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A global uniqueness result for acoustic tomography of moving fluid

A. D. Agaltsov^{1,2}

We consider a model time-harmonic wave equation of acoustic tomography of moving fluid in an open bounded domain in dimension $d \geq 2$. We give global uniqueness theorems for related inverse boundary value problem at fixed frequency.

Keywords: inverse boundary value problems, time-harmonic wave equation, acoustic tomography

Subjects: partial differential equations, mathematical physics

AMS classification: 35R30 (Inverse problems), 35Q35 (PDEs in connection with fluid mechanics)

1 Introduction

Consider the operator

$$L_{A,V} = -\Delta - 2iA(x) \cdot \nabla + V(x), \quad (1)$$

where Δ is the standard Laplacian, $x \in D$, $A \in W^{1,\infty}(D, \mathbb{R}^d)$, $V \in L^\infty(D, \mathbb{R})$, D is an open bounded domain in \mathbb{R}^d ($d \geq 2$). In the present article we study an inverse boundary value problem for the equation $L_{A,V}\psi = 0$ in D .

As in [AN], [RW], [RE] we consider the equation $L_{A,V}\psi = 0$ as a model equation for a time-harmonic ($e^{-i\omega t}$) pressure ψ in moving fluid. In this setting

$$A(x) = \frac{\omega}{c^2(x)}v(x), \quad V(x) = -\frac{\omega^2}{c^2(x)},$$

where v is the fluid velocity vector, c is the sound speed, ω is the frequency.

Suppose that 0 is not a Dirichlet eigenvalue for operator $L_{A,V}$ in D . Then the Dirichlet problem

$$\begin{cases} L_{A,V}\psi = 0 & \text{in } D, \\ \psi|_{\partial D} = f, \end{cases} \quad (2)$$

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is uniquely solvable for $\psi \in H^1(D)$ for any $f \in H^{1/2}(\partial D)$. The Dirichlet-to-Neumann map $\Lambda_{A,V}$ sends $f \in H^{1/2}(\partial D)$ to $\Lambda_{A,V}f \in H^{-1/2}(\partial D)$ defined by the formula

$$\Lambda_{A,V}f = \frac{\partial \psi}{\partial \nu} \Big|_{\partial D} + i(A \cdot \nu)f, \quad (3)$$

where ν is the unit exterior normal to ∂D and $\frac{\partial \psi}{\partial \nu} \Big|_{\partial D} \in H^{-1/2}(\partial D)$ can be defined, in particular, by the following formula:

$$\langle \frac{\partial \psi}{\partial \nu} \Big|_{\partial D}, u \rangle = \int_D \left(\nabla \psi(x) \nabla \tilde{u}(x) - 2i\tilde{u}(x)A(x) \nabla \psi(x) + \tilde{u}(x)V(x)\psi(x) \right) dx, \quad (4)$$

for $u \in H^{1/2}(\partial D)$ and arbitrary $\tilde{u} \in H^1(D)$ with $\tilde{u}|_{\partial D} = u$. Note that since ψ satisfies $L_{A,V}\psi = 0$, the right hand side of the above formula doesn't depend on the choice of \tilde{u} .

The inverse boundary value problem for equation $L_{A,V}\psi = 0$ in D consists in finding A, V from $\Lambda_{A,V}$. In the case when coefficients A, V can be complex-valued there is an obstruction to the unique solvability of this problem caused by the gauge invariance of the map $\Lambda_{A,V}$ with respect to the gauge transformations

$$\begin{cases} A \rightarrow A + \nabla \varphi, \\ V \rightarrow V - i\Delta \varphi + (\nabla \varphi)^2 + 2A \nabla \varphi, \end{cases}$$

where $\varphi \in W^{2,\infty}(D, \mathbb{C})$, $\varphi|_{\partial D} = 0$, see, e.g., [KU] ($d \geq 3$), [GT] ($d = 2$).

However, in the case of real-valued coefficients A, V there is no gauge non-uniqueness as it was observed, for example, in [AN].

In addition, in general case, under some regularity assumptions on $\partial D, A$ and V , the Dirichlet-to-Neumann map $\Lambda_{A,V}$ uniquely determines the two-form dA and the function q in D and the tangential component of A on ∂D , where

$$\begin{aligned} dA &= \sum_{1 \leq k < l \leq d} \left(\frac{\partial A^l}{\partial x_k} - \frac{\partial A^k}{\partial x_l} \right) dx_k \wedge dx_l, \\ q &= V + i \nabla \cdot A - A \cdot A, \end{aligned} \quad (5)$$

and $A = (A^1, \dots, A^d)$.

In particular, it was shown in [KU] that in dimension $d \geq 3$ the map $\Lambda_{A,V}$ uniquely determines dA and q in D if $A \in L^\infty(D, \mathbb{C}^d)$ and $V \in L^\infty(D, \mathbb{C})$. And in dimension $d = 2$ it was shown in [GT] that if D is a smooth Riemann surface with boundary (in particular, if D is a planar domain with $\partial D \in C^\infty$) then the map $\Lambda_{A,V}$ uniquely determines dA and q provided that $A \in W^{2,p}(D, \mathbb{R}^d)$, $V \in W^{1,p}(D, \mathbb{C})$, $p > 2$.

In addition, concerning the identifiability of tangential components of A on the boundary, it was proved in [BS] that if $\partial D \in C^1$ ($d \geq 3$) or $\partial D \in C^{1,\alpha}$, $\alpha \in (0, 1)$ ($d = 2$) and if $A \in C(\bar{D}, \mathbb{C}^d)$, $V \in L^\infty(D, \mathbb{C})$ then $\Lambda_{A,V}$ uniquely determines $A - \nu(A \cdot \nu)$ on ∂D , where ν is the unit exterior normal field to ∂D .

In the present article we combine the aforementioned results in order to obtain the following global uniqueness results in the case when coefficients A, V are real-valued.

Theorem 1. *Let D be a bounded simply connected domain with path connected boundary in \mathbb{R}^d ($d \geq 3$) with $\partial D \in C^1$. Let $A_1, A_2 \in W^{1,\infty}(D, \mathbb{R}^d)$ and $V_1, V_2 \in L^\infty(D, \mathbb{R})$. If $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$, then $A_1 = A_2, V_1 = V_2$.*

Theorem 2. *Let D be a bounded simply connected domain in \mathbb{R}^2 with $\partial D \in C^\infty$. Let $A_1, A_2 \in W^{2,p}(D, \mathbb{R}^d)$ and $V_1, V_2 \in W^{1,p}(D, \mathbb{R})$ with $p > 2$. If $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$ then $A_1 = A_2$ and $V_1 = V_2$.*

Theorems 1 and 2 are proved in Section 3. In Section 2 we present formulas and equations for finding A, V from $dA, q, A - \nu(A \cdot \nu)|_{\partial D}$.

2 Formulas and equations for finding A, V

In this section we suppose that D is a bounded contractible domain with path connected C^2 boundary in \mathbb{R}^d ($d \geq 2$). By contractibility we mean that there exists $F \in C^2(D \times [0, 1], D)$ such that $F_0 \equiv \tilde{x}, F_1 = \text{id}_D$, where $F_t(x) = F(x, t)$, \tilde{x} is some fixed point in D and id_D is the identity mapping on D . We also suppose that $A \in W^{2,\infty}(D, \mathbb{R}^d), V \in L^\infty(D, \mathbb{R})$. Given dA, q as in (5) in D and $A - \nu(A \cdot \nu)$ on ∂D , we can find A, V in the following way:

1. Define $\tilde{A} = (\tilde{A}^1, \dots, \tilde{A}^d) \in W^{1,\infty}(D, \mathbb{R}^d)$ by the formula

$$\tilde{A}^k = \sum_{i < j} \int_0^1 \left(\frac{\partial F_t^i}{\partial t} \frac{\partial F_t^j}{\partial x_k} - \frac{\partial F_t^j}{\partial t} \frac{\partial F_t^i}{\partial x_k} \right) \left(\frac{\partial A^j}{\partial x_i} \circ F_t - \frac{\partial A^i}{\partial x_j} \circ F_t \right) dt, \quad (6)$$

where $k = 1, \dots, d; A = (A^1, \dots, A^d), F_t = (F_t^1, \dots, F_t^d)$ and \circ denotes the composition of maps, i.e. $\frac{\partial A^j}{\partial x_i} \circ F_t(y) = \frac{\partial A^j}{\partial x_i}(F_t(y)), y \in D$.

2. Fix $x^0 \in \partial D$. Define $\varphi_0 \in C^1(\partial D)$ by the formula

$$\varphi_0(x) = \sum_{k=1}^d \int_{x^0}^x (A_\tau^k(y) - \tilde{A}_\tau^k(y)) dy_k, \quad x \in \partial D, \quad (7)$$

where $A_\tau = A - \nu(A \cdot \nu), \tilde{A}_\tau = \tilde{A} - \nu(\tilde{A} \cdot \nu), A_\tau = (A_\tau^1, \dots, A_\tau^d), \tilde{A}_\tau = (\tilde{A}_\tau^1, \dots, \tilde{A}_\tau^d), \nu$ is the unit exterior normal field to ∂D and integration is over an arbitrary C^1 curve on ∂D linking x^0 to x .

3. Find the unique generalized solution $\varphi \in W^{2,\infty}(D, \mathbb{R})$ to

$$\begin{cases} \Delta \varphi = \text{Im } q - \nabla \cdot \tilde{A} & \text{in } D, \\ \varphi|_{\partial D} = \varphi_0. \end{cases} \quad (8)$$

4. Coefficients A, V are given by the following formulas:

$$\begin{cases} A = \tilde{A} + \nabla \varphi, \\ V = q - i \Delta \varphi - i \nabla \cdot \tilde{A} + \tilde{A} \cdot \tilde{A} + 2\tilde{A} \cdot \nabla \varphi + (\nabla \varphi)^2. \end{cases}$$

This algorithm will be justified in Section 4.

3 Proofs of Theorems 1, 2

We will prove Theorems 1 and 2 simultaneously. Let D , A_1 , A_2 , V_1 , V_2 satisfy the conditions of Theorem 1 (resp. Theorem 2) and suppose that $\Lambda_{A_1, V_1} = \Lambda_{A_2, V_2}$.

Using Theorem 1.1 of [BS] we obtain that

$$(A_1 - \nu(A_1 \cdot \nu))|_{\partial D} = (A_2 - \nu(A_2 \cdot \nu))|_{\partial D}, \quad (9)$$

where ν is the unit exterior normal field to ∂D . Using Theorem 1.1 of [KU] (resp. Theorem 1.1 of [GT]) we get

$$dA_1 = dA_2 \text{ in } D, \quad (10)$$

$$q_1 = q_2 \text{ in } D, \quad (11)$$

where

$$dA_j = \sum_{1 \leq k < l \leq d} \left(\frac{\partial A_j^l}{\partial x_k} - \frac{\partial A_j^k}{\partial x_l} \right) dx_k \wedge dx_l,$$

$$q_j = V_j + i \nabla \cdot A_j - A_j \cdot A_j,$$

where $A_j = (A_j^1, \dots, A_j^d)$, $j = 1, 2$.

Since the domain D is simply connected it follows from (10) that there exists $\varphi \in W^{2,\infty}(D, \mathbb{R})$ such that

$$A_1 - A_2 = \nabla \varphi \text{ in } D. \quad (12)$$

In dimension $d = 2$ it follows from simple connectedness of D and from smoothness of ∂D that ∂D is path connected. Formulas (9), (12) and path connectedness of ∂D imply that φ is constant on ∂D .

Using (11), (12) we obtain that

$$V_1 - V_2 = -i \Delta \varphi - (\nabla \varphi)^2 + 2A_1 \nabla \varphi \text{ in } D.$$

Taking the imaginary part of this equation we obtain the equation $\Delta \varphi = 0$ in D . Since φ is constant on ∂D and $\varphi \in W^{2,\infty}(D)$ it follows that φ is constant in \overline{D} . Hence $A_1 = A_2$ and $V_1 = V_2$. Theorems 1 and 2 are proved.

4 Justification of the algorithm of Section 2

It follows from formula (6) that $d\tilde{A} = dA$. More precisely, if we denote by F^*dA the pullback of the form dA by the map F and by ι_{∂_t} we denote the interior product with the vector field $\frac{\partial}{\partial t}$ on $\{(x, t) \in D \times [0, 1]\}$, then

$$\sum_{k=1}^d \tilde{A}^k dx_k = \int_0^1 (\iota_{\partial_t} F^* dA) dt,$$

and the equality $d\tilde{A} = dA$ follows from the Cartan magic formula $\mathcal{L}_{\partial_t} = d \circ \iota_{\partial_t} + \iota_{\partial_t} \circ d$, where \mathcal{L}_{∂_t} is the Lie derivative along $\frac{\partial}{\partial t}$, d is the exterior derivative on $\{(x, t) \in D \times [0, 1]\}$ and \circ denotes the composition of maps.

Hence we can define $\varphi \in W^{2,\infty}(D, \mathbb{R})$ by the formula

$$\varphi(x) = \int_{\tilde{x}}^x \sum_{k=1}^d (A_k - \tilde{A}_k) dx_k, \quad x \in \bar{D},$$

where $\tilde{x} \in D$ is some fixed point and integration is over an arbitrary C^1 curve in D linking \tilde{x} to x . Then $\nabla\varphi = A - \tilde{A}$ in \bar{D} and this implies that $\varphi|_{\partial D}$ differs by constant from φ_0 defined in (7). We also obtain from (5) the equation

$$V = q - i\Delta\varphi - i\nabla \cdot \tilde{A} + \tilde{A} \cdot \tilde{A} + 2\tilde{A} \cdot \nabla\varphi + (\nabla\varphi)^2 \quad \text{in } D.$$

Taking into account that V is real-valued and separating the imaginary part in the latter equation we obtain (8). Since $\text{Im } q$ and $\nabla \cdot \tilde{A}$ belong to $L^\infty(D, \mathbb{R})$, the problem (8) is uniquely solvable for $\varphi \in W^{2,\infty}(D, \mathbb{R})$. Thus, the algorithm of Section 2 is justified.

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6 References

- [AN] A. D. Agaltsov, R. G. Novikov, Riemann-Hilbert problem approach for two-dimensional flow inverse scattering, *J. Math. Phys.* **55** (10), 2014, 103502
- [BS] R. M. Brown, M. Salo, Identifiability at the boundary for first-order terms, *Appl. Anal.* **85** (6–7), 2006, 735–749
- [GT] C. Guillarmou, L. Tzou, Identification of a connection from Cauchy data on a Riemann surface with boundary, *Geom. Funct. Anal.* **21** (2), 2011, 393–418
- [KU] K. Krupchyk, G. Uhlmann, Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential, *Comm. Math. Phys.* **327** (3), 2014, 993–1009
- [RW] D. Roussef, K. B. Winters, Two-dimensional vector flow inversion by diffraction tomography, *Inverse Problems* **10**, 1994, 687–697
- [RE] M. N. Rychagov, H. Ermert, Reconstruction of fluid motion in acoustic diffraction tomography, *J. Acoust. Soc. Am.* **99** (5), 1996, 3029–3035