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Universal components of random nodal sets

Damien Gayet Jean-Yves Welschinger

March 5, 2015

Abstract

We give, as L grows to infinity, an explicit lower bound of order $L^{\frac{n}{m}}$ for the expected Betti numbers of the vanishing locus of a random linear combination of eigenvectors of P with eigenvalues below L. Here, P denotes an elliptic self-adjoint pseudo-differential operator of order $m > 0$, bounded from below and acting on the sections of a Riemannian line bundle over a smooth closed n-dimensional manifold M equipped with some Lebesgue measure. In fact, for every closed hypersurface Σ of $\mathbb{R}^n$, we prove that there exists a positive constant p_Σ depending only on Σ, such that for every large enough L and every $x \in M$, a component diffeomorphic to Σ appears with probability at least p_Σ in the vanishing locus of a random section and in the ball of radius $L^{-\frac{1}{m}}$ centered at x. These results apply in particular to Laplace-Beltrami and Dirichlet-to-Neumann operators.

Keywords: Pseudo-differential operator, random nodal sets, Betti numbers.

Mathematics subject classification 2010: Primary 34L20, 58J40 ; Secondary 60D05.

Introduction

Let M be a smooth closed manifold of positive dimension n and E be a real line bundle over M. We equip M with a Lebesgue measure $|dy|$, that is a positive measure that can be locally expressed as the absolute value of some smooth volume form, and E with a Riemannian metric h_E. These induce a L^2-scalar product on the space $\Gamma(M, E)$ of smooth global sections of E which reads

$$\forall (s,t) \in \Gamma(M,E)^2, \langle s,t\rangle = \int_M h_E\big(s(y), t(y)\big)|dy|. \tag{0.1}$$

Let $P : \Gamma(M, E) \to \Gamma(M, E)$ be a self-adjoint elliptic pseudo-differential operator of positive order m which is bounded from below. The spectrum of such an operator is thus real, discrete and bounded from below. Its eigenspaces are finite dimensional with smooth eigenfunctions, see [8]. We set, for every $L \in \mathbb{R}$,

$$\mathbb{U}_L = \bigoplus_{\lambda \leq L} \ker(P - \lambda Id).$$

The dimension N_L of $\mathbb{U}_L$ satisfies Weyl's asymptotic law

$$\frac{1}{L^{\frac{n}{m}}} N_L \underset{L\to+\infty}{\to} \frac{1}{(2\pi)^n} Vol\{\xi \in T^*M, \mid \sigma_P(\xi) \leq 1\},$$

where σ_P denotes the homogenized principal symbol of P, see [8] and Definition $A.8$ of [5]. The space $\mathbb{U}_L$ inherits by restriction the L^2-scalar product (0.1) and its associated Gaussian measure defined by the density

$$\forall s \in \mathbb{U}_L,\ d\mu(s) = \frac{1}{\sqrt{\pi}^{N_L}} \exp(-\|s\|^2)|ds|, \tag{0.2}$$

where $|ds|$ denotes the Lebesgue measure of $\mathbb{U}_L$ associated to its scalar product. The measure of the discriminant

$$\Delta_L = \{s \in \mathbb{U}_L,\ s \text{ does not vanish transversally}\}$$

vanishes when L is large enough, see Lemma A.1 of [5].

Our purpose is to study the topology of the vanishing locus $s^{-1}(0) \subset M$ of a section $s \in \mathbb{U}_L$ taken at random. More precisely, for every closed hypersurface Σ of $\mathbb{R}^n$ not necessarily connected, and every $s \in \mathbb{U}_L \setminus \Delta_L$, we denote by $N_\Sigma(s)$ the maximal number of disjoint open subsets of M with the property that every such open subset U' contains a hypersurface Σ' such that $\Sigma' \subset s^{-1}(0)$ and (U', Σ') gets diffeomorphic to $(\mathbb{R}^n, \Sigma)$ (compare [6]). We then set

$$\mathbb{E}(N_\Sigma) = \int_{\mathbb{U}_L \setminus \Delta_L} N_\Sigma(s) d\mu(s) \tag{0.3}$$

the mathematical expectation of the function N_Σ. Note that when Σ is connected, the expected number of connected components diffeomorphic to Σ of the vanishing locus of a random section of $\mathbb{U}_L$ gets bounded from below by $\mathbb{E}(N_\Sigma)$.

Theorem 0.1 *Let M be a smooth closed manifold of positive dimension n, equipped with a Lebesgue measure $|dy|$. Let E be a real line bundle over M equipped with a Riemannian metric h_E. Let $P : \Gamma(M, E) \to \Gamma(M, E)$ be an elliptic pseudo-differential operator of positive order m, which is self-adjoint and bounded from below. Let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected. Then, there exists a positive constant $c_\Sigma(P)$, such that*

$$\liminf_{L\to+\infty} \frac{1}{L^{\frac{n}{m}}} \mathbb{E}(N_\Sigma) \geq c_\Sigma(P).$$

The constant $c_\Sigma(P)$ is in fact explicit, given by (2.3).

Now, as in [6], we denote by $\mathcal{H}_n$ the space of diffeomorphism classes of closed connected hypersurfaces of $\mathbb{R}^n$. For every $[\Sigma] \in \mathcal{H}_n$ and every $i \in \{0, \cdots, n-1\}$, we denote by $b_i(\Sigma) = \dim H_i(\Sigma, \mathbb{R})$ the $i-$th Betti number of Σ with real coefficients. Likewise, for every $s \in \mathbb{U}_L \setminus \Delta_L$, $b_i(s^{-1}(0))$ denotes the $i-$th Betti number of $s^{-1}(0)$, and we set

$$\mathbb{E}(b_i) = \int_{\mathbb{U}_L \setminus \Delta_L} b_i(s^{-1}(0)) d\mu(s) \tag{0.4}$$

its mathematical expectation.

Corollary 0.2 *Let M be a smooth closed manifold of positive dimension n equipped with a Lebesgue measure $|dy|$. Let E be a real line bundle over M equipped with a Riemannian metric h_E. Let $P : \Gamma(M,E) \to \Gamma(M,E)$ be an elliptic pseudo-differential operator of positive order m, which is self-adjoint and bounded from below. Then, for every $i \in \{0, \cdots, n-1\}$,*

$$\liminf_{L\to\infty} \frac{1}{L^{\frac{n}{m}}}\mathbb{E}(b_i) \geq \sum_{[\Sigma]\in\mathcal{H}_n} \sup_{\Sigma\in[\Sigma]} \big(c_\Sigma(P)\big) b_i(\Sigma).$$

Note that an upper estimate for $\mathbb{E}(b_i)$ of the same order in L is given by Theorem 0.2 of [5].

Theorem 0.1 is in fact the consequence of Theorem 0.3, which is local and more precise. Let $Met_{|dy|}(M)$ be the space of Riemannian metrics of M whose associated Lebesgue measure equals $|dy|$. For every $g \in Met_{|dy|}(M)$, every $R > 0$ and every point $x \in M$, we set

$$\begin{aligned} Prob^x_\Sigma(R) = \mu\Big\{ & s \in \mathbb{U}_L \setminus \Delta_L \mid \big(s^{-1}(0) \cap B_g(x, RL^{-\frac{1}{m}})\big) \supset \Sigma_L \\ & \text{with } \big(B_g(x, RL^{-\frac{1}{m}}), \Sigma_L\big) \text{ diffeomorphic to } (\mathbb{R}^n, \Sigma)\Big\}, \end{aligned} \tag{0.5}$$

where $B_g(x, RL^{-\frac{1}{m}})$ denotes the ball centered at x of radius $RL^{-\frac{1}{m}}$ for the metric g.

Theorem 0.3 *Under the hypotheses of Theorem 0.1, let $g \in Met_{|dy|}(M)$. Then, for every $x \in M$ and every $R > 0$,*

$$\liminf_{L\to+\infty} Prob^x_\Sigma(R) \geq p^x_\Sigma(R),$$

where for R large enough, $p_\Sigma(R) = \inf_{x\in M} p^x_\Sigma(R)$ is positive.

Again, the function p_Σ is explicit, defined by (2.2) (see also (1.7) and (1.8)). In particular, when Σ is diffeomorphic to the product of spheres $S^i \times S^{n-i-1}$, Theorem 0.4 provides explicit lower estimates for the constants $c_\Sigma(P)$ and $p_\Sigma(R)$ appearing in Theorems 0.1 and 0.3.

Theorem 0.4 *Under the hypotheses of Theorem 0.1, let $g \in Met_{|dy|}(M)$ and $c_{P,g} > 0$, $d_{P,g} > 0$ such that for every $\xi \in T^*M$,*

$$d_{P,g}^{-1} \leq \frac{\sigma_P(\xi)^{\frac{1}{m}}}{\|\xi\|} \leq c_{P,g}^{-1}.$$

Then, for every $i \in \{0, \cdots, n-1\}$ and every $R \geq \frac{48\sqrt{5}n}{c_{P,g}}$,

$$\begin{aligned} c_{S^i\times S^{n-i-1}}(P) &\geq \frac{e^{-(2\tau+1)^2}}{2^{n+1}\sqrt{\pi}Vol(B(0,48\sqrt{5}n))} c_{P,g}^n Vol_{|dy|}(M) \\ \text{and } p_{S^i\times S^{n-i-1}}(R) &\geq \frac{1}{2\sqrt{\pi}} \exp\big(-(2\tau+1)^2\big), \\ \text{where } \tau &= 20\frac{(n+6)^{11/2}}{\sqrt{\Gamma(\frac{n}{2}+1)}}\big(48n\frac{d_{P,g}}{c_{P,g}}\big)^{\frac{n+2}{2}} \exp\Big(48\sqrt{5}n^{3/2}\frac{d_{P,g}}{c_{P,g}}\Big). \end{aligned}$$

In the case of Laplace-Beltrami operators, we get in particular the following.

Corollary 0.5 *Let (M,g) be a smooth closed n-dimensional Riemannian manifold and Δ be its associated Laplace-Beltrami operator acting on functions. Then for every $i \in \{0,\cdots,n-1\}$,*

$$\liminf_{L\to+\infty} \frac{1}{\sqrt{L}^n}\mathbb{E}(b_i) \geq c_{S^i\times S^{n-i-1}}(\Delta) \geq \exp\big(-\exp(257n^{3/2})\big)Vol_g(M).$$

As a second example, Theorem 0.4 specializes to the case of the Dirichlet-to-Neumann operator on the boundary M of some $(n+1)$-dimensional compact Riemannian manifold (W,g).

Corollary 0.6 *Let (W,g) be a smooth compact Riemannian manifold of dimension $n+1$ with boundary M and let Λ_g be the associated Dirichlet-to-Neumann operator on M. Then, for every $i \in \{0,\cdots,n-1\}$,*

$$\liminf_{L\to+\infty} \frac{1}{L^n}\mathbb{E}(b_i) \geq c_{S^i\times S^{n-i-1}}(\Lambda_g) \geq \exp\big(-\exp(257n^{3/2})\big)Vol_g(M).$$

Note that the double exponential decay in Corollaries 0.5 and 0.6 has to be compared with the exponential decay observed in Proposition 0.4 of [5] and with the analogous double exponential decay already observed in Corollary 1.3 of [6].

Let us mention some related works. In [11], F. Nazarov and M. Sodin proved the existence of an equivalent of order L for the expected number of components of the vanishing locus of random eigenfunctions with eigenvalue L of the Laplace operator on the round 2-sphere. In [9], A. Lerario and E. Lundberg proved, for the Laplace operator on the round n-sphere, the existence of a positive constant c such that $\mathbb{E}(b_0) \geq c\sqrt{L}^n$ for large values of L. We got in [5] upper estimates for $\limsup_{L\to+\infty} L^{-\frac{n}{m}}\mathbb{E}(b_i)$ under the same hypotheses as Corollary 0.2, and previously obtained similar upper and lower estimates for the expected Betti numbers or N_Σ's of random real algebraic hypersurfaces of real projective manifolds (see [7], [3], [6], [4]). In [10], T. Letendre proved, under the hypotheses of Corollary 0.5, the existence of an equivalent of order $\sqrt{L}^n$ for the mean Euler characteristics (for odd n). Let us finally mention the lecture [13], where M. Sodin announces a convergence in probability for b_0 under some hypotheses, and [12], where P. Sarnak and I. Wigman announce a convergence in probability for N_Σ in the case of Laplace-Beltrami operators.

In the first section, we introduce the space of Schwartz functions of $\mathbb{R}^n$ whose Fourier transforms have supports in the compact

$$K_x = \{\xi \in T_x^*M \mid \sigma_P(\xi) \leq 1\},$$

where $x \in M$ is given and T_x^*M is identified with $\mathbb{R}^n$ via some isometry. This space appears to be asymptotically a local model for the space $\mathbb{U}_L$. Indeed, any function f in this space can be implemented in $\mathbb{U}_L$, in the sense that there exists a family of sections

$(s_L \in \mathbb{U}_L)_{L \gg 1}$ whose restriction to a ball of radius of order $L^{-\frac{1}{m}}$ centered at x converges to f after rescaling, see Corollary 1.11. The vanishing locus of f gets then implemented as the vanishing locus of the sections s_L for L large enough. The second section is devoted to the proofs of Theorems 0.1 and 0.3, and of Corollary 0.2. For this purpose we follow the approach used in [6] (see also [4]), which was itself partially inspired by the works [11] and [2], see also [9]. We begin by estimating the expected local C^1-norm of elements of $\mathbb{U}_L$, see Proposition 2.1, and then compare it with the amount of transversality of s_L. We can then prove Theorem 0.3, see §2.2, and finally Theorem 0.1 and its Corollary 0.2, see §2.3. The last section is devoted to the explicit estimates and the proofs of Theorem 0.4 and Corollaries 0.5 and 0.6.

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Contents

1 The local model and its implementation **5**
1.1 The local model 6
1.2 Quantitative transversality 7
1.3 Implementation of the local model 9

2 Probability of the local presence of a hypersurface **11**
2.1 Expected local C^1-norm of sections 11
2.2 Proof of Theorem 0.3 13
2.3 Proofs of Theorem 0.1 and Corollary 0.2 14

3 Explicit estimates **15**
3.1 Key estimates for the approximation 16
3.2 The product of spheres 18
3.3 Proofs of Theorem 0.4, Corollary 0.5 and Corollary 0.6 22

1 The local model and its implementation

In the first paragraph of this section, we associate to any closed hypersurface Σ of $\mathbb{R}^n$ and any symmetric compact subset K of $\mathbb{R}^n$ with the origin in its interior, a Schwartz function f vanishing transversally along a hypersurface isotopic to Σ and whose Fourier transform has support in K. In the third paragraph, we implement the function f in the neighbourhood of every point x_0 in M, as the limit after rescaling of a sequence of sections of $\mathbb{U}_L$. Here, K is the pull-back of K_{x_0} under some measure-preserving isomorphism between $T^*_{x_0}M$ and

$\mathbb{R}^n$. As a consequence, these sections of $\mathbb{U}_L$ vanish in a neighbourhood U_{x_0} of x_0 along a hypersurface Σ_L of M such that the pair (U_{x_0}, Σ_L) gets diffeomorphic to $(\mathbb{R}^n, \Sigma)$. The second paragraph quantifies the transversality of the vanishing of the function f and thus of the associated sequence of sections, in order to prepare the estimates of the second section which involve perturbations.

1.1 The local model

Let K be a measurable subset of $\mathbb{R}^n$ and let χ_K be its characteristic function, so that $\chi_K(\xi) = 1$ if $\xi \in K$ and $\chi_K(\xi) = 0$ otherwise. It provides the projector $f \in L^2(\mathbb{R}^n) \mapsto \chi_K f \in L^2(\mathbb{R}^n)$. After conjugation by the Fourier transform $\mathcal{F}$ of $L^2(\mathbb{R}^n)$, defined for every $f \in L^2(\mathbb{R}^n)$ and every $\xi \in \mathbb{R}^n$ by

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} f(y)dy \in L^2(\mathbb{R}^n),$$

we get the projector $\pi_K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, defined for every $f \in L^2(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ by

$$\pi_K(f)(x) = \frac{1}{(2\pi)^n} \int_{\xi\in K} \int_{y\in\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} f(y)d\xi dy.$$

Note that for $K = \mathbb{R}^n$, π_K is the identity map. Denote by $L^2_K(\mathbb{R}^n)$ the image of π_K. This is a Hilbert subspace of $L^2(\mathbb{R}^n)$, the kernel of the continuous operator $Id - \pi_K = \pi_{\mathbb{R}^n\setminus K}$. Denote by $C_0^\infty(K)$ the space of smooth functions on $\mathbb{R}^n$ whose support is included in K, by $S(\mathbb{R}^n)$ the space of Schwartz functions of $\mathbb{R}^n$ and set

$$S_K(\mathbb{R}^n) = \mathcal{F}^{-1}(C_0^\infty(K)). \tag{1.1}$$

Lemma 1.1 *Let K be a bounded measurable subset of $\mathbb{R}^n$. Then, $S_K(\mathbb{R}^n) \subset L^2_K(\mathbb{R}^n) \cap S(\mathbb{R}^n)$.*

Proof. Since K is bounded, $C_0^\infty(K) \subset S(\mathbb{R}^n)$ so that $S_K(\mathbb{R}^n) \subset \mathcal{F}^{-1}(S(\mathbb{R}^n)) = S(\mathbb{R}^n)$. Likewise, for every $f \in C_0^\infty(K)$, $\chi_K f = f$, so that by definition, $f \in L^2_K(\mathbb{R}^n)$. □

Lemma 1.2 *Let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected, and K be a bounded measurable subset of $\mathbb{R}^n$, symmetric with respect to the origin and which contains the origin in its interior. Then, there exists a hypersurface $\widetilde{\Sigma}$ of $\mathbb{R}^n$, isotopic to Σ, and a function f_Σ in $S_K(\mathbb{R}^n)$ such that f_Σ vanishes transversally along $\widetilde{\Sigma}$.*

Recall that $\widetilde{\Sigma}$ is said to be isotopic to Σ if and only if there exists a continuous family $(\phi_t)_{t\in[0,1]}$ of diffeomorphisms of $\mathbb{R}^n$ such that $\phi_0 = Id$ and $\phi_1(\Sigma) = \widetilde{\Sigma}$.

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$ be a smooth compactly supported function of $\mathbb{R}^n$ which vanishes transversally along Σ and let $\tilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ be an even function which equals 1 in a neighbourhood of the origin. For every $R > 0$, we set

$$\tilde{\chi}_R : \xi \in \mathbb{R}^n \mapsto \tilde{\chi}(\xi R^{-1}) \in \mathbb{R}.$$

Then $\mathcal{F}(f) \in S(\mathbb{R}^n)$ and $\tilde{\chi}_R\mathcal{F}(f)$ converges to $\mathcal{F}(f)$ in $S(\mathbb{R}^n)$ as R grows to infinity. Thus, $\mathcal{F}^{-1}(\tilde{\chi}_R\mathcal{F}(f))$ converges to f in $S(\mathbb{R}^n)$ as R grows to infinity, and $\mathcal{F}^{-1}(\tilde{\chi}_R\mathcal{F}(f))$ takes real values. We deduce that when R is large enough, the function $f_R = \mathcal{F}^{-1}(\tilde{\chi}_R\mathcal{F}(f))$ is real and vanishes transversally in a neighbourhood of Σ along a hypersurface isotopic to Σ. By construction, the support of $\mathcal{F}(f_R)$ is compact. By hypotheses, there exists thus $\rho > 0$ such that the function $\mathcal{F}_\rho(f_R) : \xi \in \mathbb{R}^n \mapsto \mathcal{F}(f_R)(\frac{\xi}{\rho}) \in \mathbb{R}$ has compact support in K. The function $f_\Sigma = \mathcal{F}^{-1}(\mathcal{F}_\rho(f_R))$ then belongs to $S_K(\mathbb{R}^n)$ and vanishes transversally along a hypersurface isotopic to Σ. $\square$

1.2 Quantitative transversality

We now proceed as in [6] to introduce our needed quantitative transversality estimates.

Definition 1.3 *Let W be a bounded open subset of $\mathbb{R}^n$ and $f \in S(\mathbb{R}^n)$, $n > 0$. The pair (W, f) is said to be regular if and only if zero is a regular value of the restriction of f to W and the vanishing locus of f in W is compact.*

Example 1.4 *Let $f_\Sigma \in S_K(\mathbb{R}^n) \subset S(\mathbb{R}^n)$ be a function given by Lemma 1.2. Then, there exists a tubular neighbourhood W of $\tilde{\Sigma} \subset f_\Sigma^{-1}(0)$ such that (W, f_Σ) is a regular pair in the sense of Definition 1.3.*

Definition 1.5 *For every regular pair (W, f) given by Definition 1.3, we denote by $\mathcal{T}_{(W,f)}$ the set of pairs $(\delta, \epsilon) \in (\mathbb{R}_+^*)^2$ such that*

1. *there exists a compact subset K_W of W such that $\inf_{W\setminus K_W} |f| > \delta$*
2. *$\forall z \in W,\ |f(z)| \leq \delta \Rightarrow \|d_{|z}f\| > \epsilon$, where $\|d_{|z}f\|^2 = \sum_{i=1}^n |\frac{\partial f}{\partial x_i}|^2(z)$.*

The quantities and functions that are going to appear in the proof of our theorems are the following. Let K be a bounded measurable subset of $\mathbb{R}^n$. We set, for every positive R and every $j \in \{1, \cdots, n\}$,

$$\rho_K(R) = \frac{\sqrt{2}\lfloor \frac{n}{2}+1\rfloor}{\sqrt{2\pi}^n} \inf_{t\in\mathbb{R}_+^*} \left(\Big(\frac{R+t}{t}\Big)^{\frac{n}{2}} \sum_{i=0}^{\lfloor \frac{n}{2}+1\rfloor} \frac{t^i}{i!} \Big(\sum_{\substack{(j_1,\cdots,j_i)\\ \in\{1,\cdots,n\}^i}} \int_K \prod_{k=1}^i |\xi_{j_k}|^2 |d\xi| \Big)^{\frac{1}{2}} \right) \tag{1.2}$$

$$\theta_K^j(R) = \frac{\sqrt{2}\lfloor \frac{n}{2}+1\rfloor}{\sqrt{2\pi}^n} \inf_{t\in\mathbb{R}_+^*} \left(\Big(\frac{R+t}{t}\Big)^{\frac{n}{2}} \sum_{i=0}^{\lfloor \frac{n}{2}+1\rfloor} \frac{t^i}{i!} \Big(\sum_{\substack{(j_1,\cdots,j_i)\\ \in\{1,\cdots,n\}^i}} \int_K |\xi_j|^2 \prod_{k=1}^i |\xi_{j_k}|^2 |d\xi| \Big)^{\frac{1}{2}} \right) \tag{1.3}$$

Remark 1.6 *Denoting by* $\nu(K) = \int_K |d\xi|$ *the total measure of* K *and by* $d(K) = \sup_{\xi\in K} \|\xi\|$ *we note that for every* $(j_1, \cdots, j_i) \in \{1, \cdots, n\}^i$ *and every* $j \in \{1, \cdots, n\}$,

$$\int_K \prod_{k=1}^{i} |\xi_{j_k}|^2 |d\xi| \leq d(K)^{2i}\nu(K)$$

and $\int_K |\xi_j|^2 \prod_{k=1}^{i} |\xi_{j_k}|^2 \leq d(K)^{2(i+1)}\nu(K)$. *It follows, after evaluation at* $t = R$, *that for every* $j \in \{1, \cdots, n\}$,

$$\rho_K(R) \leq \frac{1}{\sqrt{\pi}^n}\sqrt{2\nu(K)}\lfloor\frac{n}{2}+1\rfloor \exp\big(Rd(K)\sqrt{n}\big) \tag{1.4}$$

$$\theta_K^j(R) \leq \frac{1}{\sqrt{\pi}^n}\sqrt{2\nu(K)}\lfloor\frac{n}{2}+1\rfloor d(K) \exp\big(Rd(K)\sqrt{n}\big). \tag{1.5}$$

For every regular pair (W, f) we set

$$R_{(W,f)} = \sup_{z\in W} \|z\|$$

and for every bounded measurable subset K of $\mathbb{R}^n$,

$$\tau^K_{(W,f)} = \|f\|_{L^2(\mathbb{R}^n)} \inf_{(\delta,\epsilon)\in\mathcal{T}_{(W,f)}} \Big(\frac{1}{\delta}\rho_K(R_{(W,f)}) + \frac{n\sqrt{n}}{\epsilon}\sum_{i=1}^{n}\theta_K^j(R_{(W,f)})\Big) \tag{1.6}$$

$$\text{and } p^K_{(W,f)} = \frac{1}{\sqrt{\pi}} \sup_{T\in[\tau^K_{(W,f)},+\infty[} \big(1 - \frac{\tau^K_{(W,f)}}{T}\big) \int_T^{+\infty} e^{-t^2}dt. \tag{1.7}$$

Remark 1.7 *Note that* $p^K_{(W,f)} \geq \frac{1}{2\sqrt{\pi}} \exp\big(-(2\tau^K_{(W,f)}+1)^2\big)$.

Now, let Σ be a closed hypersurface of $\mathbb{R}^n$, not necessarily connected.

Definition 1.8 *Let* $\mathcal{I}^K_\Sigma$ *be the set of regular pairs* (W, f) *given by Definition 1.3 such that* $f \in S_K(\mathbb{R}^n)$ *and such that the vanishing locus of* f *in* W *contains a hypersurface isotopic to* Σ *in* $\mathbb{R}^n$. *Likewise, for every* $R > 0$, *we set*

$$\mathcal{I}^{K,R}_\Sigma = \{(W, f) \in \mathcal{I}^K_\Sigma \mid R_{(W,f)} \leq R\}.$$

Finally, for every positive R we set

$$p^K_\Sigma(R) = \sup_{(W,f)\in\mathcal{I}^{K,R}_\Sigma} p^K_{(W,f)}. \tag{1.8}$$

Remark 1.9 *It follows from Lemma 1.2 and Example 1.4 that when* R *is large enough and* K *satisfies the hypotheses of Lemma 1.2,* $\mathcal{I}^{K,R}_\Sigma$ *is not empty, so that* $p^K_\Sigma(R) > 0$.

1.3 Implementation of the local model

In this paragraph, we prove that for every $x_0 \in M$ and every measure-preserving linear isomorphism A between $\mathbb{R}^n$ and $T^*_{x_0}M$, every function f in $S_{A^*K_{x_0}}(\mathbb{R}^n)$ can be implemented in $\mathbb{U}_L$ as a sequence of sections, see Proposition 1.10. Corollary 1.11 then estimates the amount of transversality of these sections along their vanishing locus, in terms of the one of f.

Proposition 1.10 *Under the hypotheses of Corollary 0.2, let $x_0 \in M$, $\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$ be a measure-preserving chart and $\tilde{\chi}_V \in C^\infty_c(V)$ be an even function with support in V which equals 1 in a neighbourhood of 0. Then, for every $f \in S_{(d_{|x_0}\phi_{x_0}^{-1})^* K_{x_0}}(\mathbb{R}^n)$, there exists a family $(s_L)_{L\in\mathbb{R}^*_+} \in \Gamma(M, E)$ such that*

1. *for L large enough, $s_L \in \mathbb{U}_L$ and $\|s_L\|_{L^2(M)} = \|f\|_{L^2(\mathbb{R}^n)}$*
2. *the function $z \in \mathbb{R}^n \mapsto L^{-\frac{n}{2m}}\tilde{\chi}_V(L^{-\frac{1}{m}}z)(s_L \circ \phi_{x_0}^{-1})(L^{-\frac{1}{m}}z) \in \mathbb{R}$ converges to f in $S(\mathbb{R}^n)$.*

Note indeed that the isomorphism $(d_{|x_0}\phi_{x_0})^{-1} : \mathbb{R}^n \to T_{x_0}M$ defines by pull-back an isomorphism $((d_{|x_0}\phi_{x_0})^{-1})^* : T^*_{x_0}M \to \mathbb{R}^n$ that makes it possible to identify the compact

$$K_{x_0} = \{\xi \in T^*_{x_0}M \ |\sigma_P(\xi) \leq 1\} \tag{1.9}$$

with the compact $\big((d_{|x_0}\phi_{x_0})^{-1}\big)^* K_{x_0}$ of $\mathbb{R}^n$. Moreover, the Riemannian metric h_E of E given in the hypotheses of Corollary 0.2 provides a trivialization of E in the neighbourhood U_{x_0} of x_0, choosing a smaller U_{x_0} if necessary, unique up to sign. This trivialization makes it possible to identify $\tilde{\chi}_V s_L \circ \phi_{x_0}^{-1}$ with a function from V to $\mathbb{R}$.

Proof. For every $L \in \mathbb{R}^*_+$, we set

$$\tilde{s}_L : x \in U_{x_0} \mapsto L^{\frac{n}{2m}}\tilde{\chi}_V(\phi_{x_0}(x))f(L^{\frac{1}{m}}\phi_{x_0}(x)) \in E_{|x}$$

that we extend by zero to a global section of E. We denote then by s_L the orthogonal projection of $\tilde{s}_L$ in $\mathbb{U}_L \subset L^2(M, E)$. This section reads

$$s_L = \langle e_L, \tilde{s}_L\rangle = \int_M h_E\big(e_L(x, y), \tilde{s}_L(y)\big)|dy|,$$

where e_L denotes the Schwartz kernel of the orthogonal projection onto $\mathbb{U}_L$. Then, for every $z \in \mathbb{R}^n$, $L^{-\frac{1}{m}}z$ belongs to V when L is large enough and

$$\begin{aligned}
L^{-\frac{n}{2m}} s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z) &= L^{-\frac{n}{2m}} \int_M h_E\Big(e_L\big(\phi_{x_0}^{-1}(L^{-\frac{1}{m}}z), y\big), \tilde{s}_L(y)\Big)|dy| \\
&= \int_{U_{x_0}} \tilde{\chi}_V\big(\phi_{x_0}(y)\big)e_L\big(\phi_{x_0}^{-1}(L^{-\frac{1}{m}}z), y\big)f\big(L^{\frac{1}{m}}\phi_{x_0}(y)\big)(y)|dy| \\
&= L^{-\frac{n}{m}} \int_{\mathbb{R}^n} \tilde{\chi}_V(L^{-\frac{1}{m}}h)(\phi_{x_0}^{-1})^* e_L(L^{-\frac{1}{m}}z, L^{-\frac{1}{m}}h)f(h)|dh|,
\end{aligned}$$

where we performed the substitution $h = L^{\frac{1}{m}}\phi_{x_0}(y)$, so that $|dh| = L^{\frac{n}{m}}|dy|$. But from Theorem 4.4 of [8],

$$L^{-\frac{n}{m}}(\phi_{x_0}^{-1})^* e_L(L^{-\frac{1}{m}}z, L^{-\frac{1}{m}}h) \underset{L\to+\infty}{\to} \frac{1}{(2\pi)^n}\int_{K'_{x_0}} e^{i\langle z-h,\xi\rangle}|d\xi|,$$

where $K'_{x_0} = (d_{|x_0}\phi_{x_0}^{-1})^* K_{x_0}$. Moreover, there exists $\epsilon > 0$ such that this convergence holds in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ for the semi-norms family defined by the supremum of the derivatives of the functions on the bidisc $\bar{B}(\epsilon L^{\frac{1}{m}})^2$, where $\bar{B}(\epsilon L^{\frac{1}{m}})$ denotes the closed ball of $\mathbb{R}^n$ of radius $\epsilon L^{\frac{1}{m}}$, see [5]. As a consequence, after perhaps taking a smaller V so that V is contained in the ball of radius ϵ,

$$L^{-\frac{n}{m}}\tilde{\chi}_V(L^{-\frac{1}{m}}h)(\phi_{x_0}^{-1})^* e_L(L^{-\frac{1}{m}}z, L^{-\frac{1}{m}}h)f(h) \underset{L\to+\infty}{\to} \frac{1}{(2\pi)^n}\int_{K'_{x_0}} e^{i\langle z-h,\xi\rangle}f(h)|d\xi|$$

in this same sense, which implies, with z fixed, a convergence in the Schwartz space $S(\mathbb{R}^n)$. After integration, it follows that

$$L^{-\frac{n}{2m}}s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z) \underset{L\to+\infty}{\to} \frac{1}{(2\pi)^n}\int_{K'_{x_0}} e^{i\langle z,\xi\rangle}\mathcal{F}(f)(\xi)|d\xi|$$

in $C^\infty(\mathbb{R}^n)$ for our family of semi-norms on $\bar{B}(\epsilon L^{\frac{1}{m}})$. But $f \in S_{K'_{x_0}}(\mathbb{R}^n)$, so that

$$\frac{1}{(2\pi)^n}\int_{K'_{x_0}} e^{i\langle z,\xi\rangle}\mathcal{F}(f)(\xi)|d\xi| = f(z).$$

Hence, $z \mapsto L^{-\frac{n}{2m}}s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}}z)$ converges to f in $S(\mathbb{R}^n)$, which proves the second assertion.

If $\tilde{\chi}_U = \tilde{\chi}_V \circ \phi_{x_0}$, we deduce that $\|s_L\tilde{\chi}_U\|_{L^2(M)} \underset{L\to+\infty}{\to} \|f\|_{L^2(\mathbb{R}^n)}$. We still need to prove that $\|s_L(1-\tilde{\chi}_U)\|_{L^2(M)} \underset{L\to+\infty}{\to} 0$. But since s_L is the orthogonal projection of $\tilde{s}_L$ onto $\mathbb{U}_L$,

$$\|s_L\|_{L^2(M)} \leq \|\tilde{s}_L\|_{L^2(M)} \underset{L\to+\infty}{\to} \|f\|_{L^2(\mathbb{R}^n)}.$$

The result follows. □

Corollary 1.11 *Under the hypotheses of Theorem 0.3, let $x_0 \in M$ and*

$$\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$$

be a measure-preserving chart such that $A = d_{|x_0}\phi_{x_0}^{-1}$ is an isometry. Let $(W, f_\Sigma) \in \mathcal{I}_\Sigma^{A^*K_{x_0}}$ and $(\delta, \epsilon) \in \mathcal{T}_{(W,f_\Sigma)}$, see Definitions 1.5 and 1.8. Then, there exist $L_0 \in \mathbb{R}$ and $(s_L)_{L\geq L_0}$ such that for every $L \geq L_0$,

1. *$s_L \in \mathbb{U}_L$ and $\|s_L\|_{L^2(M)} \underset{L\to+\infty}{\to} \|f_\Sigma\|_{L^2(\mathbb{R}^n)}$*
2. *The vanishing locus of s_L contains a hypersurface Σ_L included in the ball $B_g(x_0, R_{(W,f_\Sigma)}L^{-\frac{1}{m}})$ such that the pair $\big(B(x_0, R_{(W,f_\Sigma)}L^{-\frac{1}{m}}), \Sigma_L\big)$ is diffeomorphic to the pair $(\mathbb{R}^n, \Sigma)$.*

3. *There exist two neighbourhoods K_L and W_L of Σ_L such that K_L is compact, W_L is open, $\Sigma_L \subset K_L \subset W_L \subset B_g(x_0, R_{(W,f_\Sigma)}L^{-\frac{1}{m}})$, $\inf_{W_L \setminus K_L} |s_L| > \delta L^{\frac{n}{2m}}$ and for every $y \in W_L$,*

$$|s_L(y)| < \delta L^{\frac{n}{2m}} \Rightarrow \|d_{|y}(s_L \circ \phi_{x_0}^{-1})\| > \epsilon L^{\frac{n+2}{2m}}.$$

Proof. Let $L_0 \in \mathbb{R}$ and $(s_L)_{L \geq L_0}$ be a family given by Proposition 1.10 for $f = f_\Sigma$. Then, the first condition is satisfied and the family of functions $z \in B(0, R_{(W,f_\Sigma)}) \mapsto L^{-\frac{n}{2m}} s_L \circ \phi_{x_0}^{-1}(L^{-\frac{1}{m}} z)$ converges to f_Σ in $C^\infty(B(0, R_{(W,f_\Sigma)}))$. Let K be the compact given by Definition 1.5, $K_L = \phi_{x_0}^{-1}(L^{-\frac{1}{m}} K)$ and $W_L = \phi_{x_0}^{-1}(L^{-\frac{1}{m}} W)$. The conditions 2. and 3. follow from this convergence and from Definition 1.5. □

2 Probability of the local presence of a hypersurface

In this section, we follow the method of [6] partially inspired by [11] and [2] (see also [9], [4]) in order to prove Theorem 0.3. If Σ is a smooth closed hypersurface of $\mathbb{R}^n$, $x \in M$ and $s_L \in \mathbb{U}_L$ be given by Proposition 1.10, vanishing transversally along Σ_L in a small ball $B(x, L^{-\frac{1}{m}})$, then we decompose any random section $s \in \mathbb{U}_L$ as $s = a s_L + \sigma$, where $a \in \mathbb{R}$ is Gaussian and σ is taken at random in the orthogonal complement of $\mathbb{R} s_L$ in $\mathbb{U}_L$. In §2.1, we estimate the average of the values of σ and its derivatives on $B(x, L^{-\frac{1}{m}})$ see Proposition 2.1. In §2.2, we prove that with a probability at least p_Σ^x independent of L, s vanishes in the latter ball along a hypersurface isotopic to Σ_L, thanks to the quantitative estimates of the transversality of s_L given by Corollary 1.11, and thanks to Proposition 2.1.

2.1 Expected local C^1-norm of sections

Recall that for $x_0 \in M$,

$$K_{x_0} = \{\xi \in T_{x_0}^* M \ |\sigma_P(\xi) \leq 1\}. \tag{2.1}$$

Proposition 2.1 *Under the hypotheses of Theorem 0.3, let $x_0 \in M$ and $\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$ be a measure-preserving map such that $A = d_{|x_0}\phi_{x_0}^{-1}$ is an isometry. Then, for every positive R and every $j \in \{1, \cdots, n\}$,*

$$\begin{aligned} \limsup_{L \to +\infty} L^{-\frac{n}{2m}} \mathbb{E}\left(\|s\|_{L^\infty(B_g(x_0, RL^{-\frac{1}{m}}))}\right) &\leq \rho_{A^* K_{x_0}}(R) \\ \text{and } \limsup_{L \to +\infty} L^{-\frac{n+2}{2m}} \mathbb{E}\left(\left\|\frac{\partial(s \circ \phi_{x_0}^{-1})}{\partial x_j}\right\|_{L^\infty(B_g(0, RL^{-\frac{1}{m}}))}\right) &\leq \theta^j_{A^* K_{x_0}}(R), \end{aligned}$$

where $\rho_{A^* K_{x_0}}$ and $\theta^j_{A^* K_{x_0}}$ are defined by (1.2) and (1.3).

Proof. Let $t \in \mathbb{R}_+^*$. When L is large enough, the ball $B(0,(R+t)L^{-\frac{1}{m}})$ of $\mathbb{R}^n$ gets included in V. From the Sobolev inequality (see §2.4 of [1]), we deduce that for every $s \in \mathbb{U}_L$, every $k > n/2$ and every $z \in B(0, RL^{-\frac{1}{m}})$,

$$|s\circ\phi_{x_0}^{-1}(z)| \leq \frac{2k}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{k} (tL^{-\frac{1}{m}})^i \left(\frac{1}{i!} \int_{B(0,(R+t)L^{-\frac{1}{m}})} |D^i(s\circ\phi_{x_0}^{-1})|^2(x)|dx| \right)^{1/2},$$

where by definition, the norm of the $i-$th derivative $D^i(s\circ\phi_{x_0}^{-1})$ of $s\circ\phi_{x_0}^{-1}$ satisfies

$$i!|D^i(s\circ\phi_{x_0}^{-1})(x)|^2 = \sum_{\substack{(j_1,\cdots,j_i)\\ \in\{1,\cdots,n\}^i}} \Big|\frac{\partial^i}{\partial x_{j_1}\cdots\partial x_{j_i}}(s\circ\phi_{x_0}^{-1})(x)\Big|^2.$$

Note indeed that the metric h_E of the bundle E makes it possible to identify $s_{|U_{x_0}}$ with a real valued function well defined up to a sign. As a consequence, we deduce from the Cauchy-Schwarz inequality

$$\begin{aligned}\mathbb{E}\Big(\|s\circ\phi_{x_0}^{-1}\|_{L^\infty(B(0,RL^{-\frac{1}{m}}))}\Big) \quad &\leq \quad \frac{2k}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{k} \frac{1}{i!}(tL^{-\frac{1}{m}})^i \\ &\left(\int_{B(0,(R+t)L^{-\frac{1}{m}})} i!\mathbb{E}(|D^i(s\circ\phi_{x_0}^{-1})|^2(x))|dx| \right)^{1/2}.\end{aligned}$$

But given $(j_1,\cdots j_i) \in \{1,\cdots,n\}^i$ and $z \in B(0,(R+t)L^{-\frac{1}{m}})$, we can choose an orthonormal basis $(s_1,\cdots,s_{N_L})$ of $\mathbb{U}_L$ such that $\frac{\partial^i}{\partial x_{j_1}\cdots\partial x_{j_i}}(s_l\circ\phi_{x_0}^{-1})(z) = 0$ for every $l > 1$. Since the spectral function reads $(x,y) \in M\times M \mapsto e_L(x,y) = \sum_{i=0}^{N_L} s_i(x)s_i^*(y)$, we deduce, using the decomposition of s in the basis $(s_1,\cdots,s_{N_L})$, that

$$\mathbb{E}\left(\Big|\frac{\partial^i}{\partial x_{j_1}\cdots\partial x_{j_i}}(s\circ\phi_{x_0}^{-1})\Big|^2(z)\right) = \left(\int_{\mathbb{R}} a^2e^{-a^2}\frac{da}{\sqrt{\pi}}\right)\frac{\partial^{2i}}{\partial x_{j_1}\cdots\partial x_{j_i}\partial y_{j_1}\cdots\partial y_{j_i}}(e_L\circ\phi_{x_0}^{-1})(z,z).$$

Choosing $k = \lfloor\frac{n}{2}+1\rfloor$ and noting that $\int_{\mathbb{R}} a^2e^{-a^2}\frac{da}{\sqrt{\pi}} = \frac{1}{2}$, we deduce that for L large enough, $\mathbb{E}\big(\|s\circ\phi_{x_0}^{-1}\|_{L^\infty(B(0,RL^{-\frac{1}{m}}))}\big)$ is bounded from above by

$$\inf_{t\in\mathbb{R}_+^*} \frac{\sqrt{2}\lfloor\frac{n}{2}+1\rfloor}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{\lfloor\frac{n}{2}+1\rfloor} \frac{1}{i!}(tL^{-\frac{1}{m}})^i \Big(\int_{B(0,(R+t)L^{-\frac{1}{m}})} \sum_{\substack{(j_1,\cdots,j_i)\\ \in\{1,\cdots,n\}^i}} \frac{\partial^{2i}e_L(x,x)}{\partial x_{j_1}\cdots\partial x_{j_i}\partial y_{j_1}\cdots\partial y_{j_i}}|dx|\Big)^{1/2}.$$

Likewise, for every $j \in \{1,\cdots,n\}$, $\mathbb{E}\Big(\|\frac{\partial(s\circ\phi_{x_0}^{-1})}{\partial z_j}\|_{L^\infty(B(0,RL^{-\frac{1}{m}}))}\Big)$ gets bounded from above by

$$\inf_{t\in\mathbb{R}_+^*} \frac{\sqrt{2}\lfloor\frac{n}{2}+1\rfloor}{Vol(B(0,tL^{-\frac{1}{m}}))^{\frac{1}{2}}} \sum_{i=0}^{\lfloor\frac{n}{2}+1\rfloor} \frac{1}{i!}(tL^{-\frac{1}{m}})^i \Big(\int_{B(0,(R+t)L^{-\frac{1}{m}})} \sum_{\substack{(j_1,\cdots,j_i)\\ \in\{1,\cdots,n\}^i}} \frac{\partial^{2i+2}e_L(x,x)}{\partial x_j\partial x_{j_1}\cdots\partial x_{j_i}\partial y_j y_{j_1}\cdots\partial y_{j_i}}|dx|\Big)^{1/2}.$$

Now, the result is a consequence of the asymptotic estimate

$$\frac{\partial^{2i} e_L(x,x)}{\partial x_{j_1}\cdots\partial x_{j_i}\partial y_{j_1}\cdots\partial y_{j_i}} \underset{L\to+\infty}{\sim} \frac{1}{(2\pi)^n} L^{\frac{n+2i}{m}} \int_{K_0} |\xi_{j_1}|^2\cdots|\xi_{j_i}|^2 |d\xi|,$$

see Theorem 2.3.6 of [5]. We used here that the balls $B_g(x_0, RL^{-\frac{1}{m}})$ and $\phi_{x_0}^{-1}(B(0, RL^{-\frac{1}{m}}))$ coincide at the first order in L. $\square$

2.2 Proof of Theorem 0.3

Let $x_0 \in M$, $R > 0$ and $A \in Isom_g(\mathbb{R}^n, T_{x_0}M)$. Let

$$\phi_{x_0} : (U_{x_0}, x_0) \subset M \to (V, 0) \subset \mathbb{R}^n$$

be a measure-preserving map such that $A = d_{|x_0}\phi_{x_0}^{-1}$. Let $(W, f_\Sigma) \in \mathcal{I}_\Sigma^{A^*K_{x_0,R}}$, $(\delta,\epsilon) \in \mathcal{T}_{(W,f_\Sigma)}$ and $(s_L)_{L\geq L_0}$ be a family given by Corollary 1.11 associated to f_Σ, where K_{x_0} is defined by (2.1). Denote by $s_L^\perp$ the hyperplane orthogonal to s_L in $\mathbb{U}_L$. Then,

$$\int_{s_L^\perp} \|s\circ\phi_{x_0}^{-1}\|_{L^\infty(B(0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}}))} d\mu(s) \leq \int_{\mathbb{U}_L} \|s\circ\phi_{x_0}^{-1}\|_{L^\infty(B(0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}}))} d\mu(s)$$

and for every $j \in \{1,\cdots,n\}$,

$$\int_{s_L^\perp} \Big\|\frac{\partial}{\partial x_j}(s\circ\phi_{x_0}^{-1})\Big\|_{L^\infty(B(0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}}))} d\mu(s) \leq \int_{\mathbb{U}_L} \Big\|\frac{\partial}{\partial x_j}(s\circ\phi_{x_0}^{-1})\Big\|_{L^\infty(B(0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}}))} d\mu(s),$$

compare the proof of Proposition 3.1 of [6]. From Proposition 2.1 and Markov's inequality we deduce that for every $T \in \mathbb{R}_+^*$,

$$\mu\Big\{s \in s_L^\perp \mid \sup_{B_g(x_0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}})} |s| \geq \frac{T\delta L^{\frac{n}{2m}}}{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}\Big\} \leq \frac{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}{T\delta}\rho_{A^*K_{x_0}}(R_{(W,f_\Sigma)}) + o(1)$$

and for every $j \in \{1,\cdots,n\}$,

$$\mu\Big\{s \in s_L^\perp \mid \sup_{B(0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}})} |\frac{\partial}{\partial x_j}(s\circ\phi_{x_0}^{-1})| \geq \frac{T\epsilon L^{\frac{n+2}{2m}}}{\sqrt{n}\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}\Big\} \leq \frac{\sqrt{n}\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}{T\epsilon}\theta^j_{A^*K_{x_0}}(R_{(W,f_\Sigma)}) + o(1).$$

It follows that the measure of the set

$$\mathcal{E}_{s_L^\perp} = \Big\{s \in s_L^\perp \mid \sup_{B_g(x_0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}})} |s| < \frac{T\delta L^{\frac{n}{2m}}}{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}} \text{ and } \sup_{B(0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}})} |d(s\circ\phi_{x_0}^{-1})| < \frac{T\epsilon L^{\frac{n+2}{2m}}}{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}\Big\}$$

satisfies

$$\mu(\mathcal{E}_{s_L^\perp}) \geq 1 - \frac{\|f_\Sigma\|_{L^2(\mathbb{R}^n)}}{T}\Big(\frac{1}{\delta}\rho_{A^*K_{x_0}}(R_{(W,f_\Sigma)}) + \frac{n\sqrt{n}}{\epsilon}\sum_{j=1}^n \theta^j_{A^*K_{x_0}}(R_{(W,f_\Sigma)})\Big) + o(1),$$

where the $o(1)$ term can be chosen independently of x_0 since M is compact. Taking the supremum over the pairs $(\delta,\epsilon)\in\mathcal{T}_{(W,f_\Sigma)}$ and passing to the liminf, we deduce from (1.6) the estimate

$$\liminf_{L\to+\infty}\mu(\mathcal{E}_{s_L^\perp})\geq 1-\frac{\tau_{(W,f_\Sigma)}^{A^*K_{x_0}}}{T}.$$

Now, let

$$\mathcal{F}_T=\Big\{a\frac{s_L}{\|s_L\|_{L^2(M)}}+\sigma\mid a>T\text{ and }\sigma\in\mathcal{E}_{s_L^\perp}\Big\}.$$

From Lemma 3.6 of [6], every section $s\in\mathcal{F}_T$ vanishes transversally in $B_g(x_0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}})$ along a hypersuface Σ_L such that $(B_g(x_0,R_{(W,f_\Sigma)}L^{-\frac{1}{m}}),\Sigma_L)$ is diffeomorphic to $(\mathbb{R}^n,\Sigma)$. Moreover, since μ is a product measure,

$$\liminf_{L\to+\infty}\mu(\mathcal{F}_T)\geq\Big(\frac{1}{\sqrt{\pi}}\int_T^{+\infty}e^{-t^2}dt\Big)\Big(1-\frac{\tau_{(W,f_\Sigma)}^{A^*K_{x_0}}}{T}\Big).$$

Taking the supremum over $T\in[\tau_{(W,f_\Sigma)},+\infty[$, we deduce from (1.7) that

$$\liminf_{L\to+\infty}Prob_{x_0,\Sigma}(R_{(W,f_\Sigma)})\geq\liminf_{L\to+\infty}\mu(\mathcal{F}_T)\geq p_{(W,f_\Sigma)}^{A^*K_{x_0}}.$$

Taking the supremum over all pairs $(W,f_\Sigma)\in\mathcal{I}_\Sigma^{A^*K_{x_0},R}$, see (1.8), and then over every $A\in Isom_g(\mathbb{R}^n,T_{x_0}M)$, we obtain Theorem 0.3 by choosing

$$p_\Sigma^x(R)=\sup_{A\in Isom_g(\mathbb{R}^n,T_xM)}(p_\Sigma^{A^*K_x}(R)). \tag{2.2}$$

Indeed, from Remark 1.9, this function is positive for R large enough. □

2.3 Proofs of Theorem 0.1 and Corollary 0.2

Proof of Theorem 0.1. Let $g\in Met_{|dy|}(M)$. For every point x in M, the supremum $\sup_{R\in\mathbb{R}_+^*}\big(\frac{1}{Vol_{eucl}(B(0,R))}p_\Sigma^x(R)\big)$ is achieved and we denote by $R_m(x)$ the smallest positive real number where it is reached. Denote by $\tilde{g}$ the normalized metric g/R_m^2. For every L large enough, let Λ_L be a maximal subset of M such that the distance between any two distinct points of Λ_L is larger than $2L^{-\frac{1}{m}}$ for $\tilde{g}$. The $\tilde{g}-$balls centered at points of Λ_L and of radius $L^{-\frac{1}{m}}$ are disjoint, whereas the ones of radius $2L^{-\frac{1}{m}}$ cover M. For every $s\in\mathbb{U}_L\setminus\Delta_L$ and every $x\in\Lambda_L$, we set $N_{x,\Sigma}(s)=1$ if $B_{\tilde{g}}(x,L^{-\frac{1}{m}})$ contains a hypersurface $\tilde{\Sigma}$ such that $\tilde{\Sigma}\subset s^{-1}(0)$ and $(B_{\tilde{g}}(x,L^{-\frac{1}{m}}),\tilde{\Sigma})$ is diffeomorphic to $(\mathbb{R}^n,\Sigma)$, and $N_{x,\Sigma}=0$ otherwise. Note that

$$\int_{\mathbb{U}_L\setminus\Delta_L}N_{x,\Sigma}(s)d\mu(s)\underset{L\to+\infty}{\sim}Prob_{x,\Sigma}(R).$$

Thus,

$$\begin{aligned}
\liminf_{L\to+\infty}\frac{1}{L^{\frac{n}{m}}}\mathbb{E}(N_\Sigma) &\geq \liminf_{L\to+\infty}\frac{1}{L^{\frac{n}{m}}}\int_{\mathbb{U}_L\setminus\Delta_L}\Big(\sum_{x\in\Lambda_L}N_{x,\Sigma}(s)\Big)d\mu(s)\\
&= \liminf_{L\to+\infty}\frac{1}{L^{\frac{n}{m}}}\sum_{x\in\Lambda_L}Prob_{x,\Sigma}(R_m(x))\\
&\geq \frac{1}{2^n}\liminf_{L\to+\infty}\sum_{x\in\Lambda_L}Vol(B_{\tilde{g}}(x,2L^{-\frac{1}{m}}))R_m^n(x)\Big(\frac{p_\Sigma^x(R_m(x))}{Vol_{eucl}B(0,R_m(x))}\Big)
\end{aligned}$$

by Theorem 0.3. Hence, we get

$$\begin{aligned}
\liminf_{L\to+\infty}\frac{1}{L^{\frac{n}{m}}}\mathbb{E}(N_\Sigma) &\geq \frac{1}{2^n}\int_M\sup_{R>0}\Big(\frac{p_\Sigma^x(R)}{Vol_{eucl}(B(0,R))}\Big)R_m^n(x)|dvol_{\tilde{g}}(x)|\\
&= \frac{1}{2^n}\int_M\sup_{R>0}\Big(\frac{p_\Sigma^x(R)}{Vol_{eucl}(B(0,R))}\Big)|dx|.
\end{aligned}$$

Theorem 0.1 can be deduced after taking the supremum over $g\in Met_{|dy|}(M)$ and choosing the quantity $c_\Sigma(P)$ to be equal to

$$c_\Sigma(P)=\frac{1}{2^n}\sup_{g\in Met_{|dy|}(M)}\int_M\sup_{R>0}\Big(\frac{p_\Sigma^x(R)}{Vol_{eucl}(B(0,R))}\Big)|dx|. \tag{2.3}$$

□

Proof of Corollary 0.2. For every $i\in\{0,\cdots,n-1\}$ and every large enough $L>0$,

$$\begin{aligned}
\mathbb{E}(b_i) &= \int_{\mathbb{U}_L\setminus\Delta_L}b_i(s^{-1}(0))d\mu(s)\\
&\geq \int_{\mathbb{U}_L\setminus\Delta_L}\Big(\sum_{[\Sigma]\in\mathcal{H}_n}N_\Sigma(s)b_i(\Sigma)\Big)d\mu(s)\\
&\geq \sum_{[\Sigma]\in\mathcal{H}_n}b_i(\Sigma)\mathbb{E}(N_\Sigma).
\end{aligned}$$

The result is a consequence of Theorem 0.1 after passing to the liminf in the latter bound.□

3 Explicit estimates

The goal of this section is to obtain explicit lower bounds for the constants $c_\Sigma(P)$ and $\inf_{x\in M}p_\Sigma^x(R)$ appearing in Theorems 0.1 and 0.3, when Σ is diffeomorphic to the product of spheres $S^{i+1}\times S^{n-i-1}$ (whose i-th Betti number is at least one). In the first paragraph, we approximate quantitatively the product of a polynomial function and a Gaussian one by a function whose Fourier transform gets compact support. We then apply this result to a particular degree four polynomial vanishing along a product of spheres to finally get Theorem 0.4, Corollary 0.5 and 0.6.

3.1 Key estimates for the approximation

Let $\tilde{\chi}_c : \mathbb{R}^n \to [0,1]$ be a smooth function with support in the ball of radius $c > 0$, such that $\chi_c = 1$ on the ball of radius $c/2$. For every $Q \in \mathbb{R}[X_1, \cdots, X_n]$ and every $\eta > 0$, we set

$$q : x \in \mathbb{R}^n \quad \mapsto \quad q(x) = Qe^{-\frac{\|x\|^2}{2}} \in \mathbb{R} \text{ and} \tag{3.1}$$

$$q_\eta^c : x \in \mathbb{R}^n \quad \mapsto \quad q_\eta^c(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\chi}_c(\eta\xi)\mathcal{F}(q(x))(\xi)e^{i\langle x,\xi\rangle}|d\xi|. \tag{3.2}$$

Note that $q_\eta^c \in S_{B(0,c/\eta)}(\mathbb{R}^n)$, see (1.1).

Proposition 3.1 *Let $Q = \sum_{I\in\mathbb{N}^n} a_I x^I \in \mathbb{R}[X_1, \cdots, X_n]$ and $c, \eta > 0$. Then,*

1. $$\|q_\eta^c - q\|_{L^\infty(\mathbb{R}^n)} \leq \sqrt{\lfloor n/2+1 \rfloor}\big(\frac{c}{2\eta}\big)^{\frac{n-2}{2}} e^{-\frac{1}{4}(\frac{c}{2\eta})^2}\big(\sum_{I\in\mathbb{N}^n} |a_I|\sqrt{I!}\big).$$

2. *For every $k \in \{1, \cdots, n\}$,*
$$\Big\|\frac{\partial q_\eta^c}{\partial x_k} - \frac{\partial q}{\partial x_k}\Big\|_{L^\infty(\mathbb{R}^n)} \leq \sqrt{\lfloor n/2+3 \rfloor}\big(\frac{c}{2\eta}\big)^{\frac{n}{2}} e^{-\frac{1}{4}(\frac{c}{2\eta})^2}\big(\sum_{I\in\mathbb{N}^n} |a_I|\sqrt{I!}\big).$$

3. $$\big\|q_\eta^c - q\big\|^2_{L^2(\mathbb{R}^n)} \leq \sqrt{2\pi}^n N(Q)\big(\sum_{I\in\mathbb{N}^n} a_I^2 I!\big)e^{-\frac{1}{2}(\frac{c}{2\eta})^2},$$

where $N(Q)$ denotes the number of monomials of Q.

Proof. For every $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |q_\eta^c(x) - q(x)| &\leq \frac{1}{(2\pi)^n}\int_{\|\xi\|\geq\frac{c}{2\eta}} |\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})|(\xi)|d\xi| \\ &\leq \frac{1}{(2\pi)^n}\sum_{I\in\mathbb{N}^n}|a_I|\int_{\|\xi\|\geq\frac{c}{2\eta}} |\mathcal{F}(x_I e^{-\frac{\|x\|^2}{2}})|(\xi)|d\xi|. \end{aligned}$$

However,

$$\mathcal{F}(x_I e^{-\frac{\|x\|^2}{2}}) = i^{|I|}\frac{\partial}{\partial\xi_I}\big(\mathcal{F}(e^{-\frac{\|x\|^2}{2}})\big) \tag{3.3}$$

$$= \sqrt{2\pi}^n i^{|I|}\frac{\partial}{\partial\xi_I}(e^{-\frac{\|\xi\|^2}{2}}) \tag{3.4}$$

$$= \sqrt{2\pi}^n i^{|I|}\prod_{j=1}^n\Big(H_{i_j}(\xi_j)e^{-\frac{\xi_j^2}{2}}\Big), \tag{3.5}$$

where we have set $I=(i_1,\cdots,i_n)$ and H_j the $j-$th Hermite polynomial. We deduce from Cauchy-Schwarz inequality that

$$\begin{aligned}|q_\eta^c(x)-q(x)| &\leq \frac{1}{\sqrt{2\pi}^n}\sum_{I\in\mathbb{N}^n}|a_I|\Big(\prod_{j=1}^n\int_{\mathbb{R}}H_{i_j}^2(\xi_j)e^{-\frac{\xi_j^2}{2}}d\xi_j\Big)^{1/2}\Big(\int_{\|\xi\|\geq\frac{c}{2\eta}}e^{-\frac{\|\xi\|^2}{2}}d\xi\Big)^{1/2}\\ &\leq \frac{1}{(2\pi)^{n/4}}\big(\sum_{I\in\mathbb{N}^n}|a_I|\sqrt{I!}\big)\sqrt{Vol(S^{n-1})}\Big(\int_{\frac{c}{2\eta}}^{+\infty}r^{n-1}e^{-\frac{r^2}{2}}dr\Big)^{1/2},\end{aligned}$$

since for every $k\in\mathbb{N}$, the dominating coefficient of $H_k(\xi)$ equals $(-1)^k$, so that an integration by parts leads to

$$\int_{\mathbb{R}}\xi^kH_k(\xi)e^{-\frac{\xi^2}{2}}d\xi=(-1)^kk!\sqrt{2\pi} \tag{3.6}$$

since Hermite polynomials are orthogonal to each other. Likewise, after integration by parts we obtain

$$\begin{aligned}\int_{\frac{c}{2\eta}}^{+\infty}r^{n-1}e^{-\frac{r^2}{2}}dr &= [-r^{n-2}e^{-\frac{r^2}{2}}]_{\frac{c}{2\eta}}^{+\infty}+(n-2)\int_{\frac{c}{2\eta}}^{+\infty}r^{n-3}e^{-\frac{r^2}{2}}dr\\ &\leq \big(\frac{c}{2\eta}\big)^{n-2}e^{-\frac{1}{2}(\frac{c}{2\eta})^2}+(n-2)\big(\frac{c}{2\eta}\big)^{n-4}e^{-\frac{1}{2}(\frac{c}{2\eta})^2}+\cdots\end{aligned}$$

From the latter we deduce, when $|\frac{c}{2\eta}|\geq 1$,

$$\int_{\frac{c}{2\eta}}^{+\infty}r^{n-1}e^{-\frac{r^2}{2}} = \lfloor\frac{n}{2}+1\rfloor\big(\frac{c}{2\eta}\big)^{n-2}e^{-\frac{1}{2}(\frac{c}{2\eta})^2}(n-2)(n-4)\cdots$$

Recall that

$$Vol(S^{n-1})=\begin{cases}\frac{\sqrt{2\pi}^n}{(n-2)(n-4)\cdots 2} & \text{if } n \text{ is even}\\ \frac{\sqrt{2}\sqrt{2\pi}^n}{\sqrt{\pi}(n-2)(n-4)\cdots 3\times 1} & \text{if } n \text{ is odd}\end{cases}$$

We thus finally get

$$\|q_\eta^c-q\|_{L^\infty}\leq\sqrt{\lfloor\frac{n}{2}+1\rfloor}\big(\frac{c}{2\eta}\big)^{\frac{n-2}{2}}e^{-\frac{1}{4}(\frac{c}{2\eta})^2}\big(\sum_{I\in\mathbb{N}^n}|a_I|\sqrt{I!}\big).$$

Likewise, for every $k\in\{1,\cdots,n\}$,

$$\begin{aligned}\Big\|\frac{\partial q_\eta^c}{\partial x_k}-\frac{\partial q}{\partial x_k}\Big\|_{L^\infty(\mathbb{R}^n)} &\leq \frac{1}{(2\pi)^n}\int_{\|\xi\|\geq\frac{c}{2\eta}}|\xi_k||\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})|(\xi)|d\xi|\\ &\leq \frac{1}{\sqrt{2\pi}^n}\sum_{I\in\mathbb{N}^n}|a_I|\big(\prod_{j=1}^n\int_{\mathbb{R}}H_{i_j}^2(\xi)e^{-\frac{\xi_j^2}{2}}dx_j\big)^{\frac{1}{2}}\big(\int_{\|\xi\|\geq\frac{c}{2\eta}}\|\xi\|^2e^{-\frac{\|\xi\|^2}{2}}d\xi\big)^{\frac{1}{2}}\\ &\leq \frac{1}{(2\pi)^{n/4}}\sum_{I\in\mathbb{N}^n}|a_I|\sqrt{I!}Vol(S^{n-1})^{\frac{1}{2}}\big(\int_{\frac{c}{2\eta}}^{+\infty}r^{n+1}e^{-r^2/2}dr\big)^{\frac{1}{2}}\\ &\leq \sqrt{\lfloor\frac{n}{2}+3\rfloor}\big(\frac{c}{2\eta}\big)^{n/2}e^{-\frac{1}{4}(\frac{c}{2\eta})^2}\big(\sum_{I\in\mathbb{N}^n}|a_I|\sqrt{I!}\big).\end{aligned}$$

Lastly,

$$\begin{aligned}
\|q_\eta^c - q\|^2_{L^2(\mathbb{R}^n)} &\leq \big\|\mathcal{F}^{-1}\big(\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})(1-\tilde{\chi}_c(\eta\xi))\big)\big\|^2_{L^2(\mathbb{R}^n)} \\
&\leq \frac{1}{(2\pi)^n}\int_{\xi\geq\frac{c}{2\eta}} |\mathcal{F}(Qe^{-\frac{\|x\|^2}{2}})|^2|d\xi| \text{ from Plancherel's equality} \\
&= \int_{\xi\geq\frac{c}{2\eta}} \big|\sum_{I\in\mathbb{N}^n} i^{|I|}a_I\prod_{j=1}^n H_{i_j}(\xi_j)e^{-\frac{\xi_j^2}{2}}\big|^2|d\xi| \text{ from (3.3)} \\
&\leq N(Q)e^{-\frac{1}{2}(\frac{c}{2\eta})^2}\sum_{I\in\mathbb{N}^n} a_I^2\prod_{j=1}^n\int_{\mathbb{R}} H_{i_j}^2(\xi_j)e^{-\frac{\xi_j^2}{2}}d\xi_j \text{ from Cauchy-Schwarz} \\
&\leq \sqrt{2\pi}^n N(Q)\big(\sum_{I\in\mathbb{N}^n} a_I^2 I!\big)e^{-\frac{1}{2}(\frac{c}{2\eta})^2} \text{ from (3.6).}
\end{aligned}$$

□

3.2 The product of spheres

For every $n > 0$ and every $i \in \{0, \cdots, n-1\}$, let

$$\begin{aligned}
Q_i : \mathbb{R}^{i+1}\times\mathbb{R}^{n-i-1} &\to \mathbb{R} && (3.7)\\
(x,y) &\mapsto (\|x\|^2-2)^2+\|y\|^2-1. && (3.8)
\end{aligned}$$

We recall that this polynomial vanishes in the ball of radius $\sqrt{5}$ along a hypersurface diffeomorphic to the product of spheres $S^i \times S^{n-i-1}$, see §2.3.2 of [6]. Let

$$q_i : (x,y)\in\mathbb{R}^{i+1}\times\mathbb{R}^{n-i-1} \mapsto Q_i(x,y)e^{-\frac{1}{2}(\|x\|^2+\|y\|^2)} \in \mathbb{R}.$$

This function belongs to the Schwartz space and has the same vanishing locus as Q_i. Let us quantify the transversality of this vanishing. We set

$$W = \{(x,y)\in\mathbb{R}^{i+1}\times\mathbb{R}^{n-i-1},\ \|x\|^2+\|y\|^2\leq 5\}.$$

Lemma 3.2 *For every $\delta \leq 1/2$,*

$$\Big(\delta e^{-5/2}, \frac{e^{-5/2}}{2}(2-\delta)\Big) \in \mathcal{T}_{(W,q_i)},$$

see Definition 1.5.

Proof. Let $(x,y)\in\mathbb{R}^{i-1}\times\mathbb{R}^{n-i-1}$ be such that $\|x\|^2+\|y\|^2\leq 5$ and $\delta\leq 1/2$. Then

$$\begin{aligned}
|q_i(x,y)| < \delta e^{-5/2} &\Rightarrow |Q_i(x,y)| < \delta \\
&\Leftrightarrow 1-\delta < (\|x\|^2-2)^2+\|y\|^2 < 1+\delta \\
&\Rightarrow \begin{cases} \|x\|^2 > 2-\sqrt{1+\delta} > 1/2 \\ \|x\|^2-2 > 1/2 \text{ or } \|y\|^2 > 1/4 \text{ since } \delta\leq 1/2. \end{cases}
\end{aligned}$$

Moreover, for every $j \in \{1, \cdots, i+1\}$,

$$\begin{aligned} |\frac{\partial q_i}{\partial x_j}| &\geq |\frac{\partial Q_i}{\partial x_j}| e^{-5/2} - |x_j| \delta e^{-5/2} \\ &\geq 4|x_j| \big| \|x\|^2 - 2 \big| e^{-5/2} - |x_j| \delta e^{-5/2} \\ &\geq |x_j| e^{-5/2} \big(4 \big| \|x\|^2 - 2 \big| - \delta \big) \end{aligned}$$

and for every $k \in \{1, \cdots, n-i-1\}$,

$$\begin{aligned} |\frac{\partial q_i}{\partial y_k}| &\geq |\frac{\partial Q_i}{\partial y_k}| e^{-5/2} - |y_k| \delta e^{-5/2} \\ &\geq |y_k| e^{-5/2} (2 - \delta). \end{aligned}$$

Summing up, we deduce

$$\begin{aligned} |d_{|(x,y)} q_i|^2 &\geq \|x\|^2 e^{-5} \big(4 \big| \|x\|^2 - 2 \big| - \delta \big)^2 + \|y\|^2 e^{-5} (2-\delta)^2 \\ &\geq \frac{e^{-5}}{2} \big(4 \big| \|x\|^2 - 2 \big| - \delta \big)^2 + \|y\|^2 e^{-5} (2-\delta)^2 \\ &\geq \frac{e^{-5}}{4} (2-\delta)^2. \end{aligned}$$

Since on the boundary of the ball W, either $\|x\|^2 \geq 7/2$ or $\|y\|^2 \geq 3/2$, the values of the function q_i are greater than $\frac{1}{2} e^{-5/2}$ and we get the result. □

We now estimate the L^2-norm of q_i.

Lemma 3.3 *For every $i \in \{0, \cdots, n-1\}$,*

$$\|q_i\|_{L^2(\mathbb{R}^n)} \leq \sqrt{\frac{3}{2}} \pi^{n/4} (n+6)^2.$$

Proof. We have

$$\begin{aligned} \|q_i\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}} \Big((\|x\|^2 - 2)^2 + \|y\|^2 - 1 \Big)^2 e^{-\|x\|^2 - \|y\|^2} dx dy \\ &= \int_{\mathbb{R}^{i+1} \times \mathbb{R}^{n-i-1}} \big(\|x\|^4 - 4\|x\|^2 + 3 + \|y\|^2 \big)^2 e^{-\|x\|^2 - \|y\|^2} dx dy \\ &\leq \sqrt{\pi}^{n-i-1} \int_{\mathbb{R}^{i+1}} (\|x\|^8 + 16\|x\|^4) e^{-\|x\|^2} dx \\ &\quad + \sqrt{\pi}^{i+1} \int_{\mathbb{R}^{n-i-1}} \big(\|y\|^4 + 6\|y\|^2 + 9 \big) e^{-\|y\|^2} dy \\ &\quad + 2 \left(\int_{\mathbb{R}^{i+1}} \|x\|^4 e^{-\|x\|^2} dx \right) \left(\int_{\mathbb{R}^{n-i-1}} (\|y\|^2 + 3) e^{-\|y\|^2} dy \right). \end{aligned}$$

Now,

$$\begin{aligned} \int_{\mathbb{R}^{i+1}} (\|x\|^8 + 16\|x\|^4) e^{-\|x\|^2} dx &= \frac{1}{2} Vol(S^i) \int_0^{+\infty} (t^4 + 16t^2) t^{\frac{i-1}{2}} e^{-t} dt \\ &= \frac{1}{2} Vol(S^i) \Big(\Gamma(\frac{i+9}{2}) + 16 \Gamma(\frac{i+5}{2}) \Big) \\ &\leq \frac{17}{2} Vol(S^i) \Gamma(\frac{i+9}{2}) \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{n-i-1}} \big(\|y\|^4+6\|y\|^2+9\big)e^{-\|y\|^2}dy &= \frac{1}{2}Vol(S^{n-i-2})\int_0^{+\infty}(t^2+6t+9)t^{\frac{1}{2}(n-i-3)}e^{-t}dt\\ &= \frac{1}{2}Vol(S^{n-i-2})\Big(\Gamma(\frac{n-i+3}{2})+6\Gamma(\frac{n-i+1}{2})+9\Gamma(\frac{n-i-1}{2})\Big)\\ &\leq \frac{25}{2}Vol(S^{n-i-2}).\end{aligned}$$

Likewise

$$\begin{aligned}\int_{\mathbb{R}^{i+1}} \|x\|^4 e^{-\|x\|^2}dx &= \frac{1}{2}Vol(S^i)\int_0^{+\infty} t^{\frac{i+3}{2}}e^{-t}dt\\ &= \frac{1}{2}Vol(S^i)\Gamma(\frac{i+5}{2}),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{n-i-1}} (\|y\|^2+3)e^{-\|y\|^2}dy &= \frac{1}{2}Vol(S^{n-i-2})\int_0^{+\infty}(t+3)t^{\frac{n-i-3}{2}}e^{-t}dt\\ &= \frac{1}{2}Vol(S^{n-i-2})\big(\Gamma(\frac{n-i+1}{2})+3\Gamma(\frac{n-i-1}{2})\big)\\ &\leq \frac{7}{2}Vol(S^{n-i-2})\Gamma(\frac{n-i+1}{2}).\end{aligned}$$

Finally, since

$$Vol(S^i)=\frac{2\pi^{\frac{i+1}{2}}}{\Gamma(\frac{i+1}{2})} \text{ and } Vol(S^{n-i-2})=\frac{2\pi^{\frac{n-i-1}{2}}}{\Gamma(\frac{n-i-1}{2})},$$

we get

$$\begin{aligned}\|q_i\|^2_{L^2(\mathbb{R}^n)} &\leq \sqrt{\pi}^n\left(17\frac{\Gamma(\frac{i+9}{2})}{\Gamma(\frac{i+1}{2})}+25\frac{\Gamma(\frac{n-i+3}{2})}{\Gamma(\frac{n-i-1}{2})}+14\frac{\Gamma(\frac{n-i+1}{2})\Gamma(\frac{i+5}{2})}{\Gamma(\frac{n-i-1}{2})\Gamma(\frac{i+1}{2})}\right)\\ &\leq \sqrt{\pi}^n\Big(\frac{17}{16}(i+7)^4+\frac{25}{4}(n-i+1)^2+\frac{7}{4}(n-i-1)(i+3)^2\Big)\\ &\leq \frac{3}{2}\sqrt{\pi}^n(n+6)^4,\end{aligned}$$

since $n+6\geq 7$, so that $\frac{25}{4}(n-i+1)^2\leq\frac{25}{4\times 49}(n+6)^4$ and $\frac{7}{4}(n-i-1)(i+3)^2\leq\frac{1}{4}(n+6)^4$.
□

We now approximate q_i by a function whose Fourier transform has compact support. For every $i\in\{0,\cdots,n-1\}$ and $c>0$, we set

$$q_{i,c}: x\in\mathbb{R}\mapsto q_{i,c}(x) = q^c_{i,\eta}(\eta x) \tag{3.9}$$

$$= \frac{1}{\eta^n}\int_{\mathbb{R}^n}\tilde{\chi}_c(\xi)\mathcal{F}(Q_i e^{-\frac{\|x\|^2}{2}})(\frac{\xi}{\eta})e^{i\langle x,\xi\rangle}|d\xi|\in\mathbb{R}, \tag{3.10}$$

see (3.1). By construction, $q_{i,c}$ belongs to the Schwartz space of $\mathbb{R}^n$ and its Fourier transform has support in the ball of radius c, so that with the notations of §1.1, $q_{i,c}\in S_{B(0,c)}(\mathbb{R}^n)$.

Corollary 3.4 *For every $i \in \{0, \cdots, n-1\}$, every $c > 0$ and every $\eta \leq \frac{c}{48n}$, $q_{i,c}$ vanishes in the ball $W_\eta = \{x \in \mathbb{R}^n, \|x\|^2 \leq 5/\eta^2\}$ along a hypersurface diffeomorphic to $S^i \times S^{n-i-1}$. Moreover,*

$$\Big(\frac{e^{-5/2}}{4}, \frac{\eta}{\sqrt{2}} e^{-5/2}\Big) \in \mathcal{T}_{(W_\eta, q_{i,c})}$$

and

$$\|q_{i,c}\|_{L^2(\mathbb{R}^n)} \leq \frac{3}{2\eta^{n/2}} \pi^{n/4} (n+6)^2.$$

Proof. The polynomial Q_i reads

$$Q_i(x,y) = \sum_{k=1}^{i+1} x_k^4 + 2 \sum_{1 \leq j < k \leq n} x_j^2 x_k^2 - 4 \sum_{k=1}^{i+1} x_k^2 + \sum_{k=1}^{n-i-1} y_k^2 + 3.$$

We deduce, with the notations of Proposition 3.1,

$$\begin{aligned} \sum_{I \in \mathbb{N}^n} |a_I| \sqrt{I!} &= (i+1)\sqrt{4!} + 4\binom{i+1}{2} + 4\sqrt{2}(i+1) + (n-i-1)\sqrt{2} + 3 \\ &\leq 5n + 2n^2 + 8n + 3 \leq 18n^2 \end{aligned}$$

and

$$\begin{aligned} \sum_{I \in \mathbb{N}^n} a_I^2 I! &= (i+1)4! + 16\binom{i+1}{2} + 32(i+1) + 2(n-i-1) + 9 \\ &\leq 24n + 8n^2 + 34n + 9 \leq 75n^2, \end{aligned}$$

whereas

$$\begin{aligned} N(Q_i) &= (i+1) + \binom{i+1}{2} + (i+1) + (n-i-1) + 1 \\ &\leq 2n + 1 + \frac{n(n-1)}{2} \leq 3n^2. \end{aligned}$$

Noting that $\sqrt{\lfloor \frac{n}{2} + 1 \rfloor} \leq \sqrt{\lfloor \frac{n}{2} + 3 \rfloor} \leq 2\sqrt{n}$, that $\left(\frac{c}{2\eta}\right)^{\frac{n-2}{2}} \leq \left(\frac{c}{2\eta}\right)^{\frac{n}{2}}$ as soon as $\frac{c}{2\eta} \geq 1$, and that

$$\frac{5}{2} \ln n + \frac{n}{2} \ln(\frac{c}{2\eta}) \leq 3n(\frac{c}{2\eta})$$

under the same hypothesis, we deduce from Proposition 3.1 that when $\eta \leq \frac{c}{48n}$,

$$\|q_{i,c}(x) - q_i(\eta x)\|_{L^\infty(\mathbb{R}^n)} \leq 36 e^{-\frac{1}{8}(\frac{c}{2\eta})^2} \leq 36 e^{-72n^2}$$

and for every $k \in \{1, \cdots, n\}$,

$$\Big\| \frac{\partial q_{i,c}}{\partial x_k}(x) - \eta \frac{\partial q_i}{\partial x_k}(\eta x) \Big\|_{L^\infty(\mathbb{R}^n)} \leq 36 \eta e^{-72n^2}.$$

From Lemma 3.2 follows, choosing $\delta = 1/2$, that for every $x \in \mathbb{R}^n$ such that $\|x\|^2 \leq 5/\eta^2$ and every $\eta \leq \frac{c}{48n}$,

$$\begin{aligned} q_{i,c}(x) \leq \frac{e^{-5/2}}{4} \quad &\Rightarrow \quad q_i(\eta x) \leq \frac{e^{-5/2}}{2} \\ &\Rightarrow \quad |d_{|\eta x} q_i| > 3\frac{e^{-5/2}}{4} \\ &\Rightarrow \quad |d_{|x} q_{i,c}| > \eta \frac{e^{-5/2}}{\sqrt{2}}, \end{aligned}$$

since

$$\begin{aligned} |d_{|x} q_{i,c}| &\geq \eta |d_{|\eta x} q_i| - |d_{|x} q_{i,c} - \eta d_{|\eta x} q_i| \\ &> \eta \frac{3e^{-5/2}}{4} - \sqrt{\sum_{k=1}^{n} \Big| \frac{\partial q_{i,c}}{\partial x_k}(x) - \eta \frac{\partial q_i}{\partial x_k}(\eta x) \Big|^2} \\ &\geq \eta \big(\frac{3e^{-5/2}}{4} - 36\sqrt{n} e^{-72n^2} \big) \\ &> \eta \frac{e^{-5/2}}{\sqrt{2}}. \end{aligned}$$

From Lemma 3.6 of [6], $q_{i,c}$ vanishes in the ball W_η along a hypersurface diffeomorphic to $S^i \times S^{n-i-1}$ and by definition, $(\frac{e^{-5/2}}{4}, \eta \frac{e^{-5/2}}{\sqrt{2}}) \in \mathcal{T}_{(W_\eta, q_{i,c})}$ if $\eta \leq \frac{c}{48n}$.

Lastly, we estimate the L^2-norm of $q_{i,c}$. By Proposition 3.1 and the bounds given above, $\|q^c_{i,\eta} - q_i\|^2_{L^2(\mathbb{R}^n)} \leq \sqrt{2\pi}^n 225 n^4 e^{-288n^2}$, so that

$$\begin{aligned} \|q_{i,c}\|_{L^2(\mathbb{R}^n)} &= \frac{1}{\eta^{n/2}} \|q^c_{i,\eta}\|_{L^2(\mathbb{R}^n)} \\ &\leq \frac{1}{\eta^{n/2}} (\|q_i\|_{L^2(\mathbb{R}^n)} + \|q^c_{i,\eta} - q_i\|_{L^2(\mathbb{R}^n)}) \\ &\leq \frac{1}{\eta^{n/2}} \Big(\sqrt{\frac{3}{2}} \pi^{n/4} (n+6)^2 + (\sqrt{2\pi}^n 225 n^4 e^{-288n^2})^{1/2} \Big) \text{ by Lemma 3.3} \\ &\leq \frac{3}{2\eta^{n/2}} \pi^{n/4} (n+6)^2. \end{aligned}$$

□

3.3 Proofs of Theorem 0.4, Corollary 0.5 and Corollary 0.6

Proof of Theorem 0.4. Let us choose $c = c_{P,g}$ and $\eta = \frac{c_{P,g}}{48n}$ see the definition (3.10) of $q_{i,c}$. It follows from Corollary 3.4 that $R \geq \frac{48\sqrt{5}n}{c_{P,g}}$, $(W_\eta, q_{i,c}) \in \mathcal{I}^{B(0,c_{P,g}),R}_{S^i \times S^{n-1-i}}$ so that for any $x \in M$ and any $A \in Isom_g(\mathbb{R}^n, T_x M)$,

$$(W_\eta, q_{i,c}) \in \mathcal{I}^{A^* K_x, R}_{S^i \times S^{n-1-i}}.$$

Indeed,

$$B(0, c_{P,g}) \subset A^* K_x \subset B(0, d_{P,g}).$$

From Remark 1.7, we get that for every $x \in M$ and every $R \geq \frac{48\sqrt{5}n}{c_{P,g}}$,

$$p^x_{S^i \times S^{n-i-1}}(R) \geq \frac{1}{2\sqrt{\pi}} \exp(-(2\tau+1)^2).$$

From (1.4), (1.5), (1.6) and Corollary 3.4 with $\eta = \frac{c_{P,g}}{48n}$, using that $\nu(A^*K_x) \leq Vol(B(0, d_{P,g}))$, we deduce

$$\begin{aligned}
\tau &\leq \frac{3}{2}\pi^{n/4}(n+6)^2\big(\frac{48n}{c_{P,g}}\big)^{n/2}\Big(\frac{4}{e^{-5/2}}\frac{1}{\sqrt{\pi}^n}\sqrt{2Vol(B(0,d_{P,g}))}\lfloor\frac{n}{2}+1\rfloor \exp\big(48\sqrt{5}n\sqrt{n}\frac{d_{P,g}}{c_{P,g}}\big) \\
&+ \frac{48n\sqrt{2}}{e^{-5/2}c_{P,g}}n\sqrt{n}\frac{n}{\sqrt{\pi}^n}\sqrt{2Vol(B(0,d_{P,g}))}\lfloor\frac{n}{2}+1\rfloor d_{P,g}\exp\big(48\sqrt{5}n\sqrt{n}\frac{d_{P,g}}{c_{P,g}}\big)\Big) \\
&\leq \frac{3}{4\pi^{n/4}}(n+6)^3(48n)^{n/2}\sqrt{2Vol(B(0,1))}\big(\frac{d_{P,g}}{c_{P,g}}\big)^{n/2}\exp\big(48\sqrt{5}n^{3/2}\frac{d_{P,g}}{c_{P,g}}\big) \\
&\quad \big(4e^{5/2}+\sqrt{2}e^{5/2}n^{5/2}(48n)\frac{d_{P,g}}{c_{P,g}}\big) \\
&\leq 20\frac{(n+6)^{11/2}}{\sqrt{\Gamma(\frac{n}{2}+1)}}\big(48n\frac{d_{P,g}}{c_{P,g}}\big)^{\frac{n+2}{2}}\exp\big(48\sqrt{5}n^{3/2}\frac{d_{P,g}}{c_{P,g}}\big).
\end{aligned}$$

The estimate for $c_{[S^i \times S^{n-i-1}]}$ follows from the above estimate with $R = 48\sqrt{5}\frac{n}{c_{P,g}}$, see (2.3). □

Proof of Corollary 0.5. If P is the Laplace-Beltrami operator associated to a metric g on M, then we choose as the Lebesgue measure $|dy|$ on M the measure $|dvol_g|$ associated to g, so that $g \in Met_{|dy|}(M)$ and the principal symbol of P equals $\xi \in T^*M \mapsto \|\xi\|^2 \in \mathbb{R}$. Theorem 0.4 then applies with $m = 2$ and $c_{P,g} = d_{P,g} = 1$ and we deduce, using $\Gamma(\frac{n}{2}+1) \geq 1/2$, that

$$\begin{aligned}
\tau &\leq 20\frac{(7n)^{11/2}}{\sqrt{\Gamma(\frac{n}{2}+1)}}(48n)^{\frac{n+2}{2}}\exp(108n^{3/2}) \\
&\leq \exp\big(\ln(20\sqrt{2}) + \frac{11}{2}\ln 7 + \frac{n+2}{2}\ln 48 + \frac{13}{2}\ln n + \frac{n}{2}\ln n + 108n^{3/2}\big) \\
&\leq \exp(18 + \frac{17}{2}(n-1) + \frac{n}{2}(2\sqrt{n}-1) + 108n^{3/2}) \\
&\leq \exp(127n^{3/2}).
\end{aligned}$$

Theorem 0.4 then provides for every $i \in \{0, \cdots, n-1\}$,

$$\begin{aligned}
\big(Vol_g(M)\big)^{-1}c_{[S^i \times S^{n-i-1}]}(P) &\geq \exp\big(-(2\tau+1)^2 - (n+1)\ln 2 - \frac{1}{2}\ln\pi \\
&\quad -n\ln(48\sqrt{5}n) - \ln(\pi^{n/2}) + \ln(\Gamma(n/2+1))\big) \\
&\geq \exp\big(-(2\tau+1)^2 - 3/2 - 6\ln n - n\ln n\big) \\
&\geq \exp\Big(-\exp(256n^{3/2}) - \exp\big(\ln(17/2) + \ln n + \ln(\ln n)\big)\Big) \\
&\geq \exp(-\exp(257n^{3/2}).
\end{aligned}$$

□

Remark 3.5 *Under the assumptions of Corollary 0.5, we get likewise for $R \geq 48\sqrt{5}n$,*

$$\begin{aligned} \inf_{x\in M} \left(p^x_{S^i\times S^{n-i-1}}(R)\right) &\geq \frac{1}{2\sqrt{\pi}} \exp(-\exp(256n^{3/2})) \\ &\geq \exp(-\exp(257n^{3/2})). \end{aligned}$$

Proof of Corollary 0.6. If P denotes the Dirichlet-to-Neumann operator on M, then the principal symbol of P equals $\xi \in T^*M \mapsto \|\xi\| \in \mathbb{R}$. Theorem 0.4 then applies with $m = 1$ and $c_{P,g} = d_{P,g} = 1$. Thus, the proof is the same as the one of Corollary 0.5. □

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