,\varepsilon} g_{*,\varepsilon}.$$

(2) *Moreover,*

$$\lim_{\varepsilon\to 0} \frac{1 - e_{l,\varepsilon}}{1 - e_{r,\varepsilon}} = \frac{\alpha}{1-\alpha}.$$

Remark 3.6. As it was observed in [22], and this remains true in our case as well, the limit (3.9) surely exists whenever the perturbations open up holes whose size is first order in ε.
We also note that condition **(B4)** is satisfied, for instance, when the map T is continuous in the neighborhood of the infinitesimal holes and we perturb it with an additive noise.

3.5. Technical lemmas and the proof of Theorem 3.4.

Lemma 3.7. *We have $\int_{\Omega_\varepsilon} \mu_\varepsilon(H_{l,\omega}) d\theta_\varepsilon(\omega) = \int_{\Omega_\varepsilon} \mu_\varepsilon(H_{r,\omega}) d\theta_\varepsilon(\omega)$, where $d\mu_\varepsilon = \rho_\varepsilon dm$.*

Proof. We have

$$\mu_\varepsilon(I_l) = \int_{\Omega_\varepsilon} \mu_\varepsilon(I_l) d\theta_\varepsilon(\omega) = \int_{\Omega_\varepsilon} \mu_\varepsilon(X_{l,\omega}) d\theta_\varepsilon(\omega) + \int_{\Omega_\varepsilon} \mu_\varepsilon(H_{l,\omega}) d\theta_\varepsilon(\omega). \tag{3.10}$$

Since $T_\omega^{-1}(I_l) = X_{l,\omega} \cup H_{r,\omega}$, we also have

$$\int_{\Omega_\varepsilon} \mu_\varepsilon(T_\omega^{-1} I_l) d\theta_\varepsilon(\omega) = \int_{\Omega_\varepsilon} \mu_\varepsilon(X_{l,\omega}) d\theta_\varepsilon(\omega) + \int_{\Omega_\varepsilon} \mu_\varepsilon(H_{r,\omega}) d\theta_\varepsilon. \tag{3.11}$$

Using (3.10), (3.11) and that μ_ε is $\mathfrak{T}_\varepsilon$-invariant the lemma follows. □

Lemma 3.8. *There exists a constant K, depending only on the map T, such that for $0 \le n < \infty$ there exists σ, depending on n, and ε_σ, depending on σ, such that $\forall\varepsilon \le \varepsilon_\sigma$ and $\tilde{\sigma} \le \sigma$, we have*

(1) *$\mathcal{L}_\varepsilon^n 1$ is a C^1-function on $I_{\tilde{\sigma}} = [h_* - \tilde{\sigma}, h_* + \tilde{\sigma}]$, $* \in \{l, r\}$;*
(2) *for any $x, y \in I_{\tilde{\sigma}}$, we have*

$$|\mathcal{L}_\varepsilon^n 1(x) - \mathcal{L}_\varepsilon^n 1(y)| \le K \cdot |x - y|;$$

in particular,

$$|\mathcal{L}_\varepsilon^n 1(x) - \mathcal{L}_\varepsilon^n 1(y)| \le K \cdot \tilde{\sigma}.$$

Proof. By condition (A4) we have for every $n > 0$, $(T_0^n C_0) \cap H_0 = \emptyset$. In particular, for a fixed $n \in \mathbb{N}$, we can choose $\sigma' > 0$ so small so that, for $0 \le k \le n$, $(T_0^k C_0) \cap (h_* - \sigma', h_* + \sigma') = \emptyset$. We now prove that there is a possibly smaller subinterval $(h_* - \sigma, h_* + \sigma)$, $\sigma \le \sigma'$ upon which (1) and (2) of the lemma are satisfied. Let us give the explicit form of this operator.

The iterates of $\mathcal{L}_\varepsilon$ are given by:

$$(\mathcal{L}_\varepsilon^n 1) = \int \cdots \int \mathcal{L}_{\omega_1}\left(\mathcal{L}_{\omega_2}\left(\cdots \mathcal{L}_{\omega_n}(1)\right)\right) d\theta_\varepsilon(\omega_1)\cdots d\theta_\varepsilon(\omega_n).$$

Let us call I_{l,ω_j} the l-domain of injectivity of the map T_{ω_j} and call T_{l,ω_j}^{-1} the inverse of T_{ω_j} restricted to I_{l,ω_j}. By $T_{l,0}$ we denote the restriction of the unperturbed map to its l-th interval of monotonicity; finally for simplicity of notation we drop the suffix l, r from the intervals of the partition. We have:

$$\Psi_{\omega_n,\cdots,\omega_1}(x) := \mathcal{L}_{\omega_n}\left(\mathcal{L}_{\omega_{n-1}}\left(\cdots \mathcal{L}_{\omega_1}(1)\right)\right)(x) =$$

$$\sum_{k_1,\ldots,k_n} \frac{1}{|D(T_{\omega_n}\circ\cdots\circ T_{\omega_1})((T_{k_1,\omega_1}^{-1}\circ\cdots\circ T_{k_n,\omega_n}^{-1})(x))|} \times \mathbf{1}_{\Omega_{\omega_1,\cdots,\omega_n}^{k_1,\cdots,k_n}}(x).$$

The sets

$$\Omega_{\omega_1,\cdots,\omega_n}^{k_1,\cdots,k_n} :=$$
$$T_{k_1,\omega_1}^{-1}\circ\cdots\circ T_{k_{n-1},\omega_{n-1}}^{-1} I_{k_n,\omega_n} \cap T_{k_1,\omega_1}^{-1}\circ\cdots\cap T_{k_1,\omega_1}^{-1} I_{k_2,\omega_2}\cap I_{k_1,\omega_1}$$

are intervals and they give a partition mod-0 of $I=[0,1]$; moreover the image

$$H_{\omega_1,\cdots,\omega_n}^{k_1,\cdots,k_n} := T_{\omega_n}\circ\cdots\circ T_{\omega_1}\Omega_{\omega_1,\cdots,\omega_n}^{k_1,\cdots,k_n}$$
$$= T_{\omega_n} I_{k_n,\omega_n} \cap T_{\omega_n}T_{\omega_{n-1}} I_{k_1,\omega_{k-1}} \cap\cdots\cap T_{\omega_n}T_{\omega_{n-1}}\circ\cdots\circ T_{\omega_1} I_{k_1,\omega_1}$$

for a given n-tuple $\{k_n,\cdots,k_1\}$ is a connected interval. We will prove, by contradiction, that the function $\Psi_{\omega_n,\cdots,\omega_1}(x)$ is smooth in a neighborhood of h_*, actually C^1, when h_* is in the interior of one or several of the images $H_{\omega_1,\cdots,\omega_n}^{k_1,\cdots,k_n}$ described above. If $\Psi_{\omega_n,\cdots,\omega_1}(x)$ is not smooth in a neighborhood of h_* on any interval $\mathcal{O}\supset h_*$, we can find a sequence $(\omega_n,\cdots\omega_{n-l})$, $1\le l\le n-1$ and at least an endpoint of one of the domain of injectivity of $T_{\omega_{n-l}}$, call it $c_{\omega_{n-l}}^*$, such that $T_{\omega_n}\circ\cdots\circ T_{\omega_{n-l}}(c_{\omega_{n-l}}^*)\in\mathcal{O}$. We now proceed by induction and we begin by showing that Ψ_ω is smooth in a neighborhood of h_* for any choice of ω. If not there will a point $c_{l,\omega}$ (see above, where we now mean that this point is one of the two boundaries of the interval $I_{l,\omega}$), such that $T_{\omega_l}c_{l,\omega}\in(h_*-\sigma'/2, h_*+\sigma'/2)$. Let us now take the corresponding point $c_{l,0}$ of T which will belongs to C_0. We have

$$|T_{l,0}(c_{l,0}) - T_{l,\omega}(c_{l,\omega})| \le |T_{l,0}(c_{l,0}) - \bar{T}_{l,0}(c_{l,\omega})| + |\bar{T}_{l,0}(c_{l,\omega}) - T_{l,\omega}(c_{l,\omega})|$$

By the uniform conditions (3.5) and (3.6) we can take ε smaller that a certain $\varepsilon_{\sigma'}$ depending on σ' such that

$$|T_{l,0}(c_{l,0}) - \bar{T}_{l,0}(c_{l,\omega})| \le \sup_{l,\ x\in I_{l,0}} |D\bar{T}_{l,0}||c_{l,0}-c_{l,\omega}| \le \sigma'/2$$

and

$$|\bar{T}_{l,0}(c_{l,\omega}) - T_{l,\omega}(c_{l,\omega})| \le \sigma'/2$$

and this implies

$$|T_{l,0}(c_{l,0}) - h_*| \le \sigma'$$

which contradicts the above statement $T^nC_0\cap H_0=\emptyset$. Fix $\sigma=\sigma'/2$; we now continue the induction process by supposing that for any sequence $(\omega_1,\cdots,\omega_n)$ the function $\Psi_{\omega_n,\cdots,\omega_1}$ is smooth on the interval $(h_*-\sigma, h_*+\sigma)$ and we want to prove that the function $\Psi_{\omega_{n+1},\cdots,\omega_1}$ is still smooth for the sequence $(\omega_1,\cdots,\omega_{n+1})$. For that it will be enough to repeat the previous argument by noticing that

$$\Psi_{\omega_{n+1},\cdots,\omega_1} = \mathcal{L}_{\omega_{n+1}}\Psi_{\omega_n,\cdots,\omega_1}.$$

This shows that $\Psi_{\omega_n,\cdots,\omega_1}(x)$ is smooth in a neighborhood of h_*. We now take two points x, y in the interval $(h_* - \sigma, h_* + \sigma)$; we introduce the notation $T^{-1}_{\bar{\omega}_n} := T^{-1}_{k_1,\omega_1} \circ \cdots \circ T^{-1}_{k_n,\omega_n}$ and we compute

$$|\Psi_{\omega_n,\cdots,\omega_1}(x) - \Psi_{\omega_n,\cdots,\omega_1}(y)| \leq$$

$$\sum_{k_1,\ldots,k_n} \left| \frac{1}{|D(T_{\omega_n} \circ \cdots \circ T_{\omega_1})(T^{-1}_{\bar{\omega}_n}(x))|} - \frac{1}{|D(T_{\omega_n} \circ \cdots \circ T_{\omega_1})(T^{-1}_{\bar{\omega}_n}(y))|} \right| \tag{3.12}$$

where we skip the characteristic function since both x, y will be in the same $\Omega^{k_1,\cdots,k_n}_{\omega_1,\cdots,\omega_n}$. We have

$$\begin{aligned} &D\left[\frac{1}{|D(T_{\omega_n} \circ \cdots \circ T_{\omega_1})(z)}\right] \\ &= \sum_{k=0}^{n-1} \frac{D^2 T_{\omega_{n-k}}\left(\prod_{l=1}^{n-1-k} T_{\omega_{n-l}} z\right)}{\left[DT_{\omega_{n-k}}\left(\prod_{l=1}^{n-1-k} T_{\omega_{n-l}} z\right)\right]^2 \prod_{j=0}^{k} DT_{\omega_{n-j+1}}\left(\prod_{l=1}^{n-j} T_{\omega_{n-l}} z\right)}. \end{aligned} \tag{3.13}$$

Since

$$\sup_{\omega, x \in I_{k,\omega}} \left| \frac{D^2 T_\omega(x)}{DT_\omega(x)} \right| \leq C_1 < \infty,$$

which follows from our assumptions on the map T and its perturbations, and

$$\inf_{\omega, x \in I_{k,\omega}} |DT_\omega(x)| \geq \beta > 2,$$

the sum in (3.13) will be bounded by C_1 times the sum of a geometric series of common ratio β^{-1}: we call C_2 the upper bound thus found. Therefore, by the mean value theorem, we have

$$(3.12) \leq C_2 \sum_{k_1,\cdots,k_n} |T^{-1}_{\bar{\omega}_n}(x) - T^{-1}_{\bar{\omega}_n}(y)| \leq |x-y| \sum_{k_1,\cdots,k_n} \frac{1}{|D(T_{\omega_1} \circ \cdots \circ T_{\omega_n})(T^{-1}_{\bar{\omega}_n}(\zeta))|}$$

where $\zeta \in (x, y)$. We can replace ξ with x (or y) by a standard distortion argument which works in our case by the assumptions we have on the maps T_ω. By modifying the constant C_2 into a constant C_3 which takes into account the distortion factor we finally have

$$(3.12) \leq C_3 |x-y| \sum_{k_1,\cdots,k_n} \frac{1}{|D(T_{\omega_1} \circ \cdots \circ T_{\omega_n})(T^{-1}_{\bar{\omega}_n}(x))|}. \tag{3.14}$$

We now integrate (3.14) over the noise and get, for any $x, y \in I_{\tilde{\sigma}}$,

$$|\mathcal{L}^n_\varepsilon 1(x) - \mathcal{L}^n_\varepsilon 1(y)| \leq C_3 \sigma \mathcal{L}^n_\varepsilon 1 \leq C_3 \sigma (1 + B),$$

where $\mathcal{L}^n_\varepsilon 1 \leq 1 + B$ is obtained from the Lasota-Yorke inequality in Proposition 3.2. The Lemma finally follows by choosing $K = C_3(1 + B)$. □

In [22], the authors dealt with the deterministic version of this section of our paper. In [22], the notion of the *postcritical set* of a map was used extensively to identify the location of discontinuities of the invariant density. However, in the random setting, the notion of a postcritical set of $\mathfrak{T}_\varepsilon$ does not exist. Following the ideas of our Lemma 3.8 above, we develop an approach which is suitable for our random setting. We first recall a useful representation of a 1-dimensional function

of bounded variation[6]. For $f \in BV(I)$, choose a version of f with regular discontinuities: for each x, $f(x) = (\lim_{y\to x^-} f(y) + \lim_{y\to x^+} f(y))/2$. Then, f can be uniquely decomposed as $f = f^{reg} + f^{sal}$, where the regular term f^{reg} is continuous and of bounded variation, with $V(f^{reg}) \le V(f)$, while the singular (or saltus) part f^{sal} is the sum of jumps

$$f^{sal} = \sum_{u\in S} s_u H_u,$$

where S is a countable set, and

$$H_u(x) = \begin{cases} -1 & \text{if } x < u \\ -\frac{1}{2} & \text{if } x = u \\ 0 & \text{if } x > u \end{cases}.$$

This representation imposes the boundary condition $f^{sal}(1) = 0$. Moreover, the variation of the singular part satisfies

$$V(f^{sal}) = \sum_{u\in S} |s_u| \le V(f).$$

Call

$$F_{n,\varepsilon} := \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{L}_\varepsilon^k 1 := \frac{1}{n}\sum_{k=0}^{n-1} \int_{\bar\Omega_\varepsilon} \mathcal{L}_{\bar\omega}^k 1 d\theta_\varepsilon^\infty(\bar\omega),$$

where $\mathcal{L}_{\bar\omega}^k = \mathcal{L}_{\omega_1}\cdots\mathcal{L}_{\omega_k}$. Note that, by (**R1**), we have

$$\lim_{n\to\infty} ||F_{n,\varepsilon} - \rho_\varepsilon||_1 = 0.$$

We have already shown in Lemma 3.8 that the set of discontinuities of $\mathcal{L}_{\bar\omega}^n$ is given by

$$\mathcal{S}_{n,\bar\omega} = \cup_{j=0}^{n-1} \cup_{k_{n-j}} T_{\omega_n} T_{\omega_{n-1}} \cdots T_{\omega_{n-j}} \partial I_{k_{n-j},\omega_{n-j}}$$

where k_{n-j} runs over the domains of injectivity of $T_{\omega_{n-j}}$ and ∂I denotes the endpoint of the connected interval I. We rewrite this set as

$$\mathcal{S}_{n,\bar\omega} = \cup_{j=0}^{n-1} \hat{\mathcal{S}}_{j,\bar\omega}, \quad \text{where} \quad \hat{\mathcal{S}}_{j,\bar\omega} := \cup_{k_{n-j}} T_{\omega_n} T_{\omega_{n-1}} \cdots T_{\omega_{n-j}} \partial I_{k_{n-j},\omega_{n-j}}.$$

We also write

$$\tilde{\mathcal{S}}_n := \cup_{\bar\omega} \mathcal{S}_{n,\bar\omega} \quad \text{and} \quad \hat{\mathcal{S}}_j = \cup_{\bar\omega} \hat{\mathcal{S}}_{j,\bar\omega}.$$

Notice that $\hat{S}_j \subset \tilde{\mathcal{S}}_n$. Finally, we write

$$\mathcal{S} := \cup_{n\ge 1} \tilde{\mathcal{S}}_n.$$

Lemma 3.9.

(1) *$\tilde{\mathcal{S}}_n$ is a finite set and $\mathcal{S}$ is a countable set.*
(2) *$[0,1] \setminus \tilde{\mathcal{S}}_n$ is a finite collection of open intervals.*
(3) *For any $\bar\omega := (\omega_n, \ldots, \omega_1)$, $T_{\bar\omega}$ is C^2 on each open interval belonging to $[0,1] \setminus \tilde{\mathcal{S}}_n$.*

Proof. Since Ω_ε is a finite set, (1) follows by definition of $\tilde{\mathcal{S}}_n$ and $\mathcal{S}$. (2) is a consequence of (1). (3) follows by definition of $\tilde{\mathcal{S}}_n$. □

[6]The usefulness of such a representation in studying stability and *response theory* of 1-dimensional invariant densities was popularized by Baladi [5].

Remark 3.10. As we have mentioned in footnote 4, if one relaxes the assumption that Ω_ε is finite, even to the case where Ω_ε is countable but not finite, then intractable complications arise. Indeed, if Ω_ε is countable but not finite, one can construct an example of a random metastable system such that in an interval, $[u, v]$, say, each rational point is in the postcritical set of a different T_ω. For a system like this, the complement set $[0, 1] \setminus \tilde{\mathcal{S}}_n$ becomes uncountable, and its intersection with $[u, v]$ will not contain any interval[7]. This makes it impossible to obtain regularity properties of ρ_ε^{reg}, the regular part of the stationary density ρ_ε.

Lemma 3.11. *The discontinuities of $F_{n,\varepsilon}$ belong to $\tilde{\mathcal{S}}_n$.*

Proof. We prove the lemma by contradiction. Namely, let a be a discontinuity point for $F_{n,\varepsilon}$, and suppose that $a \notin \tilde{\mathcal{S}}_n$,. Then $\forall \bar{\omega}$ and $0 \le k \le n-1$, we have that

$$\lim_{x \to a} \mathcal{L}_{\bar{\omega}}^k 1(x) = \mathcal{L}_{\bar{\omega}}^k 1(a). \tag{3.15}$$

Since, for sufficiently small ϵ, each $\mathcal{L}_\omega$, $\omega \in \Omega_\varepsilon$, satisfies a Lasota-Yorke inequality with uniform constants (see Proposition 3.2)

$$V\mathcal{L}_\omega f \le \eta V f + B_0 ||f||_1,$$

we concatenate the transfer operators and get the existence of a positive constant C', independent of the realization, for which

$$||\mathcal{L}_{\bar{\omega}}^k 1||_\infty \le ||\mathcal{L}_{\bar{\omega}}^k 1||_{BV} \le C'.$$

Therefore we can use (3.15) and apply the Lebesgue dominated convergence theorem to get

$$\begin{aligned} F_{n,\varepsilon}(a) &= \frac{1}{n} \sum_{k=0}^{n-1} \int_{\bar{\Omega}_\varepsilon} \mathcal{L}_{\bar{\omega}}^k 1(a) = \frac{1}{n} \sum_{k=0}^{n-1} \int_{\bar{\Omega}_\varepsilon} \lim_{x \to a} \mathcal{L}_{\bar{\omega}}^k 1(x) \\ &= \lim_{x \to a} \frac{1}{n} \sum_{k=0}^{n-1} \int_{\bar{\Omega}_\varepsilon} \mathcal{L}_{\bar{\omega}}^k 1(x) = \lim_{x \to a} F_{n,\varepsilon}(x). \end{aligned}$$

Thus, a contradiction. □

Let

$$\#u = \inf\{j \ge 1;\ u \in \hat{\mathcal{S}}_j\}.$$

Lemma 3.12.

(1) *The discontinuity set of ρ_ε is subset of $\mathcal{S}$. If we write $\rho_\varepsilon^{sal} := \sum_{u \in \mathcal{S}} s_u H_u$, then $\sum_{u \in \mathcal{S}, \#u > m} |s_u| \le \eta^{-m} C'$.*

(2) *ρ_ε^{reg} is Lipschitz continuous.*

---

[7]One can easily construct a random system with such a bad behaviour. For example, let us write the formula of the first branch for the map T_0 in Figure 2. We have $T_0(x) := 3x$ for $0 \le x \le 1/6$. For a fixed rational number ε, Let $\Omega_\varepsilon = \mathbb{Q} \cap [0, \varepsilon]$. Then, each $\omega \in \Omega_\varepsilon$ is a rational number. Further, define the first branch of $T_\omega(x) := (3 + 6\omega)x$, for $0 \le x \le 1/6$. Then $\omega + 1/2$ is in the postcritical set of T_ω. In particular $\tilde{S}_1 \cap [1/2, 1/2 + \varepsilon] = \{\omega + 1/2\}_{\omega \in \Omega_\varepsilon}$. Consequently, $[0, 1] \setminus (\tilde{S}_1 \cap [1/2, 1/2 + \varepsilon])$ is an uncountable set and it does not contain any interval.

Proof. Since $F_{n,\varepsilon}$ is a function of bounded variation, $F_{n,\varepsilon}$ can be decomposed as $F_{n,\varepsilon} = F^{reg}_{n,\varepsilon} + F^{sal}_{n,\varepsilon}$. By Lemma 3.11, the discontinuity set of $F_{n,\varepsilon}$ is contained in $\tilde{\mathcal{S}}_n \subset \mathcal{S}$. Then we can write the saltus part of $F_{n,\varepsilon}$ as

$$F^{sal}_{n,\varepsilon} = \sum_{u\in\mathcal{S}} s_{u,n} H_u.$$

We first show that $\sum_{\#u>m} |s_{u,n}|$ decays exponentially fast in m. To show this, observe that if $m \geq n$ then $\sum_{u\in S;\ \#u>m} |s_{u,n}| = 0$ and

$$\sum_{u\in\mathcal{S};\ \#u>0} |s_{u,n}| \leq V(F_{n,\varepsilon}) \leq V(\mathcal{L}^k_\varepsilon 1) \leq C'.$$

Moreover,

$$F_{n,\varepsilon} = \frac{n-1}{n}\mathcal{L}_\varepsilon F_{n-1,\varepsilon} + \frac{1}{n}.$$

Now for $\#u > 1$, i.e. $u \notin \hat{\mathcal{S}}_1$, which means that $\forall\omega$, each preimage $T^{-1}_{k,\omega}$ is continuous at u. Therefore the contributions to the jump $s_{u,n}$ come from the jumps of $F_{n-1,\varepsilon}$ and are located at the points v such that $T_\omega v = u$, precisely

$$|s_{u,n}| \leq \frac{n-1}{n}\frac{1}{\eta}\int \sum_{v\in\mathcal{S};T_{\omega'}v=u} |s_{v,n-1}| d\theta_\varepsilon(\omega').$$

Then if $0 < m < n$ we have

$$\begin{aligned}\sum_{\#u>m} |s_{u,n}| &\leq \sum_{\#u>m} \frac{n-1}{n}\frac{1}{\eta}\int \sum_{v\in\mathcal{S};T_{\omega'}v=u} |s_{v,n-1}| d\theta_\varepsilon(\omega') \\ &\leq \frac{n-1}{n}\frac{1}{\eta}\sum_{\#u>m-1} |s_{u,n-1}| \leq \cdots \\ &\leq \frac{n-m}{n}\frac{1}{\eta^m}\sum_{\#u>0} |s_{u,n-m}| \leq \frac{1}{\eta^m}C'.\end{aligned}$$

By using a diagonalization argument, we may find a subsequence n_j such that for each $u \in \mathcal{S}$, s_{u,n_j} converges as $n_j \to \infty$ to some number, which we call s'_u. Let us construct the function

$$F^{sal}_\varepsilon := \sum_{u\in\mathcal{S}} s'_u H_u.$$

It is clear that $F^{sal}_{n,\varepsilon}$ converges in L^1 to F^{sal}_ε and that for each m: $\sum_{u\in\mathcal{S},\#u>m} |s'_u| \leq \eta^{-m}C'$. We now show that F^{sal}_ε coincides with ρ^{sal}_ε and F^{reg}, a limit point of $F^{reg}_{n,\varepsilon}$, coincides with ρ^{reg}_ε. Indeed, by (3) of Lemma 3.9, $F_{n,\varepsilon}$ is C^1 on the open intervals of $[0,1] \setminus \tilde{\mathcal{S}}_n$. Therefore, on $[0,1] \setminus \tilde{\mathcal{S}}_n$, the derivative of $F_{n,\varepsilon}$ coincides with the derivative of $F^{reg}_{n,\varepsilon}$ (since $F^{sal}_{n,\varepsilon}$ is constant on each interval of $[0,1] \setminus \tilde{\mathcal{S}}_n$). Then, using the arguments in Lemma 3.8, in particular estimates (3.13) and (3.14), we obtain a uniform estimate on the derivative of $F^{reg}_{n,\varepsilon}$ on $[0,1] \setminus \tilde{\mathcal{S}}_n$. By (2) Lemma 3.9, $[0,1]\setminus\tilde{\mathcal{S}}_n$ consists of a finite number of intervals. Thus, since by definition $F^{reg}_{n,\varepsilon}$ is a continuous function, we conclude that $F^{reg}_{n,\varepsilon}$ is uniformly Lipschitz on $[0,1]$. Consequently, by the Arzelà-Ascoli theorem, we can find a continuous function F^{reg} such that some subsequence of $\{F^{reg}_{n_j,\varepsilon}\}$ converges in L^∞, and hence in L^1, to F^{reg}. By the uniqueness of the decomposition $\rho_\varepsilon = \rho^{sal}_\varepsilon + \rho^{reg}_\varepsilon$, we get that $F^{reg} = \rho^{reg}_\varepsilon$ and $F^{sal} = \rho^{sal}_\varepsilon$. □

Lemma 3.13.

(1) *For any $\tau > 0$ there exist an n, $0 \leq n < \infty$, a σ depending on n, and an ε_σ, depending on σ, such that $\forall \varepsilon \leq \varepsilon_\sigma$ and $\tilde{\sigma} \leq \sigma$, we have*

$$V_{I_{\tilde{\sigma}}} \rho_\varepsilon^{sal} < \eta^{-n} C' < \tau,$$

where $I_{\tilde{\sigma}} = [h_* - \tilde{\sigma}, h_* + \tilde{\sigma}]$, $* \in \{l, r\}$ and C' is the constant which appears in Lemma 3.12.

(2) *For any $x, y \in I_{\tilde{\sigma}}$, there exists a uniform constant K such that*

$$|\rho_\varepsilon^{reg}(x) - \rho_\varepsilon^{reg}(y)| \leq K \cdot |x - y|.$$

Proof. Choose n large enough so that $\eta^{-n} C' < \tau$. Then, using (1) of Lemma 3.8, for $n > 0$ such that $\eta^{-n} C' < \tau$ we can find a σ, depending on n, and an ε_σ, depending on σ, such that $\forall \varepsilon \leq \varepsilon_\sigma$ and $\tilde{\sigma} \leq \sigma$, $\mathcal{L}_\varepsilon^n 1$ is a C^1-function on $I_{\tilde{\sigma}}$. Consequently, using Lemma 3.12 with $m = n$, we obtain part (1) of the lemma. Part (2) of the lemma is a consequence of part (2) of Lemma 3.12. □

We now observe that by the Lasota-Yorke inequality of Proposition 3.2 and the compactness of the unit closed ball of $|| \cdot ||_{BV}$ in L^1, there are subsequences of values $\tilde{\varepsilon} \in \Omega_\varepsilon$ for which $\rho_{\tilde{\varepsilon}}$ converges in the L^1 norm to the density of an acim on I. Obviously, such a density will surely be a convex combination of ρ_l and ρ_r. We would like to show that those accumulation points are always the same so that ρ_ε will admit a limit when $\varepsilon \to 0$. The first step is to characterize the coefficients of the convex combination in terms of the behavior of the density in the neighborhoods of the holes. We first have:

Lemma 3.14. *Let ρ_0 be the accumulation point of the convergent subsequence $\rho_{\tilde{\varepsilon}}$ with $\tilde{\varepsilon} \in \Omega_\varepsilon$; then there exists $0 \leq \alpha \leq 1$ such that:*[8]

(1) $\rho_0 = \alpha \rho_l + (1 - \alpha) \rho_r$;

(2) $\lim\limits_{\tilde{\varepsilon} \to 0} \sup\limits_{\omega \in \Omega_{\tilde{\varepsilon}}} \sup\limits_{x \in H_{l,\omega}} |\rho_{\tilde{\varepsilon}}(x) - \alpha \rho_l(x)| = 0$;

(3) $\lim\limits_{\tilde{\varepsilon} \to 0} \sup\limits_{\omega \in \Omega_{\tilde{\varepsilon}}} \sup\limits_{x \in H_{r,\omega}} |\rho_{\tilde{\varepsilon}}(x) - (1 - \alpha) \rho_r(x)| = 0$.

Proof. We argued above that (1) is true. The proof of (3) will be identical to that of (2). Thus, we only prove (2). Observe that for all $\omega \in \Omega_{\tilde{\varepsilon}}$ we have $H_{l,\omega} \subseteq H_{l,\tilde{\varepsilon}}$. Therefore,

$$\sup_{\omega \in \Omega_{\tilde{\varepsilon}}} \sup_{x \in H_{l,\omega}} |\rho_{\tilde{\varepsilon}}(x) - \alpha \rho_l(x)| \leq \sup_{x \in H_{l,\tilde{\varepsilon}}} |\rho_{\tilde{\varepsilon}}(x) - \alpha \rho_l(x)|.$$

Thus, it is enough to prove that

$$\lim_{\tilde{\varepsilon} \to 0} \sup_{x \in H_{l,\tilde{\varepsilon}}} |\rho_{\tilde{\varepsilon}}(x) - \alpha \rho_l(x)| = 0. \tag{3.16}$$

We will prove (3.16) by contradiction. For this purpose we will suppose that the there exists a $C > 0$ and a subsequence $\varepsilon' \to 0$ of $\tilde{\varepsilon}$ such that for each ε' there is an $x_{\varepsilon'} \in H_{l,\varepsilon'}$, $x_{\varepsilon'} \to h_l$ with $|\rho_{\varepsilon'}(x_{\varepsilon'}) - \alpha \rho_l(x_{\varepsilon'})| > C$. By Lemma 1 of [22], for sufficiently small $\bar{\sigma} > 0$, for any $x \in [h_l - \bar{\sigma}, h_l + \bar{\sigma}] := I_{\bar{\sigma}}$ and $x_{\varepsilon'} \in I_{\bar{\sigma}}$ we have

$$|\alpha \rho_l(x_{\varepsilon'}) - \alpha \rho_l(x)| < 2C/5. \tag{3.17}$$

[8]Later in the proof of (1) of Theorem 3.4, we will find $\frac{\alpha}{1-\alpha}$ explicitly. In particular, we will show that $l.a.h.r = \frac{\alpha}{1-\alpha}$.

Choose n big enough so that $C'\eta^{-n} < C/5$. By Lemma 3.13 we can find a σ depending on n, and an ε'_σ, depending on σ, such that $\forall \varepsilon' \leq \varepsilon'_\sigma$ and $\tilde{\sigma} \leq \sigma$, we have

$$V_{I_{\tilde{\sigma}}} \rho^{sal}_{\varepsilon'} < C/5,$$

and for any $x, y \in I_{\tilde{\sigma}}$, there exists a uniform constant K such that

$$|\rho^{reg}_{\varepsilon}(x) - \rho^{reg}_{\varepsilon}(y)| \leq K \cdot |x - y| \leq K\tilde{\sigma}.$$

We make sure to take $\tilde{\sigma}$ small enough such that $K\tilde{\sigma} < C/5$, and $\tilde{\sigma} \leq \bar{\sigma}$. Then for $x, x_{\varepsilon'} \in I_{\tilde{\sigma}}$ we have

$$|\rho_{\varepsilon'}(x_{\varepsilon'}) - \rho_{\varepsilon'}(x)| \leq |\rho^{sal}_{\varepsilon'}(x_{\varepsilon'}) - \rho^{sal}_{\varepsilon'}(x)| + |\rho^{reg}_{\varepsilon'}(x_{\varepsilon'}) - \rho^{reg}_{\varepsilon'}(x)| < 2C/5 \tag{3.18}$$

Thus, for any $x \in I_{\tilde{\sigma}}$ and $x_{\varepsilon'} \in I_{\tilde{\sigma}}$, by (3.17), (3.18) and the assumption that $|\rho_{\varepsilon'}(x_{\varepsilon'}) - \alpha\rho_l(x_{\varepsilon'})| > C$, we obtain

(3.19)
$$\begin{aligned} |\rho_{\varepsilon'}(x) - \alpha\rho_l(x)| &\geq |\rho_{\varepsilon'}(x_{\varepsilon'}) - \alpha\rho_l(x_{\varepsilon'})| - |\alpha\rho_l(x) - \alpha\rho_l(x_{\varepsilon'})| - |\rho_{\varepsilon'}(x_{\varepsilon'}) - \rho_{\varepsilon'}(x)| \\ &\geq C - 2C/5 - 2C/5 = C/5. \end{aligned}$$

The estimate in (3.19) contradicts the fact that $\lim_{\tilde{\varepsilon}\to 0} ||\rho_{\tilde{\varepsilon}} - \rho_0||_1 = 0$. Therefore, (3.16) holds and the lemma follows. □

We are now ready to prove Theorem 3.4.

Proof. (of Theorem 3.4) Let $\rho_0 = \alpha\rho_l + (1-\alpha)\rho_r$ be the limit of $\rho_{\tilde{\varepsilon}}$ obtained in (1) of Lemma 3.14. We will show that $\alpha/(1-\alpha) = l.a.h.r.$ This will imply the first part of Theorem 3.4. We first observe that

(3.20)
$$\begin{aligned} &\int_{\Omega_{\tilde{\varepsilon}}} \mu_{\tilde{\varepsilon}}(H_{l,\omega}) d\theta_{\tilde{\varepsilon}}(\omega) = \int_{\Omega_{\tilde{\varepsilon}}} \int_{H_{l,\omega}} \rho_{\tilde{\varepsilon}} dx d\theta_{\tilde{\varepsilon}}(\omega) \\ &= \int_{\Omega_{\tilde{\varepsilon}}} \int_{H_{l,\omega}} \alpha\rho_l dx d\theta_{\tilde{\varepsilon}}(\omega) + \int_{\Omega_{\tilde{\varepsilon}}} \int_{H_{l,\omega}} (\rho_{\tilde{\varepsilon}} - \alpha\rho_l) dx d\theta_{\tilde{\varepsilon}}(\omega) \\ &= \alpha \int_{\Omega_{\tilde{\varepsilon}}} \mu_l(H_{l,\omega}) d\theta_{\tilde{\varepsilon}}(\omega) + O\left(\sup_{\omega\in\Omega_{\tilde{\varepsilon}}} \sup_{x\in H_{l,\omega}} |\rho_{\tilde{\varepsilon}}(x) - \alpha\rho_l(x)| \right) \int_{\Omega_{\tilde{\varepsilon}}} m(H_{l,\omega}) d\theta_{\tilde{\varepsilon}}(\omega). \end{aligned}$$

Let h_l be the infinitesimal hole in I_l. By condition **(A4)** ρ_0 is continuous on h_l. This implies $\lim_{\tilde{\varepsilon}\to 0} \frac{\mu_l(H_l,\omega)}{m(H_l,\omega)} = \rho_l(h_l) > 0$ by condition **(A5)**. Therefore, by using (2) Lemma 3.14 and (3.20), we obtain

$$\int_{\Omega_{\tilde{\varepsilon}}} \mu_{\tilde{\varepsilon}}(H_{l,\omega}) d\theta_{\tilde{\varepsilon}}(\omega) = \alpha \int_{\Omega_{\tilde{\varepsilon}}} \mu_l(H_{l,\omega}) d\theta_{\tilde{\varepsilon}}(\omega) + o(1) \cdot \int_{\Omega_{\tilde{\varepsilon}}} \mu_l(H_{l,\omega}) d\theta_{\tilde{\varepsilon}}(\omega). \tag{3.21}$$

Similarly we can obtain

$$\int_{\Omega_{\tilde{\varepsilon}}} \mu_{\tilde{\varepsilon}}(H_{r,\omega}) d\theta_{\tilde{\varepsilon}}(\omega) = (1-\alpha) \int_{\Omega_{\tilde{\varepsilon}}} \mu_r(H_{r,\omega}) d\theta_{\tilde{\varepsilon}}(\omega) + o(1) \cdot \int_{\Omega_{\tilde{\varepsilon}}} \mu_r(H_{r,\omega}) d\theta_{\tilde{\varepsilon}}(\omega). \tag{3.22}$$

Using Lemma 3.7 together with equations (3.21) and (3.22) complete the proof of Theorem 3.4. □

4. DISCUSSION

In this paper we studied both open and metastable systems. In Theorem 2.1, we rigorously derived first-order asymptotic escape rate formulae for piecewise expanding maps of the interval with *random holes*. In [27] the authors studied a *deterministic* sequence of *nested* holes and derived a first-order asymptotic escape rate formulae for piecewise expanding maps of the interval with holes. In this paper the sequence of holes is *random*. Moreover, the random holes we consider are *not necessarily* nested. Consequently our work generalizes the pioneering work of [27] to the random setting. It would be interesting to explore whether one can obtain higher order approximations in the random setting, even for a simpler class of maps, such as circle maps. A study for higher order approximations of escape rates for deterministic holes using circle maps was carried in [14]. Other possible generalizations, such as higher dimensional systems, were discussed earlier in the introduction.

In the second part of the paper we studied the problem of approximating the unique stationary density for a random system which initially contains two separate ergodic components. We have shown in Theorem 3.4 that the unique stationary density of the random system can be approximated by a weighted combination of the two initially separate ergodic components of the deterministic system. Furthermore, we have shown that the weights are given by the ratio of the escape rates on the individual ergodic components. A deterministic perturbation version of the above result was first studied in [22] for piecewise expanding maps. A version of [22] for intermittent maps was obtained in [3]. It is worth restating (see introduction) that it would be also interesting to obtain a random version in the case of intermittent maps.

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References

1. Afraimovich, V. and Bunimovich, L.: *Which hole is leaking the most: a topological approach to study open systems.* Nonlinearity 23 (2010), no. 3, 643–656.
2. Alves, J. and Vilarinho, H.: *Strong stochastic stability for non-uniformly expanding maps.* To appear in Ergod. Th. Dynam Sys. DOI: http://dx.doi.org/10.1017/S0143385712000077.
3. Bahsoun, W. and Vaienti, S.: *Metastability of certain intermittent maps.* Nonlinearity, 2012, vol. 25, 107–124.
4. Baladi, V.: *Positive transfer operators and decay of correlations.* Advanced Series in Nonlinear Dynamics, 16. World Sci. Publ., NJ, 2000.
5. Baladi, V.: *On the susceptibility function of piecewise expanding interval maps.* Comm. Math. Phys. 275 (2007) 839–859.
6. Baladi, V. and Young, L.-S.: *On the spectra of randomly perturbed expanding maps.* Comm. Math. Phys. 156 (1993), no. 2, 355–385.
7. Boyarsky, A. and Góra, P.: *Laws of Chaos.* Brikhäuser, Boston, 1997.
8. Bunimovich, L., and Yurchenko, A.: *Where to place a hole to achieve a maximal escape rate.* Israel J. Math. 182 (2011), 229–252.
9. Collet, P., Martínez, S. and Schmitt, B.: *On the enhancement of diffusion by chaos, escape rates and stochastic instability.* Trans. Am. Math. Soc. 351, (1999), 2875–97.
10. Dellnitz, M., Froyland, G., Horenkamp, C., Padberg-Gehle, K., and Sen Gupta, A.: *Seasonal variability of the subpolar gyres in the Southern Ocean: a numerical investigation based on transfer operators.* Nonlinear Processes in Geophysics, 16:655-664, (2009).

11. Dellnitz, M., Junge, O., Koon, W.S., Lekien, F., Lo, M. W., Marsden, J., Padberg, K., Preis, R., Ross, S., and Thiere, B.: *Transport in dynamical astronomy and multibody problems.* Internat. J. Bifur. Chaos Appl. Sci. Engrg., 15(3): 699–727, 2005.
12. Demers, M., Wright, P.: *Behavior of the Escape Rate Function in Hyperbolic Dynamical Systems.* Nonlinearity 25 (2012), 2133-2150.
13. Demers, M. and Young, L-S.: *Escape rates and conditionally invariant measures.* Nonlinearity 19 (2006), 377-397.
14. Dettmann, C., *Open circle maps: Small hole asymptotics* Nonlinearity 26 (2013), 307–317.
15. Dettmann, C.: *Recent advances in open billiards with some open problems.* In Frontiers in the study of chaotic dynamical systems with open problems, Ed. Z. Elhadj and J. C. Sprott, World Sci Publ., 2010.
16. Dolgopyat, D. and Wright, P.: *The Diffusion Coefficient for Piecewise Expanding Maps of the Interval with Metastable States.* Stoch. Dyn. 12 (2012), no. 1, 1150005, 13 pp.
17. Ferguson, A. and Pollicott, M.: *Escape rates for Gibbs measures.* Ergodic Theory Dynam. Systems., 32, (2012), 961–988.
18. Froyland, G., Padberg, K., England, M. H., and Treguier, A. M., *Detecting coherent oceanic structures via transfer operator.* Physical Review Letters, 98:224503, (2007).
19. Froyland, G., Murray, R. and Stancevic, O.: *Spectral degeneracy and escape dynamics for intermittent maps with a hole.* Nonlinearity, 24, (2011), no. 9, 2435–2463.
20. Froyland, G. and Stancevic, O.: *Metastability, Lyapunov exponents, escape rates, and topological entropy in random dynamical systems.* To appear in Stochastics and Dynamics.
21. Gaspard, P. and Gilbert, T: *Heat Conduction and Fouriers Law by Consecutive Local Mixing and Thermalization*, Phys. Rev. Lett. 101, (2008), 020601.
22. González Tokman, C., Hunt, B. R. and Wright, P.: *Approximating invariant densities of metastable systems.* Ergodic Theory Dynam. Systems, 31 (2011), no. 5, 1345–1361.
23. Góra, P.: *Graph-theoretic bound on number of A.C.I.M. for random transformation.* Proc. Amer. Math. Soc. 116 (1992), no. 2, 401–410.
24. Góra, P., Boyarsky, A., and Eslami, P.: *Metastable systems as random maps.* To appear in Internat. J. Bifur. Chaos Appl. Sci. Engrg.
25. Keller, G.: *Rare events, exponential hitting times and extremal indices via spectral perturbation.* Dynamical Systems, 27, (2012), no. 1, 11–27.
26. Keller, G. and Liverani, C.: *Stability of the spectrum for transfer operators.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 1, 141-152.
27. Keller, G. and Liverani, C.: *Rare events, escape rates, quasistationarity: some exact formulae.* J. Stat. Phys., 135 (2009), no. 3, 519-534.
28. Lasota, A. and Yorke, J.: *On the existence of invariant measures for piecewise monotonic transformations.* Trans. Amer. Math. Soc. 186 (1973), 481–488.
29. Liverani, C. and Maume-Deschamps, V.: *Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set.* Ann. Inst. H. Poincaré Probab. Statist. 39 (2003), no. 3, 385–412.
30. Santitissadeekorn, N., Froyland, G., and Monahan, A.: *Optimally coherent sets in geophysical flows: A new approach to delimiting the stratospheric polar vortex.* Phys. Rev. E, (2010), 056311-6.
31. Schütte, C., Huisinga, W., and Deuflhard, P.: *Transfer operator approach to conformational dynamics in biomolecular systems.* In Ergodic theory, analysis, and efficient simulation of dynamical systems, pages 191–223. Springer, Berlin, 2001.
32. Shen, W.: *On stochastic stability of non-uniformly expanding interval maps.* arXiv:1107.2537.
33. Waalkens, H., Burbanks, A. and Wiggins, S.: *Escape from planetary neighbourhoods,* Mon. Not. R. Astron. Soc., 361, (2005), 763–775.

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