

Minimum d -transversals for the maximum weight-stable set in trees

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- 1 Definitions
- 2 Initial results
- 3 Minimum d -transversal in weighted-trees
- 4 Conclusion

Definitions

$G = (V, E)$ simple finite

$|V| = n, |E| = m$

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- $\alpha_w(G)$: weighted-stability number
- $\mu(G)$: matching number

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- A vertex which is neither forced nor excluded is *free*

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- $\tau_{wd}(G)$ minimum weight of a minimum size d -transversal

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Results for unweighted graphs

Theorem (E. Boros, M.C. Golumbic, V.E. Levit 2002)

- v is forced if and only if $\alpha(G - v) = \alpha(G) - 1$.

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- v is forced if and only if $\alpha(G - v) = \alpha(G) - 1$.
- v is free if and only if $\alpha(G - N_G[v]) = \alpha(G) - 1$.
- it is \mathcal{NP} -complete to decide the status of v .

Results for unweighted graphs

Theorem

For all fixed d , $d \in \{1, 2, \dots, \alpha(G)\}$, finding a minimum d -transversal is \mathcal{NP} -hard even for line graphs of bipartite graphs.

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consequence of \mathcal{NP} -hardness minimum d -transversal for maximum matchings (Zenklusen et al. 2009)

Results for unweighted graphs

Theorem

Finding a minimum d -transversal is polynomial for bipartite graphs.

M.-C. Costa, D. de Werra, ChP 2010

Results for weighted graphs

Theorem

Finding a minimum weight minimum d -transversal is \mathcal{NP} -hard in bipartite graphs.

R. Zenklusen 2010, S. Toubaline 2010

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Weighted trees

Property

Using dynamic programming determining the status of $v \in V$ is polynomial.

Weighted trees

T^r tree rooted at $r \in V$

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$$\alpha_w(T^r(v)) = \max\{p^r(v, 0), p^r(v, 1)\}$$

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- v is excluded if and only if $p^v(v, 1) < p^v(v, 0)$;
- v is free if and only if $p^v(v, 1) = p^v(v, 0)$.

Weighted trees

We suppose

- v is free $\forall v \in V$
- $w_s > 0$

Weighted trees

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Weighted trees

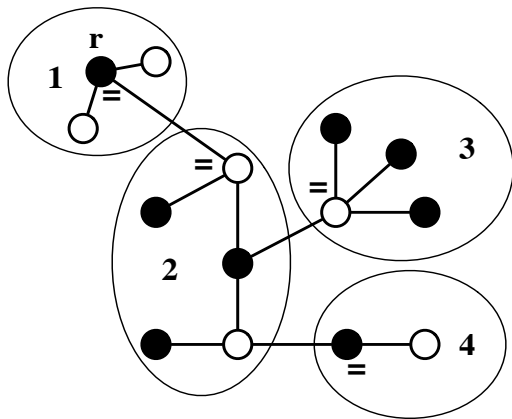
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$|V_i^r| \geq 2$ and $T[V_i^r]$ connected \implies tree

Weighted trees



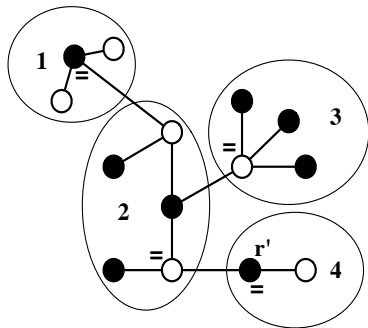
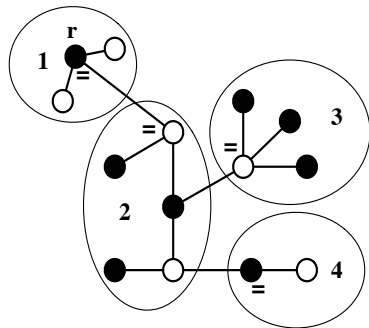
Weighted trees

Property

$$\forall r, r' : \mathcal{V}^r = \mathcal{V}^{r'}$$

$\implies \mathcal{V} = \{V_1, \dots, V_q\}$ the partition of V

Weighted trees



Weighted trees

$V = B \cup W$: bipartition into black vertices or white vertices

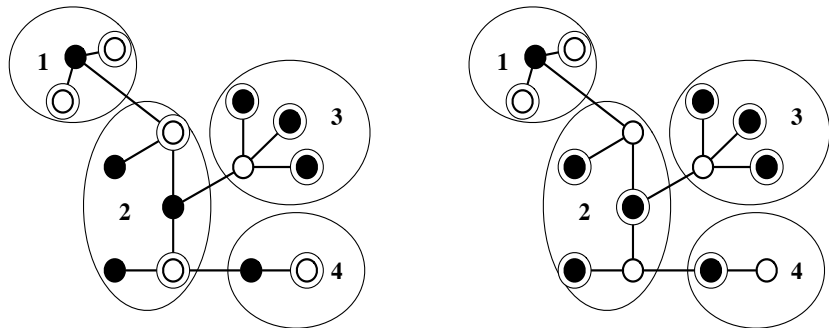
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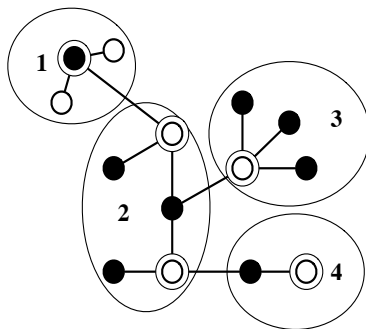
Let T be a weighted tree containing only free vertices, then S is a maximum-weight stable set if and only if S is stable and for any $i \in \{1, \dots, q\}$ either $S \cap V_i = B \cap V_i$ or $S \cap V_i = W \cap V_i$.

Weighted trees



maximum-weight stable sets

Weighted trees



not stable set

Weighted trees

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- $A^* = \{(V_i, V_j) \mid \exists vu \in E, v \in V_i \cap B, u \in V_j \cap W\}$

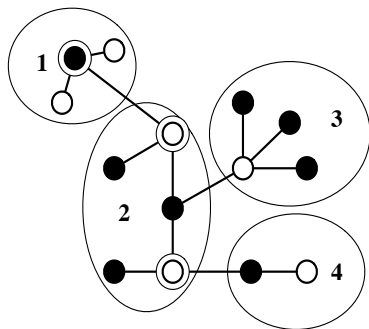
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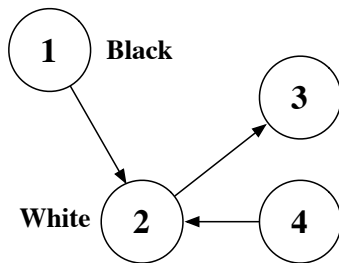
- $V^* = \{V_1, \dots, V_q\}$
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T^* is a directed tree.

Weighted trees



not stable set



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partial order (V, \preceq)

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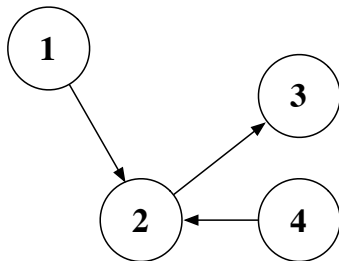
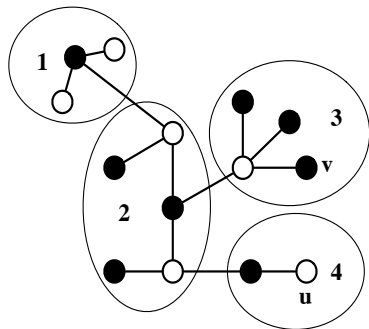
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- $u, v \in V_i$ or
- $u \in V_i, v \in V_j, i \neq j$, a (directed) path from V_i to V_j in T^*

Weighted trees



Weighted trees

Property

Let T be a weighted tree containing only free vertices. For any $0 \leq d \leq d_{\max}$, T' is a minimal (inclusionwise) d -transversal if and only if it consists of d disjoint pairs (u, v) such that $u \preceq v$

Weighted trees

Sketch of the proof :

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- 4 $> d$ disjoint pairs $(u, v) : T'$ not inclusionwise minimal
- 5 $< d$ disjoint pairs $(u, v) : \text{cannot be a } d\text{-transversal}$

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- 3 if $d > |f|$: take the $|f|$ forced vertices;
 $d - |f|$ disjoint pairs (u, v) , $u \preceq v$: matching of size $d - |f|$ in \tilde{G} ;
 $\tau_d(G) = 2d - f$

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- 4 set $w(e) = w(u) + w(v)$ for $e = (u, v) \in \tilde{E}$

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- 5 compute a minimum weight matching of size d

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- minimum size d -transversal and minimum weight minimum size d -transversal are polynomial for weighted trees (NP -hard for weighted bipartite graphs)
- polynomial for weighted cobipartite graphs
 NP -hard for weighted split graphs

Thank you