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# PERFORMANCE ANALYSIS OF THE GLRT-BASED ARRAY RECEIVERS FOR THE DETECTION OF A KNOWN SIGNAL CORRUPTED BY NONCIRCULAR INTERFERENCE

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## ABSTRACT

This paper presents a performance analysis of likelihood ratio test (LRT)-based and generalized likelihood ratio test (GLRT)-based array receivers for the detection of a known signal corrupted by a potentially noncircular interference. Studying the distribution of the statistics associated with the LRT and GLRT, expressions of the probability of detection ( $P_D$ ) and false alarm ( $P_{FA}$ ) are given. In particular, an exact closed-form expression of  $P_D$  and  $P_{FA}$  are given for two LRT-based receivers and asymptotic (with respect to the data length) closed-form expression are given for  $P_D$  and  $P_{FA}$  for four GLRT-based receivers. Finally illustrative examples are presented in order to strengthen the obtained results.

**Index Terms**— Detection, generalized likelihood ratio test (GLRT), receiver operating characteristics (ROC), non-circular, rectilinear, interference, widely linear.

## 1. INTRODUCTION

The detection of a known signal with unknown parameters in the presence of noise plus interference (called total noise) whose covariance matrix is unknown is an important problem which has received much attention these last decades for applications such as radar, satellite localization or time acquisition in radio communications. However, most of the proposed detectors assume implicitly or explicitly a second order (SO) circular (or proper) total noise and become suboptimal in the presence of SO noncircular (or improper) sources of interference which may be potentially omnipresent in the aforementioned applications.

For this reason, some detectors in SO noncircular context have been introduced, but under the restrictive condition of either a known signal with known parameters (e.g., [1, 2]) or a random signal [3]. Despite these works, the major issue of practical uses consisting in detecting a known signal with unknown parameters in the presence of an arbitrary unknown SO noncircular total noise has been investigated to the best of our knowledge only in [4, 5] for completely or partially known propagation channel respectively. But no comprehensive performance analysis of these GLRT detectors has been investigated in these works. In particular, only a Monte Carlo

simulation exhibiting the non probability detection for specific false alarm and signal to interference plus noise ratio (SINR) was presented in [5].

The purpose of this paper is to present a comprehensive performance analysis of some detectors proposed in [5]. The paper is organized as follows. The observation model and the statement of the problem are given in Section 2. A review of the LRT and some GLRT detectors are given in Section 3. A performance analysis of these detectors is presented and illustrated in Sections 4 and 5 respectively.

## 2. HYPOTHESES AND PROBLEM FORMULATION

### 2.1. Hypotheses

Let us consider an array of  $N$  narrow-band sensors. Each sensor is assumed to receive a known linearly modulated digital signal<sup>1</sup> corrupted by a potentially noncircular total noise composed of interference and background noise. The signal of interest satisfies the Nyquist condition and is composed of  $K$  real-valued<sup>2</sup> known symbols  $a_k$ .

Under these assumptions, after matched filtering and sampling at the symbol rate<sup>3</sup> the vector of complex amplitudes of the signals at the output of these sensors,  $(\mathbf{x}_k)_{k=1,\dots,K}$  can be written as follows

$$\mathbf{x}_k = \rho_s e^{i\phi_s} a_k \mathbf{s} + \mathbf{n}_k, \quad (1)$$

where  $\mathbf{s}$  is the steering vector of the known signal, such that its first component is equal to one.  $\rho_s$  and  $\phi_s$  control the amplitude and the phase of the known signal on the first sensor respectively, and  $\mathbf{n}_k$  are the samples of the zero-mean total noise at the output of the matched filter.

<sup>1</sup>This signal may either correspond to a training sequence in a radio communication link, a binary coding signal over the coherent processing interval in radar applications, or a PN code over a symbol period for DS-CDMA networks or GPS systems.

<sup>2</sup>Note that this assumption is not so restrictive since rectilinear signals such as DS-BPSK signals in particular, are currently used in a large domain of practical applications. Extension to complex-valued symbols leads to more involved derivations of some GLRT receivers, but the analysis of Section 4 extends straightforwardly.

<sup>3</sup>Note that the samples  $\mathbf{x}_k$  are sufficient statistics for the detection problem when the total noise is whitely Gaussian distributed only.

## 2.2. Second order statistics of the data

The SO statistics of the potentially noncircular data  $\mathbf{x}_k$  are defined by

$$\begin{aligned}\mathbf{R}_x(k) &\stackrel{\text{def}}{=} \mathbb{E}(\mathbf{x}_k \mathbf{x}_k^H) = \pi_s(k) \mathbf{s} \mathbf{s}^H + \mathbf{R}_n(k), \\ \mathbf{C}_x(k) &\stackrel{\text{def}}{=} \mathbb{E}(\mathbf{x}_k \mathbf{x}_k^T) = \pi_s(k) e^{2i\phi_s} \mathbf{s} \mathbf{s}^T + \mathbf{C}_n(k),\end{aligned}$$

where  $\pi_s(k) \stackrel{\text{def}}{=} \rho_s^2 a_k^2$  since  $a_k$  is deterministic.

## 2.3. Problem formulation

The problem addressed in this paper is the detection problem with two hypotheses  $H_0$  and  $H_1$ , respectively associated with the presence of total noise  $\mathbf{n}_k$  only and signal plus total noise in the data  $(\mathbf{x}_k)_{k=1,\dots,K}$  based on the GLRT.

$$\begin{aligned}H_0 : \quad & \mathbf{x}_k = \mathbf{n}_k, & k = 1, \dots, K \\ H_1 : \quad & \mathbf{x}_k = \rho_s e^{i\phi_s} a_k \mathbf{s} + \mathbf{n}_k, & k = 1, \dots, K.\end{aligned} \quad (2)$$

To derive GLRT-based receivers, we need the following theoretical assumptions<sup>4</sup> which are not necessarily verified or required in practice.

A.1: the matrices  $\mathbf{R}_n(k)$  and  $\mathbf{C}_n(k)$  do not depend on  $k$

A.2: the samples  $(\mathbf{n}_k)_{k=1,\dots,K}$  are independent zero-mean Gaussian and possibly noncircular.

Under these conditions, the parameters of the distribution of  $(\mathbf{x}_k)_{k=1,\dots,K}$  are  $(\rho_s, \phi_s)$ ,  $\mathbf{s}$  and  $(\mathbf{R}_n, \mathbf{C}_n)$ <sup>5</sup>. As each of these parameters may be either known or unknown, depending on the application, different GLRT-based receivers has been derived in [5].

## 3. REVIEW OF THE LRT AND GLRT RECEIVERS

### 3.1. Clairvoyant receivers

We first consider the unrealistic case of completely known parameters. According to the statistical theory of detection [6, Ch. 3], the optimal detector is the LRT receiver that consists in comparing to a threshold, the likelihood ratio  $\text{LR}(\mathbf{x}, K)$  defined by

$$\text{LR}(\mathbf{x}, K) \stackrel{\text{def}}{=} \frac{p[(\mathbf{x}_k)_{k=1,\dots,K}/H_1]}{p[(\mathbf{x}_k)_{k=1,\dots,K}/H_0]}.$$

With assumptions A.1 and A.2, it is straightforward to prove [5], that the LRT receiver decides  $H_1$  if the statistic  $\text{OPT}(\mathbf{x}, K)$  defined by

$$\text{OPT}(\mathbf{x}, K) \stackrel{\text{def}}{=} \tilde{\mathbf{w}}_o^H \hat{\mathbf{r}}_{\mathbf{x},a} \quad (3)$$

is greater than a specific threshold, where  $\tilde{\mathbf{w}}_o \stackrel{\text{def}}{=} \mathbf{R}_n^{-1} \tilde{\mathbf{s}}_\phi$  is the so-called widely linear spatial matched filter (SMF)

<sup>4</sup>These assumptions are not critical in the sense that the GLRT-based receivers derived under these assumptions still provide good decision performance even if most of the latter are not verified in practice.

<sup>5</sup> $\mathbf{R}_n$  and  $\mathbf{C}_n$  are supposed arbitrary Hermitian and complex symmetric structured matrices respectively.

[7],  $\tilde{\mathbf{s}}_\phi \stackrel{\text{def}}{=} [e^{i\phi_s} \mathbf{s}^T, e^{-i\phi_s} \mathbf{s}^H]^T$ ,  $\mathbf{R}_{\tilde{n}} \stackrel{\text{def}}{=} \mathbb{E}[\tilde{\mathbf{n}}_k \tilde{\mathbf{n}}_k^H]$  with  $\tilde{\mathbf{n}}_k \stackrel{\text{def}}{=} [\mathbf{n}_k^T, \mathbf{n}_k^H]^T$  and  $\hat{\mathbf{r}}_{\mathbf{x},a} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \tilde{\mathbf{x}}_k a_k$  where  $\tilde{\mathbf{x}}_k \stackrel{\text{def}}{=} [\mathbf{x}_k^T, \mathbf{x}_k^H]^T$ . In the particular case of a SO circular total noise ( $\mathbf{C}_n = \mathbf{O}$ ), the statistic  $\text{OPT}(\mathbf{x}, K)$  (3) reduces to the conventional one defined by

$$\text{CONV}(\mathbf{x}, K) \stackrel{\text{def}}{=} 2\Re[\mathbf{w}_c^H \hat{\mathbf{r}}_{\mathbf{x},a}], \quad (4)$$

with  $\mathbf{w}_c \stackrel{\text{def}}{=} e^{i\phi_s} \mathbf{R}_n^{-1} \mathbf{s}$  and  $\hat{\mathbf{r}}_{\mathbf{x},a} \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K \mathbf{x}_k a_k$ .

## 3.2. GLRT receivers

In most of situations of practical interest, the parameters  $(\rho_s, \phi_s)$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  are unknown, while for some applications the steering vector  $\mathbf{s}$  is known or unknown (see applications given in [5]). Thereby, we resort to GLRT approach where we maximize  $p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_1]$  and  $p[(\mathbf{x}_k)_{k=1,\dots,K}; \boldsymbol{\theta}_0]$  with respect to the unknown parameters  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_0$  under  $H_1$  and  $H_0$  respectively, and use the resulting LR (denoted  $\text{GLR}(\mathbf{x}, K)$ ) as a decision statistic. Depending on the unknown parameters  $\boldsymbol{\theta}_1$  and  $\boldsymbol{\theta}_0$ , different expressions of  $\text{GLR}(\mathbf{x}, K)$  have been derived in [5]. For example for  $(\rho_s, \phi_s)$  unknown only,

$$2 \ln[\text{GLR}(\mathbf{x}, K)] = K \hat{\mathbf{r}}_{\mathbf{x},a}^H \mathbf{R}_n^{-1} \mathbf{S} (\mathbf{S}^H \mathbf{R}_n^{-1} \mathbf{S})^{-1} \mathbf{S}^H \mathbf{R}_n^{-1} \hat{\mathbf{r}}_{\mathbf{x},a} \quad (5)$$

with  $\mathbf{S} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{s} & \mathbf{0} \\ \mathbf{0} & \mathbf{s}^* \end{pmatrix}$ .

## 4. PERFORMANCE ANALYSIS

### 4.1. Clairvoyant receivers

To be able to quantify and to compare the performance of the previous clairvoyant receivers, we assume in this subsection that the propagation channel has no delay spread and that the total noise  $\mathbf{n}_k$  is composed of a BPSK interference with equiprobable symbols  $b_k \in \{-1, +1\}$ , plus background noise  $\mathbf{n}'_k$  uncorrelated with each other. Under these assumptions  $\mathbf{n}_k$  is written as

$$\mathbf{n}_k = \rho_1 e^{i\phi_1} b_k \mathbf{j}_1 + \mathbf{n}'_k, \quad (6)$$

where  $\mathbf{j}_1$  is the steering vector of the interference whose first component is equal to one, and where  $(\mathbf{n}'_k)_{k=1,\dots,K}$  are spatially white zero-mean circularly Gaussian independent distributed random variables (RV) with  $\mathbb{E}(\mathbf{n}'_k \mathbf{n}'_k^H) = \eta_2 \mathbf{I}$ .

The probability of detection and false alarm associated with the threshold  $\lambda$  are given respectively by

$$\begin{aligned}P_D &= P[\text{OPT}(\mathbf{x}, K) > \lambda/H_1] = \\ &P[\rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \tilde{\mathbf{w}}_o^H \tilde{\mathbf{s}}_\phi + \rho_1 \tilde{\mathbf{w}}_o^H \mathbf{j}_\phi \frac{1}{K} \sum_{k=1}^K a_k b_k + \frac{1}{K} \sum_{k=1}^K a_k \tilde{\mathbf{w}}_o^H \mathbf{n}'_k > \lambda] \\ &= P[\beta + \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k + n'_K > \lambda] \quad (7) \\ P_{FA} &= P[\text{OPT}(\mathbf{x}, K) > \lambda/H_0] \\ &= P[\rho_1 \tilde{\mathbf{w}}_o^H \mathbf{j}_\phi \frac{1}{K} \sum_{k=1}^K a_k b_k + \frac{1}{K} \sum_{k=1}^K a_k \tilde{\mathbf{w}}_o^H \mathbf{n}'_k > \lambda]\end{aligned}$$

$$= P[\alpha(\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k + n'_K > \lambda] \quad (8)$$

with  $\tilde{\mathbf{j}}_\phi \stackrel{\text{def}}{=} [e^{i\phi_1} \mathbf{j}_1^T, e^{-i\phi_1} \mathbf{j}_1^T]^T$ ,  $\alpha \stackrel{\text{def}}{=} \frac{\rho_1}{K} (\sum_{k=1}^K a_k^2)^{1/2} \tilde{\mathbf{w}}_o^H \tilde{\mathbf{j}}_\phi$ ,  $\beta \stackrel{\text{def}}{=} \rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \tilde{\mathbf{w}}_o^H \tilde{\mathbf{s}}_\phi = \rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \tilde{\mathbf{s}}_\phi^H \mathbf{R}_n^{-1} \tilde{\mathbf{s}}_\phi > 0$  and  $n'_K \stackrel{\text{def}}{=} \frac{1}{K} \sum_{k=1}^K a_k \tilde{\mathbf{w}}_o^H \tilde{\mathbf{n}}'_k$  which is a real-valued zero-mean Gaussian RV with variance  $\sigma^2 = \frac{\eta_2}{K^2} (\sum_{k=1}^K a_k^2) \|\tilde{\mathbf{w}}_o\|^2$ . Conditioning  $P_D$  (7) and  $P_{FA}$  (8) on the different equiprobable symbols  $(b_1, \dots, b_K)$ , we obtain by the total probability formula

$$P_D = \frac{1}{2^K} \sum_{i=1}^{2^K} Q \left( \frac{\lambda - \beta - \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k^{(i)}}{\sigma} \right), \quad (9)$$

$$P_{FA} = \frac{1}{2^K} \sum_{i=1}^{2^K} Q \left( \frac{\lambda - \alpha (\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k^{(i)}}{\sigma} \right), \quad (10)$$

where  $(b_1^{(i)}, \dots, b_K^{(i)})_{i=1, \dots, 2^K}$  denote the  $2^K$  different binary symbols  $(b_1, \dots, b_K)$  and  $Q(x) \stackrel{\text{def}}{=} \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ .

Expressions (9) and (10) of  $P_D$  and  $P_{FA}$  are valid for the conventional receiver (4) as well, with now

$\alpha \stackrel{\text{def}}{=} \frac{2\rho_1}{K} (\sum_{k=1}^K a_k^2)^{1/2} \Re(e^{i\phi_1} \mathbf{w}_c^H \mathbf{j}_1)$ ,  $\beta \stackrel{\text{def}}{=} 2\rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \Re(e^{i\phi_s} \mathbf{w}_c^H \mathbf{s}) = 2\rho_s (\frac{1}{K} \sum_{k=1}^K a_k^2) \mathbf{s}^H \mathbf{R}_n^{-1} \mathbf{s} > 0$  and  $n'_K \stackrel{\text{def}}{=} \frac{2}{K} \Re(\sum_{k=1}^K a_k \mathbf{w}_c^H \mathbf{n}'_k)$  which is a zero-mean Gaussian RV with variance  $\sigma^2 = \frac{2\eta_2}{K^2} (\sum_{k=1}^K a_k^2) \|\mathbf{w}_c\|^2$ .

Numerical computation of (9) and (10) are computationally costly for large values of  $K$ . But for these values, using the central limit theorem, the distribution of the RV  $(\sum_{k=1}^K a_k^2)^{-1/2} \sum_{k=1}^K a_k b_k$  can be approximated by a zero-mean and unit variance Gaussian distribution. In this case (9) and (10) become respectively

$$P_D \approx Q \left( \frac{\lambda - \beta}{\sqrt{\alpha^2 + \sigma^2}} \right) \quad (11)$$

$$P_{FA} \approx Q \left( \frac{\lambda}{\sqrt{\alpha^2 + \sigma^2}} \right). \quad (12)$$

This gives the closed-form expression of the ROC of clairvoyant receivers (3) and (4)

$$P_D \approx Q \left( Q^{-1}(P_{FA}) - \sqrt{\text{SINR}} \right) \quad (13)$$

where  $\text{SINR} = \frac{\beta^2}{\alpha^2 + \sigma^2}$  is  $K$  times the mean (with respect to  $a_k$ ) of the SINR at the output of the widely linear SMF  $\tilde{\mathbf{w}}_o$  and of the linear SMF  $\mathbf{w}_c$ , respectively. Computation and comparison of these SINR are done in [7] for BPSK signal of interest and not reported here for want of space.

## 4.2. GLRT receivers

Depending on the unknown parameters  $\theta$  among  $(\rho_s, \phi_s)$ ,  $\mathbf{s}$  and  $(\mathbf{R}_n, \mathbf{C}_n)$ , we consider the following practical four cases:

$C_1$ :  $(\rho_s, \phi_s)$  is unknown,  $\mathbf{s}$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  are known

$C_2$ :  $(\rho_s, \phi_s)$  and  $\mathbf{s}$  are unknown,  $(\mathbf{R}_n, \mathbf{C}_n)$  is known

$C_3$ :  $(\rho_s, \phi_s)$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  are unknown,  $\mathbf{s}$  is known

$C_4$ :  $(\rho_s, \phi_s)$ ,  $\mathbf{s}$  and  $(\mathbf{R}_n, \mathbf{C}_n)$  are unknown.

For each of these cases, the probability density function of  $(\mathbf{x}_k)_{k=1, \dots, K}$  under  $H_0$  and  $H_1$  is the same, except that the value of the unknown parameter vector  $\theta$  is different. Decompose the general unknown parameter  $\theta$  in  $\theta_s$  and  $\theta_r$  that collect the unknown parameters among  $(\rho_s, \phi_s)$ ,  $\mathbf{s}$  and  $(\mathbf{R}_n, \mathbf{C}_n)$ , respectively.  $s$  and  $r$  represent the dimensions of the real-valued vector unknown parameters  $\theta_s$  and  $\theta_r$ . Note that  $(\mathbf{R}_n, \mathbf{C}_n)$  is of no concern and is sometimes referred to as a nuisance parameter. For identifiability reasons, we must use reparameterizations, to get the parameters  $\theta_s$  and  $\theta_r$  that are given for each cases by  $\theta_s = [\rho_s \cos(\phi_s), \rho_s \sin(\phi_s)]^T$  with  $s = 2$  for  $C_1$  and  $C_3$  and  $\theta_s = [\Re(\rho_s e^{i\phi_s} \mathbf{s})^T, \Im(\rho_s e^{i\phi_s} \mathbf{s})^T]^T$  with  $s = 2N$  for  $C_2$  and  $C_4$ . There is no nuisance parameter for  $C_1$  and  $C_2$ .  $\theta_r = [(\mathbf{R}_n)_{i,i}, \Re((\mathbf{R}_n)_{i,j}), \Im((\mathbf{R}_n)_{i,j})$  for  $1 \leq i < j \leq N$ ,  $\Re((\mathbf{C}_n)_{i,j}), \Im((\mathbf{C}_n)_{i,j})$  for  $1 \leq i \leq j \leq N]^T$  with  $r = N(2N + 1)$  for  $C_3$  and  $C_4$ . In this situation, detection problem (2) can be recast as the following composite hypothesis testing problem [8], [6, Ch. 6]

$$\begin{aligned} H_0 : & p[(\mathbf{x}_k)_{k=1, \dots, K}; \theta_s = \mathbf{0}, \theta_r] \\ H_1 : & p[(\mathbf{x}_k)_{k=1, \dots, K}; \theta_s \neq \mathbf{0}, \theta_r]. \end{aligned} \quad (14)$$

With these notations, GLR( $\mathbf{x}, K$ ) becomes

$$\text{GLR}(\mathbf{x}, K) = \frac{p[(\mathbf{x}_k)_{k=1, \dots, K}; \hat{\theta}_{s_1}, \hat{\theta}_{r_1}]}{p[(\mathbf{x}_k)_{k=1, \dots, K}; \hat{\theta}_{s_0}, \hat{\theta}_{r_0}]},$$

with  $\theta_{s_0} = \mathbf{0}$  and  $(\hat{\theta}_{s_1}, \hat{\theta}_{r_1})$  and  $\hat{\theta}_{r_0}$  are the maximum likelihood estimates of  $(\theta_s, \theta_r)$  and  $\theta_r$  under  $H_1$  and  $H_0$ , respectively.

The exact distribution of GLR( $\mathbf{x}, K$ ) under  $H_1$  and  $H_0$  for the true distribution of the data appears prohibitive to compute. For example, the derivation of the distribution of simplest statistic (5) after conditioning on the symbols  $(b_1, \dots, b_K)$ , comes down to derive the distribution of the Hermitian form  $\mathbf{z}^H \mathbf{\Omega} \mathbf{z}$  where  $\mathbf{z}$  is a zero-mean circular Gaussian RV with covariance matrix arbitrary with respect to  $\mathbf{\Omega}$ , for which no simple closed-form expression is available. But asymptotically with respect to  $K$  and under the assumptions for which the GLRT has been derived, i.e., according to a noncircular Gaussian distribution of the data, Wilk's theorem with nuisance parameters [10, p.132] can be applied. Thus the following convergence in distribution follows:

$$2 \ln[\text{GLR}(\mathbf{x}, K)] \xrightarrow{L} \chi^2(s) \quad \text{under } H_0, \quad (15)$$

where  $\chi^2(s)$  denotes the chi-squared distribution with  $s$  degrees of freedom. Under  $H_1$ , it may be shown [9, Ch. 23.7] under the assumption<sup>6</sup> that  $\theta_s$  can take values near  $\mathbf{0}$ , that  $2 \ln[\text{GLR}(\mathbf{x}, K)]$  is approximately distributed as

$$2 \ln[\text{GLR}(\mathbf{x}, K)] \stackrel{a}{\sim} \chi^2(s, \mu_K) \quad \text{under } H_1, \quad (16)$$

<sup>6</sup>The following more formal definition is given in [6, A. 6A]:  $\|\theta_s\|^2 = c/\sqrt{K}$  for some constant  $c$ , where  $\theta_s$  is embedded in an adequate sequence indexed by  $K$ .

where  $\chi^2(s, \mu_K)$  denotes the noncentral chi-squared distribution with  $s$  degrees of freedom and noncentrality parameter  $\mu_K$ , which is a measure of discrimination between the two hypotheses given by [8] and [6, Ch. 6] given by

$$\mu_K = \theta_s^T [\mathbf{I}_{s,s}(\mathbf{0}, \theta_r) - \mathbf{I}_{s,r}(\mathbf{0}, \theta_r) \mathbf{I}_{r,r}^{-1}(\mathbf{0}, \theta_r) \mathbf{I}_{r,s}(\mathbf{0}, \theta_r)] \theta_s,$$

where  $\theta_s$  and  $\theta_r$  are the true values under  $H_1$ , and the terms in the brackets are given by partitioning the Fisher information matrix of  $(\mathbf{x}_k)_{k=1,\dots,K}$  for  $\theta = (\theta_s^T, \theta_r^T)^T$  as

$$\mathbf{I}_K(\theta) = \begin{pmatrix} \mathbf{I}_{s,s}(\theta_s, \theta_r) & \mathbf{I}_{s,r}(\theta_s, \theta_r) \\ \mathbf{I}_{r,s}(\theta_s, \theta_r) & \mathbf{I}_{r,r}(\theta_s, \theta_r) \end{pmatrix}.$$

We prove that  $\mu_K = (\sum_{k=1}^K a_k^2) \rho_s^2 \tilde{\mathbf{s}}_\phi^H \mathbf{R}_n^{-1} \tilde{\mathbf{s}}_\phi$  for the four cases. Since the asymptotic distribution of  $2 \ln[\text{GLR}(\mathbf{x}, K)]$  under  $H_0$  does not depend on any unknown parameter, the detector is a constant false alarm rate (CFAR) detector. But in general this CFAR property holds only for large data ( $K \gg 1$ ).

We note that this performance analysis cannot be used to obtain the asymptotic distributions of  $2 \ln[\text{GLR}(\mathbf{x}, K)]$  derived under the circular Gaussian distribution of the data, but applied under noncircular Gaussian distribution of the data described in [5].

## 5. ILLUSTRATIONS

To illustrate the analysis of performance of Section 4, we consider an array of  $N = 2$  omnidirectional sensors equispaced half a wavelength apart. The phase and the direction of arrival with respect to broadside, of both the BPSK signal of interest and the BPSK interference are assumed constant over a burst of  $K$  symbols. They take the following values:  $\phi_s = 0$ ,  $\theta_s = 0$ ,  $\phi_1 = \pi/4$  and  $\theta_1 = \pi/9$ . The input SNR and interference to noise ratio (INR) are defined by  $\text{SNR} = \rho_s^2/\eta_2$  and  $\text{INR} = \rho_1^2/\eta_2$ , respectively.

Fig.1 shows the empirical, the exact and approximate theoretical ROCs of the optimal (3) and conventionnel (4) clairvoyant receivers for  $K = 4$  and  $K = 64$ .

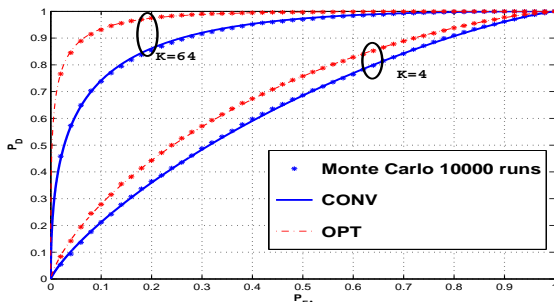


Fig.1 Theoretical and empirical ROCs for SNR = -15dB and INR = 5dB.

We see that the optimal receiver largely outperforms conventional ones and that the two theoretical (exact and approximate) ROCs coincide for both values of  $K$ . In fact, approximations (11) and (12) of (9) and (10), respectively, remain very accurate (to two significant digits) for  $K = 4$  in all the scenarios.

Fig.2 shows the empirical and the asymptotic theoretical ROCs of the four GLRT receivers compared to the exact theoretical ROC of the optimal clairvoyant receiver described in Section 3.

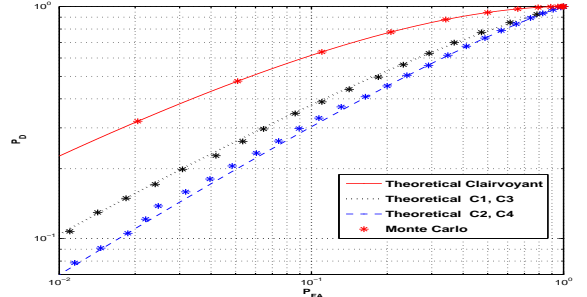


Fig.2 Asymptotic theoretical and empirical ROC for SNR = -20dB, INR = 0dB and  $K = 64$ .

Comparing the five ROCs, we see that the clairvoyant receiver outperforms the four GLRT receivers, the performance improves with the knowledge of the steering vector  $\mathbf{s}$ , whereas for the validity conditions of our analysis (i.e., very weak SNR and large data size), the knowledge of  $(\mathbf{R}_n, \mathbf{C}_n)$  does not improve the performance. Furthermore, we observe that the empirical ROCs fit the asymptotic theoretical ones for the relatively small data length  $K = 64$ .

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