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Asymptotics of the critical time in Wiener sausage percolation with a small radius

Dirk Erhard *
Julien Poisat ‡

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Abstract

We consider a continuum percolation model on $\mathbb{R}^d$, where $d \geq 4$. The occupied set is given by the union of independent Wiener sausages with radius r running up to time t and whose initial points are distributed according to a homogeneous Poisson point process. It was established in a previous work by Erhard, Martínez and Poisat [4] that (1) if r is small enough there is a non-trivial percolation transition in t occuring at a critical time $t_c(r)$ and (2) in the supercritical regime the unbounded cluster is unique. In this paper we investigate the asymptotic behaviour of the critical time when the radius r converges to 0. The latter does not seem to be deducible from simple scaling arguments. We prove that for $d \geq 4$, there is a positive constant c such that $c^{-1}\sqrt{\log(1/r)} \leq t_c(r) \leq c\sqrt{\log(1/r)}$ when $d = 4$ and $c^{-1}r^{(4-d)/2} \leq t_c(r) \leq c\, r^{(4-d)/2}$ when $d \geq 5$, as r converges to 0. We derive along the way moment estimates on the capacity of Wiener sausages, which may be of independent interest.

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*Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK,
D.Erhard@warwick.ac.uk

‡CEREMADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 PARIS CEDEX 16 - FRANCE
poisat@ceremade.dauphine.fr

1 Introduction

Notation. For every $d \geq 1$, we denote by Leb_d the Lebesgue measure on $\mathbb{R}^d$, whereas $|\cdot|$ denotes the cardinality of a set. Let $c_{\mathrm{vol}} = \mathrm{Leb}_d(\mathcal{B}(0,1))$. For $a \in \mathbb{R}$, we denote by $\lceil a \rceil$ its upper integer part. The symbol $||\cdot||$ stands for the Euclidean norm on $\mathbb{R}^d$ and the symbol $|\cdot|_1$ stands for the ℓ_1 norm on $\mathbb{Z}^d$. The open ball with center z and radius r is denoted by $\mathcal{B}(z,r)$, and the closed ball by $\overline{\mathcal{B}}(z,r)$. We denote by $G : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ the Green function of the standard Brownian motion. Throughout the paper the letter c will be used to denote a constant whose precise value is irrelevant (and possibly depending on the dimension) and which may change from line to line.

1.1 Introduction to the model

Let $\mathcal{E}$ be a Poisson point process with intensity $\lambda\,\mathrm{Leb}_d$, where $\lambda > 0$. Conditionally on $\mathcal{E}$, we define a collection of independent Brownian motions $\{(B_t^x)_{t\geq 0},\, x \in \mathcal{E}\}$ such that for each $x \in \mathcal{E}$, $B_0^x = x$ and $(B_t^x - x)_{t\geq 0}$ is independent of $\mathcal{E}$. We refer the reader to Section 1.4 in [4] for a rigorous construction. We denote by P and E the probability measure and expectation of Brownian motion, respectively. Given a d-dimensional Brownian motion $(B_t^x)_{t\geq 0}$ that starts at $x \in \mathbb{R}^d$, we denote by $W_{[0,t]}^{x,r} = \bigcup_{0\leq s\leq t} \mathcal{B}(B_s^x, r)$ the corresponding Wiener sausage running up to time t. When it is more convenient, we shall use P_x for a Brownian motion started at x, and we remove the superscript x from B or W. Also, we will use the symbol $\widetilde{\mathrm{P}}$ to refer to an independent copy of a Brownian motion. If A is an event, then $\mathrm{E}(\,\cdot\,; A)$ stands for $\mathrm{E}(\,\cdot\,\mathbb{1}_A)$. Finally, we use the letter $\mathbb{P}$ for the law of the whole process that is formed by the Poisson points and the Brownian motions.

We study for $t, r \geq 0$ the *occupied* set, which is defined by

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} W_{[0,t]}^{x,r}, \qquad \mathcal{O}_t := \mathcal{O}_{t,0}. \tag{1.1}$$

The rigorous construction found in [4] yields ergodicity of $\mathcal{O}_{t,r}$ with respect to shifts in space. For $d \geq 4$, the model also appears in Černý, Funken and Spodarev [2] where it describes the target detection area of a network of mobile sensors initially distributed at random and moving according to Brownian motions. In a similar spirit Kesidis, Konstantopoulos and Phoha [8] study the detection time of a particle that is placed at the origin. Note that at time $t = 0$, the set reduces to a collection of balls with randomly located centers: this goes under the name of Boolean percolation model and was first introduced by Gilbert [6] to study infinite communication networks. We refer to Meester and Roy [11] for an introductory overview of this model.

Two points x and y of $\mathbb{R}^d$ are said to be *connected* in $\mathcal{O}_{t,r}$ if and only if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A subset of $\mathcal{O}_{t,r}$ is connected if and only if all of its points are pairwise connected. In the following a connected subset of $\mathcal{O}_{t,r}$ is called a component. A component $\mathcal{C}$ is bounded if there exists $R > 0$ such that $\mathcal{C} \subseteq \mathcal{B}(0,R)$. Otherwise, the component is said to be unbounded. A *cluster* is a connected component which is maximal in the sense that it is not strictly contained in another connected component. Denote by $\mathcal{C}(x)$ the set of points in $\mathcal{E}$ which are connected to x through $\mathcal{O}_{t,r}$.

A set is said to percolate if it contains an unbounded connected component. In [4] it was shown that $\mathcal{O}_{t,r}$ undergoes a non-trivial percolation phase transition for all $d \geq 2$. More precisely it was shown that if $d \in \{2,3\}$, then for all $\lambda > 0$ there exists $t_c(\lambda) \in (0,\infty)$ such that for all $t < t_c(\lambda)$ the set $\mathcal{O}_t$ only contains bounded connected components, whereas for $t > t_c(\lambda)$, the set $\mathcal{O}_t$ percolates with a unique unbounded cluster. What happens at criticality is still unknown. In essence the same result holds for $d \geq 4$. However, due to the fact that the paths of two independent Brownian motions do not intersect (except at a possibly common starting point), the set $\mathcal{O}_{t,r}$ almost surely (a.s.) does not percolate for all $t \geq 0$. Therefore, the radius r needs to be chosen positive. In this case, denote by $\lambda_c(r)$ the critical value such that the set $\mathcal{O}_{0,r}$ a.s. percolates for all $\lambda > \lambda_c(r)$, and a.s. does not for $\lambda < \lambda_c(r)$, see Section 3.3 in Meester and Roy [11]. Theorem 1.3 in [4] states that when $r > 0$ and $\lambda < \lambda_c(r)$, then there is a critical time $t_c(\lambda, r) \in (0,\infty)$ which separates a percolation regime ($t > t_c(\lambda, r)$) from a non-percolation regime ($t < t_c(\lambda, r)$). Equivalently, a phase transition occurs when λ is fixed and the radius is chosen smaller than a critical radius $r_c(\lambda)$. This formulation is somewhat more relevant for what comes next.

1.2 Main Result

In this work we study the behaviour of the critical time as the radius converges to 0 and the intensity is kept fixed, say $\lambda = 1$. For this reason, we shall now write $t_c(r)$ instead of $t_c(1, r)$. Let us mention that no simple scaling argument seems to immediately yield bounds on $t_c(r)$. Indeed, since for each d there are three parameters (λ, t and r), it is not possible to scale two parameters independently of the third parameter. We expect that $t_c(r)$ goes to infinity as $r \to 0$ since $t_c(0) = \infty$. However, this is not an immediate consequence of continuity since the event $\{\mathcal{O}_t$ does not percolate$\}$ is not the increasing union of the events $\{\mathcal{O}_{t,r}$ does not percolate$\}$ for $r > 0$. The following theorem determines at which speed the convergence takes place.

Theorem 1.1. *Let $d \geq 4$. There is a constant c and an $r_0 \in (0,1)$ such that for all $r \leq r_0$,*

$$\begin{cases} c^{-1}\sqrt{\log(1/r)} \leq t_c(r) \leq c\sqrt{\log(1/r)}, & \text{if } d = 4, \\ c^{-1} r^{(4-d)/2} \leq t_c(r) \leq c\, r^{(4-d)/2}, & \text{if } d \geq 5. \end{cases} \tag{1.2}$$

1.3 Discussion

Items (1)–(3) below contain comments about the result. Items (4)–(6) are general comments about the model.

(1) For completeness, we state that $r \mapsto t_c(r)$ stays bounded as $r \to 0$ when $d \in \{2,3\}$, since, by monotonicity, $\limsup_{r\to 0} t_c(r) \leq t_c(0) < \infty$. This follows from [4, Theorem 2]. Continuity at $r = 0$ is not immediate, but we expect that this follows from a finite-box criterion of percolation. Theorem 1.1 shows in particular that when $d \geq 4$ the critical time is continuous at $r = 0$, since $t_c(0) = \infty$.

(2) One motivation to study the small radius asymptotics of the critical time is to gain a better understanding of the percolation mechanisms when $d \geq 4$. Indeed, when $d \in \{2,3\}$ percolation can occur because two independent Brownian motions that start close to each

other eventually intersect, see [4, Lemma 5.1]. This argument however breaks down when $d \geq 4$. The proof of Theorem 1.1 gives some insight on how percolation occurs in that case.

(3) The proof of our result makes use of first and second moment estimates on the capacity of a Wiener sausage, that we derive in Section 5. When $d = 4$, these are more subtle and therefore require a more careful analysis than in the high dimensional case $d \geq 5$. This is due to the logarithmic correction in the increase of the mutual intersection local time in four dimensions. We conjecture that these estimates carry over to the simple random walk case. In this framework however, Rath and Sapozhnikov [16] already obtained estimates when $d \geq 5$, so that only the critical dimension $d = 4$ would provide new insights. Let us mention that while preparing this manuscript we were getting aware of a work in progress by van den Berg, Bolthausen and den Hollander [1] who developed simultaneously to us capacity estimates that are similar in spirit.

(4) In the limit $t \nearrow \infty$, the model exhibits long-range dependence. Indeed, if A_1 and A_2 are two bounded sets, then

$$\mathrm{Cov}(\mathbb{1}_{\{A_1 \cap \mathcal{O}_t \neq \emptyset\}}, \mathbb{1}_{\{A_2 \cap \mathcal{O}_t \neq \emptyset\}}) \sim c \ \mathrm{dist}(A_1, A_2)^{2-d}, \tag{1.3}$$

as $\mathrm{dist}(A_1, A_2) \nearrow \infty$. This follows from these three facts:

(i) The number of trajectories intersecting a set E is Poisson distributed with parameter $\int_{\mathbb{R}^d} \mathrm{d}x \mathrm{P}(B^x_{[0,\infty)} \cap E \neq \emptyset)$.

(ii) $\mathrm{P}(B^x_{[0,\infty)} \cap (A_1 \cup A_2) \neq \emptyset) - \mathrm{P}(B^x_{[0,\infty)} \cap A_1 \neq \emptyset) - \mathrm{P}(B^x_{[0,\infty)} \cap A_2 \neq \emptyset) = -\mathrm{P}(B^x_{[0,\infty)} \cap A_1 \neq \emptyset, B^x_{[0,\infty)} \cap A_2 \neq \emptyset)$.

(iii) $\mathrm{P}(B^x_{[0,\infty)} \cap A_1 \neq \emptyset, B^x_{[0,\infty)} \cap A_2 \neq \emptyset) \sim \|x\|^{2-d} \mathrm{cap}(A_1)\mathrm{cap}(A_2) \ \mathrm{dist}(A_1, A_2)^{2-d}$, as $\|x\| \nearrow \infty$.

Note that the correlation exponent coincides with that of interlacement percolation, see Item (5) below.

(5) We expect that, as (i) $t \nearrow \infty$ (ii) $\lambda \searrow 0$ and (iii) λt stays constant, our model shares features with a continuous version of random interlacements. The latter is given by the random subset obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, when started from the uniform distribution and running up to time uN^d as $N \nearrow \infty$, see Sznitman [18]. Indeed, one can argue that in the regime described above, the number of Brownian trajectories entering a set A is a Poisson random variable with intensity $c \ \lambda t \ \mathrm{cap}(A)$, which is a key feature of random interlacements. Consequently, the product of λ and t serves as an intensity parameter.

(6) Peres, Sinclair, Sousi and Stauffer [13, 14] also study a system of points randomly distributed in space and moving according to Brownian motions. However, instead of only looking at $\mathcal{O}_{t,r}$, they also look at $\Sigma_{t,r} = \cup_{x \in \mathcal{E}} \mathcal{B}(B^x_t, r)$ at *each fixed time* t. Nevertheless, in contrast to our setting, they choose r large enough such that $\Sigma_{t,r}$ contains an unbounded cluster for all $t \geq 0$. In these papers the focus is on three aspects:

(i) detection (the first time that a target point is contained in $\Sigma_{t,r}$);

(ii) coverage (the first time that all points inside a finite box are contained in $\mathcal{O}_{t,r}$);

(iii) detection by the unbounded cluster (the time it takes until a target point belongs to the unbounded cluster of $\Sigma_{t,r}$).

1.4 Open questions

(1) Do the upper and lower bounds in Theorem 1.1 match? More precisely, is there a $c_* \in (0,\infty)$ such that

$$\lim_{r\to 0} t_c(r)/f(r) = c_*, \quad \text{with} \quad f(r) = \begin{cases} r^{(4-d)/2}, & d \geq 5, \\ \sqrt{\log(1/r)}, & d = 4? \end{cases} \tag{1.4}$$

(2) Given f as in (1.4), Theorem 1.1 implies that there is a $c_0 > 0$ such that for all $c \geq c_0$ the occupied set $\mathcal{O}_{t(r),r}$ has an unbounded component when $t(r) = cf(r)$ and r is chosen small enough. In particular in that regime there is a large but finite box $B(t(r))$ that intersects the unbounded cluster. Denote the size of that box by $L(t(r))$ and consider $\mathfrak{O}_{L,t(r)} = L(t(r))^{-1}\mathcal{O}_{t(t),r}$. Is it possible to obtain a description of $\lim_{r\to 0}\mathfrak{O}_{L,t(r)}$, if such limit exists? If yes, the constant c could be regarded as an intensity parameter of a Brownian percolation model that undergoes a non-trivial percolation phase transition even when $d \geq 4$.

(3) When $r = 0$, is there a way to define the model conditionally on 0 being in an unbounded cluster? One way to go would be to do the conditioning when $t = c\ f(r)$, with $r > 0$ and c large enough and then let r tend to 0. This way is reminiscent of the construction of the incipient infinite cluster in Bernoulli percolation, see Kesten [9].

1.5 Outline

In Section 2, we recall facts about the Green function and the Newtonian capacity. Section 3 contains the proof of the lower bound, which is guided by the following idea: suppose that the origin is contained in the occupied set, then perform a tree-like exploration of the cluster containing the origin and dominate it by a sub-critical Galton-Watson branching process. Extinction of the Galton-Watson process implies non-percolation of the cluster. Section 4 contains the proof of the upper bound, which consists in the following coarse-graining procedure: (i) we split space in an infinite collection of balls all having a radius of the order $\sqrt{t}$, (ii) each ball is shown to contain with high probability the starting point of a Wiener sausage whose Newtonian capacity is large enough, and (iii) provided t is large enough, these Wiener sausages form an unbounded connected component. Finally, Section 5 contains the proof of several capacity estimates that we use along Sections 3 and 4.

2 Preliminaries on Green function and capacity

In this section we introduce the notion of capacity. We refer the reader to Mörters and Peres [12] and Port and Stone [15] for more detailed surveys on this subject. Let $d \geq 3$ and

denote by Γ the Gamma function. The Green function associated with Brownian motion on $\mathbb{R}^d$ is defined as

$$G(x,y) = \frac{\Gamma(d/2-1)}{2\pi^{d/2}||x-y||^{d-2}}, \quad x,y \in \mathbb{R}^d, \tag{2.1}$$

which can also be rewritten as

$$G(x,y) = \int_0^\infty \mathrm{P}(B_s^x \in \mathrm{d}y)\,\mathrm{d}s. \tag{2.2}$$

Definition 2.1. *Let $A \subset \mathbb{R}^d$ be a Borel set. The energy of a finite Borel measure ν on A is defined as*

$$\mathcal{I}(\nu) = \int_A \int_A G(x,y)\nu(\mathrm{d}x)\nu(\mathrm{d}y) \tag{2.3}$$

and the Newtonian capacity of A is defined as

$$\mathrm{cap}(A) = [\inf_\nu \mathcal{I}(\nu)]^{-1}, \tag{2.4}$$

where the infimum is over all probability measures on A.

Let A, A' be bounded Borel sets. The function $A \mapsto \mathrm{cap}(A)$ is non-decreasing in A, satisfies the scaling relation

$$\mathrm{cap}(aA) = a^{d-2}\mathrm{cap}(A), \quad a > 0, \tag{2.5}$$

and the union bound

$$\mathrm{cap}(A \cup A') + \mathrm{cap}(A \cap A') \leq \mathrm{cap}(A) + \mathrm{cap}(A'). \tag{2.6}$$

Given a bounded set $A \subset \mathbb{R}^d$, let τ_A be the last exit time of A (with the convention that $\tau_A = 0$ if the Brownian motion does not visit the set A). There exists a finite measure e_A on A, the equilibrium measure of A, such that for any Borel set $\Lambda \subseteq A$ and every $x \in \mathbb{R}^d$ (see Chapter 3, Theorem 2.1 in [15]),

$$\mathrm{P}_x(B_{\tau_A} \in \Lambda, \tau_A > 0) = \int_\Lambda G(x,y)e_A(\mathrm{d}y) \tag{2.7}$$

and such that

$$\mathrm{cap}(A) = e_A(A). \tag{2.8}$$

It moreover has an interpretation in terms of hitting probabilities:

$$\lim_{||x||\to\infty} ||x||^{d-2}\mathrm{P}(B^x_{[0,\infty)} \cap A \neq \emptyset) = \frac{\mathrm{cap}(A)}{\kappa_d}, \quad A \subset \mathbb{R}^d \text{ bounded Borel set}, \tag{2.9}$$

where $\kappa_d = 2\pi^{d/2}/\Gamma(d/2-1)$ is the capacity of the unit ball (see Chapter 3, Theorem 1.10 in [15]).

3 Proof of the lower bound

In this section we use the following important lemma, whose proof is deferred to Section 5:

Lemma 3.1. *Let $d \geq 4$, $t_0 > 1$ and $r_0 \in (0,1)$. There is a constant $c > 0$ such that for all $t \geq t_0$ and all $r \in (0, r_0)$,*

$$\mathrm{E}\Big[\mathrm{cap}(W^{0,r}_{[0,t]})^2\Big] \leq c \times \begin{cases} t^2\, r^{2(d-4)} & \text{if } d \geq 5 \\ \Big(\frac{t}{\log(tr^{-2})}\Big)^2 & \text{if } d = 4. \end{cases} \tag{3.1}$$

3.1 Case $d \geq 5$

We use a technique that has been used in the context of Boolean percolation, which consists of exploring the cluster containing the origin and comparing it to a Galton-Watson branching process. For simplicity, we assume that there is a Poisson point at the origin (which is justified in the proof of Proposition 3.2 below). For that purpose we introduce $\mathbb{P}^0$ the law of our process after addition of a Brownian motion at the origin. The Wiener sausages intersecting the Wiener sausage starting at the origin are called first generation sausages, all other sausages intersecting the first generation sausages constitute the second generation sausages, and so on. If the process becomes extinct, the cluster contains finitely many Poisson points, which proves non-percolation.

Let us define

$$N(t,r) = \mathbb{E}^0\Big|\{x \in \mathcal{E} \setminus \{0\} : W^{x,r}_{[0,t]} \cap W^{0,r}_{[0,t]} \neq \emptyset\}\Big|, \tag{3.2}$$

and recall that a Galton-Watson process a.s. becomes extinct if the average number of offsprings is smaller than or equal to 1.

The idea sketched in the paragraph above is summarized in the following Proposition.

Proposition 3.2. *If $N(t,r) < 1$, then $t \leq t_c(r)$.*

Proof. First, we justify why it is enough to consider the cluster containing the Wiener sausage starting from the origin (even though this can be considered standard, we have not found a rigorous argument in the literature). Indeed, since $\mathcal{E}$ is a Poisson point process, $\mathbb{P}^0$ coincides with the Palm version of $\mathbb{P}$, see Proposition 13.1.VII in Daley and Vere-Jones [3]. By definition of the Palm measure, for all bounded Borel sets $A \subseteq \mathbb{R}^d$

$$\mathbb{P}^0(|\mathcal{C}(0)| < \infty) = \frac{1}{\mathrm{Leb}_d(A)}\mathbb{E}\Big\{\sum_{x \in \mathcal{E}\cap A} \mathbf{1}\{|\mathcal{C}(x)| < \infty\}\Big\}. \tag{3.3}$$

Therefore, if $\mathbb{P}^0(|\mathcal{C}(0)| < \infty) = 1$ then by choosing a sequence of Borel sets $(A_n)_{n\in\mathbb{N}}$ increasing to $\mathbb{R}^d$, we get that $\mathbb{P}$-a.s. all clusters are finite, which proves non-percolation.

We now sequentially explore the points in $\mathcal{C}(0)$. To that end we define

$$G_0 = \{0\}, \qquad G_{k+1} = \Big\{x \in \mathcal{E} \setminus \cup_{i=0}^k G_i \colon W^{x,r}_{[0,t]} \bigcap \Big(\bigcup_{y \in G_k} W^{y,r}_{[0,t]}\Big) \neq \emptyset\Big\}, \quad k \in \mathbb{N}_0 \tag{3.4}$$

and $G_{k+1} = \emptyset$ if $G_k = \emptyset$. The restriction that G_{k+1} does not intersect $\cup_{i=0}^k G_i$ avoids double counting of a Poisson point when exploring the cluster. Note that

$$\mathcal{C}(0) = \cup_{k\geq 0} G_k, \qquad \text{and} \quad \mathbb{E}^0(|G_1|) = N(t,r). \tag{3.5}$$

We write for $y \in G_k$,

$$G_{k+1}(y) = \Big\{x \in \mathcal{E} \setminus \cup_{i=1}^k G_i \colon W^{x,r}_{[0,t]} \bigcap W^{y,r}_{[0,t]} \neq \emptyset\Big\}, \tag{3.6}$$

so that $G_{k+1} = \cup_{y\in G_k} G_{k+1}(y)$. Conditionally on $\cup_{i=0}^k G_i$ and for all $y \in G_k$, $G_{k+1}(y)$ is stochastically dominated by a Poisson random variable with parameter equal to $N(t,r)$, because of the geometrical constraint $G_{k+1} \cap (\cup_{i=0}^k G_i) = \emptyset$. Since $N(t,r) < 1$ by assumption, the process $(G_k)_{k\geq 0}$ is dominated by a sub-critical Galton-Watson process. Consequently, $t \leq t_c(r)$ which finishes the proof. □

We are left with estimating $N(t,r)$. This is done in the following Lemma.

Lemma 3.3. *Let $d \geq 5$, $t_0 > 1$ and $r_0 \in (0,1)$. There exists a constant c such that for all $t > t_0$ and $r \in (0, r_0)$,*

$$N(t,r) \leq c\, t^2\, r^{d-4}. \tag{3.7}$$

Proof of Lemma 3.3. Let A be a bounded Borel set of $\mathbb{R}^d$ and $M = \sup_{y \in A} \|y\|$. We first provide an upper bound on

$$N(A) = \{x \in \mathcal{E} \setminus \{0\} \colon B^x_{[0,t]} \cap A \neq \emptyset\}. \tag{3.8}$$

In a second step we will choose $A = W^{0,2r}_{[0,t]}$ and average over $W^{0,2r}_{[0,t]}$, which will eventually yield an upper bound on $N(t,r)$. Write

$$\begin{aligned} \mathbb{E}[|N(A)| \,|\, \mathcal{E}] &= |(\mathcal{E} \setminus \{0\}) \cap A| + \sum_{x \in \mathcal{E} \cap A^c} \mathbb{P}(B^x_{[0,t]} \cap A \neq \emptyset \mid \mathcal{E}) \\ &= |(\mathcal{E} \setminus \{0\}) \cap A| + S_- + S_+, \end{aligned} \tag{3.9}$$

where

$$S_- = \sum_{\substack{x \in \mathcal{E} \cap A^c \\ \|x\| \leq 3M}} \mathbb{P}(B^x_{[0,t]} \cap A \neq \emptyset \mid \mathcal{E}), \qquad S_+ = \sum_{\substack{x \in \mathcal{E} \cap A^c \\ \|x\| > 3M}} \mathbb{P}(B^x_{[0,t]} \cap A \neq \emptyset \mid \mathcal{E}). \tag{3.10}$$

We first bound S_-. Using (2.7), we get

$$S_- \leq \sum_{\substack{x \in \mathcal{E} \cap A^c \\ \|x\| \leq 3M}} \int_A G(x,y) e_A(\mathrm{d}y) = c \sum_{\substack{x \in \mathcal{E} \cap A^c \\ \|x\| \leq 3M}} \int_A \|x-y\|^{2-d} e_A(\mathrm{d}y). \tag{3.11}$$

Hence, an application of Fubini-Tonelli yields

$$S_- \leq c \int_A \sum_{\substack{x \in \mathcal{E} \cap A^c \\ \|x\| \leq 3M}} \|x-y\|^{2-d} e_A(\mathrm{d}y). \tag{3.12}$$

We now average over $\mathcal{E}$. First note that for all $y \in A$,

$$\{x \in \mathcal{E} \cap A^c \colon \|x\| \leq 3M\} \subseteq \overline{\mathcal{B}}(y, 4M). \tag{3.13}$$

Therefore, for all $y \in A$,

$$\mathbb{E}\left[\sum_{\substack{x \in \mathcal{E} \cap A^c \\ \|x\| \leq 3M}} \|x-y\|^{2-d} \right] \leq c \int_0^{4M} \varrho^{2-d} \times (\varrho^{d-1} \mathrm{d}\varrho) = cM^2. \tag{3.14}$$

Thus, using (2.8),

$$\mathbb{E}[S_-] \leq c\, M^2 \int_A e_A(\mathrm{d}y) = c\, \mathrm{cap}(A)\, M^2. \tag{3.15}$$

Let us now turn to bound S_+ from above. Observe that if $\|x\| > 3M$ and $B^x_{[0,t]}$ intersects A, then B^x must shorten its distance from the origin to $\|x\|/2$ (for that it has to travel a

distance of at least $\|x\|/2$) and then intersect A in the remaining time interval. Note that after the first step, the Brownian motion is at distance at least $\|x\|/2$ from the origin. We can therefore write, using the Markov property and (2.7),

$$\begin{aligned} S_+ &\le \sum_{\substack{x\in\mathcal{E}\cap A^c\\ \|x\|>3M}} \mathbb{P}\Big(\sup_{s\in[0,t]}\|B_s^0\|\ge \|x\|/2\,\Big|\,\mathcal{E}\Big) \sup_{z\colon \|z\|=\|x\|/2} \mathbb{P}\Big(B^z_{[0,\infty)}\cap A\neq\emptyset\,\Big|\,\mathcal{E}\Big),\\ &= c\sum_{\substack{x\in\mathcal{E}\cap A^c\\ \|x\|>3M}} \mathbb{P}\Big(\sup_{s\in[0,t]}\|B_s^0\|\ge \|x\|/2\,\Big|\,\mathcal{E}\Big) \sup_{z\colon \|z\|=\|x\|/2} \int_A \|z-y\|^{2-d} e_A(\mathrm{d}y). \end{aligned} \tag{3.16}$$

Note that in the right-hand side, $\|z-y\|\ge\|z\|-\|y\|\ge\frac{\|x\|}{2}-M$. We average over $\mathcal{E}$ to obtain

$$\mathbb{E}[S_+]\le c\,\mathrm{cap}(A)\int_{3M/2}^\infty \mathrm{P}\Big(\sup_{s\in[0,t]}\|B_s^0\|\ge\varrho\Big)\,(\varrho-M)^{2-d}\,\varrho^{d-1}\,\mathrm{d}\varrho. \tag{3.17}$$

Let us choose $A=W^{0,2r}_{[0,t]}$, so that $N(t,r)=\mathbb{E}^0|N(A)|$ and, from (3.9)–(3.10),

$$N(t,r)=\mathbb{E}^0\Big[|(\mathcal{E}\setminus\{0\})\cap W^{0,2r}_{[0,t]}|\Big]+\mathbb{E}^0[S_-]+\mathbb{E}^0[S_+]. \tag{3.18}$$

In this context, $M=\sup_{s\in[0,t]}\|B_s^0\|$*. We first focus on $\mathbb{E}^0[S_-]$. From (3.15) we get

$$\mathbb{E}^0[S_-]\le c\,\mathrm{E}\Big[\sup_{s\in[0,t]}\|B_s^0\|^2\,\mathrm{cap}\Big(W^{0,2r}_{[0,t]}\Big)\Big]. \tag{3.19}$$

By the Cauchy-Schwartz inequality,

$$\mathbb{E}^0[S_-]\le c\,\mathrm{E}\Big[\sup_{s\in[0,t]}\|B_s^0\|^4\Big]^{\frac12}\,\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,2r}_{[0,t]}\Big)^2\Big]^{\frac12}. \tag{3.20}$$

Using Lemma 3.1 and the scale invariance of Brownian motion, we get

$$\mathbb{E}^0[S_-]\le c\,t^2\,r^{d-4}. \tag{3.21}$$

Next, we estimate $\mathrm{E}^0[S_+]$. From (3.17) and Fubini-Tonelli we obtain that

$$\begin{aligned} \mathbb{E}^0[S_+]\le c\int_0^\infty \mathrm{E}\Big[\Big(\varrho-\sup_{s\in[0,t]}\|B_s^0\|\Big)^{2-d}\mathrm{cap}\Big(W^{0,2r}_{[0,t]}\Big)\mathbf{1}\Big\{\varrho\ge\tfrac32\sup_{s\in[0,t]}\|B_s^0\|\Big\}\Big]\\ \times\mathrm{P}\Big(\sup_{s\in[0,t]}\|B_s^0\|\ge\varrho\Big)\varrho^{d-1}\mathrm{d}\varrho. \end{aligned} \tag{3.22}$$

The change of variables $\varrho=u\sqrt{t}$ gives

$$\begin{aligned} \mathbb{E}^0[S_+]\le c\,t\int_0^\infty \mathrm{E}\Big[\Big(u-\sup_{s\in[0,t]}\tfrac{\|B_s^0\|}{\sqrt{t}}\Big)^{2-d}\mathrm{cap}\Big(W^{0,2r}_{[0,t]}\Big)\mathbf{1}\Big\{u\ge\tfrac32\sup_{s\in[0,t]}\tfrac{\|B_s^0\|}{\sqrt{t}}\Big\}\Big]\\ \times\mathrm{P}\Big(\sup_{s\in[0,1]}\|B_s^0\|\ge u\Big)u^{d-1}\mathrm{d}u. \end{aligned} \tag{3.23}$$

*We should actually write $M=\sup_{s\in[0,t]}\|B_s^0\|+r$. The reader may check that this slight abuse is harmless.

By the Cauchy-Schwartz inequality and Lemma 3.1,

$$\begin{aligned}
&\mathbb{E}_0[S_+] \\
&\leq c\,t^2\,r^{d-4}\int_0^\infty \mathrm{E}\Big[\Big(u-\sup_{s\in[0,t]}\tfrac{\|B_s^0\|}{\sqrt{t}}\Big)^{2(2-d)}\mathbf{1}\Big\{u\geq\tfrac{3}{2}\sup_{s\in[0,t]}\tfrac{\|B_s^0\|}{\sqrt{t}}\Big\}\Big]^{\frac12}\mathrm{P}\Big(\sup_{s\in[0,1]}\|B_s^0\|\geq u\Big)u^{d-1}\mathrm{d}u \\
&= c\,t^2\,r^{d-4}\int_0^\infty \mathrm{E}\Big[\Big(u-\sup_{s\in[0,1]}\|B_s^0\|\Big)^{2(2-d)}\mathbf{1}\Big\{u\geq\tfrac{3}{2}\sup_{s\in[0,1]}\|B_s^0\|\Big\}\Big]^{\frac12}\mathrm{P}\Big(\sup_{s\in[0,1]}\|B_s^0\|\geq u\Big)u^{d-1}\mathrm{d}u.
\end{aligned} \tag{3.24}$$

Here, we have used Brownian scale invariance to obtain the last equality. All what remains to prove is that the integral on the right hand side of (3.24) is finite. To that end note that $\int_0^\infty \mathrm{P}(\sup_{s\in[0,1]}\|B_s^0\|\geq u)u^{d-1}\mathrm{d}u$ is finite. Hence, by Jensen's inequality applied to the function $x\mapsto\sqrt{x}$, this integral is at most

$$c\left\{\int_0^\infty \mathrm{E}\Big[\Big(u-\sup_{s\in[0,1]}\|B_s^0\|\Big)^{2(2-d)}\mathbf{1}\Big\{u\geq\tfrac{3}{2}\sup_{s\in[0,1]}\|B_s^0\|\Big\}\Big]\mathrm{P}\Big(\sup_{s\in[0,1]}\|B_s^0\|\geq u\Big)u^{d-1}\mathrm{d}u\right\}^{\frac12}. \tag{3.25}$$

By a change of variables, the term in the square root equals

$$\mathrm{E}\left[\int_{\frac32}^\infty \sup_{s\in[0,1]}\|B_s^0\|^{4-d}\widetilde{\mathrm{P}}\Big(\sup_{s\in[0,1]}\|\widetilde{B}_s^0\|\geq u\sup_{s\in[0,1]}\|B_s^0\|\Big)(u-1)^{2(2-d)}u^{d-1}\mathrm{d}u\right], \tag{3.26}$$

which is finite since (i) the probability is bounded by 1, (ii) there exists a constant $c\in(0,\infty)$ such that $\mathrm{E}[\sup_{s\in[0,1]}\|B_s^0\|^{4-d}]\leq c\,\mathrm{E}[\|B_1^0\|^{4-d}]$, which is finite, and (iii) $\int_{3/2}^\infty u^{3-d}\mathrm{d}u$ is finite. This proves that

$$\mathbb{E}^0[S_+]\leq c\,t^2\,r^{d-4}. \tag{3.27}$$

To complete the proof, observe that

$$\mathbb{E}^0\Big[\Big|\Big(\mathcal{E}\setminus\{0\}\Big)\cap W_{[0,t]}^{0,2r}\Big|\Big]=\mathrm{E}\Big[\mathrm{Leb}_d\Big(W_{[0,t]}^{0,2r}\Big)\Big]=c\,r^d\mathrm{E}\Big[\mathrm{Leb}_d\Big(W_{[0,tr^{-2}]}^{0,1}\Big)\Big]=c\;t\,r^{d-2}\leq c\,t^2\,r^{d-4}. \tag{3.28}$$

where in the two last equalities we have used Brownian scale invariance and the fact that $\mathrm{E}[\mathrm{Leb}_d(W_{[0,t]}^{0,1})]\overset{t\to\infty}{\sim}\mathrm{cap}(\overline{\mathcal{B}}(0,1))\,t/2$, see [17, 5]. Combining (3.18), (3.21), (3.27) and (3.28), we get the result. □

Proof of the lower bound in Theorem 1.1. Combine Proposition 3.2 and Lemma 3.3. □

3.2 Case $d=4$

The proof strategy is the same as for $d\geq5$. However, Lemma 3.3 is adapted as follows:

Lemma 3.4. *Let $d=4$, $t_0>1$ and $r_0\in(0,1)$. There exists a constant c such that for all $t>t_0$ and $r\in(0,r_0)$,*

$$N(t,r)\leq c\,\frac{t^2}{\log(tr^{-2})}. \tag{3.29}$$

Proof of Lemma 3.4. The proof is similar to that of Lemma 3.3, except for the following point: the argument to prove that (3.26) is finite needs to be adapted, since (iii) below (3.26) does not hold anymore. Instead, we may estimate using Doob's inequality

$$\widetilde{\mathrm{P}}\Big(\sup_{s\in[0,1]}\|\widetilde{B}^0_s\| \geq u \sup_{s\in[0,1]}\|B^0_s\|\Big) \leq e^{-\frac{1}{2}u^2 \sup_{s\in[0,1]}\|B^0_s\|^2}. \tag{3.30}$$

Thus,

$$\begin{aligned}(\mathrm{P}\times\widetilde{\mathrm{P}})\Big(\sup_{s\in[0,1]}\|\widetilde{B}^0_s\| \geq u \sup_{s\in[0,1]}\|B^0_s\|\Big) \leq \mathrm{E}\Big[e^{-\frac{1}{2}u^2 \sup_{s\in[0,1]}\|B^0_s\|^2}\Big] &\leq \mathrm{E}\Big[e^{-\frac{1}{2}u^2\|B^0_1\|^2}\Big] \\ &= \frac{c}{(1+u^2)^2},\end{aligned} \tag{3.31}$$

which is enough to prove that the integral in (3.26) converges. □

4 Proof of the upper bound

In this section we prove the upper bound in Theorem 1.1, that is if we set

$$t = c_* \times \begin{cases} r^{(4-d)/2} & \text{if } d \geq 5 \\ \sqrt{\log(1/r)} & \text{if } d = 4 \end{cases}, \quad r \in (0,1), \tag{4.1}$$

then there exists c_* large enough such that for r small enough, there is percolation.

The proof is organized as follows. In Section 4.1, we use a coarse-graining procedure to prove the existence of an unbounded component with a positive probability. More precisely, we divide space into boxes indexed by $\mathbb{Z}^d$ and we define a notion of *good* boxes, as well as a way to connect good boxes. Provided the box at the origin is good, we explore the cluster of good boxes connected to the origin and prove that with positive probability, this cluster is unbounded. This implies percolation. The procedure relies on two estimates, one on the probability for the box at the origin to be good (Lemma 4.1), the other one on the probability of two neighboring good boxes being connected to each other (Lemma 4.2). These estimates are proven in Section 4.2.

In this section we use the following notation: for $c, r, t > 0$, we let

$$A_t(c,r;x) = \Big\{W^{x,r}_{[0,t]} \subseteq \mathcal{B}\Big(x, c\sqrt{t}\Big)\Big\}, \qquad A_t(c,r) = A_t(c,r;0). \tag{4.2}$$

4.1 Coarse-graining procedure

Parameters are now chosen as in (4.1). Let $c_B > 0$ be a parameter to be determined later. Let us consider for $z \in \mathbb{Z}^d$ the ball $\mathcal{B}_z = \mathcal{B}(2zc_B\sqrt{t}, c_B\sqrt{t})$. In the following we identify $\mathbb{Z}^2 \times \{0\}^{d-2}$ with $\mathbb{Z}^2$. We are going to prove that if c_* is large enough and r is small enough then percolation occurs by using only Wiener sausages from the two-dimensional coarse-grained model $\cup_{z\in\mathbb{Z}^2}\mathcal{B}_z$.

Definition of a *good* box. We say that $z \in \mathbb{Z}^2$ is good if there exists $x \in \mathcal{E} \cap \mathcal{B}_z$ such that

(Condition 1) The event $A_{t/2}(c_B, r; x)$ is fulfilled,

(Condition 2) $\mathrm{cap}(W^{x,r}_{[0,t/2]}) \geq \frac{1}{2}\mathrm{E}[\mathrm{cap}(W^{0,r}_{[0,t/2]}); A_{t/2}(c_B, r)]$.

Construction of a cluster on the coarse-grained model. We now define a cluster on this two-dimensional lattice, starting from 0. Let us define iteratively a sequence $(\mathbf{C}_n)_{n\geq 0}$, where $\mathbf{C}_n$ is a subset of the ball $\mathbf{B}^1_n = \{z \in \mathbb{Z}^2 \colon |z|_1 = n\}$, in the following way:
Initialization. If 0 is good then set $\mathbf{C}_0 = \{0\}$, pick one of the Poisson points in $\mathcal{E} \cap \mathcal{B}_0$ satisfying conditions 1 and 2 above (let us say, the closest to 0), and call it $e(0)$. If 0 is not good then set $\mathbf{C}_0 = \emptyset$.
Iteration. Let $n \geq 0$. If $\mathbf{C}_n = \emptyset$, then set $\mathbf{C}_{n+1} = \emptyset$. Otherwise, for each $z \in \mathbf{C}_n$, there is by construction a point $e(z) \in \mathcal{E} \cap \mathcal{B}_z$ satisfying conditions 1 and 2 which, if $n \geq 1$, is connected to $\{e(x), x \in \mathbf{C}_j, 0 \leq j \leq n-1\}$. Let $x \in \mathbf{B}^1_{n+1}$. We declare that $x \in \mathbf{C}_{n+1}$ if and only if there exists $z \in \mathbf{C}_n$ such that $|z - x|_1 = 1$ and there exists $y \in \mathcal{E} \cap \mathcal{B}_x$ satisfying conditions 1 and 2 (this means in particular that x is good), for which

$$\text{(Condition 3)} \qquad W^{y,r}_{[t/2,t]} \bigcap \Big(\cup_{z \in \mathbf{C}_n \colon |z-x|_1 = 1} W^{e(z),r}_{[0,t]} \Big) \neq \emptyset. \tag{4.3}$$

Among all such possible points y, denote by $e(x)$ the one which is closest to the center of $\mathcal{B}_x$.
We denote by $\mathbf{C} = \cup_{n\geq 0}\mathbf{C}_n$ the full cluster, which may be finite or infinite. By construction $\cup_{z\in\mathbf{C}} W^{e(z),r}_{[0,t]}$ is a connected component. Therefore, this implies that if $\mathbf{C}$ is infinite then there is an unbounded connected component in the original model. We are going to prove below that $\mathbf{C}$ is indeed infinite with positive probability, provided c_* is large enough and r is small enough.

For the rest of the proof we rely on the following two key lemmas, which will be proven in Section 4.2.

Lemma 4.1. *Let $d \geq 4$ and fix $c_B > 0$. The probability that 0 is good converges to 1 as t goes to ∞.*

Lemma 4.2. *Let $d \geq 4$, fix $c_B > 0$, let $n \in \mathbb{N}$ and $y \in \mathbf{B}^1_n$. On the event $\{\exists x \in \mathbf{C}_{n-1}$ with $|x - y|_1 = 1\}$, we have for t large enough,*

$$\mathbb{P}(y \notin \mathbf{C}_n | W^{e(z)}, z \in \mathbf{C}_j, j \leq n-1) \leq \exp\{-c_*^2 \theta(c_B)\}, \tag{4.4}$$

where $\theta(c_B)$ becomes positive for c_B small enough.

We now explain how to conclude the proof with these two lemmas at hand. For this, we use the so-called standard Peierls contour argument, see Grimmett [7, Proof of Theorem 1.10]. In what follows, a $*$-path of length $N \geq 2$ is a vector $(x_i)_{1\leq i\leq N} \in (\mathbb{Z}^2)^N$ such that $|x_{i+1} - x_i|_\infty = 1$ for all $1 \leq i < N$. If $x_N = x_1$ and for all $1 \leq i, j < N$ with $i \neq j$, $x_i \neq x_j$, then the $*$-path is said to be a $*$-contour. This contour contains $x \in \mathbb{Z}^2$ if x belongs to the bounded component delimited by the contour and x is not on the contour. Denote by $\partial_{\mathrm{ext}}\mathbf{C}$ the exterior boundary of $\mathbf{C}$, that is the set of vertices in the boundary which are the starting point of an infinite non-intersecting nearest neighbor path with no vertex in $\mathbf{C}$.

An induction argument which mimicks the construction of $\mathbf{C}$ shows that if $|\mathbf{C}| < \infty$, then $\partial_{\text{ext}}\mathbf{C}$ is a $*$-contour. We may write

$$\begin{aligned}\mathbb{P}(|\mathbf{C}| < \infty) &\leq \mathbb{P}(0 \text{ is not good}) + \sum_{N\geq 4} \mathbb{P}(|\partial_{\text{ext}}\mathbf{C}| = N) \\ &\leq \mathbb{P}(0 \text{ is not good}) + \sum_{N\geq 4} e^{-c_*\theta(c_B)N} C_N,\end{aligned} \tag{4.5}$$

where C_N is the number of $*$-contours of length N containing the origin. To obtain the second line in (4.5) we first sum over all possible realisations of $\partial_{\text{ext}}\mathbf{C}$ and thereafter we use Lemma 4.2 in combination with the conditional independence of events of the form $\{y_i \in \partial_{\text{ext}}\mathbf{C}\}_{1\leq i\leq N}$ given the realisations of all Wiener sausages contained in the set delimited by the contour $\partial_{\text{ext}}\mathbf{C}$. By a standard counting argument (see Grimmett [7, Proof of Theorem 1.10]) it can be seen that $C_N \leq N\ 7^N$. We obtain

$$\mathbb{P}(|\mathbf{C}| < \infty) \leq \mathbb{P}(0 \text{ is not good}) + c \sum_{N\geq 4} N \left(7e^{-c_*\theta(c_B)}\right)^N. \tag{4.6}$$

We conclude as follows. First, fix c_B such that $\theta(c_B)$ is positive. Then, choose c_* so large that the sum in the r.h.s of (4.6) is smaller than $1/4$. Finally, choose r small enough (therefore t large enough) such that, by Lemma 4.1, $\mathbb{P}(0 \text{ is not good}) \leq 1/4$. At the end, we get $\mathbb{P}(|\mathbf{C}| = \infty) \geq 1/2$, which finishes the proof.

4.2 Proof of Lemmas 4.2 and 4.1.

Throughout this section we shall make use of the following key lemma, which estimates the mean capacity of a Wiener sausage confined to a ball with radius of order $\sqrt{t}$. Its proof is deferred to Section 5.

Lemma 4.3. *For all $d \geq 4$, there exists a positive constant c such that for all $t \geq 1$ and all $r \in (0,1)$,*

$$\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big); A_t(c_B, r)\Big] \geq c\ \mathrm{P}(A_t(c_B,r))^2 \ \times \begin{cases} t\ r^{d-4} & \text{if } d \geq 5 \\ \frac{t}{\log(tr^{-2})} & \text{if } d = 4. \end{cases} \tag{4.7}$$

We start with Lemmas 4.4 and 4.5, which are preparatory lemmas. Lemma 4.4 gives a lower bound on the probability that a Wiener sausage has a capacity larger than a fraction of its mean capacity, when it is confined to a ball of order $\sqrt{t}$. Lemma 4.5 gives a lower bound on the probability that a Wiener sausage intersects a set that is at a distance of order $\sqrt{t}$ from its starting point.

Lemma 4.4. *Let $d \geq 4$. Abbreviate by A the event $A_t(c_B, r)$, see Equation (4.2). There is a constant c such that,*

$$\mathrm{P}\Big(\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big) \geq \tfrac{1}{2}\ \mathrm{E}\Big(\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big); A\Big); A\Big) \geq c\ \Phi(c_B)^4(1 + o(1)), \tag{4.8}$$

where $\Phi(c_B) = \mathrm{P}\big(\sup_{s\in[0,1]} ||B^0_s|| \leq c_B\big)$ and the $o(1)$ term tends to zero as t tends to infinity.

Proof. The Paley-Zigmund inequality (more precisely: a slight generalization thereof) states that

$$\mathrm{P}\Big(\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big) \geq \tfrac{1}{2}\,\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big); A\Big]; A\Big) \geq \tfrac{1}{4}\frac{\Big(\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big); A\Big]\Big)^2}{\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big)^2\Big]}. \tag{4.9}$$

By Lemma 3.1, there is a constant c such that for all $r > 0$ small enough

$$\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big)^2\Big] \leq c \begin{cases} t^2\, r^{2(d-4)}, & \text{if } d \geq 5, \\ \Big(\frac{t}{\log(tr^{-2})}\Big)^2, & \text{if } d = 4. \end{cases} \tag{4.10}$$

Moreover, by Lemma 4.3 there is a constant $c > 0$ such that

$$\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big); A\Big] \geq c\,\mathrm{P}(A)^2 \begin{cases} t\, r^{d-4}, & \text{if } d \geq 5, \\ \frac{t}{\log(tr^{-2})}, & \text{if } d = 4. \end{cases} \tag{4.11}$$

Furthermore, by the scale invariance of Brownian motion

$$\mathrm{P}(A) = \mathrm{P}\Big(W^{0,r/\sqrt{t}}_{[0,1]} \subseteq \mathcal{B}(0, c_B)\Big) = \mathrm{P}\Big(\sup_{s\in[0,1]} ||B^0_s|| \leq c_B + \frac{r}{\sqrt{t}}\Big) = \Phi(c_B)(1 + o(1)). \tag{4.12}$$

Hence, equations (4.9)–(4.12) yield the claim. □

Given a measurable set $A \subseteq \mathbb{R}^d$ we denote by A^r its r-thickening, i.e.,

$$A^r = \bigcup_{x\in A} \mathcal{B}(x, r). \tag{4.13}$$

Lemma 4.5. *There is a finite constant $c > 0$ such that the following estimate holds uniformly for all $r \in (0, 1)$ and all measurable sets A such that $A \subseteq \mathcal{B}(0, 6c_B\sqrt{t})$,*

$$\mathrm{P}\Big(W^{0,r}_{[0,t]} \cap A \neq \emptyset\Big) \geq t^{-d/2+1}\mathrm{cap}(A^r)\Big(\frac{c}{c_B^{d-2}} - \frac{1}{(2\pi)^{d/2}}\Big). \tag{4.14}$$

Proof. Note that

$$\begin{aligned} &\mathrm{P}\Big(W^{0,r}_{[0,t]} \cap A \neq \emptyset\Big) \\ &\quad = \mathrm{P}\Big(W^{0,r}_{[0,\infty)} \cap A \neq \emptyset\Big) - \mathrm{P}\Big(\inf\Big\{s > 0\colon W^{0,r}_{[0,s]} \cap A \neq \emptyset\Big\} \in (t, \infty)\Big), \end{aligned} \tag{4.15}$$

so that it is enough to find a lower bound for the first term on the right hand side of (4.15) and an upper bound for the second term on the right hand side of (4.15). Let e_{A^r} be the equilibrium measure of A^r. The identity in (2.7) yields

$$\mathrm{P}\Big(W^{0,r}_{[0,\infty)} \cap A \neq \emptyset\Big) = \mathrm{P}\Big(B^0_{[0,\infty)} \cap A^r \neq \emptyset\Big) = \int_{A^r} G(0, y)e_{A^r}(dy). \tag{4.16}$$

Hence, using that $G(0, y) = c\,||y||^{2-d}$ and $e_{A^r}(A^r) = \mathrm{cap}(A^r)$, the integral on the right hand side of (4.16) may be bounded from below by

$$c \inf_{y\in A^r} ||y||^{2-d}\mathrm{cap}(A^r). \tag{4.17}$$

Since $A^r \subseteq \mathcal{B}(0, 6c_B\sqrt{t} + r)$, we see that there is a constant $c > 0$ such that (4.17) is at least

$$c\, t^{-d/2+1} c_B^{2-d} \mathrm{cap}(A^r). \tag{4.18}$$

This is the desired lower bound for the first term on the right hand side of (4.15). An application of the Markov property shows that the second term on the right hand side of (4.15) may be written as

$$\begin{aligned} &\mathrm{E}\Big[\mathbb{1}\Big\{B^0_{[0,t]} \cap A^r = \emptyset\Big\} \widetilde{\mathrm{P}}_{B^0_t}\Big(\widetilde{B}_{[0,\infty)} \cap A^r \neq \emptyset\Big)\Big] \\ &\quad = \mathrm{E}\Big[\mathbb{1}\Big\{B^0_{[0,t]} \cap A^r = \emptyset\Big\} \int_{A^r} G(B^0_t, y)\, e_{A^r}(\mathrm{d}y)\Big], \end{aligned} \tag{4.19}$$

where the equality is a consequence of (2.7) and $\widetilde{B}$ is a Brownian motion independent of B^0. Hence, (4.19) is bounded from above by

$$\int_{A^r} \mathrm{E}[G(B^0_t, y)]\, e_{A^r}(\mathrm{d}y). \tag{4.20}$$

We obtain by the Markov property applied to B^0 at time t,

$$\mathrm{E}(G(B^0_t, y)) = \int_t^\infty \mathrm{P}(B^0_s \in \mathrm{d}y)\, \mathrm{d}s = \int_t^\infty \frac{1}{(2\pi s)^{d/2}} e^{-\|y\|^2/2s}\, \mathrm{d}s. \tag{4.21}$$

Using the substitution $w = ||y||^2/2s$, we see that the right hand side of (4.21) equals

$$\int_0^{\|y\|^2/2t} w^{d/2-2} e^{-w}\, \mathrm{d}w \times \frac{\|y\|^{2-d}}{2\pi^{d/2}}, \tag{4.22}$$

which is bounded from above by $(2\pi)^{-d/2} t^{-d/2+1}$ (by bounding the exponential factor by 1). Therefore, (4.20) is bounded from above by

$$\mathrm{cap}(A^r) t^{-d/2+1} (2\pi)^{-d/2}. \tag{4.23}$$

Combining (4.15) with (4.18) and (4.23) yields the claim. □

Proof of Lemma 4.2. Let $n \in \mathbb{N}$, $y \in \mathbf{B}^1_n$ and abbreviate $\bar{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot \mid W^{e(z)}, z \in \mathbf{C}_j, j \leq n-1)$. Suppose that there exists $z \in \mathbf{C}_{n-1}$ such that $|y - z|_1 = 1$. Let $x \in \mathcal{E} \cap \mathcal{B}_y$. We first give a lower bound on the probability that x satisfies the conditions 1–3 of Section 4.1, that is

$$p_0 := \bar{\mathbb{P}}\left(\begin{array}{c} \mathrm{cap}(W^{x,r}_{[0,t/2]}) \geq \frac{1}{2}\mathrm{E}[\mathrm{cap}(W^{0,r}_{[0,t/2]}); A_{t/2}(c_B, r)], A_{t/2}(c_B, r; x), \\ W^{x,r}_{[t/2,t]} \bigcap \Big(\cup_{z \in \mathbf{C}_{n-1} :\, |z-y|_1 = 1} W^{e(z),r}_{[0,t]} \Big) \neq \emptyset \end{array} \right). \tag{4.24}$$

Using the Markov property on B^x at time $t/2$ and that $B^x_{t/2} \in \mathcal{B}(2yc_B\sqrt{t}, 2c_B\sqrt{t})$ this probability can be bounded from below by

$$\begin{aligned} &\mathrm{P}\Big(\mathrm{cap}\Big(W^{x,r}_{[0,t/2]}\Big) \geq \tfrac{1}{2}\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t/2]}\Big); A_{t/2}(c_B, r)\Big], A_{t/2}(c_B, r; x)\Big) \\ &\qquad \times \inf_{x_0 \in \mathcal{B}(2yc_B\sqrt{t}, 2c_B\sqrt{t})} \bar{\mathbb{P}}\Big(W^{x_0,r}_{[0,t/2]} \bigcap \Big(\cup_{z \in \mathbf{C}_{n-1} :\, |z-y|_1 = 1} W^{e(z),r}_{[0,t]} \Big) \neq \emptyset\Big). \end{aligned} \tag{4.25}$$

Using Lemma 4.4 on the first factor and Lemma 4.5 on the second factor and noticing that for all $x_0 \in \mathcal{B}(2yc_B\sqrt{t}, 2c_B\sqrt{t})$, $\cup_{z\in\mathbf{C}_{n-1}\colon |z-y|_1=1} W^{e(z),r}_{[0,t/2]} \subseteq \mathcal{B}(x_0, 6c_B\sqrt{t})$, we get that this probability is larger than

$$\varphi(c_B)t^{-d/2+1}\mathrm{cap}\Big(\cup_{z\in\mathbf{C}_{n-1}\colon |z-y|_1=1} W^{e(z),r}_{[0,t/2]}\Big). \tag{4.26}$$

Here, $\varphi(c_B) := c\ \Phi(c_B)^4\Big(\frac{c}{c_B^{d-2}} - \frac{1}{(2\pi)^{d/2}}\Big)(1+o(1))$ is positive provided c_B is small enough. By construction, we know that

$$\mathrm{cap}\Big(\cup_{z\in\mathbf{C}_{n-1}\colon |z-y|_1=1} W^{e(z),r}_{[0,t/2]}\Big) \geq \tfrac{1}{2}\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t/2]}\Big); A_{t/2}(c_B,r)\Big]. \tag{4.27}$$

Recalling Lemma 4.3 and Equation (4.1), we obtain

$$p_0 \geq c\, c_*^2\varphi(c_B)\Phi(c_B)^2 t^{-d/2}(1+o(1)). \tag{4.28}$$

Therefore, the number of points in $\mathcal{E}\cap\mathcal{B}_y$ satisfying conditions 1–3 is a Poisson random variable with parameter

$$p_0 \times \mathrm{Leb}(\mathcal{B}_y) \geq c_*^2\ \theta(c_B), \qquad \text{with} \quad \theta(c_B) = c\ c_B^d\varphi(c_B)\Phi(c_B)^2, \tag{4.29}$$

which concludes the proof. □

Proof of Lemma 4.1. If $x \in \mathcal{E}\cap\mathcal{B}_0$ then the probability that x satisfies conditions 1–2 in Section 4.1 is larger than $c\ \Phi(c_B)^4(1+o(1))$, by Lemma 4.4. Therefore, the number of such points is a Poisson random variable with parameter:

$$c\ c_B^d\ \Phi(c_B)^4\ t^{d/2}(1+o(1)), \tag{4.30}$$

which goes to ∞ as $t\to\infty$. This concludes the proof. □

5 Capacity estimates

5.1 Green function estimates

Lemma 5.1. *Let $d \geq 4$ and $t_0 > 1$. There exists a positive constant c such that for all $t \geq t_0$,*

$$\mathrm{E}\Big[\int_{[t,2t]^2}\int_{\mathcal{B}(0,1)^2} G(B_u^0+z, B_v^0+z')\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}u\,\mathrm{d}v\Big] \leq \begin{cases} c\ t, & \text{if } d\geq 5,\\ c\ t\log t, & \text{if } d=4.\end{cases} \tag{5.1}$$

Proof. **Case $d \geq 5$.**
We start with two estimates. First, let $0 \leq u \leq 1$. We claim that

$$\mathrm{E}(G(B_u^0, z)) \leq G(0,z) \quad \text{for all } z\in\mathbb{R}^d. \tag{5.2}$$

Indeed, an application of the Markov property in the second equality shows that

$$\mathrm{E}(G(B_u^0,z)) = \int_0^\infty \mathrm{E}[\widetilde{\mathrm{P}}_{B_u^0}(\widetilde{B}_s\in\mathrm{d}z)]\,\mathrm{d}s = \int_0^\infty \mathrm{P}(B_{u+s}^0\in\mathrm{d}z)\,\mathrm{d}s = \int_u^\infty \mathrm{P}(B_s^0\in\mathrm{d}z)\,\mathrm{d}s. \tag{5.3}$$

Since the right hand side is bounded from above by $G(0,z)$ we obtain the claim. Now, let $u>1$. In this case we claim that there is a constant $c>0$ such that

$$\mathrm{E}(G(B_u^0,z)) \le c\,u^{1-d/2} \quad \text{for all } z\in\mathbb{R}^d. \tag{5.4}$$

This is a direct consequence of (4.21)–(4.22). To make use of the inequalities (5.2) and (5.4) we write the left hand side in (5.1) as a sum of three terms:

$$\begin{aligned}
(1) &= \mathrm{E}\left[\int_t^{2t}\int_t^{(v-1)\vee t}\int_{\mathcal{B}(0,1)^2} G(B_u^0+z,B_v^0+z')\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}u\,\mathrm{d}v\right],\\
(2) &= \mathrm{E}\left[\int_t^{2t}\int_{(v+1)\wedge 2t}^{2t}\int_{\mathcal{B}(0,1)^2} G(B_u^0+z,B_v^0+z')\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}u\,\mathrm{d}v\right],\\
(3) &= \mathrm{E}\left[\int_t^{2t}\int_{(v-1)\vee t}^{(v+1)\wedge 2t}\int_{\mathcal{B}(0,1)^2} G(B_u^0+z,B_v^0+z')\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}u\,\mathrm{d}v\right].
\end{aligned} \tag{5.5}$$

We first estimate the third term. Note that for all $x,y\in\mathbb{R}^d$ the relation $G(x,y)=G(0,y-x)$ holds. Hence, a change in the order of integration together with equation (5.2) and the fact that $B_v^0-B_u^0$ has the same distribution as $B_{|v-u|}^0$ show that

$$(3) \le \int_{\mathcal{B}(0,1)^2}\int_t^{2t}\int_{v-1}^{v+1} G(0,z-z')\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z\,\mathrm{d}z' \le 2t\int_{\mathcal{B}(0,1)^2} G(0,z-z')\,\mathrm{d}z\,\mathrm{d}z'. \tag{5.6}$$

Hence, it suffices to show that the integral on the right-hand side of (5.6) converges. For that, first note that by the representation of the Green function in (2.1), there is $c>0$ such that the right hand of (5.6) is at most

$$2ct\int_{\mathcal{B}(0,1)}\int_{\mathcal{B}(z',1)} \|z-z'\|^{2-d}\mathrm{d}z\,\mathrm{d}z' = 2ct\int_{\mathcal{B}(0,1)^2}\|z\|^{2-d}\,\mathrm{d}z\,\mathrm{d}z', \tag{5.7}$$

where we made the substitution $\zeta=z-z'$ to obtain the last equality. The convergence of the inner integral on the right-hand side of (5.7) is now a standard fact. Thus, there is $c>0$ such that $(3)\le ct$. It remains to show that the first and the second term in (5.5) give the correct contribution. Equation (5.4) yields

$$(1) \le c\int_{\mathcal{B}(0,1)^2}\int_t^{2t}\int_t^{(v-1)\vee t} |v-u|^{1-d/2}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z\,\mathrm{d}z', \tag{5.8}$$

for some constant c. A simple computation now shows that there is indeed a constant $c>0$ such that for all $t\ge 0$ the bound $(1)\le ct$ holds. The argument for (2) in (5.5) is similar and will therefore be omitted. This finishes the proof in this case.

Case $d=4$. The proof works almost verbatim as in the previous case. The only difference is that (5.8) becomes

$$\int_{\mathcal{B}(0,1)^2}\int_t^{2t}\int_t^{(v-1)\vee t} |v-u|^{-1}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}z\,\mathrm{d}z', \tag{5.9}$$

which is upper bounded by $ct\log t$. We omit the details. □

5.2 Lower bounds. Proof of Lemma 4.3.

The proof of Lemma 4.3 makes use of the variational representation (2.4). This representation implies that to prove Lemma 5.1 it suffices to construct a measure, which is close to the "true" minimizer in (2.4). It will turn out that it suffices to choose a measure that measures the local time of the Brownian motion in a neighborhood of a given set. In this way Green function estimates enter naturally into the picture. They are provided by Lemma 5.1.

Proof of Lemma 4.3. We start with the case $d \geq 5$.
1st Step: Let $r = 1$ and ν be the probability measure supported on $W^{0,1}_{[0,t]}$ and defined by

$$\nu(A) = \frac{1}{c_{\mathrm{vol}}(t/4)} \int_{t/4}^{t/2} \int_{\mathcal{B}(0,1)} \mathbb{1}\{B^0_s + z \in A\} \, \mathrm{d}s \, \mathrm{d}z, \quad A \text{ Borel-measurable.} \tag{5.10}$$

Note that by the variational formula (2.4) for the capacity

$$\mathrm{E}\Big[\mathrm{cap}\Big(W^{0,1}_{[0,t]}\Big); A_t(c_B, 1)\Big] \geq \mathrm{E}[\mathcal{I}(\nu)^{-1}; A_t(c_B, 1)], \tag{5.11}$$

where

$$\mathcal{I}(\nu) = \frac{1}{c^2_{\mathrm{vol}}(t/4)^2} \int_{[t/4,t/2]^2} \int_{\mathcal{B}(0,1)^2} G(B^0_u + z, B^0_v + z') \, \mathrm{d}z \, \mathrm{d}z' \mathrm{d}u \, \mathrm{d}v. \tag{5.12}$$

An application of the Cauchy-Schwarz inequality shows that

$$\mathrm{E}[\mathcal{I}(\nu)^{-1}; A_t(c_B, 1)] \geq \mathrm{E}[\mathcal{I}(\nu)]^{-1} \mathrm{P}(A_t(c_B, 1))^2. \tag{5.13}$$

Finally, by Equation (5.1), there is a constant $c > 0$ such that the right hand side of (5.13) is bounded from below by $ct\mathrm{P}(A_t(c_B, 1))^2$. This yields the claim in the case $r = 1$.
2nd Step: Let now $r > 0$ be chosen arbitrarily. Note that

$$\begin{aligned} \mathrm{E}\Big[\mathrm{cap}\Big(W^{0,r}_{[0,t]}\Big); A_t(c_B, r)\Big] &= \mathrm{E}\Big[\mathrm{cap}\Big(\frac{r}{r} W^{0,r}_{[0,t]}\Big); A_t(c_B, r)\Big] \\ &= r^{d-2} \mathrm{E}\Big[\mathrm{cap}\Big(W^{0,1}_{[0,t/r^2]}\Big); A_{tr^{-2}}(c_B r^{-1}, 1)\Big], \end{aligned} \tag{5.14}$$

where we used Brownian scaling and the capacity scaling relation (2.5) to obtain the second equality. Using the result for the case $r = 1$ and noting that $\mathrm{P}(A_{tr^{-2}}(c_B r^{-1}, 1)) = \mathrm{P}(A_t(c_B, r))$ finishes the proof for $d \geq 5$.

The proof in the case $d = 4$ works along similar lines, the only difference being that the application of Lemma 5.1 is adapted. □

5.3 Second moment estimates. Proof of Lemma 3.1

5.3.1 Case $d \geq 5$

Proof. **1st Step:** In this step we prove Lemma 3.1 under the assumption $r = 1$. First note that by Equation (2.6)

$$\mathrm{cap}\Big(W^{0,1}_{[0,t]}\Big) \leq \mathrm{cap}\Bigg(\bigcup_{i=1}^{\lceil t \rceil} W^{0,1}_{[(i-1),i]}\Bigg) \leq \sum_{i=1}^{\lceil t \rceil} \mathrm{cap}\Big(W^{0,1}_{[(i-1),i]}\Big), \tag{5.15}$$

so that by the independence of $B_i^0 - B_{i-1}^0$ and $B_j^0 - B_{j-1}^0$ for all $j \neq i$, $i, j \in \{1, 2, \dots, \lceil t \rceil\}$,

$$\begin{aligned} &\mathrm{E}\Big[\mathrm{cap}\Big(W_{[0,t]}^{0,1}\Big)^2\Big] \\ &\leq \sum_{\substack{i,j=1 \\ i\neq j}}^{\lceil t \rceil} \mathrm{E}\Big[\mathrm{cap}\Big(W_{[(i-1),i]}^{0,1}\Big)\Big] \times \mathrm{E}\Big[\mathrm{cap}\Big(W_{[(j-1),j]}^{0,1}\Big)\Big] + \sum_{i=1}^{\lceil t \rceil} \mathrm{E}\Big[\mathrm{cap}\Big(W_{[(i-1),i]}^{0,1}\Big)^2\Big]. \end{aligned} \tag{5.16}$$

Consequently, by the stationarity of Brownian motion in time and by the Cauchy-Schwarz inequality, the right hand side of (5.16) is bounded from above by

$$\lceil t \rceil^2 \times \mathrm{E}\Big[\mathrm{cap}\Big(W_{[0,1]}^{0,1}\Big)^2\Big]. \tag{5.17}$$

To see that the expectation on the right hand side of (5.17) is finite, note that by the scaling relation (2.5) for any $R > 0$ the identity $\mathrm{cap}(\mathcal{B}(0,R)) = R^{d-2}\mathrm{cap}(\mathcal{B}(0,1))$ holds. Since $\mathrm{E}(\sup_{s\in(0,1)} \|B_s\|^{d-2}) < \infty$, the desired finiteness readily follows. This proves Lemma 3.4 in the case $r = 1$.

2nd Step: In this step we treat the general case. To that end, note that by Brownian scaling and by the scaling relation (2.5),

$$\mathrm{cap}\Big(W_{[0,t]}^{0,r}\Big) = \mathrm{cap}\Big(\frac{r}{r} W_{[0,t]}^{0,r}\Big) \overset{\text{(law)}}{=} r^{d-2}\mathrm{cap}\Big(W_{[0,tr^{-2}]}^{0,1}\Big). \tag{5.18}$$

The claim follows from equation (5.18) in combination with the first step. □

5.3.2 Case $d = 4$

The proof is based on methods presented in [10, Chapter 10]. Fix $t > 0$, let B be the Brownian motion driving $W_{[0,t]}^{0,1}$.

Proof. We give the proof in the case $r = 1$. A scaling argument as in (5.18) yields the general case. We denote by Z_t the random variable

$$Z_t = \inf_{y \in W_{[0,t]}^{0,1}} \int_0^t \int_{\mathcal{B}(0,1)} G(y, B_u + z)\, \mathrm{d}u\, \mathrm{d}z \tag{5.19}$$

and we write for a constant $c_0 > 0$ to be determined later,

$$\begin{aligned} \mathrm{E}\Big[\mathrm{cap}\Big(W_{[0,1]}^{0,1}\Big)^2\Big] &= \mathrm{E}\Big[\mathbb{1}\{Z_t \leq c_0 \log t\}\, \mathrm{cap}\Big(W_{[0,1]}^{0,1}\Big)^2\Big] + \mathrm{E}\Big[\mathbb{1}\{Z_t > c_0 \log t\}\, \mathrm{cap}\Big(W_{[0,1]}^{0,1}\Big)^2\Big] \\ &=: (1) + (2). \end{aligned} \tag{5.20}$$

Note that by an application of the Cauchy-Schwarz inequality,

$$(1) \leq \mathrm{P}(Z_t \leq c_0 \log t)^{1/2} \mathrm{E}\Big[\mathrm{cap}\Big(W_{[0,t]}^{0,1}\Big)^4\Big]^{1/2}. \tag{5.21}$$

To estimate the right hand side in (5.21) we use the a priori estimate

$$\mathrm{E}\Big[\mathrm{cap}\Big(W_{[0,t]}^{0,1}\Big)^4\Big] \leq c\, t^4, \tag{5.22}$$

which may be proven as the corresponding second moment estimate in (5.16) or via a scaling argument using Brownian scaling and capacity scaling. Hence, it rests to estimate the probability appearing on the right hand side of (5.21). This is the content of the following lemma, which will be proven at the end of this section.

Lemma 5.2. *For all $\alpha > 0$, there are $c_\alpha, t_\alpha > 0$ such that for all $c_0 \le c_\alpha$ and $t \ge t_\alpha$*

$$\mathrm{P}(Z_t \le c_0 \log t) \le t^{-\alpha}. \tag{5.23}$$

Thus, for $\alpha > 0$ and c_0 and t chosen according to Lemma 5.2 we obtain that

$$(1) \le c\Big(\frac{t}{\log t}\Big)^2. \tag{5.24}$$

It remains to estimate (2). We write (2) as (2a) + (2b), where

$$\begin{aligned}(2a) &= \mathrm{E}\Big[\mathbb{1}\{Z_t > c_0 \log t\}\ \mathbb{1}\{B_{[0,t]} \subseteq \mathcal{B}(0,t)\}\ \mathrm{cap}\Big(W^{0,1}_{[0,t]}\Big)^2\Big],\\ (2b) &= \mathrm{E}\Big[\mathbb{1}\{Z_t > c_0 \log t\}\ \mathbb{1}\{B_{[0,t]} \not\subseteq \mathcal{B}(0,t)\}\ \mathrm{cap}\Big(W^{0,1}_{[0,t]}\Big)^2\Big].\end{aligned} \tag{5.25}$$

Note that by the Cauchy-Schwarz inequality,

$$(2b) \le \mathrm{P}(B_{[0,t]} \not\subseteq \mathcal{B}(0,t))^{\frac{1}{2}}\ \mathrm{E}\Big[\mathrm{cap}\Big(W^{0,1}_{[0,t]}\Big)^4\Big]^{\frac{1}{2}}\ \le c\ e^{-ct}\ t^2, \tag{5.26}$$

for some constant $c > 0$. To estimate $(2a)$ let $B^{(1)}$ and $B^{(2)}$ be two Brownian motions independent of B and of each other, introduce the near-intersection local times

$$T^i_t = \int_0^t \mathrm{d}u \int_0^\infty \mathrm{d}s\ \mathbb{1}\{\|B^{(i)}_s - B_u\| \le 1\}, \quad i \in \{1,2\} \tag{5.27}$$

and set

$$T_t = T^1_t\ T^2_t. \tag{5.28}$$

In what follows, the notation $\mathrm{P}_{x,x}$ indicates that $B^{(1)}$ and $B^{(2)}$ start from x. Note that as a consequence of (2.9),

$$\mathrm{cap}\Big(W^{0,1}_{[0,1]}\Big)^2 = c \lim_{||x||\to\infty} ||x||^4 \mathrm{P}_{x,x}\Big(\bigcap_{i=1,2}\Big\{B^{(i)}_{[0,\infty)} \cap W^{0,1}_{[0,t]} \ne \emptyset\Big\}\Big). \tag{5.29}$$

We now derive an upper bound for the probability appearing in (5.29) Using the relation

$$\mathrm{P}_{x,x}\Big(\bigcap_{i=1,2}\Big\{B^{(i)}_{[0,\infty)} \cap W^{0,1}_{[0,t]} \ne \emptyset\Big\}\Big) = \mathrm{P}_{x,x}(T_t > 0), \tag{5.30}$$

we can write

$$\mathrm{P}_{x,x}\Big(\bigcap_{i=1,2}\Big\{B^{(i)}_{[0,\infty)} \cap W^{0,1}_{[0,t]} \ne \emptyset\Big\}\Big) = \frac{\mathrm{E}_{x,x}(T_t)}{\mathrm{E}_{x,x}(T_t \mid T_t > 0)}. \tag{5.31}$$

To estimate the conditional expectation in (5.31) we introduce for $i \in \{1,2\}$ the stopping times

$$\tau^i = \inf\Big\{u \geq 0\colon\ B_u^{(i)} \in W_{[0,t]}^{0,1}\Big\} \tag{5.32}$$

Note that almost surely $\mathbb{1}\{T_t > 0\} = \mathbb{1}\{\tau^1, \tau^2 < \infty\}$. Consequently, we may write, using the strong Markov property at times τ^1 and τ^2 in the second equality,

$$\begin{aligned}
&\mathrm{E}_{x,x}(T_t \mid T_t > 0)\\
&= \mathrm{E}_{x,x}\bigg(\int_{[0,t]^2}\int_{\tau^1}^{\infty}\int_{\tau^2}^{\infty}\prod_{i=1}^{2}\mathbb{1}\{\|B_{s_i}^{(i)} - B_{u_i}\| \leq 1\}\ \Big|\ T_t > 0\bigg)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}u_1\mathrm{d}u_2\\
&= \mathrm{E}_{x,x}\bigg(\int_{[0,t]^2}\int_{\mathcal{B}(0,1)^2}\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{2}\widetilde{P}_{B_{\tau^i}^{(i)}}(\widetilde{B}_{s_i} - B_{u_i} \in \mathrm{d}z_i)\ \Big|\ T_t > 0\bigg)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}u_1\mathrm{d}u_2\\
&= \mathrm{E}_{x,x}\bigg(\int_{[0,t]^2}\int_{\mathcal{B}(0,1)^2}\prod_{i=1}^{2}G(B_{\tau^i}^{(i)}, B_{u_i} + z_i)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}u_1\mathrm{d}u_2\ \Big|\ T_t > 0\bigg).
\end{aligned} \tag{5.33}$$

Hence, on the event $\{Z_t > c_0 \log t\}$ we get that

$$\mathrm{E}_{x,x}(T_t \mid T_t > 0) \geq (c_0 \log t)^2. \tag{5.34}$$

Fatou's lemma in combination with (5.29) and (5.31) yields

$$(2a) \leq c \liminf_{\|x\|\to\infty} \|x\|^4 \mathrm{E}\bigg(\mathbb{1}\{Z_t > c_0 \log t\}\mathbb{1}\{B_{[0,t]} \subseteq \mathcal{B}(0,t)\}\frac{\mathrm{E}_{x,x}(T_t)}{\mathrm{E}_{x,x}(T_t \mid T_t > 0)}\bigg). \tag{5.35}$$

Consequently, using (5.34),

$$(2a) \leq \frac{c}{(c_0 \log t)^2} \liminf_{\|x\|\to\infty} \|x\|^4 \mathrm{E}(\mathbb{1}\{B_{[0,t]} \subseteq \mathcal{B}(0,t)\}\mathrm{E}_{x,x}(T_t)). \tag{5.36}$$

Moreover,

$$\mathrm{E}[\mathbb{1}\{B_{[0,t]} \subseteq \mathcal{B}(0,t)\}\ \mathrm{E}_{x,x}(T_t)] = \mathrm{E}\bigg[\mathbb{1}\{B_{[0,t]} \subseteq \mathcal{B}(0,t)\}\bigg(\int_0^t\int_{\mathcal{B}(0,1)} G(x, B_u + z)\mathrm{d}u\mathrm{d}z\bigg)^2\bigg]. \tag{5.37}$$

Since $G(x,y) = c\,\|x-y\|^{-2}$, there is a $c > 0$ such that for all $\|x\| \geq 2t+1$ the right hand side of (5.37) is smaller than $c\ \|x\|^{-4}\ t^2$. Plugging this estimate back in into (5.36) shows that

$$(2a) \leq c\bigg(\frac{t}{\log t}\bigg)^2. \tag{5.38}$$

Finally, (5.20), (5.24), (5.26) and (5.38) finish the proof. □

Proof of Lemma 5.2. Define for $1 \leq i \leq \lceil t \rceil$,

$$Z_t^i = \inf_{i-1<s\leq i, \|z\|\leq 1}\int_0^t\int_{\mathcal{B}(0,1)} G(B_s + z, B_u + z')\ \mathrm{d}u\ \mathrm{d}z' \tag{5.39}$$

and note that for $i \le \lceil t/2 \rceil$,

$$\begin{aligned} \mathrm{P}(Z_t^i \le c\log t) &\le \mathrm{P}\Bigg(\inf_{i-1<s\le i, \|z\|\le 1} \int_{i-1}^{i-1+t/2} \int_{\mathcal{B}(0,1)} G(B_s+z, B_u+z')\, \mathrm{d}u\, \mathrm{d}z' \le c\log t \Bigg) \\ &= \mathrm{P}(Z_{t/2}^1 \le c\log t). \end{aligned} \tag{5.40}$$

Since a similar argument shows that the same is true for $i > \lceil t/2 \rceil$, we may deduce that

$$\begin{aligned} \mathrm{P}(Z_t \le c\log t) &\le \sum_{i=1}^{\lceil t \rceil} \mathrm{P}(Z_t^i \le c\log t) \\ &\le \lceil t \rceil\, \mathrm{P}(Z_{t/2}^1 \le c\log t). \end{aligned} \tag{5.41}$$

Therefore, it is enough to prove that for all α, there exists a c_α such that for all t large enough,

$$\mathrm{P}(Z_t^1 \le c_\alpha \log t) \le t^{-\alpha}. \tag{5.42}$$

Let $\alpha > 0$. We decompose

$$\begin{aligned} \mathrm{P}(Z_t^1 \le c\log t) &= \mathrm{P}\Big(Z_t^1 \le c\log t, \sup_{s\le 1} \|B_s\| > \log t \Big) + \mathrm{P}\Big(Z_t^1 \le c\log t, \sup_{s\le 1} \|B_s\| \le \log t \Big) \\ &:= (1) + (2). \end{aligned} \tag{5.43}$$

By a standard Gaussian estimate, $(1) \le t^{-\alpha}$ for t large enough, so we may solely focus on estimating (2).

We first estimate (2) when Z_t^1 is replaced by $Z_{\zeta_t}^1$, where the random times (ζ_t) are defined by

$$\zeta_t = \inf\{u \ge 0 \colon \exists s \le 1, \|z\|, \|z'\| \le 1, \|(B_u+z) - (B_s+z')\| > t\}. \tag{5.44}$$

Thus, ζ_t is the first time that B reaches distance t from some point inside $W_{[0,1]}^{0,2}$.

Note that the ζ_t's are in general not stopping times. Indeed, knowing whether $\zeta_t \le 1/2$, for instance, requires all the information of the Brownian motion up to time 1. Note also that it is enough to consider $t = 2^\ell$, with $\ell \in \mathbb{N}$. To that purpose, abbreviate $\zeta^{(k)} = \zeta_{2^k}$, for $k \in \mathbb{N}$.

Let $\delta > 0$. We are first going to prove that if $\sup_{s\le 1} \|B_s\| \le \log t$ and $k \ge k_0 := \left\lceil 3 + \frac{\log(\log t + 2)}{\log 2} \right\rceil$, then the following implication holds:

$$\zeta^{(k)} - \zeta^{(k-1)} < \delta 2^{2k} \Rightarrow \sup_{0\le u\le \delta 2^{2k}} \|B_{\zeta^{(k-1)}+u} - B_{\zeta^{(k-1)}}\| \ge 2^{k-2}. \tag{5.45}$$

(Note that k_0 grows with ℓ, rougly like $\log \ell$ only). Indeed, by definition, there exists $s \in [0,1]$ and $z, z' \in \overline{\mathcal{B}}(0,1)$ such that $\|B_{\zeta^{(k)}} + z - (B_s + z')\| = 2^k$. Then

$$\Big| \|B_{\zeta^{(k)}}\| - 2^k \Big| \le \log t + 2, \quad k \in \mathbb{N}. \tag{5.46}$$

Therefore, we obtain

$$\begin{aligned}\|B_{\zeta^{(k)}} - B_{\zeta^{(k-1)}}\| &\geq \|B_{\zeta^{(k)}}\| - \|B_{\zeta^{(k-1)}}\| \\ &\geq 2^k - 2^{k-1} - 2(\log t + 2) \\ &= 2^{k-1} - 2(\log t + 2),\end{aligned} \tag{5.47}$$

which is larger than 2^{k-2} if and only if $2(\log t + 2) \leq 2^{k-2}$, i.e. $k \geq k_0$. This proves the implication in (5.45).

Even if the ζ_t's are not stopping times, we may use the Markov property at times $\zeta^{(k)}$ on the event $\{\sup_{0\leq s\leq 1} \|B_s\| \leq \log t\}$ for $k \geq k_0$, i.e., if $k \geq k_0$, $0 < t_0 < t_1 < \ldots < t_n$, $n \geq 1$, and $A_0, \ldots, A_n$ be Borel sets, then

$$\begin{aligned}&\mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t, B_{[0,\zeta^{(k)}]} \in \ \cdot\ , B_{\zeta^{(k)}+t_0} \in A_0, \ldots, B_{\zeta^{(k)}+t_n} \in A_n\Big) \\ &= \mathrm{E}\Big[\mathbb{1}\Big\{\sup_{s\leq 1} \|B_s\| \leq \log t, B_{[0,\zeta^{(k)}]} \in \cdot\ \Big\}\widetilde{\mathrm{P}}_{B_{\zeta^{(k)}}}(\widetilde{B}_{t_0} \in A_0, \ldots, \widetilde{B}_{t_n} \in A_n)\Big].\end{aligned} \tag{5.48}$$

Indeed, the l.h.s. writes

$$\int_u \mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t, B_{[0,u]} \in \cdot, \zeta^{(k)} \in \mathrm{d}u, B_{u+t_0} \in A_0, \ldots, B_{u+t_n} \in A_n\Big). \tag{5.49}$$

Note that for $u, v \leq 1$, $z, z' \in \mathcal{B}(0,1)$, $\|B_u - B_v + z - z'\| \leq 2(\log t + 1)$ and if $k \geq k_0$ then $2^k \geq 8(\log t + 2) > 2(\log t + 1)$, so necessarily $\zeta^{(k)} \geq 1$. This implies that we can apply the Markov property at u in (5.49), which yields (5.48).

Write $\varepsilon = \varepsilon(\delta) = \mathrm{P}(\sup_{0\leq u\leq \delta} \|B_u\| > 1/4)$, and define

$$I_k = \mathbb{1}\{\zeta^{(k)} - \zeta^{(k-1)} < \delta 2^{2k}\}, \quad 1 \leq k \leq \ell, \qquad \mathcal{I}_\ell = \sum_{k=1}^{\ell} I_k. \tag{5.50}$$

Note that by Brownian scaling, for $k \geq 1$,

$$\widetilde{\mathrm{P}}_{B_{\zeta^{(k)}}}\Big(\sup_{0\leq u\leq \delta 2^{2(k+1)}} \|\widetilde{B}_u - B_{\zeta^{(k)}}\| > 2^{k-1}\Big) = \varepsilon. \tag{5.51}$$

Using (5.45), (5.48),(5.51), we may write for $k > k_0$,

$$\begin{aligned}&\mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t,\ B_{[0,\zeta^{(k-1)}]} \in \ \cdot, \zeta^{(k)} - \zeta^{(k-1)} < \delta 2^{2k}\Big) \\ &\qquad\qquad \leq \varepsilon \mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t,\ B_{[0,\zeta^{(k-1)}]} \in \cdot\Big).\end{aligned} \tag{5.52}$$

We may then prove by iteration that for all $(i_n)_{k_0\leq n\leq k}$ in $\{0,1\}^{k-k_0+1}$,

$$\mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t, I_n = i_n, k_0 \leq n \leq k\Big) \leq \mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t\Big)\varepsilon^{\sum_{k_0\leq n\leq k} i_n}. \tag{5.53}$$

Therefore, for ℓ large enough and ε small enough we have

$$\begin{aligned}\mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t, \mathcal{I}_\ell \geq \ell/2\Big) &\leq \mathrm{P}\Big(\sup_{s\leq 1} \|B_s\| \leq \log t, \sum_{k=k_0+1}^{\ell} I_k \geq \frac{\ell}{2} - k_0\Big) \\ &\leq (2\sqrt{\varepsilon})^\ell \varepsilon^{-(k_0+1)} \leq 2^{-\ell\alpha} = t^{-\alpha}.\end{aligned} \tag{5.54}$$

On the other hand, on the event $\{\mathcal{I}_\ell < \ell/2\}$, we get

$$\begin{aligned} Z^1_{\zeta^{(\ell)}} = \inf_{\substack{0\le s\le 1\\ \|z\|\le 1}} \int_0^{\zeta^{(\ell)}} \int_{\|z'\|\le 1} G(B_s+z, B_u+z')\,\mathrm{d}u\,\mathrm{d}z' &\ge c\sum_{k=1}^{\ell} 2^{-2k}[\zeta^{(k)}-\zeta^{(k-1)}] \\ &\ge c \sum_{1\le k\le \ell:\ I_k=0} 2^{-2k}[\zeta^{(k)}-\zeta^{(k-1)}] \\ &= c\,\delta\ \log t. \end{aligned} \tag{5.55}$$

The first inequality in (5.55) is obtained by cutting the integral over $[0,\zeta^{(\ell)}]$ into subintervals $[\zeta^{(k-1)},\zeta^{(k)}]$, $1\le k\le \ell$, and using (2.1) as well as (5.44) to bound the Green function from below. We may now conclude thanks to (5.54)–(5.55). For that first fix $\alpha>0$, then choose $\varepsilon>0$ in accordance with (5.54); this determines δ by the line preceding (5.50), which by (5.55) provides a value for c_α.

It remains to replace $(Z^1_{\zeta_t})$ by (Z^1_t). This is done by observing that

$$\mathrm{P}(Z^1_t \le c\log t) \le \mathrm{P}\Big(Z^1_{\zeta_{t^{\frac14}}} \le c\log t\Big) + \mathrm{P}\Big(\zeta_{t^{\frac14}} > t\Big). \tag{5.56}$$

From the previous observations, the first term on the right hand side of (5.56) decreases faster than any power of t provided c is chosen small enough. The second term decreases faster than any power of t by standard small ball estimates. This finally yields the result. □

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